Chapter 9
Central Forces and Angular Momentum

I have been dealing mainly with problems involving one-dimensional motion. Since nature is three-dimensional, I want to look at solutions of Schrödinger's equation in three dimensions. I consider only problems having *spherical symmetry*, that is, potentials that are a function of r only. These correspond to *central forces*. Moreover, for the most part, I restrict the discussion to bound state problems, such as the important problem of determining the bound states of the hydrogen atom. Problems related to continuum states will be discussed in the context of scattering theory in Chap. 18. Since angular momentum is conserved for central forces, we are led naturally to the quantum theory of angular momentum. *It is important to remember, however, that the results to be derived for the angular momentum operator are valid independent of the specific nature of the interaction potential, spherically symmetric or not.*

9.1 Classical Problem

In classical physics, the concept of angular momentum plays a critical role in central force motion. A particle having mass m, velocity $\mathbf{v}$, and momentum $\mathbf{p} = m\mathbf{v}$ moving in a central potential $V(r)$ experiences a force given by

$$\mathbf{F} = -\nabla V(r) = -\frac{dV(r)}{dr}\mathbf{u}_r, \tag{9.1}$$

where $\mathbf{u}_r$ is a unit vector in the $\mathbf{r}$ direction. The torque $\boldsymbol{\tau}$ on the particle vanishes,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{0}, \tag{9.2}$$

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which implies that the angular momentum ,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{9.3}$$

is a constant of the motion. Moreover,

$$\mathbf{r} \cdot \mathbf{L} = 0; \tag{9.4}$$

the motion of the particle is in a plane perpendicular to **L**.

For the moment, I assume that $\mathbf{L} = L\mathbf{u}_z$, so that the motion is in the $x - y$ plane. Using polar coordinates r and ϕ in this plane, I construct the displacement and velocity vectors,

$$\mathbf{r} = r\left(\cos\phi\mathbf{u}_x + \sin\phi\mathbf{u}_y\right) = r\mathbf{u}_r, \tag{9.5a}$$

$$\begin{aligned}\mathbf{v} = \dot{\mathbf{r}} &= \dot{r}\left(\cos\phi\mathbf{u}_x + \sin\phi\mathbf{u}_y\right) + r\left(-\sin\phi\mathbf{u}_x + \cos\phi\mathbf{u}_y\right)\dot{\phi} \\ &= \dot{r}\mathbf{u}_r + r\dot{\phi}\mathbf{u}_\phi,\end{aligned} \tag{9.5b}$$

where

$$\mathbf{u}_r = \cos\phi\mathbf{u}_x + \sin\phi\mathbf{u}_y, \tag{9.6a}$$

$$\mathbf{u}_\phi = -\sin\phi\mathbf{u}_x + \cos\phi\mathbf{u}_y, \tag{9.6b}$$

and $\mathbf{u}_\phi$ is a unit vector in the direction of increasing ϕ. The kinetic energy is

$$T = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\right]. \tag{9.7}$$

Using the relationship

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} = mr^2\dot{\phi}\mathbf{u}_z, \tag{9.8}$$

I can rewrite the kinetic energy as

$$T = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2}, \tag{9.9}$$

where

$$p_r = \mathbf{p} \cdot \mathbf{u}_r = m\dot{r} \tag{9.10}$$

is the *radial momentum*. Although Eq. (9.9) was derived assuming that $\mathbf{L} = L\mathbf{u}_z$, it remains valid for arbitrary directions of **L** if Eq. (9.5a) is replaced by

$$\mathbf{r} = r\left(\sin\theta\cos\phi\mathbf{u}_x + \sin\theta\sin\phi\mathbf{u}_y + \cos\theta\mathbf{u}_z\right) = r\mathbf{u}_r, \tag{9.11}$$

where (r, θ, ϕ) are now spherical coordinates.

The first term in Eq. (9.9) is the radial and the second term the angular or rotational motion contribution to the kinetic energy. The total energy is

$$E = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + V(r), \tag{9.12}$$

such that

$$\frac{1}{2}m\dot{r}^2 = E - \left[V(r) + \frac{L^2}{2mr^2}\right]. \tag{9.13}$$

In other words, the *radial* motion is determined by the so-called *effective potential* defined by

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}. \tag{9.14}$$

The effective potential is extremely useful in analyzing problems involving central forces, as you shall see in Chap. 10.

