Chapter 13
(A) Review of Basic Concepts (B) Feynman Path Integral Approach (C) Bell's Inequalities Revisited

Chapters 1–12 form what could be the basis for a one semester course in quantum mechanics. Before moving on to advanced topics such as time-independent perturbation theory, the variational method, the WKB approximation, and irreducible tensor operators, it can't hurt to review the basic concepts of quantum mechanics that I have covered up to this point. I also use this chapter to discuss an alternative formulation of quantum mechanics given by Feynman based on path-integrals. Finally I present a proof of Bell's theorem and see how it can be tested.

13.1 Review of Basic Concepts

13.1.1 Postulates

The postulates of quantum mechanics are:

1. The absolute square of the wave function $|\psi(\mathbf{r},t)|^2$ corresponds to the probability density of finding the particle at position $\mathbf{r}$ at time t.
2. To each physical observable in classical mechanics, there corresponds a Hermitian operator in quantum mechanics.
3. The time dependence of $\psi(\mathbf{r},t)$ is governed by the Schrödinger equation,

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=\hat{H}\psi(\mathbf{r},t) \tag{13.1}$$

   where $\hat{H}$ is the energy operator of the system.
4. The only possible outcome of a measurement on a *single* quantum system of a physical observable associated with a given Hermitian operator is one of the eigenvalues of the operator.

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5. The fifth postulate can take different forms. One way to state the postulate is that the Poisson brackets of two dynamic variables of classical mechanics are replaced by the $(i\hbar)^{-1}$ times the commutator of the corresponding Hermitian operators in quantum mechanics, the postulate I used when discussing Dirac notation. An alternative and equivalent postulate is that the Fourier transform of $\psi(\mathbf{r},t)$, denoted by $\Phi(\mathbf{p},t)$, corresponds to the wave function in momentum space, the postulate I used in the wave function approach.

The fundamental equation of quantum mechanics is the time-dependent Schrödinger equation,

$$\begin{aligned} i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} &= \hat{H}\psi(\mathbf{r},t) = \left[\frac{\hat{p}^2}{2m}+\hat{V}\right]\psi(\mathbf{r},t) \\ &= \left[\frac{-\hbar^2\nabla^2}{2m}+V(\mathbf{r})\right]\psi(\mathbf{r},t)\,; \end{aligned} \tag{13.2a}$$

$$\begin{aligned} i\hbar\frac{\partial\psi(x,t)}{\partial t} &= \hat{H}\psi(x,t) = \left[\frac{\hat{p}_x^2}{2m}+\hat{V}\right]\psi(x,t) \\ &= \left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t)\,. \end{aligned} \tag{13.2b}$$

That is, matter is described by a wave equation, in which $|\psi(\mathbf{r},t)|^2$ is given the interpretation of the probability density to find a particle at position $\mathbf{r}$ at time t. I list equations in both one and three dimensions.

13.1.2 Wave Function Approach

A solution of the time-dependent Schrödinger equation is

$$\psi(\mathbf{r},t) = e^{-iEt/\hbar}\psi(\mathbf{r})\,; \tag{13.3a}$$

$$\psi(x,t) = e^{-iEt/\hbar}\psi(x)\,, \tag{13.3b}$$

provided

$$\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r})\,; \tag{13.4a}$$

$$\hat{H}\psi(x) = E\psi(x)\,. \tag{13.4b}$$

Thus, if you solve the *time-independent* Schrödinger equation and find the eigenfunctions $\psi_E(\mathbf{r})$ $[\psi_E(x)]$ and the eigenvalues E, you have a complete solution to the problem. In other words,

$$\psi(\mathbf{r},t) = \sum_E b_E e^{-iEt/\hbar} \psi_E(\mathbf{r}); \tag{13.5a}$$

$$\psi(x,t) = \sum_E b_E e^{-iEt/\hbar} \psi_E(x) \tag{13.5b}$$

is a general solution of the time-dependent Schrödinger equation, provided Eqs. (13.4) hold. I can equally write this as

$$\psi(\mathbf{r},t) = \sum_E b_E(t) \psi_E(\mathbf{r}); \tag{13.6a}$$

$$\psi(x,t) = \sum_E b_E(t) \psi_E(x), \tag{13.6b}$$

with

$$b_E(t) = e^{-iEt/\hbar} b_E. \tag{13.7}$$

In this form it is clear that the probability to be in an eigenstate,

$$P_E = |b_E(t)|^2 = |b_E|^2, \tag{13.8}$$

is independent of time. The quantity $b_E(t)$ is referred to as the *state amplitude*.

Although the probability to be in a given eigenstate is constant in time, the wave function squared or probability density evolves in time, since

$$|\psi(\mathbf{r},t)|^2 = \sum_{E,E'} b_E^* b_{E'} \psi_E^*(\mathbf{r}) \psi_{E'}(\mathbf{r}) e^{i(E-E')t/\hbar}; \tag{13.9a}$$

$$|\psi(x,t)|^2 = \sum_{E,E'} b_E^* b_{E'} \psi_E^*(x) \psi_{E'}(x) e^{i(E-E')t/\hbar}, \tag{13.9b}$$

is a function of time, in general. The relative phases of the state amplitudes give rise to the time dependence.

To solve a dynamic problem in which you are given $\psi(\mathbf{r},0)$ [$\psi(x,0)$], you first find the eigenfunctions and eigenenergies of the Hamiltonian. Then you set

$$\psi(\mathbf{r},0) = \sum_E b_E \psi_E(\mathbf{r}); \tag{13.10a}$$

$$\psi(x,0) = \sum_E b_E \psi_E(x), \tag{13.10b}$$

which lets you calculate

$$b_E = (\psi_E, \psi(\mathbf{r},0)) = \int d\mathbf{r} \psi_E^*(\mathbf{r}) \psi(\mathbf{r},0); \tag{13.11a}$$

$$b_E = (\psi_E, \psi(x,0)) = \int dx \psi_E^*(x) \psi(x,0) \tag{13.11b}$$

and

$$\psi(\mathbf{r}, t) = \sum_E b_E e^{-iEt/\hbar} \psi_E(\mathbf{r}); \tag{13.12a}$$

$$\psi(x, t) = \sum_E b_E e^{-iEt/\hbar} \psi_E(x). \tag{13.12b}$$

Example 1 A particle having mass m moves in an infinite square well potential located between $x = 0$ and $x = L$. At $t = 0$,

$$\psi(x, 0) = \begin{cases} \sqrt{\frac{1}{a}}; & \frac{L}{2} - \frac{a}{2} \le x \le \frac{L}{2} + \frac{a}{2} \\ 0; & \text{otherwise} \end{cases}, \tag{13.13}$$

with $a < L$. Find $\psi(x, t)$.

The eigenfunctions and eigenvalues are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right); \qquad n = 1, 2, 3, \ldots \tag{13.14a}$$

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}. \tag{13.14b}$$

Therefore

$$\begin{aligned} b_n &= \int_0^L dx \psi_n^*(x) \psi(x, 0) = \sqrt{\frac{1}{a}} \sqrt{\frac{2}{L}} \int_{\frac{L}{2}-\frac{a}{2}}^{\frac{L}{2}+\frac{a}{2}} dx \sin\left(\frac{n\pi x}{L}\right) \\ &= -\sqrt{\frac{2L}{a}} \frac{1}{n\pi} \left\{\cos\left[\frac{n\pi}{2}\left(1 + \frac{a}{L}\right)\right] - \cos\left[\frac{n\pi}{2}\left(1 - \frac{a}{L}\right)\right]\right\} \end{aligned} \tag{13.15}$$

and

$$\psi(x, t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} b_n \exp\left(-i\frac{\hbar^2 n^2 \pi^2}{2mL^2} t\right) \sin\left(\frac{n\pi x}{L}\right). \tag{13.16}$$

Example 2 Sometimes you can read off the expansion coefficients by inspection. Imagine in the previous problem that

$$\psi(x, 0) = \frac{1}{\sqrt{2}} \sqrt{\frac{2}{L}} \left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]. \tag{13.17}$$

Clearly $b_1 = b_2 = \frac{1}{\sqrt{2}}$ and all other b_n are zero, such that

$$\psi(x, t) = \frac{1}{\sqrt{2}} \sqrt{\frac{2}{L}} \begin{bmatrix} \exp\left(-i\frac{\hbar^2\pi^2}{2mL^2} t\right) \sin\left(\frac{\pi x}{L}\right) \\ + \exp\left(-i\frac{4\hbar^2\pi^2}{2mL^2} t\right) \sin\left(\frac{2\pi x}{L}\right) \end{bmatrix}. \tag{13.18}$$

Note that $|\psi(x,t)|^2$ is a function of time,

$$|\psi(x,t)|^2 = \frac{1}{L}\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{3\hbar^2\pi^2}{2mL^2}t\right)\right]. \tag{13.19}$$

13.1.3 Quantum-Mechanical Current Density

The total probability is conserved for a single-particle quantum system. The probability current density,

$$\mathbf{J}(\mathbf{r},t) = \frac{i\hbar}{2m}\left[\psi(\mathbf{r},t)\nabla\psi^*(\mathbf{r},t) - \psi^*(\mathbf{r},t)\nabla\psi(\mathbf{r},t)\right]; \tag{13.20a}$$

$$J_x(x,t) = \frac{i\hbar}{2m}\left[\psi(x,t)\frac{\partial\psi^*(x,t)}{\partial x} - \psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x}\right], \tag{13.20b}$$

satisfies an equation of continuity

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla\cdot\boldsymbol{J}(\mathbf{r},t) = 0; \tag{13.21a}$$

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial J_x(x,t)}{\partial x} = 0, \tag{13.21b}$$

where

$$\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2; \tag{13.22a}$$

$$\rho(x,t) = |\psi(x,t)|^2 \tag{13.22b}$$

is the probability density.