9.2 Quantum Problem

9.2.1 Angular Momentum

The classical definition of angular momentum can be taken over to the quantum domain; that is, a Hermitian angular momentum *operator* in quantum mechanics can be defined as

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = \hat{L}_x\mathbf{u}_x + \hat{L}_y\mathbf{u}_y + \hat{L}_z\mathbf{u}_z, \tag{9.15}$$

where

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y; \tag{9.16a}$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z; \tag{9.16b}$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x. \tag{9.16c}$$

You can show easily that $\hat{\mathbf{L}}$ is Hermitian (recall that the product of any two Hermitian operators is Hermitian if the operators commute). Since the angular momentum is a constant of the motion, we expect it to commute with the Hamiltonian. Before proving this, let me establish some basic commutation properties of the angular momentum operators. The commutator of $\hat{L}_x$ and $\hat{L}_y$ is

$$\left[\hat{L}_x, \hat{L}_y\right] = \left[\left(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right), \left(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z\right)\right]$$
$$= \hat{y}\hat{p}_x\left[\hat{p}_z, \hat{z}\right] + \hat{p}_y\hat{x}\left[\hat{z}, \hat{p}_z\right] = i\hbar\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right) = i\hbar\hat{L}_z. \tag{9.17}$$

In obtaining this result you do not have to worry about the order of *commuting* operators. I can cycle this relation by letting $x \to y, y \to z, z \to x$, to obtain

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x; \tag{9.18a}$$

$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y. \tag{9.18b}$$

The different components of the angular momentum operator do *not* commute, so it is *not* possible to measure two components simultaneously. If we knew all three components of the angular momentum simultaneously, it would imply that we could simultaneously measure both the position and momentum of the particle precisely, which would constitute a violation of the uncertainty principle.

Other useful commutation relations are:

$$\left[\hat{L}_x, \hat{x}\right] = 0; \quad \left[\hat{L}_x, \hat{y}\right] = i\hbar\hat{z}; \quad \left[\hat{L}_x, \hat{z}\right] = -i\hbar\hat{y}; \tag{9.19a}$$

$$\left[\hat{L}_x, \hat{p}_x\right] = 0; \quad \left[\hat{L}_x, \hat{p}_y\right] = i\hbar\hat{p}_z; \quad \left[\hat{L}_x, \hat{p}_z\right] = -i\hbar\hat{p}_y; \tag{9.19b}$$

$$\left[\hat{L}_x, \hat{\mathbf{p}}^2\right] = \left[\hat{L}_x, \hat{p}_x^2\right] + \left[\hat{L}_x, \hat{p}_y^2\right] + \left[\hat{L}_x, \hat{p}_z^2\right] = 0; \tag{9.19c}$$

$$\left[\hat{L}_x, \hat{V}\right] = \left[\left(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right), \hat{V}\right] = y\left[\hat{p}_z, V(r)\right] - z\left[\hat{p}_y, V(r)\right]$$
$$= \frac{\hbar}{i}\left[y\frac{\partial V(r)}{\partial z} - z\frac{\partial V(r)}{\partial y}\right]$$
$$= \frac{\hbar}{i}\left[y\frac{dV(r)}{dr}\frac{\partial r}{\partial z} - z\frac{dV(r)}{dr}\frac{\partial r}{\partial y}\right]$$
$$= \frac{\hbar}{i}\frac{dV(r)}{dr}\left[y\frac{z}{r} - z\frac{y}{r}\right] = 0, \tag{9.19d}$$

plus terms with $x \to y, y \to z, z \to x$. The fact that $r = \sqrt{x^2 + y^2 + z^2}$ was used in deriving the last commutation relation, in which I also replaced $\hat{V}$ with $V(r)$ by assuming implicitly that each commutator acted on a function of $\mathbf{r}$. As a consequence of the commutator relations,

$$\left[\hat{\mathbf{L}}, \hat{H}\right] = \left[\hat{\mathbf{L}}, \frac{\hat{p}^2}{2m} + \hat{V}\right] = 0. \tag{9.20}$$

The angular momentum commutes with the Hamiltonian and is a constant of the motion.

Since the individual components of $\hat{\mathbf{L}}$ do not commute it is not possible to find simultaneous eigenfunctions of $\hat{L}_x, \hat{L}_y, \hat{L}_z$. As you will see, the eigenvalues of *one* component of $\hat{\mathbf{L}}$ are not sufficient to uniquely label the degenerate energy eigenfunctions of a Hamiltonian in problems having spherical symmetry. A new operator is needed that commutes with both $\hat{\mathbf{L}}$ and $\hat{H}$. An operator that satisfies these requirements is the square of the angular momentum operator, defined by

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \tag{9.21}$$

since

$$\left[\hat{\mathbf{L}}, \hat{L}^2\right] = 0; \tag{9.22a}$$

$$\left[\hat{L}^2, \hat{p}^2\right] = 0, \tag{9.22b}$$

in general, and

$$\left[\hat{L}^2, \hat{V}\right] = 0; \tag{9.23a}$$

$$\left[\hat{L}^2, \hat{H}\right] = 0, \tag{9.23b}$$

for spherically symmetric potentials. In effect, eigenvalues of the operator $\hat{L}^2$ determine what values of the magnitude of the angular momentum can be measured in a quantum system. Since

$$\left[\hat{L}^2, \hat{H}\right] = 0; \quad \left[\hat{\mathbf{L}}, \hat{H}\right] = 0; \quad \left[\hat{\mathbf{L}}, \hat{L}^2\right] = 0, \tag{9.24}$$

it *is* possible to find simultaneous eigenfunctions of $\hat{L}^2$, $\hat{H}$, and (any) one component of $\hat{\mathbf{L}}$ for a spherically symmetric potential. It turns out that the eigenvalues of $\hat{H}, \hat{L}^2$, and (any) one component of $\hat{\mathbf{L}}$ can be used to uniquely label the eigenfunctions of the Hamiltonian associated with spherically symmetric potentials.

9.2.1.1 Eigenfunctions of $\hat{L}^2$ and $\hat{\mathbf{L}}$

As you might imagine it is not especially convenient to get eigenfunctions of $\hat{L}^2$ and $\hat{\mathbf{L}}$ in rectangular coordinates. Spherical coordinates are the natural venue. There is still a lot of algebra involved. In coordinate space,

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = \frac{\hbar}{i}\mathbf{r} \times \boldsymbol{\nabla}. \tag{9.25}$$

I can express $\mathbf{r}$ and ∇ in spherical coordinates as

$$\mathbf{r} = r\mathbf{u}_r \tag{9.26}$$

and

$$\nabla = \mathbf{u}_r \frac{\partial}{\partial r} + \mathbf{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{u}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{9.27}$$

where the $\mathbf{u}$'s are orthogonal unit vectors,

$$\mathbf{u}_r = \sin\theta\cos\phi\mathbf{u}_x + \sin\theta\sin\phi\mathbf{u}_y + \cos\theta\mathbf{u}_z; \tag{9.28a}$$

$$\mathbf{u}_\theta = \cos\theta\cos\phi\mathbf{u}_x + \cos\theta\sin\phi\mathbf{u}_y - \sin\theta\mathbf{u}_z; \tag{9.28b}$$

$$\mathbf{u}_\phi = -\sin\phi\mathbf{u}_x + \cos\phi\mathbf{u}_y. \tag{9.28c}$$

As a consequence,

$$\begin{aligned}\hat{\mathbf{L}} &= \frac{\hbar}{i} r\mathbf{u}_r \times \left(\mathbf{u}_r \frac{\partial}{\partial r} + \mathbf{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{u}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \\ &= \frac{\hbar}{i}\left(\mathbf{u}_\phi \frac{\partial}{\partial \theta} - \mathbf{u}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right),\end{aligned} \tag{9.29}$$

such that

$$\hat{\mathbf{L}} = -i\hbar \left[\begin{array}{c} \left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\mathbf{u}_x \\ +\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\mathbf{u}_y + \frac{\partial}{\partial\phi}\mathbf{u}_z \end{array}\right]. \tag{9.30}$$

Therefore,

$$\hat{L}_x = -\frac{\hbar}{i}\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right); \tag{9.31a}$$

$$\hat{L}_y = \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right); \tag{9.31b}$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{9.31c}$$

The operator $\hat{L}_z$ has a simple form in spherical coordinates owing to the fact that θ is measured from the z-axis.

I still need an expression for $\hat{L}^2$. I use Eq. (9.29) to write

$$\begin{aligned}\hat{L}^2\psi &= \hat{\mathbf{L}}\cdot\hat{\mathbf{L}}\psi = \frac{\hbar}{i}\hat{\mathbf{L}}\cdot\left(\mathbf{u}_\phi\frac{\partial\psi}{\partial\theta} - \mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right) \\ &= -\hbar^2\left(\mathbf{u}_\phi\frac{\partial}{\partial\theta} - \mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\mathbf{u}_\phi\frac{\partial\psi}{\partial\theta} - \mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right).\end{aligned} \tag{9.32}$$

It is important to realize that the $\mathbf{u}_\phi$ and $\mathbf{u}_\theta$ are functions of θ and ϕ, with

$$\frac{\partial \mathbf{u}_\theta}{\partial \theta} = -\sin\theta\cos\phi\mathbf{u}_x - \sin\theta\sin\phi\mathbf{u}_y - \cos\theta\mathbf{u}_z = -\mathbf{u}_r; \tag{9.33a}$$

$$\frac{\partial \mathbf{u}_\theta}{\partial \phi} = -\cos\theta\sin\phi\mathbf{u}_x + \cos\theta\cos\phi\mathbf{u}_y = \cos\theta\mathbf{u}_\phi; \tag{9.33b}$$

$$\frac{\partial \mathbf{u}_\phi}{\partial \theta} = 0; \tag{9.33c}$$

$$\frac{\partial \mathbf{u}_\phi}{\partial \phi} = -\cos\phi\mathbf{u}_x - \sin\phi\mathbf{u}_y, \tag{9.33d}$$