13.1.4 Eigenfunctions and Eigenenergies

It is important to recognize that *each potential energy function gives rise to its own set of eigenfunctions*. Thus, the eigenfunctions of a free particle are infinite, plane mono-energetic waves, while those of any bound state problem are localized in space. But what about the unbound state eigenfunctions of the hydrogen atom? They are not localized states, nor are they plane wave states. Instead they must be found by solving the Schrödinger equation for positive energies. Similarly, for the problem of a one-dimensional well and particle energies $E > 0$, the solutions of Shcrödinger's equation are sines and cosines in each region, but the eigenfunction is not a single plane wave state. In three-dimensional problems with spherical symmetry, there is a separate effective potential problem to solve for each value of ℓ.

13.1.5 Operators

The expectation value of an operator for a quantum system is defined by

$$\left\langle \hat{A} \right\rangle = \int d\mathbf{r}\, \psi^* (\mathbf{r}, t)\, \hat{A} \psi (\mathbf{r}, t) \,. \tag{13.23}$$

For operators with no explicit time dependence,

$$i\hbar \frac{\partial \left\langle \hat{A} \right\rangle}{\partial t} = \left\langle \left[\hat{A}, \hat{H} \right] \right\rangle \tag{13.24}$$

where $\hat{H}$ is the Hamiltonian operator. The time dependence in $\left\langle \hat{A} \right\rangle$ arises from the time dependence in $\psi (\mathbf{r}, t)$. If $\left[\hat{A}, \hat{H} \right] = 0$, this implies that the dynamic variable associated with $\hat{A}$ is a constant of the motion. For an operator that doesn't commute with the Hamiltonian, its expectation value is not constant, in general.

The expectation values obey the laws of classical physics. This is known as *Ehrenfest's theorem*. That is

$$\frac{d \langle \hat{\mathbf{r}} \rangle}{dt} = \frac{1}{i\hbar} \left\langle \left[\hat{\mathbf{r}}, \hat{H} \right] \right\rangle = \frac{1}{i\hbar} \left\langle \left[\hat{\mathbf{r}}, \frac{\hat{p}^2}{2m} \right] \right\rangle = \frac{\langle \hat{\mathbf{p}} \rangle}{m}; \tag{13.25a}$$

$$\begin{aligned} \frac{d \langle \hat{\mathbf{p}} \rangle}{dt} &= \frac{1}{i\hbar} \left\langle \left[\hat{\mathbf{p}}, \hat{H} \right] \right\rangle = \frac{1}{i\hbar} \langle [\hat{\mathbf{p}}, V(\mathbf{r})] \rangle \\ &= - \langle \nabla V(\mathbf{r}) \rangle = \left\langle \hat{\mathbf{F}} \right\rangle, \end{aligned} \tag{13.25b}$$

where $\hat{\mathbf{F}} = - \nabla V(\mathbf{r})$ can be thought of the operator associated with the force acting on the particle.

13.1.5.1 Commuting Operators: Simultaneous Eigenfunctions

Two Hermitian operators possess simultaneous eigenfunctions if and only if they commute. Moreover, any operator that commutes with the Hamiltonian is a constant of the motion. Constants of the motion can be used to label eigenfunctions of the Hamiltonian. In general, whenever there is a dynamic constant of the motion such as momentum or angular momentum, there is an energy degeneracy associated with that conserved quantity. In that case the eigenvalues of the conserved operators can be used to label different eigenfunctions that have the same energy. For example, for the free particle, there is infinite degeneracy for any non zero energy since the direction of the momentum can be in any direction. However, by using simultaneous eigenfunctions of both the energy and momentum (momentum is conserved), you can uniquely label each energy eigenfunction. Similarly, in problems with spherical symmetry, angular momentum is conserved and you can label states by the eigenvalues associated with the commuting operators $\hat{L}^2$, $\hat{L}_z$, and $\hat{H}$. There is always a $(2\ell + 1)$ energy degeneracy associated with each state of a given ℓ. In general, the energy in problems with spherical symmetry depends on both the ℓ quantum number and some additional quantum number—only in problems with some extra symmetry such as hydrogen and the 3-d harmonic oscillator does the energy depend only on a single quantum number n. For example, in an infinite spherical potential well, for each ℓ, there is a set of possible energies for each ℓ that can be labeled by the zeroes of the spherical Bessel functions. In that case, the energy is determined by both ℓ and a quantum number n for which $j_\ell(k_n a) = 0$, where $n = 1, 2, 3, \ldots$ labels the zeroes of the Bessel function.

13.1.6 Measurement in Quantum Mechanics

What truly sets quantum mechanics apart from classical physics is the existence of a single quantum system. There is no classical analogue of a single quantum system in a superposition state. In contrast to a closed classical system of particles and fields for which the energy is constant, it is impossible to assign a unique energy to a single quantum system that is in a superposition state of two or more energies. A measurement of the energy will yield one of the energies in the superposition state, but we don't know which one. If you measure the dynamic variable associated with a Hermitian operator for any single quantum system in a superposition state of eigenfunctions of that operator, you get one and only one eigenvalue of that operator. Large numbers of measurements on identically prepared quantum systems are needed to map out the probability function for each of the eigenvalues.

13.1.7 Dirac Notation

Although the wave function usually gives us some information about the spatial probability distribution, it is possible to formulate quantum mechanics in a somewhat more abstract formalism using Dirac notation. In Dirac notation each Hermitian operator has its own set of eigenkets. These kets can be represented as column vectors with a 1 in one position and zeroes everywhere else. Each operator is diagonal in its own basis with the diagonal elements simply the eigenvalues of the operator. In some cases, as for the harmonic oscillator or for angular momentum, it is possible to define ladder operators and obtain the eigenenergies without solving Schrödinger's equation. However in most cases, this is impossible and one reverts to solving eigenvalue equations using the wave function formalism. The real import of Dirac notation is that different representations, such as momentum and coordinate, appear on an equal footing.

By calculating matrix elements of the momentum operator in the coordinate representation I could establish a connection between the wave function formalism and Dirac notation. To calculate matrix elements of the momentum operator in the coordinate representation, I used the fifth postulate and the Poisson bracket of x and p_x to obtain the commutation relation between $\hat{x}$ and $\hat{p}_x$ and used the commutator to show that

$$\langle \mathbf{r}| \hat{\mathbf{p}} \left|\mathbf{r}'\right\rangle = \frac{\hbar}{i}\nabla_r \delta(\mathbf{r}-\mathbf{r}'). \tag{13.26}$$

As a consequence, matrix elements of the entire Hamiltonian could be evaluated in the coordinate basis. The Hamiltonian is not diagonal in the coordinate representation; however when the energy eigenkets are expanded in terms of the coordinate representation, the expansion coefficients turn out to be the wave function in coordinate space,

$$\psi_E(\mathbf{r}) = \langle \mathbf{r} \,| E\rangle \,. \tag{13.27}$$

I will often use Dirac notation. Remember, however, that one must often revert to the wave function formalism to carry out any calculations. The time-dependent Schrödinger equation can be written in Dirac notation as

$$i\hbar\frac{d\,|\psi\,(t)\rangle}{dt} = \hat{H}\,|\psi\,(t)\rangle = \left[\frac{\hat{p}^2}{2m} + \hat{V}\right]|\psi\,(t)\rangle \,. \tag{13.28}$$

The most general solution is

$$|\psi\,(t)\rangle = \sum_n b_n e^{-iE_nt/\hbar}\,|E_n\rangle \,, \tag{13.29}$$

provided

$$\hat{H}\,|E_n\rangle = E_n\,|E_n\rangle \,. \tag{13.30}$$

Remember that the eigenkets are just column vectors with a 1 in one position and zeroes everywhere else and that $|\psi\,(t)\rangle$ is a column vector with $b_n e^{-iE_nt/\hbar}$ in the nth position.

In contrast to the wave function approach, if you are given $|\psi\,(0)\rangle$ in terms of the eigenkets,

$$|\psi\,(0)\rangle = \sum_n b_n\,|E_n\rangle \,, \tag{13.31}$$

you already know all the b_n's. If you were given the initial wave function, you must calculate the b_n's as before. In other words,

$$b_n = \langle E_n | \psi(0) \rangle = \int d\mathbf{r} \langle E_n | \mathbf{r} \rangle \langle \mathbf{r} | \psi(0) \rangle = \int d\mathbf{r} \psi^*_{E_n}(\mathbf{r}) \psi(\mathbf{r}, 0) . \tag{13.32}$$

Example: As an example of the use of Dirac notation, consider a hydrogen atom (neglecting spin) in an external magnetic field. The Hamiltonian for the electron is

$$\hat{H} = \hat{H}_0 + \beta_0 \mathbf{B} \cdot \hat{\mathbf{L}}/\hbar, \tag{13.33}$$

where $\hat{H}_0$ is the Hamiltonian in the absence of the magnetic field and β_0 is the Bohr magneton. As long as the field is along the z axis, it is simple to find the eigenfunctions and eigenenergies since the Y_ℓ^m's are still eigenfunctions. The energies are simply shifted by $\beta_0 mB$, where m is the magnetic quantum number. However if the field is not along the z axis, it is not at all obvious how to solve for the eigenfunctions.

In Dirac notation, it is trivial to obtain the eigenkets. The eigenkets are simply $|n\ell m_B\rangle$, where $m_B\hbar$ is the eigenvalue associated with the component of $\mathbf{L}$ along the direction of the magnetic field. The eigenenergies are shifted by $\beta_0 m_B B$. Of course, the eigenfunctions are still not simple to obtain since $\langle \hat{\mathbf{r}} | \ell m_B \rangle$ is not a spherical harmonic. However, if we express the Hamiltonian in terms of the (complete) set of eigenkets $|n\ell m_z\rangle$ which is easy to do, then the eigenkets in the $|n\ell m_B\rangle$ basis can be obtained by diagonalizing the Hamiltonian, just as was done in Sect. 11.4.3 for the operator $\hat{L}_x$. The eigenfunctions are obtained as linear combinations of the spherical harmonics having the same ℓ, but different m.

13.1.8 Heisenberg Representation

As a second example of the use of Dirac notation, I would like to discuss the *Heisenberg representation* or *Heisenberg picture.* The Schrödinger operators I have been using are time-independent. The expectation values of these operators are time-dependent, in general, owing to the time-dependence of the wave functions or state vectors associated with a quantum system. In the Heisenberg representation, these roles are reversed. The state vector of the system becomes constant in time, whereas the Heisenberg operators are functions of time, in general.

There is a unitary operator that connects the two representations. For a quantum system characterized by a Hamiltonian $\hat{H}$, I define the state vector in the Heisenberg representation by

$$|\psi\rangle^H = e^{\frac{i}{\hbar}\hat{H}t} |\psi(t)\rangle = e^{\frac{i}{\hbar}\hat{H}t} e^{-\frac{i}{\hbar}\hat{H}t} |\psi(0)\rangle = |\psi(0)\rangle , \tag{13.34}$$

which is constant in time. The expectation value of an operator $\hat{O}$ is

$$\begin{aligned}\langle\hat{O}\rangle &= \langle\psi(t)|\,\hat{O}|\psi(t)\rangle = \langle\psi(0)|\,e^{\frac{i}{\hbar}\hat{H}t}\hat{O}e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle \\ &= \langle\psi(0)|\,e^{\frac{i}{\hbar}\hat{H}t}\hat{O}e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle =^{H} \langle\psi|\,\hat{O}^{H}(t)|\psi\rangle^{H},\end{aligned} \tag{13.35}$$

where

$$\hat{O}^{H}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{O}e^{-\frac{i}{\hbar}\hat{H}t}. \tag{13.36}$$

The Heisenberg operator $\hat{O}^{H}(t)$ is time-dependent, in general and obeys the equation of motion

$$i\hbar\frac{d\hat{O}^{H}(t)}{dt} = -e^{\frac{i}{\hbar}\hat{H}t}\hat{H}\hat{O}e^{-\frac{i}{\hbar}\hat{H}t} + e^{\frac{i}{\hbar}\hat{H}t}\hat{O}\hat{H}e^{-\frac{i}{\hbar}\hat{H}t} = [\hat{O}^{H}(t),\hat{H}]. \tag{13.37}$$

The advantage of the Heisenberg picture is that the operators often obey equations of motion that are identical to their classical counterparts. Thus, Ehrenfest's theorem holds directly for the Heisenberg operators. It is important to note that commutation laws such as $[\hat{x}^{H}(t),\hat{p}^{H}(t)] = i\hbar$ remain valid in the Heisenberg picture, but that $[\hat{x}^{H}(t),\hat{p}^{H}(t')] \neq i\hbar$, in general, if $t \neq t'$. The time-independent operator $\hat{H}$ is the same in both the Schrödinger and Heisenberg representations.

13.2 Feynman Path-Integral Approach

In 1948 Richard Feynman published a paper in *Reviews of Modern Physics* entitled *Space-Time Approach to Non-Relativistic Quantum Mechanics*.[1] In this paper, Feynman formulated an alternative theory of quantum mechanics based on *propagators*, rather than the Schrödinger equation. As Feynman noted in his introduction, "The formulation is equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view." It turns out that the propagator or *path-integral* approach is now used routinely in both non-relativistic and relativistic quantum mechanics.[2] The approach is related to the WKB method that is discussed in Chap. 16 in that both effectively involve what is called a *stationary phase approximation*.

Feynman's approach is based on the classical *action*. The classical action S for a particle having mass m moving in a potential $V(\mathbf{r})$ from position $\mathbf{r}_a$ at time t_a to position $\mathbf{r}_b$ at time t_b is defined by

[1] R. P. Feynman, Reviews of Modern Physics, Vol. 20, pp. 367–387 (1948).

[2] A general discussion of the path-integral approach can be found in L. S. Schulman, *Techniques and Applications of Path Integration* (Dover Publications, Mineola, N.Y., 2005).

$$S(\mathbf{r}_a, t_a; \mathbf{r}_b, t_b) = \int_{t_a}^{t_b} L(\mathbf{r}, \dot{\mathbf{r}})\, dt, \tag{13.38}$$

where

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2} - V(\mathbf{r}), \tag{13.39}$$

is the associated Lagrangian. The position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ appearing in these equations are implicit functions of time. The *Principle of Least Action* states that of all the possible classical paths that take the particle from position $\mathbf{r}_a$ at time t_a to position $\mathbf{r}_b$ at time t_b, the path that makes S an extremum corresponds to the actual dynamics of the particle. It is not overly difficult to show that the requirement $\delta S = 0$, where δ corresponds to a variational derivative over particle trajectories having fixed endpoints $(\mathbf{r}_a, t_a)$ and $(\mathbf{r}_b, t_b)$, leads to Lagrange's equation for the particle.

In analogy with this concept, Feynman postulated that the wave function in quantum mechanics evolves as

$$\psi(\mathbf{r}, t) = \int_{-\infty}^{\infty} K(\mathbf{r}, t; \mathbf{r}', t')\psi(\mathbf{r}', t')d\mathbf{r}', \tag{13.40}$$

where the *propagator* is $K(\mathbf{r}_b, t_b; \mathbf{r}_a, t_a)$ is *assumed* to be given by

$$\begin{aligned} K(b, a) \equiv K(\mathbf{r}_b, t_b; \mathbf{r}_a, t_a) &= \frac{1}{N}\int_{\mathbf{r}_a}^{\mathbf{r}_b} \exp\left(\frac{i}{\hbar}S(\mathbf{r}_a, t_a; \mathbf{r}_b, t_b)\right)\mathcal{D}\mathbf{r}(t) \\ &\equiv \frac{1}{N}\int_{\mathbf{r}_a}^{\mathbf{r}_b} \exp\left(\frac{i}{\hbar}S(b, a)\right)\mathcal{D}\mathbf{r}(t), \end{aligned} \tag{13.41}$$

where N is a normalization constant. The *functional differential* $\mathcal{D}\mathbf{r}(t)$ is *not* an integral over position; it defines an operation in which the integrand is summed over all *classical* paths from $(\mathbf{r}_a, t_a)$ to $(\mathbf{r}_b, t_b)$.

Of course, you can postulate anything you want. Only if the postulates are consistent with experiment can they form the basis for a useful theory. In Feynman's case, he showed (by an argument reproduced in the Appendix) that his formulation was totally equivalent to that described by Schrödinger's equation. Moreover his formalism can offer some computational advantages for calculations in both non-relativistic and relativistic quantum mechanics. I will show how the propagator can be calculated and then evaluate it for a free particle.

The Feynman approach is similar to the Principle of Least Action, but there is a fundamental difference. In the Principle of Least Action, one path is picked out by demanding that the action is an extremum. On the other hand, the propagator in Eq. (13.41) involves a sum over *all* classical paths from $(\mathbf{r}_a, t_a)$ to $(\mathbf{r}_b, t_b)$. In quantum mechanics it is impossible to deterministically define the path of a particle. However, owing to the fact that $\hbar$ appears in the denominator of the phase in the exponent in Eq. (13.41), the phase is expected to be large. In that case, the major contribution to the integral will be along the path for which the phase is an extremum (path of stationary phase), a condition on the action similar to that encountered in the classical case. For all other paths, the exponential factor oscillates rapidly and leads to destructive interference when the integral in Eq. (13.41) is evaluated.