such that

$$\begin{aligned}
\hat{L}^2\psi &= -\hbar^2\left(\mathbf{u}_\phi\frac{\partial}{\partial\theta} - \mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\mathbf{u}_\phi\frac{\partial\psi}{\partial\theta} - \mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\\
&= -\hbar^2\mathbf{u}_\phi\cdot\begin{bmatrix} \mathbf{u}_\phi\frac{\partial^2\psi}{\partial\theta^2} \\ +\mathbf{u}_r\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \mathbf{u}_\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\end{bmatrix}\\
&\quad +\hbar^2\frac{\mathbf{u}_\theta}{\sin\theta}\cdot\begin{bmatrix} \left(-\cos\phi\mathbf{u}_x - \sin\phi\mathbf{u}_y\right)\frac{\partial\psi}{\partial\theta} + \mathbf{u}_\phi\frac{\partial^2\psi}{\partial\theta\partial\phi} \\ -\cos\theta\mathbf{u}_\phi\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\end{bmatrix}\\
&= -\hbar^2\left[\frac{\partial^2\psi}{\partial\theta^2} + \cot\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right],
\end{aligned} \tag{9.34}$$

which implies that

$$\begin{aligned}
\hat{L}^2 &= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].
\end{aligned} \tag{9.35}$$

In deriving Eq. (9.34), I used the fact that the unit vectors is spherical coordinates are orthogonal, along with the identity $\mathbf{u}_\theta\cdot\left(\cos\phi\mathbf{u}_x + \sin\phi\mathbf{u}_y\right) = \cos\theta$.

I want to get the simultaneous eigenfunctions of $\hat{L}^2$ and one component of $\hat{\mathbf{L}}$. Since $\hat{L}_z$ has the simplest form, I choose it. It is easy to solve for eigenfunctions Φ_m of $\hat{L}_z$ using

$$\hat{L}_z\Phi_m = \frac{\hbar}{i}\frac{\partial\Phi_m}{\partial\phi} = m\hbar\Phi_m, \tag{9.36}$$

where m labels the eigenvalue of $\hat{L}_z$ having value $m\hbar$. The solution is

$$\Phi_m(\phi) = e^{im\phi}. \tag{9.37}$$

The quantity m is not arbitrary; whenever the azimuthal angle ϕ increases by 2π, $\Phi_m(\phi)$ must return to its same value. This can happen only if m is an integer (positive, negative, or zero). Thus the normalized eigenfunctions are

$$\Phi_m(\phi) = \sqrt{\frac{1}{2\pi}}e^{im\phi}; \qquad m = 0, \pm 1, \pm 2, \ldots \tag{9.38}$$

and the eigenvalues of $\hat{L}_z$ are integral multiples of $\hbar$. The quantum number m is referred to as the *magnetic quantum number*, for reasons that will become apparent when I consider the Zeeman effect in Chap. 21.

There is already an important difference from the classical case where, for a given angular momentum **L**, the z component of angular momentum can take on *continuous* values from $-L$ to L. In quantum mechanics, the z-component (any component, for that matter) of angular momentum can take on only integral multiples of $\hbar$.

The eigenvalue equation for $\hat{L}^2$ is

$$\hat{L}^2\Theta_{\ell m}(\theta,\phi) = \hbar^2\ell(\ell+1)\Theta_{\ell m}(\theta,\phi), \tag{9.39}$$

where ℓ is totally arbitrary at this point (i.e., it need not be an integer). I assume a solution of the form

$$\Theta_{\ell m}(\theta,\phi) = G_{\ell m}(\theta)e^{im\phi}, \tag{9.40}$$

which is guaranteed to be a simultaneous eigenfunction of $\hat{L}_z$. Substituting this trial solution into Eq. (9.39) and using Eq. (9.35), I find that $G_{\ell m}(\theta)$ satisfies the ordinary differential equation

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dG_{\ell m}(\theta)}{d\theta} - \frac{m^2}{\sin^2\theta}G_{\ell m}(\theta)\right] = \hbar^2\ell(\ell+1)G_{\ell m}(\theta). \tag{9.41}$$

By setting $x = \cos\theta$ and $G_{\ell m}(\theta) \equiv P_\ell^m(x)$, I can transform this equation into

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_\ell^m(x)}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m(x) = 0, \tag{9.42}$$

which is known as *Legendre's equation*. The only solutions of Legendre's equation that are regular (do not diverge) at $x = \pm 1$ $[\theta = 0, \pi]$ are the so-called *associated Legendre polynomials* $P_\ell^m(x)$ for which ℓ is a positive integer or zero that is greater than or equal to $|m|$ {Mathematica symbol LegendreP[ℓ, m, x]}. In other words, the physically acceptable solutions have

$$\ell = 0, 1, 2, \ldots . \tag{9.43}$$

and, for each value of ℓ,

$$m = -\ell, -\ell + 1, \ldots \ell - 1, \ell . \tag{9.44}$$

The eigenvalues of $\hat{L}^2$ are

$$L^2 = \hbar^2 \ell (\ell + 1), \qquad \ell = 0, 1, 2, . \tag{9.45}$$

The quantum number ℓ is referred to as the *azimuthal* or *angular momentum quantum number*.