In order to evaluate the propagator, the time integral can be broken down into multiple steps, with the number of steps eventually approaching infinity. From the definition given in Eq. (13.38), it is clear that

$$S(b,a) = S(c,a) + S(b,c). \tag{13.42}$$

Since $S(b,a)$ is the *classical* action, it contains no quantum-mechanical operators. As a consequence

$$\begin{aligned}\exp\left(\frac{i}{\hbar}S(b,a)\right) &= \exp\left(\frac{i}{\hbar}\left[S(c,a)+S(b,c)\right]\right)\\ &= \exp\left(\frac{i}{\hbar}S(b,c)\right)\exp\left(\frac{i}{\hbar}S(c,a)\right).\end{aligned} \tag{13.43}$$

The integral in Eq. (13.41) can be written as an integral from the $\mathbf{r}_a$ to $\mathbf{r}_c$ multiplied by an integral from $\mathbf{r}_c$ to $\mathbf{r}_b$. To allow for all classical paths, I must integrate over all possible intermediate locations $\mathbf{r}_c$; that is,

$$\begin{aligned}K(b,a) &= \frac{1}{N}\int_{\mathbf{r}_a}^{\mathbf{r}_b}\exp\left(\frac{i}{\hbar}S(b,a)\right)\mathcal{D}\mathbf{r}(t)\\ &= \frac{1}{N}\int d\mathbf{r}_c\int_{\mathbf{r}_c}^{\mathbf{r}_b}\mathcal{D}\mathbf{r}_2(t)\int_{\mathbf{r}_a}^{\mathbf{r}_c}\mathcal{D}\mathbf{r}_1(t)\exp\left(\frac{i}{\hbar}S(b,c)\right)\exp\left(\frac{i}{\hbar}S(c,a)\right)\\ &= \frac{1}{N}\int d\mathbf{r}_c K(b,c)K(c,a).\end{aligned} \tag{13.44}$$

The integral is over all space.

I can now continue this process, dividing the interval from $\mathbf{r}_a$ to $\mathbf{r}_b$ into n sections, each having a temporal duration ϵ. At the end of the calculation a limit is taken in which $n \to \infty$ and $\epsilon \to 0$, with the product $n\epsilon \to (t_b - t_a)$. Since each time interval interval becomes infinitely small, I can use Eqs. (13.38) and (13.41) to estimate the propagator in the interval from $(\mathbf{r}_i, t_i)$ to $(\mathbf{r}_{i+1}, t_{i+1})$ as

$$\begin{aligned}K(i+1,i) &= \frac{1}{N}\int_{\mathbf{r}_i}^{\mathbf{r}_{i+1}}\exp\left(\frac{i}{\hbar}\int_{t_i}^{t_{i+1}}L\left(\mathbf{r},\dot{\mathbf{r}}\right)dt\right)\mathcal{D}\mathbf{r}(t)\\ &\approx \frac{1}{N}\exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{\mathbf{r}_i+\mathbf{r}_{i+1}}{2},\frac{\mathbf{r}_{i+1}-\mathbf{r}_i}{\epsilon}\right)\right],\end{aligned} \tag{13.45}$$

since the classical path from $(\mathbf{r}_i, t_i)$ to $(\mathbf{r}_{i+1}, t_{i+1})$ reduces to an infinitesimal interval for which $\mathbf{r}\left(t_j\right) \approx (\mathbf{r}_i + \mathbf{r}_{i+1})/2$ and $\dot{\mathbf{r}}\left(t_j\right) = \mathbf{v}\left(t_j\right) \approx (\mathbf{r}_{i+1} - \mathbf{r}_i)/\epsilon$. It then follows that the total propagator is equal to

$$K(b,a) = \lim_{\substack{\epsilon\to 0 \\ n\to\infty \\ n\epsilon\to(t_b-t_a)}} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{3n}{2}} \int d\mathbf{r}_1 \ldots d\mathbf{r}_{n-1}$$

$$\times \prod_{j=0}^{n-1} \exp\left[\frac{i\epsilon}{\hbar} L\left(\frac{\mathbf{r}_j + \mathbf{r}_{j+1}}{2}, \frac{\mathbf{r}_{j+1} - \mathbf{r}_j}{\epsilon}\right)\right], \tag{13.46}$$

where $\mathbf{r}_0 = \mathbf{r}_a$, $\mathbf{r}_n = \mathbf{r}_b$. The normalization factor that was used,

$$N = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}, \tag{13.47}$$

is derived in the Appendix.

As an example, I recalculate the one-dimensional free particle propagator already obtained in Eq. (3.48). The one-dimensional propagator is given by

$$K(b,a) = \lim_{\substack{\epsilon\to 0 \\ n\to\infty \\ n\epsilon\to(t_b-t_a)}} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{n}{2}} \int_{-\infty}^{\infty} dx_1 \ldots dx_{n-1}$$

$$\times \prod_{j=0}^{n-1} \exp\left[\frac{i\epsilon}{\hbar} L\left(\frac{x_j + x_{j+1}}{2}, \frac{x_{j+1} - x_j}{\epsilon}\right)\right], \tag{13.48}$$

where the free-particle Lagrangian for a particle having mass m is

$$L\left(\frac{x_j + x_{j+1}}{2}, \frac{x_{j+1} - x_j}{\epsilon}\right) = m\frac{v_j^2}{2} = m\frac{\left(x_{j+1} - x_j\right)^2}{2\epsilon^2}. \tag{13.49}$$

Using the fact that

$$\int_{-\infty}^{\infty} dx_j \exp\left[\frac{im}{2\hbar\epsilon}\left(x_j - x_{j-1}\right)^2\right] \exp\left[\frac{im}{2(n-j)\hbar\epsilon}\left(x_n - x_j\right)^2\right]$$

$$= \left(\frac{2\pi i\hbar\epsilon}{m}\frac{n-j}{n-j+1}\right)^{1/2} \exp\left[\frac{im}{2\hbar\epsilon\,(n-j+1)}\left(x_n - x_{j-1}\right)^2\right], \tag{13.50}$$

you can show by successive integrations starting with the integral over x_{n-1} that

$$K(b,a) = \lim_{\substack{\epsilon\to 0\\ n\to\infty\\ n\epsilon\to(t_b-t_a)}} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{n}{2}} \prod_{j=1}^{n-1}\left(\frac{2\pi i\hbar\epsilon}{m}\frac{n-j}{n-j+1}\right)^{1/2} \times \exp\left[\frac{im}{2n\epsilon\hbar}(x_n - x_0)^2\right]. \tag{13.51}$$

Since $x_0 = x_a$, $x_n = x_b$, and

$$\prod_{j=1}^{n-1}\left(\frac{n-j}{n-j+1}\right)^{1/2} = \sqrt{\frac{1}{n}}, \tag{13.52}$$

it then follows that

$$K(b,a) = \left(\frac{m}{2\pi i\hbar\,(t_b - t_a)}\right)^{1/2} \exp\left[\frac{im}{2\hbar}\frac{(x_b - x_a)^2}{(t_b - t_a)}\right], \tag{13.53}$$

which agrees with Eq. (3.48).

13.3 Bell's Theorem

13.3.1 Proof of Bell's Theorem

In Chap. 1, I promised a more detailed discussion of Bell's inequalities, once electron spin was introduced. I now make good on that promise. The proof of Bell's Theorem has nothing to do directly with quantum mechanics. It is based solely on measurements on correlated systems. I shall refer to each system as a "particle," but this need not be the case. I assume that we have two particles (A and B) and two detectors (1 and 2). The same property of each particle is measured at each detector, particle A is measured at detector 1 and particle B at detector 2. The measurements are assumed to be correlated and the correlation is assumed to occur as the result of some *hidden variable* encoded in the particles. The detectors are separated by a large distance, insofar as the measurement of one of the particles at one of the detectors cannot influence the measurement of the other particle at the other detector (this assumption implies that only *local hidden variables,* created in the particles at their creation, are used to explain the correlations between the particles).

I now assume that each detector can result only in a measurement of ± 1 when measuring the property of the particle. The value that is measured will depend on the angular orientation of the detector. Moreover, I assume that the measurements are perfectly anti-correlated for the same orientation of the detectors. That is, if detector 1 measures a value $+1$ for an orientation α of detector 1, then detector 2 must measure a value of -1 for the same orientation α of detector 2. This correlation is assumed to occur as a result of a hidden variable λ.

The proof of Bell's Theorem then follows directly.[3] Let me denote the orientation angle α of detector 1 by $\theta_{1\alpha}$ and the orientation angle β of detector 2 by $\theta_{2\beta}$. Then, according to my assumptions, the only possible measurements are

$$A(\theta_{1\alpha},\lambda) = \pm 1; \qquad B(\theta_{2\beta},\lambda) = \pm 1, \tag{13.54}$$

with

$$A(\theta_{1\alpha},\lambda) = -B(\theta_{2\alpha},\lambda). \tag{13.55}$$

It is assumed that the measurements are controlled by a hidden variable λ, with $0 \le \lambda \le 1$, governed by some distribution $\rho(\lambda)$ with

$$\int_0^1 \rho(\lambda)\, d\lambda = 1. \tag{13.56}$$

I now define $E\left(\theta_{1\alpha},\theta_{2\beta}\right)$ as the expectation value of the product of the two measurements. (It is important to recognize that $E\left(\theta_{1\alpha},\theta_{2\beta}\right)$ is *not* a probability—it can be negative.) Then

$$\begin{aligned} E\left(\theta_{1\alpha},\theta_{2\beta}\right) &= \int_0^1 d\lambda\, \rho(\lambda)\, A(\theta_{1\alpha},\lambda)B(\theta_{2\beta},\lambda) \\ &= -\int_0^1 d\lambda\, \rho(\lambda)\, A(\theta_{1\alpha},\lambda)A(\theta_{1\beta},\lambda), \end{aligned} \tag{13.57}$$

where the second line is obtained using Eq. (13.55). As a consequence, it follows that[4]

$$\begin{aligned} &E\left(\theta_{1\alpha},\theta_{2\beta}\right) - E\left(\theta_{1\alpha},\theta_{2\gamma}\right) \\ &\quad = -\int_0^1 d\lambda\, \rho(\lambda)\left[A(\theta_{1\alpha},\lambda)A(\theta_{1\beta},\lambda) - A(\theta_{1\alpha},\lambda)A(\theta_{1\gamma},\lambda)\right]. \end{aligned} \tag{13.58}$$

Using the fact that $\left[A(\theta_{1\beta},\lambda)\right]^2 = 1$, I can rewrite this equation as

$$\begin{aligned} &E\left(\theta_{1\alpha},\theta_{2\beta}\right) - E\left(\theta_{1\alpha},\theta_{2\gamma}\right) \\ &\quad = \int_0^1 d\lambda\, \rho(\lambda)\, A(\theta_{1\alpha},\lambda)A(\theta_{1\beta},\lambda)\left[A(\theta_{1\beta},\lambda)A(\theta_{1\gamma},\lambda) - 1\right]. \end{aligned} \tag{13.59}$$

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[3] John Bell, *On the Einstein Podolsky Rosen Paradox*, Physics **1,** 195–200 (1964).

[4] Some authors object to Bell using the same value of λ for different measurements. See, for example, Karl Hess, *Einstein Was Right!* (CRC Press, Boca Raton, FL, 2015).

Since the product $\left|A(\theta_{1\alpha},\lambda)A(\theta_{1\beta},\lambda)\right| \leq 1$ and $1 - A(\theta_{1\beta},\lambda)A(\theta_{1\gamma},\lambda) \geq 0$, I know that

$$\left|\int_0^1 d\lambda\, \rho(\lambda)\, A(\theta_{1\alpha},\lambda)A(\theta_{1\beta},\lambda)\left[A(\theta_{1\beta},\lambda)A(\theta_{1\gamma},\lambda) - 1\right]\right| \\ \leq \int_0^1 d\lambda\, \rho(\lambda)\left[1 - A(\theta_{1\beta},\lambda)A(\theta_{1\gamma},\lambda)\right] = 1 + E\left(\theta_{1\beta},\theta_{2\gamma}\right), \quad (13.60)$$

which, when combined with Eq. (13.59), yields

$$\left|E\left(\theta_{1\alpha},\theta_{2\beta}\right) - E\left(\theta_{1\alpha},\theta_{2\gamma}\right)\right| - E\left(\theta_{1\beta},\theta_{2\gamma}\right) \leq 1. \quad (13.61)$$

Equation (13.61) is a statement of Bell's Theorem. If measurements on any perfectly anti-correlated system of two particles violate this inequality, then the correlations cannot be attributed to a local hidden variable.

Often a modified form of Bell's inequalities is used,[5]

$$\left|E(\theta_{1\alpha},\theta_{2\beta}) - E(\theta_{1\alpha},\theta_{2\beta'}) + E(\theta_{1\alpha'},\theta_{2\beta}) + E(\theta_{1\alpha'},\theta_{2\beta'})\right| \leq 2. \quad (13.62)$$

which does not depend on the condition that the events at similarly oriented detectors be perfectly correlated or anti-correlated.

13.3.2 Electron Spin Measurements

Electron spin measurements satisfy all the requirements given above for perfectly anti-correlated measurements if the two electrons are prepared in the spin singlet state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right), \quad (13.63)$$

where the first arrow in each ket refers to electron 1 and the second to electron 2.[6] The electrons are assumed to move in the $\pm z$ directions. The expectation value of the product of spin measurements when detector 1 is oriented at an angle $\theta_{1\alpha}$ relative to the x-axis and detector 2 is oriented at an angle $\theta_{2\beta}$ relative to the x-axis is given by

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[5] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt (1969), *Proposed experiment to test local hidden-variable theories*, Physical Review Letters **23**, 880–4 (1969).

[6] Although the singlet state is written for a quantization axis in the z direction, the same state is realized for *any* quantization axis.

$$E\left(\theta_{1\alpha},\theta_{2\beta}\right)=\left\langle \begin{array}{c}\boldsymbol{\sigma}_A\cdot\left(\cos\theta_{1\alpha}\mathbf{u}_x+\sin\theta_{1\alpha}\mathbf{u}_y\right)\\ \times\boldsymbol{\sigma}_B\cdot\left(\cos\theta_{2\beta}\mathbf{u}_x+\sin\theta_{2\beta}\mathbf{u}_y\right)\end{array}\right\rangle$$
$$=\left\langle\left(\sigma_{Ax}\cos\theta_{1\alpha}+\sigma_{Ay}\sin\theta_{1\alpha}\right)\ \left(\sigma_{Bx}\cos\theta_{2\beta}+\sigma_{By}\sin\theta_{2\beta}\right)\right\rangle, \quad (13.64)$$

where

$$\boldsymbol{\sigma}_j=\sigma_{jx}\mathbf{u}_x+\sigma_{jy}\mathbf{u}_y+\sigma_{jz}\mathbf{u}_z;\quad j=A,B, \quad (13.65)$$

and $\sigma_{j\nu}$ $\{\nu=x,y,z\}$ is a Pauli spin matrix acting in the space of spin j $\{j=A,B\}$. In other words, the expectation value of spin (in units of $\hbar/2$) for a single spin system measured relative to a quantization direction $\mathbf{u}$ is $\langle\boldsymbol{\sigma}\cdot\mathbf{u}\rangle$. Calculation of the expectation value given in Eq. (13.64) for the state vector given in Eq. (13.63) is straightforward. For example,

$$\langle\sigma_{Ax}\sigma_{Bx}\rangle=\frac{1}{2}\left(\langle\uparrow\downarrow|-\langle\downarrow\uparrow|\right)\sigma_{Ax}\sigma_{Bx}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right)$$
$$=\frac{1}{2}\left(\langle\uparrow\downarrow|-\langle\downarrow\uparrow|\right)\sigma_{Ax}\left(|\uparrow\uparrow\rangle-|\downarrow\downarrow\rangle\right)$$
$$=\frac{1}{2}\left(\langle\uparrow\downarrow|-\langle\downarrow\uparrow|\right)\left(|\downarrow\uparrow\rangle-|\uparrow\downarrow\rangle\right)=-1. \quad (13.66)$$

Similarly,

$$\left\langle\sigma_{Ax}\sigma_{By}\right\rangle=0; \quad (13.67a)$$
$$\left\langle\sigma_{Ay}\sigma_{Bx}\right\rangle=0; \quad (13.67b)$$
$$\left\langle\sigma_{Ay}\sigma_{By}\right\rangle=-1. \quad (13.67c)$$

Substituting these results into Eq. (13.64), I find

$$E\left(\theta_{1\alpha},\theta_{2\beta}\right)=-\left(\cos\theta_{1\alpha}\cos\theta_{2\beta}+\sin\theta_{1\alpha}\sin\theta_{2\beta}\right)=-\cos\left(\alpha-\beta\right). \quad (13.68)$$

By combining Eqs. (13.61) and (13.68), I arrive at Bell's inequality for the spin system,

$$\left|\cos\left(\alpha-\beta\right)-\cos\left(\alpha-\gamma\right)\right|+\cos\left(\beta-\gamma\right)\leq 1. \quad (13.69)$$

If I take $\alpha=0$, $\beta=3\pi/4$, $\gamma=\pi/4$, then the left-hand side of this equation is equal to $\sqrt{2}$ which violates the inequality. The correlations that occur in quantum mechanics cannot be explained by a local hidden variable theory.

In experiments, the correlation functions $E\left(\theta_{1\alpha},\theta_{2\beta}\right)$ are not measured directly. What is measured are individual and coincidence counts at the two detectors. However it is possible to relate the coincidence counts to the correlation functions to obtain a form of Bell's inequalities for the measurable quantities. A violation of

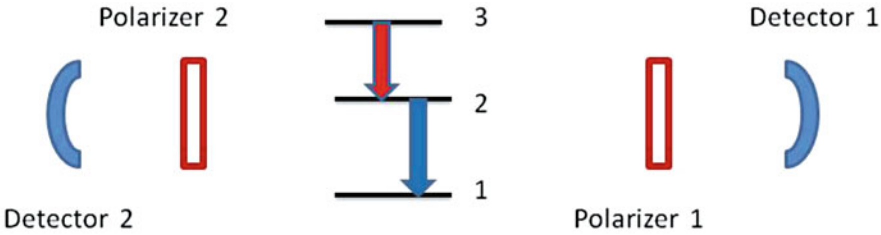

Fig. 13.1 Cascade emission to test Bell's inequalities. The emission on the 3–2 transition propagating in one direction is correlated with the radiation emitted on the 2–1 transition propagating in the opposite direction. Measurements are made for different settings of the polarizers

Bell's inequality has been observed using electron spin states in nitrogen vacancy centers in diamond with detectors placed 1.3 km apart.[7] The experimental spin measurements agree with the quantum predictions.

13.3.3 Photon Polarization Measurements

Experimentally it proved easier to use photons rather than spin to look for violations of Bell's inequalities. To understand how an experiment using photons can be used in a Bell's experiment, look at the atomic level scheme shown in Fig. 13.1. Atoms are prepared in level three and then undergo *spontaneous emission* that takes them to levels 2 and 1 via a cascade process in which a single photon is emitted on each transition (it is actually better to say that a *single photon state* is emitted on each transition since the radiation emerges as a pulse, not as a monochromatic plane wave state). There is no classical analogue of a single photon state, which makes it ideal to use in experiments to test Bell's inequalities.