Before looking at the eigenfunctions in more detail, I can summarize the results so far. Classically the magnitude squared of the angular momentum L^2 can take on any value from zero to infinity and L_z can take on *continuous* values from $-L$ to L. In quantum mechanics, the eigenvalues of $\hat{L}^2$ are limited to the set of discrete (quantized) values $0\hbar^2, 2\hbar^2, 6\hbar^2, \ldots \ell (\ell + 1) \hbar^2$. For each value of ℓ, the eigenvalues of $\hat{L}_z$ (or of any component of angular momentum, for that matter) vary from $-\ell\hbar$ to $\ell\hbar$ in integral steps of $\hbar$. In other words, for each value of ℓ, there $(2\ell + 1)$ values of m that are allowed.

Moreover, in quantum mechanics, we cannot know the *vector* angular momentum exactly since the components of $\hat{\mathbf{L}}$ do not commute. We can specify the magnitude of the angular momentum and one of its components, say L_z, but then there is uncertainty in both L_x and L_y. All we know about these other components is that they are constrained by

$$L_x^2 + L_y^2 = L^2 - L_z^2 . \tag{9.46}$$

In other words, there is an *uncertainty cone* of L_x and L_y having radius $\left[\ell (\ell + 1) - m^2\right]^{1/2} \hbar$ for a given value of ℓ and m. I will return to a discussion of the physical interpretation of angular momentum in quantum mechanics after I look at the eigenfunctions.

You might be wondering about the result that the magnitude of the angular momentum is quantized in units of $\sqrt{\ell (\ell + 1)}\hbar$ and not $\ell\hbar$. This follows from solving the eigenvalue equation, but there does not seem to be a simple *geometric* interpretation of this result as there was for L_z. Based on the uncertainty principle, you can rule out the possibility that $L^2 = (\ell\hbar)^2$. If this were the case for integer ℓ and if the maximum value of m is equal to ℓ, then the state of maximum ℓ would have no uncertainty in $L_x^2 + L_y^2$, which is impossible since $\hat{L}_z$ does not commute with $\hat{L}_x$ and $\hat{L}_y$. You can also derive the result if you assume that ℓ is integral and that m takes on integral values from $-\ell$ to ℓ. With this assumption, for a spherically symmetric state

$$\left\langle L^2\right\rangle = 3\left\langle L_z^2\right\rangle = \frac{3\hbar^2}{2\ell+1}\sum_{-\ell}^{\ell} m^2 = \ell\left(\ell+1\right)\hbar^2. \tag{9.47}$$

However this is a bit of a swindle since I am trying to understand how the magnitude of the angular momentum is quantized and I have already implicitly assumed it to be the case by assuming that m takes on integral values from $-\ell$ to ℓ.

The *normalized* simultaneous eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ are the so-called spherical harmonics $Y_\ell^m\left(\theta,\phi\right)$ defined by

$$Y_\ell^m\left(\theta,\phi\right) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi} \tag{9.48}$$

(*Mathematica* symbol SphericalHarmonicY[ℓ, m, θ, ϕ]). Thus

$$\hat{L}^2 Y_\ell^m\left(\theta,\phi\right) = \hbar^2\ell\left(\ell+1\right)Y_\ell^m\left(\theta,\phi\right); \qquad \ell = 0,1,2,\ldots \tag{9.49a}$$

$$\hat{L}_z Y_\ell^m\left(\theta,\phi\right) = m\hbar Y_\ell^m\left(\theta,\phi\right); \qquad m = 0,\pm1,\pm2\ldots\pm\ell. \tag{9.49b}$$

The first few $Y_\ell^m\left(\theta,\phi\right)$ are

$$Y_0^0\left(\theta,\phi\right) = \sqrt{\frac{1}{4\pi}}; \tag{9.50a}$$

$$Y_1^0\left(\theta,\phi\right) = \sqrt{\frac{3}{4\pi}}\cos\theta; \tag{9.50b}$$

$$Y_1^{\pm1}\left(\theta,\phi\right) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}; \tag{9.50c}$$

$$Y_2^0\left(\theta,\phi\right) = \sqrt{\frac{5}{4\pi}}\left(\frac{3\cos^2\theta-1}{2}\right); \tag{9.50d}$$

$$Y_2^{\pm1}\left(\theta,\phi\right) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}; \tag{9.50e}$$

$$Y_2^{\pm2}\left(\theta,\phi\right) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm2i\phi}; \tag{9.50f}$$

$$Y_\ell^{-m} = (-1)^m\left(Y_\ell^{-m}\right)^*. \tag{9.50g}$$

The $Y_\ell^m\left(\theta,\phi\right)$ are orthonormal,

$$\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\, d\theta\left[Y_\ell^m\left(\theta,\phi\right)\right]^* Y_{\ell'}^{m'}\left(\theta,\phi\right) = \delta_{\ell,\ell'}\delta_{m,m'}. \tag{9.51}$$

Under an inversion of coordinates, $\mathbf{r} \to -\mathbf{r}$, the angles change by $\theta \to \pi - \theta, \phi \to \phi + \pi$, and

$$Y_\ell^m(\pi - \theta, \phi + \pi) = (-1)^\ell Y_\ell^m(\theta, \phi). \tag{9.52}$$

Thus, the *parity* of $Y_\ell^m(\theta, \phi)$ is $(-1)^\ell$; that is, the $Y_\ell^m(\theta, \phi)$ are also simultaneous eigenfunctions of the parity operator. This must be the case since the $\hat{L}^2$, $\hat{L}_z$, and the parity operator commute and the eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ are nondegenerate.