The polarization of the radiation depends on the angular momentum associated with each level and the direction of emission. If an $\ell = 0 \to 1 \to 0$ cascade is chosen, the radiation emitted on *each* transition is *unpolarized* in *all* directions, but the polarization of the successively emitted single photon states is *correlated*. In other words, if the polarization of the radiation emitted on the 3–2 transition is not measured, then measurements of the radiation on the 2–1 transition will show it to be unpolarized in any direction. The situation changes if polarizers are placed between the atoms and the detectors, and the outcome of the measurement on the 2–1 transition is correlated with that on the 3–2 transition. For example, when the detectors are placed along opposite directions from the source and the polarizers

[7] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau and R. Hanson, *Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres,* Nature **526**, 682–686 (2015).

have the same orientation relative to the axis between the detectors, if a photon is detected at detector 1 then there is a 100% probability that a photon detected at detector 2 will have the same polarization, assuming perfect detectors (e.g. in a given direction, a linearly polarized photon emitted on the 3–2 transition is correlated with a photon emitted in the opposite direction on the 3–2 transition having the *same* linear polarization). In an ideal Bell's experiment, the direction of the polarizers is chosen *after* the radiation is emitted so the orientation of the detectors cannot affect the emission process.[8]

With ideal measurements,

$$E(\theta_{1\alpha}, \theta_{2\beta}) = \cos\left[2\left(\theta_{1\alpha} - \theta_{2\beta}\right)\right]. \tag{13.70}$$

For a relative angle of zero or π between the polarizers there is perfect correlation between the detectors, $E = 1$, while for relative polarizer angles of $\pi/2$ and $3\pi/2$ the events at the detectors are perfectly anti-correlated, $E = -1$. Substituting Eq. (13.70) into Eq. (13.62), I obtain

$$\left|\begin{array}{c} \cos\left[2\left(\theta_{1\alpha} - \theta_{2\beta}\right)\right] - \cos\left[2(\theta_{1\alpha} - \theta_{2\beta'})\right] \\ + \cos\left[2(\theta_{1\alpha'} - \theta_{2\beta})\right] + \cos\left[2(\theta_{1\alpha'} - \theta_{2\beta'})\right] \end{array}\right| \le 2. \tag{13.71}$$

An optimal geometry for violating Bell's inequality is one in which

$$\phi = \left|\theta_{2\beta} - \theta_{1\alpha}\right| = \left|\theta_{2\beta'} - \theta_{1\alpha'}\right| = \left|\theta_{2\beta} - \theta_{1\alpha'}\right| = \left|\theta_{2\beta'} - \theta_{1\alpha}\right|/3. \tag{13.72}$$

For $\phi = \pi/8$ or $3\pi/8$, the left-hand side of Eq. (13.71) takes on its maximum value of $2\sqrt{2}$ and violates the inequality. The experimental measurements agree with the quantum predictions and violate Bell's inequalities.

There is another way to see the inconsistency of local hidden variable theories that does not make direct use of the inequality given in Eq. (13.62).[9] Suppose that the radiation is emitted in opposite directions along the x-axis and that each detector is oriented in the $y - z$ plane so that its axis makes an angle of 0, $2\pi/3$ or $4\pi/3$ relative to the z-axis; that is, each detector has three possible positions. If a photon gets through the detector, I denote this by a y (yes) and if it does not get through

[8] S. J. Freedman and J. F. Clauser, *Experimental Test of Local Hidden-Variable Theories*, Physical Review Letters **28**, 938–941 (1972); E. S. Fry and R. C. Thompson, R. C. (1976), *Experimental Test of Local Hidden Variables Theories*, Physical Review Letters **37**, 465-468 (1976); A. Aspect, P. Grangier, and G. Roger, *Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities*, Physical Review Letters **49**, 91–94 (1982); W. Tittel, J. Brendel, H. Zbinden, N. Gisin, *Violation of Bell inequalities by photons more than 10 km apart*, Physical Review Letters **81**, 3563–3566 (1998); J-A. Larsson, M. Giustina, J. Kofler, B. Wittmann, R. Ursin, and S. Ramelow, *Bell violation with entangled photons, free of the coincidence-time loophole*. Physical. Review A **90,** 032107 (2014).

[9] This argument follows that given by Robert Adair in *The Great Design, Particles, Fields, and Creation* (Oxford University Press, New York, 1987), pp. 185–187.

by an n (no). For a given orientation of a single detector, 50% of the photons get through, on average. Thus if the detectors are at the same angles, there will be 50% yy and 50% nn and 0% yn or ny. When the detectors are at different angles, if the first photon gets through the first detector (which has a 50% probability), the second photon must have the same polarization as the first and there is a 25% [$\cos^2(2\pi/3)$ or $\cos^2(4\pi/3)$] chance that it gets through the second detector and a 75% that it does not. Thus the probabilities are 0.125 (0.5×0.25) for yy and 0.375 (0.5×0.75) for yn. On the other hand, if the first photon does not get through the first detector (which has a 50% probability), the second photon must have its polarization perpendicular to that of the first photon and there is a 75% [$\cos^2(\pi/6)$ or $\cos^2(5\pi/6)$] chance that it gets through the second detector and a 25% that it does not. Thus the probabilities are 0.375 (0.5×0.75) for ny and 0.125 (0.5×0.25) for nn. I now show how these results lead to a contradiction if a local hidden variables theory is used.

Assume now that each photon that is emitted already has its polarization encoded. When the detectors are aligned, there is a 100% correlation between the two photons, which implies that the photons carry the same code. Since there are three detector positions and two possible outcomes for each position (y or n), there are eight possible codes for *each* photon ($yyy, yyn, yny, ynn, nyy, nny, nyn, nnn$). For example, yny implies that a photon will get through the detector for $\theta = 0, 4\pi/3$ but not for $\theta = 2\pi/3$. Since n and y are equally likely for a given position, this implies that the probability for each of $yyn, yny, ynn, nyy, nny, nyn$ must be equal (call it β) and that for each of yyy, nnn must be equal (call it α). Set the detectors at a relative angle of $2\pi/3$ and try to calculate α and β. Let $W(a, b, c)$ represent the probability that we get the result a when the first detector is at 0, b when it is at $2\pi/3$, and c when it is at $4\pi/3$. In this example, only the a and b indices are relative—the third index can be either y or n and must be summed over for a specific choice of the first two indices. Thus,

$$W(yy) = W(yyy) + W(yyn) = 0.125 = \alpha + \beta, \tag{13.73a}$$

$$W(nn) = W(nny) + W(nnn) = 0.125 = \alpha + \beta, \tag{13.73b}$$

$$W(yn) = W(yny) + W(ynn) = 0.375 = 2\beta, \tag{13.73c}$$

$$W(ny) = W(nyy) + W(nyn) = 0.375 = 2\beta. \tag{13.73d}$$

For example, the first line asks what is the probability that both photons get through the detectors (which was found to be 0.125 when the detectors are at a relative angle of $2\pi/3$) and the third line that the first photon gets through and the second one doesn't (which was found to be 0.375 when the detectors are at a relative angle of $2\pi/3$). Solving for α and β gives $\beta = 0.375/2$ and $\alpha = -0.125/2$. Since negative probabilities are not allowed, we are led to a contradiction. Of course, inequality (13.62) can be viewed as a statement that negative probabilities are not allowed.

13.3.4 Quantum Teleportation

Entangled Bell states can be used as the basis for *quantum teleportation*.[10] In classical digital communication, messages to be transmitted are encoded into a stream of zeroes or ones, so-called classical bits of information. Such communication is not secure insofar as it can be intercepted by an "eavesdropper." To avoid such problems, it is now possible to send encoded messages using quantum bits or *qubits* of information over distances as large as 1200 km.[11]

A qubit is a quantum superposition state. For example, suppose Alice (in such examples, it seems that it is *Alice* who always sends information to *Bob*, trying to avoid eavesdropping by *Eve*) wants to send the superposition state

$$|\psi\rangle_1 = (\alpha\,|H\rangle_1 + \beta\,|V\rangle_1) \tag{13.74}$$

to Bob. You can think of the states $|H\rangle_1$ and $|V\rangle_1$ as two orthogonal polarization states (horizontal and vertical) of a single photon state; the subscript 1 indicates that this is the first photon state in this teleportation scheme. To initiate the teleportation protocol, she creates an entangled Bell state of two other single photon states, using a method such as cascade emission. For example, the entangled Bell state might be

$$|\psi\rangle_{23} = \frac{1}{\sqrt{2}}\left(|H\rangle_2\,|H\rangle_3 + |V\rangle_2\,|V\rangle_3\right). \tag{13.75}$$

The subscripts 2 and 3 indicate that these states belong to the second and third single photon states. The third single photon state is sent to Bob and the second is kept by Alice. Alice now has two, single photon states (1 and 2), while Bob has one (3).

The entire state of the system of quantum bit and Bell state is

$$|\psi\rangle_T = \frac{1}{\sqrt{2}}\left(|H\rangle_2\,|H\rangle_3 + |V\rangle_2\,|V\rangle_3\right)(\alpha\,|H\rangle_1 + \beta\,|V\rangle_1)\,. \tag{13.76}$$

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[10] See, for example, *The Physics of Quantum Information*, edited by Dirk Bouwmeester, Artur Ekert, and Anton Zeilinger (Springer-Verlag, Berlin, 2000), and references therein.