I also list a few properties of the associated Legendre polynomials $P_\ell^m(x)$. For $m = 0$, the associated Legendre polynomials reduce to the Legendre polynomials $P_\ell(x)$ (Mathematica symbol LegendreP[ℓ, x]); that is, $P_\ell^0(x) = P_\ell(x)$, which satisfies the differential equation

$$\frac{d}{dx}\left[\left(1 - x^2\right)\frac{dP_\ell(x)}{dx}\right] + \left[\ell(\ell + 1)\right]P_\ell(x) = 0, \tag{9.53}$$

and

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}\left(x^2 - 1\right)^\ell \quad \text{(Rodrigues formula)}; \tag{9.54a}$$

$$(\ell + 1)P_{\ell+1} - (2\ell + 1)xP_\ell + \ell P_{\ell-1} = 0; \tag{9.54b}$$

$$\left(1 - x^2\right)\frac{dP_\ell}{dx} = -\ell x P_\ell + \ell P_{\ell-1}; \tag{9.54c}$$

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell + 1}\delta_{\ell,\ell'}; \tag{9.54d}$$

$$\int_0^\pi \sin\theta\, d\theta P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell + 1}\delta_{\ell,\ell'}; \tag{9.54e}$$

$$\frac{1}{\sqrt{1 - 2xq + q^2}} = \sum_{\ell=0}^{\infty} q^\ell P_\ell(x); \quad \text{(generating function)}; \tag{9.54f}$$

$$P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{3x^2 - 1}{2}; \quad P_3(x) = \frac{5x^3 - 3x}{2}. \tag{9.54g}$$

The Legendre polynomials are defined such that

$$P_\ell(\pm 1) = (\pm 1)^\ell, \tag{9.54h}$$

implying that $P_\ell(\cos\theta) = 1$ for $\theta = 0$ and $(-1)^\ell$ for $\theta = \pi$.

The associated Legendre polynomials can be obtained from the Legendre polynomials via

$$P_\ell^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_\ell(x); \quad m \geq 0; \tag{9.55a}$$

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x) \qquad m \geq 0; \tag{9.55b}$$

$$\int_{-1}^{1} dx P_\ell^m(x) P_{\ell'}^m(x) = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell,\ell'}. \tag{9.55c}$$

Note that for odd m the associated Legendre "polynomials" are not polynomials at all since they contain the factor $\left(1-x^2\right)^{m/2}$.

9.2.2 Physical Interpretation of the Spherical Harmonics

I now return to a discussion of the physics. It should be clear by now that the angular momentum operator plays an important role in central force problems since it is a constant of the motion. In the classical problem, suppose the angular momentum is in the positive z direction, $\mathbf{L} = L\mathbf{u}_z$. This means the orbit is in the xy plane and there is ϕ dependence in a *specific* orbit, e.g. an elliptical orbit in which the semi-major axis is along x. For a given value of energy and angular momentum, the ϕ dependence is determined by the initial conditions. If, on the other hand, we average over *all* possible initial conditions having the same energy E and angular momentum $\mathbf{L} = L\mathbf{u}_z$, there *cannot* be any ϕ dependence owing to the overall symmetry about the z-axis. In some sense, quantum mechanics does this averaging for you with respect to the eigenfunctions. This is the reason why $\left|Y_\ell^m(\theta,\phi)\right|^2$ is independent of ϕ.

You will see in the next chapter that the eigenfunctions of the Hamiltonian for spherically symmetric potentials that are simultaneous eigenfunction of $\hat{L}^2$ and $\hat{L}_z$ can be written quite generally as

$$\psi_{E\ell m}(\mathbf{r}) = R_{E\ell}(r) Y_\ell^m(\theta,\phi), \tag{9.56}$$

where $R_{E\ell}(r)$ is a radial wave function. Since $\left|Y_\ell^m(\theta,\phi)\right|^2$ is independent of ϕ , I can define an angular probability distribution for the polar angle θ by

$$W_{\ell m}(\theta) = 2\pi \sin\theta \left|Y_\ell^m(\theta,\phi)\right|^2. \tag{9.57}$$

This distribution is normalized,

$$\int_0^\pi W_{\ell m}(\theta)\, d\theta = 1. \tag{9.58}$$

It is fairly amazing that, for fixed ℓ and m, the angular probability distribution $W_{\ell m}(\theta)$ associated with an eigenfunction of a spherically symmetric potential is the *same* for *all* central forces, independent of the energy.

It is a simple matter to plot $W_{\ell m}(\theta)$ as a function of θ. Graphs of $W_{\ell m}(\theta)$ are shown as the solid red curves in Figs. 9.1, 9.2, and 9.3 for $\ell = 50$ and $m = 50, 25, 0$, respectively. There are $(\ell - |m|)$ zeroes in $W_{\ell m}(\theta)$ for $0 < \theta < \pi$, reflecting the fact that there are $(\ell - |m|)$ nodes in $P_\ell^m(\cos\theta)$. This is similar to what we found for the energy eigenfunctions for one-dimensional potentials. In this case, however, there are zero nodes for $|m| = \ell$ and a new node appears with each *decrease* in the value of $|m|$.

The question that remains, however, is "What is the physical significance of these curves?" To answer this question, I can make a comparison with the corresponding classical problem. In the limit of large quantum numbers, the quantum probability distribution, averaged over oscillations in the classically allowed region, should be approximately equal to the classical probability distribution, *averaged over all possible initial conditions consistent with the constant values of energy, magnitude of angular momentum, and z-component of angular momentum.*

To compare the classical and quantum probability distributions, I must specify the values of the angular momentum that are consistent with the conserved quantities of the quantum problem. In other words, I set

$$L = \sqrt{\ell(\ell+1)}\hbar \tag{9.59}$$

and

$$L_z = m\hbar. \tag{9.60}$$

With these values, the classical angular momentum can be located anywhere on an uncertainty cone (see Fig. 9.4) for which

$$L_x^2 + L_y^2 = L^2 - L_z^2. \tag{9.61}$$

Figure 9.4 can help you to understand the nature of the classical motion. Since the motion is in a plane perpendicular to $\mathbf{L}$, for any position of $\mathbf{L}$ on the uncertainty cone, the motion must be confined to polar angles

$$\pi/2 - \cos^{-1}(L_z/L) \le \theta \le \pi/2 + \cos^{-1}(L_z/L), \tag{9.62}$$

so that the classically allowed region is

$$\pi/2 - \cos^{-1}\left(m/\sqrt{\ell(\ell+1)}\right) \le \theta \le \pi/2 + \cos^{-1}\left(m/\sqrt{\ell(\ell+1)}\right) \tag{9.63}$$

when Eqs. (9.59) and (9.60) are used.

It can be shown (see Appendix) that the classical polar angle probability distribution is given by

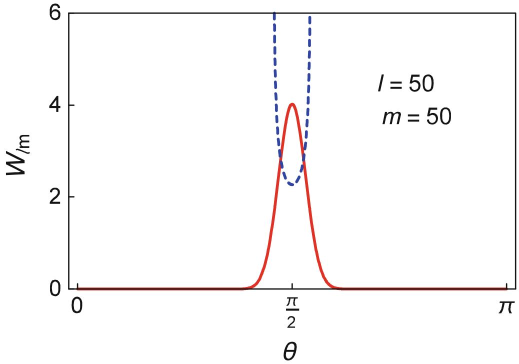

Fig. 9.1 Polar angle probability distribution $W_{\ell m}$ as a function of θ for $\ell = 50$ and $m = 50$. The solid curve is the exact, quantum result and the dashed curve is the classical probability distribution