[11] See, for example, Xiao-Song Ma, Thomas Herbst, Thomas Scheidl, Daqing Wang, Sebastian Kropatschek, William Naylor, Bernhard Wittmann, Alexandra Mech, Johannes Kofler, Elena Anisimova, Vadim Makarov, Thomas Jennewein, Rupert Ursin and Anton Zeilinger, *Quantum teleportation over 143 kilometres using active feed-forward*, Nature **489**, 269–273 (2012); Raju Valivarthi, Marcel.li Grimau Puigibert, Qiang Zhou, Gabriel Aguilar, Varun Verma, Francesco Marsili, Matthew D. Shaw, Sae Woo Nam, Daniel Oblak and Wolfgang Tittel, *Quantum teleportation across a metropolitan fibre network,* Nature Photonics **10**, 676–680 (2016); J. Yin et al., *Satellite-based entanglement distribution over 1200 kilometers*, Science **356**, 1140-1144 (2017).

Alice now makes a measurement of the state of her two single photon states in a Bell-state basis.[12] That is, she measures *one* of the four states

$$|\psi\rangle_{A1} = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 + |V\rangle_1 |V\rangle_2\right); \tag{13.77a}$$

$$|\psi\rangle_{A2} = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |H\rangle_2 - |V\rangle_1 |V\rangle_2\right); \tag{13.77b}$$

$$|\psi\rangle_{A3} = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 + |V\rangle_1 |H\rangle_2\right); \tag{13.77c}$$

$$|\psi\rangle_{A4} = \frac{1}{\sqrt{2}}\left(|H\rangle_1 |V\rangle_2 - |V\rangle_1 |H\rangle_2\right). \tag{13.77d}$$

Alice's measurement projects Bob's single photon state into one of the states (see problems)

$$|\psi\rangle_{B1} = \alpha\,|H\rangle_3 + \beta\,|V\rangle_3\,; \tag{13.78a}$$

$$|\psi\rangle_{B2} = \alpha\,|H\rangle_3 - \beta\,|V\rangle_3\,; \tag{13.78b}$$

$$|\psi\rangle_{B3} = \beta\,|H\rangle_3 + \alpha\,|V\rangle_3\,; \tag{13.78c}$$

$$|\psi\rangle_{B4} = \beta\,|H\rangle_3 - \alpha\,|V\rangle_3\,. \tag{13.78d}$$

Bob and Alice share a publicly accessible *key distribution* that correlates each Bell state in Eqs. (13.77) with the corresponding single photon state in Eqs. (13.78). Alice sends Bob two classical bits of information over a public line to tell him which Bell state measurement she made (for example, $\{0,0\}$ could correspond to an $A1$ measurement, $\{0,1\}$ to an $A2$ measurement, etc.) Bob then knows what to do to recover the initial quantum bit. For example, if Alice measured the state given in Eq. (13.77a), he already has the desired qubit. If she measured the state given in Eq. (13.77c), he carries out the unitary *quantum gate operation*

$$|\psi\rangle'_{B1} = \sigma_x\,|\psi\rangle_{B3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \alpha\,|H\rangle_3 + \beta\,|V\rangle_3\,, \tag{13.79}$$

in which he swaps the horizontal and vertical polarization components of his state. Other states are treated in a similar fashion.

[12]For example, she can make such a Bell state measurement by entangling her single photon states using beam splitters [Dik Bouwmeester, Jian-Wei Pan, Klaus Mattle, Manfred Eibl, Harald Weinfurter and Anton Zeilinger, *Experimental quantum teleportation,* Nature **390**, 575–579 (1997)] or nonlinear crystals [Yoon-Ho Kim, Sergei Kulik, and Yanhua Shih, *Quantum Teleportation of a Polarization State with a Complete Bell State Measurement*, Physical Review Letters **86**, 1370–1374 (2001)].

If the eavesdropper Eve intercepts the classical communication between Alice and Bob, it is useless to her since she doesn't have the single photon state that was sent to Bob. Moreover, if she intercepts the single photon state and makes a measurement on it, she will necessarily corrupt that superposition state. It will be evident to Alice and Bob that someone has been listening in.

13.3.5 Why the Big Fuss?

There are thousands of papers written on Bell's inequalities and more appear regularly. Many people believe in the validity of Bell's theorem and some don't. The carrying out of "loophole-free" experiments demonstrating violations of Bell's inequalities has enhanced or even made some scientific careers. The New York Times seems to be enamored with experiments that prove "Einstein was wrong" and confirm "spooky action at a distance." It seems that people are either fascinated, disturbed, or dissatisfied (or some combination of these) with experiments that have shown violations of Bell's inequalities.

I am not of the opinion that all this attention is merited. In the spin experiment, for example, each observer detects spin up or spin down 50% of the time, on average, independent of the orientation of her detector. It is only when the observers use classical communication channels to compare the results of their measurements do they find that there are correlations. How do they explain such correlations? To say that one measurement *influenced* the other is not particularly meaningful. In effect, the best answer to this question is "the reason there are correlations is because there are correlations." It is the same as asking people why two neutral objects attract one another. Attributing this attraction to "gravity" does not in any way explain the attraction. The laws of nature are what they are. People have become familiar with gravitational attraction, but do not experience the effects of quantum correlations in their everyday lives. If quantum mechanics is a correct theory, there will be violations of Bell's inequalities even if the proof of Bell's theorem has flaws. This does not prevent people from looking for alternative theories, as Einstein did in formulating an alternative theory of gravity in his Theory of General Relativity, but so far there has not yet been a theory to replace quantum mechanics. I believe that quantum mechanics remains in incomplete theory insofar as it does not address the dynamics of wave function collapse.

13.4 Summary

This chapter serves as a bridge between the introductory course material and the more advanced applications of the basic theory. A brief review was presented. I have also taken the opportunity to present two additional topics. The first was an alternative approach to the quantum theory based on the Feynman's path integrals and the second a more detailed treatment of Bell's theorem. These are both interesting topics, but the results that were derived are not used in the remainder of this text.

13.5 Appendix: Equivalence of Feynman Path Integral and Schrödinger Equation

In this Appendix, I show that Schrödinger's equation can be derived from the Feynman propagator. I work in one-dimension to simplify matters, but the results can be generalized easily to three dimensions. In the Feynman approach, the wave function at time t is related to that at time t' by the integral equation

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,t;x',t')\psi(x',t')dx', \tag{13.80}$$

where the propagator is defined in Eq. (13.41). To derive a differential equation for $\psi(x,t)$ I consider an infinitesimal time interval ϵ with $t = t' + \epsilon$ for which the propagator can be approximated as

$$\begin{aligned} K(x,t;x',t') &= \frac{1}{N}\exp\left[\frac{i}{\hbar}\int_{t'}^{t'+\epsilon} L(x,\dot{x})\,dt''\right] \\ &\approx \frac{1}{N}\exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{x+x'}{2},\frac{x-x'}{\epsilon}\right)\right], \end{aligned} \tag{13.81}$$

since $\dot{x}(t'') \approx [x(t'+\epsilon) - x(t')]/\epsilon$ and $x(t'') \approx [x(t'+\epsilon) + x(t')]/2$. The quantity N is a normalization constant, whose value is derived below. In this limit

$$\psi(x,t+\epsilon) = \frac{1}{N}\int_{-\infty}^{\infty} \exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{x+x'}{2},\frac{x-x'}{\epsilon}\right)\right]\psi(x',t)dx'. \tag{13.82}$$

I choose a specific form for the Lagrangian consistent with the Hamiltonian I have used in discussing the Schrödinger equation, namely

$$L(x,\dot{x}) = \frac{m\dot{x}^2}{2} - V(x), \tag{13.83}$$

which corresponds to a particle having mass m moving in the potential $V(x)$. With this Lagrangian,

$$\psi(x,t+\epsilon) = \frac{1}{N}\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\left(\frac{m(x-x')^2}{2\epsilon} - \epsilon V\left(\frac{x+x'}{2}\right)\right)\right]\psi(x',t)dx', \tag{13.84}$$

or, setting $x - x' = y$,

$$\psi(x, t + \epsilon) = \frac{1}{N}\int_{-\infty}^{\infty} \exp\left(i\frac{my^2}{2\hbar\epsilon}\right) \exp\left[-i\epsilon V\left(x - \frac{y}{2}\right)/\hbar\right] \psi(x - y, t)dy. \tag{13.85}$$

The first exponent blows up as $\epsilon \rightarrow 0$. As a consequence the exponential will oscillate rapidly and lead to destructive interference for the integral except in a small interval about $y = 0$. I keep the first exponential intact, but expand the remaining factors in a power series in both ϵ and y to arrive at

$$\begin{aligned}\psi(x, t + \epsilon) = \frac{1}{N}\int_{-\infty}^{\infty} \exp\left(i\frac{my^2}{2\hbar\epsilon}\right) [1 - i\epsilon V(x)/\hbar] \\ \times\left[\psi(x, t) - \frac{\partial\psi(x, t)}{\partial x}y + \frac{1}{2}\frac{\partial^2\psi(x, t)}{\partial x^2}y^2\right] dy.\end{aligned} \tag{13.86}$$

All higher order terms in the expansion vanish in the limit that $\epsilon \rightarrow 0$. If I keep only the lead term in the expansion, I find

$$\psi(x, t + \epsilon) \approx \frac{1}{N}\int_{-\infty}^{\infty} \exp\left(i\frac{my^2}{2\hbar\epsilon}\right) \psi(x, t)dy = \frac{1}{N}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\psi(x, t), \tag{13.87}$$

where the integral can be evaluated using contour integrals in the complex plane or taken from integral tables. For this equation to be valid as $\epsilon \rightarrow 0$, I must require that

$$N = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}. \tag{13.88}$$

Using the relationships

$$\int_{-\infty}^{\infty} \exp\left(i\frac{my^2}{2\hbar\epsilon}\right) dy = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}; \tag{13.89a}$$

$$\int_{-\infty}^{\infty} \exp\left(i\frac{my^2}{2\hbar\epsilon}\right) ydy = 0; \tag{13.89b}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \exp\left(i\frac{my^2}{2\hbar\epsilon}\right) y^2dy &= \frac{2\hbar\epsilon}{i}\frac{d}{dm}\int_{-\infty}^{\infty} \exp\left(i\frac{my^2}{2\hbar\epsilon}\right) dy \\ &= -\frac{\hbar\epsilon}{i}\sqrt{\frac{2\pi i\hbar\epsilon}{m^{3/2}}},\end{aligned} \tag{13.89c}$$

and Eq. (13.88) in Eq. (13.86), and taking the limit $\epsilon \rightarrow 0$, I find

$$\frac{\partial\psi(x,t)}{\partial t} = \lim_{\epsilon\to 0}\frac{\psi(x,t+\epsilon)-\psi(x,t)}{\epsilon} = -\frac{1}{2}\frac{\hbar}{im}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{i}{\hbar}V(x)\,\psi(x,t), \tag{13.90}$$

or

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t), \tag{13.91}$$

which is Schrödinger's equation.

13.6 Problems

1–2. Write an expression in the form of Eq. (13.48) for the propagator of a particle having mass m moving in a uniform gravitational field for which the classical Hamiltonian is $H = p^2/2m + mgx$. In this case the calculation is more complicated than it is for a free particle, but it can be shown that[13]

$$K(b,a) = \left(\frac{m}{2\pi i\hbar\tau}\right)^{1/2} \times \exp\left\{i\frac{m\tau}{2\hbar}\left[\left(\frac{x_b-x_a}{\tau}\right)^2 - g\,(x_b+x_a) - \frac{g^2\tau^2}{12}\right]\right\},$$

where $\tau = (t_b - t_a)$. Show that the kernel is equal to

$$K(b,a) = \left(\frac{m}{2\pi i\hbar\tau}\right)^{1/2}\exp\left[iS_{cl}(b,a)/\hbar\right],$$

where $S_{cl}(b,a)$ is the classical action.

Using this kernel with

$$\psi\,(x,0) = \frac{1}{\pi^{1/4}\sigma^{1/2}}e^{-(x-x_0)^2/\left(2\sigma^2\right)}e^{ik_0x},$$

prove that

$$|\psi(x,t)|^2 = \left(\frac{1}{\pi\sigma(t)^2}\right)^{1/2} e^{-\left(x-x_0-v_0t-gt^2/2\right)^2/\sigma(t)^2},$$

[13] See, for example, S. Huerfano, S. Sahu, and M. Socolovsky, *Quantum Mechanics and the Weak Equivalence Principle,* International Journal of Pure and Applied Mathematics **49**, 153–166 (2008).

where

$$\sigma(t)^2 = \sigma^2 + \left(\frac{\hbar t}{m\sigma}\right)^2$$

and $v_0 = \hbar k_0/m$. In other words, in a uniformly accelerating reference frame, the particle spreads as if it were a free particle.

3–4. Write an expression in the form of Eq. (13.48) for the propagator of a particle having mass m moving in a 1-D simple harmonic potential for which the classical Hamiltonian is $H = p^2/2m + m\omega^2 x^2/2$. Again, the calculation is more complicated than it is for a free particle, but it can be shown that[14]

$$K(b,a) = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega\tau)}\right)^{1/2} \times \exp\left[\frac{im\omega}{2\hbar \sin(\omega\tau)}\left[\left(x_b^2 + x_a^2\right)\cos(\omega\tau) - 2x_b x_a\right]\right],$$

where $\tau = (t_b - t_a)$. Show that the kernel is equal to

$$K(b,a) = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega\tau)}\right)^{1/2} \exp\left[iS_{cl}(b,a)/\hbar\right],$$

where $S_{cl}(b,a)$ is the classical action.

Using this kernel, find $|\psi(\xi,t)|^2$, given

$$\psi(\xi,0) = \frac{1}{\pi^{1/4}} e^{-(\xi-\xi_0)^2/2},$$

where $\xi = \sqrt{m\omega/\hbar}x$ is a dimensionless variable, and show that your answer agrees with Eq. (7.61b).

5. Suppose that two electrons are emitted in the correlated spin state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

and propagate in opposite directions, $\pm\mathbf{u}_r$, where

$$\mathbf{u}_r = \sin\theta\cos\phi\,\mathbf{u}_x + \sin\theta\sin\phi\,\mathbf{u}_y + \cos\theta\,\mathbf{u}_z$$

[14] See, for example, K. Hira, *Derivation of the harmonic oscillator propagator using the Feynman path integral and recursive relations*, European Journal of Physics **34**, 777–785 (2013).

is a unit vector, and r, θ, and ϕ are spherical coordinates. If the detectors are located in planes perpendicular to $\mathbf{u}_r$ such that the orientation of the detectors is given by

$$\mathbf{u}_A(\theta, \phi, \psi_A) = \cos\psi_A \mathbf{u}_\theta + \sin\psi_A \mathbf{u}_\varphi;$$
$$\mathbf{u}_B(\theta, \phi, \psi_B) = \cos\psi_B \mathbf{u}_\theta + \sin\psi_B \mathbf{u}_\varphi.$$

Prove that

$$\langle(\boldsymbol{\sigma}_A \cdot \mathbf{u}_A)(\boldsymbol{\sigma}_B \cdot \mathbf{u}_B)\rangle = -\cos(\psi_A - \psi_B),$$

where

$$\boldsymbol{\sigma}_j = \sigma_{jx}\mathbf{u}_x + \sigma_{jy}\mathbf{u}_y + \sigma_{jz}\mathbf{u}_z; \quad j = A, B,$$

and $\sigma_{j\nu}$ $\{\nu = x, y, z\}$ is a Pauli spin matrix acting in the space of spin j $\{j = A, B\}$.

6. Under a rotation of the quantization axis, the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)$$

is transformed into

$$|\psi\rangle' = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1'|\downarrow\rangle_2' - |\downarrow\rangle_1'|\uparrow\rangle_2'\right)$$

where

$$\begin{pmatrix}|\uparrow\rangle_{1,2}' \\ |\downarrow\rangle_{1,2}'\end{pmatrix} = \underset{\sim}{U}\begin{pmatrix}|\uparrow\rangle_{1,2} \\ |\downarrow\rangle_{1,2}\end{pmatrix}$$

and $\underset{\sim}{U}$ is a unitary matrix having determinant equal to 1. Prove that, $|\psi\rangle' = |\psi\rangle$; that is, the spins are always anti-correlated for the singlet state, independent of the choice of quantization axis, as you would expect.

7. Make a contour plot of the left side of Eq. (13.69) with axes $x = \beta - \alpha$ and $y = \gamma - \alpha$ to determine the range of detector angles for which the Bell's inequality is violated.

8. Make a contour plot of the left side of Eq. (13.71) with axes $\theta_{1\alpha'}$ and $\theta_{2\beta}$ for $\theta_{1\alpha} = 0$ and $\theta_{2\beta'} = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$ to determine the range of detector angles for which the Bell's inequality is violated.

9. The excited state of a *quantum dot* can decay into a superposition of two, nearly degenerate ground states $|1\rangle$ and $|2\rangle$. If the decay is to state $|1\rangle$, the radiation has horizontal (H) polarization and, if the decay is to state $|2\rangle$, the radiation has

vertical (V) polarization along a given direction of emission. Following emission, the radiation and the quantum dot are in the entangled state[15]

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1, H\rangle - i\,|2, V\rangle\right).$$

If a polarizer is placed in the path of the emitted radiation at an angle θ relative to the x axis and a signal is detected, what is the resultant state vector for the quantum dot? Experimentally, an effective polarization angle can be associated with the superposition state of the quantum dot and the state of the quantum dot can be read out using auxiliary laser pulses. By using different rotation angles for the polarizer and using different rotations of the quantum dot's polarization, it is possible to use the entangled state to demonstrate a violation of a Bell's inequality.

10. Show that when Alice makes measurements of the quantum state given in Eq. (13.76) using the Bell states given in Eq. (13.77), Bob's single photon state is projected into the states given in Eq. (13.78).

[15] J. R. Schaibley, A. P. Burgers, G. A. McCracken, L.-M. Duan, P. R. Berman, A. S. Bracker, D. Gammon, L. Sham, and D. G. Steel, *Entanglement between a Single Electron Spin Confined to an InAs Quantum Dot and a Photon*, Physical Review Letters **110**, 167401 pp. 1–5 (2013).