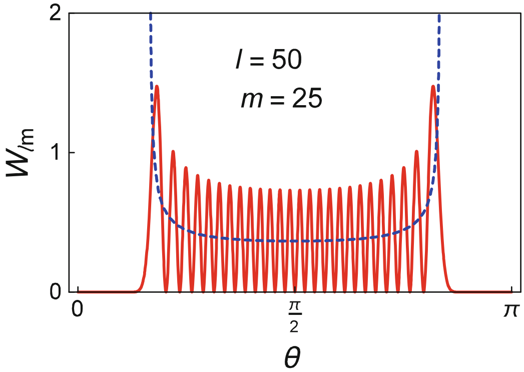

Fig. 9.2 Same as Fig. 9.1, but with $\ell = 50$ and $m = 25$

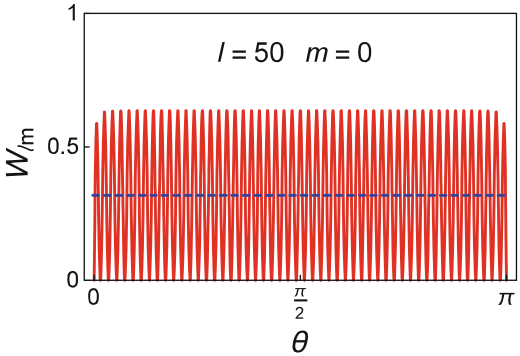

Fig. 9.3 Same as Fig. 9.1, but with $\ell = 50$ and $m = 0$

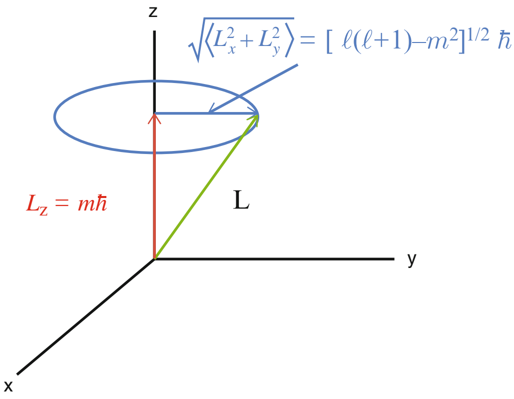

Fig. 9.4 Uncertainty cone for the x and y components of angular momentum

$$W_{\ell m}(\theta)^{\text{class}} = \frac{\sin\theta}{\pi\left\{\frac{\ell(\ell+1)-m^2}{\ell(\ell+1)} - \cos^2\theta\right\}^{1/2}}, \tag{9.64}$$

restricted to positive values of the term in curly brackets. Owing to the symmetry, the classical motion, averaged over all values of $L_x^2 + L_y^2$, consistent with Eq. (9.61), cannot depend on the azimuthal angle ϕ, even if there is a ϕ dependence for *specific* orbits.

In Figs. 9.1, 9.2 and 9.3, the classical distribution function is shown as the dashed blue curves. For $m = 50$, the motion is constrained to be very close to the xy plane ($\theta = \pi/2$) since this corresponds to $\mathbf{L} \approx L_z \mathbf{u}_z$. For $m = 25$, the values of θ are restricted to the classically allowed regime

$$0.52 \leq \theta \leq 2.62. \tag{9.65}$$

When $m = 0$, the angular momentum lies in xy plane so, for a given value of $\mathbf{L} \approx L_x \mathbf{u}_z + L_y \mathbf{u}_z$, the classical motion must be in a plane perpendicular to the xy plane. In this case, $W_{\ell 0}(\theta)^{\text{class}} = 1/\pi$; the classical angle distribution, averaged over all possible initial conditions, is constant. Since the motion is in a plane perpendicular to the xy plane and since an average over all initial conditions is taken, the resultant time-averaged angular probability distribution must be independent of θ. For values of $\ell \gg 1$, the quantum probability distribution, averaged over oscillations in the classically allowed region, is approximately equal to the classical probability distribution.

Returning to the quantum problem, I note that the eigenfunction with $m = \ell$ comes closest to the classical state in which the angular momentum is in the z direction. In other words, the eigenfunction with $m = \ell$ minimizes the angle of uncertainty cone. For this state,

$$\left|Y_{\ell}^{\ell}(\theta,\phi)\right|^{2} \sim \sin^{2\ell}\theta; \tag{9.66a}$$

$$W_{\ell m}(\theta) = 2\pi \sin\theta \left|Y_{\ell}^{m}(\theta,\phi)\right|^{2} \sim \sin^{2\ell+1}\theta. \tag{9.66b}$$

Thus, $\left|Y_{\ell}^{\ell}(\theta,\phi)\right|^{2}$ is peaked about $\theta = \pi/2$, and the sharpness of the peak increases with increasing ℓ. This corresponds to the fact that the classical orbit is constrained to be very close to the xy plane. In fact, I can estimate how far the orbit strays from the xy plane by calculating

$$\langle\theta\rangle = \frac{\int_0^{\pi} d\theta\, \theta \sin^{2\ell+1}\theta}{\int_0^{\pi} d\theta \sin^{2\ell+1}\theta} = \frac{\pi}{2} \tag{9.67a}$$

$$\Delta\theta = \left(\frac{\int_0^{\pi} d\theta\, \theta^2 \sin^{2\ell+1}\theta}{\int_0^{\pi} d\theta \sin^{2\ell+1}\theta} - \frac{\pi^2}{4}\right)^{1/2} \approx \frac{1}{\sqrt{2\ell+1}}. \tag{9.67b}$$

The integral in the equation for $\Delta\theta$ can be evaluated exactly in terms of hypergeometric functions, but has been approximated for large ℓ. You can also determine the dependence on ℓ by looking at the value $\Delta\theta \sim (\theta - \pi/2)$ for which the function $\sin^{2\ell+1}\theta$ is equal to $1/2$.

The uncertainty in the magnitude of $\mathbf{L}$, ΔL, is of order of the radius of the uncertainty cone,

$$\Delta L \approx \sqrt{\left\langle \hat{L}_x^2 + \hat{L}_y^2 \right\rangle} = \sqrt{[\ell(\ell+1) - \ell^2]}\hbar = \hbar\sqrt{\ell} \tag{9.68}$$

Therefore

$$\Delta L \Delta\theta \approx \hbar\sqrt{\frac{\ell}{2\ell+1}} \tag{9.69}$$

which is an *angular momentum—angle uncertainty relation*, although it is not a strict uncertainty relation since there is no Hermitian operator that corresponds to angle. For values of $m \neq \ell$, the values of both ΔL and $\Delta\theta$ increase.

The key point to remember is that the motion in the θ direction is, for the most part, restricted to a range of angles given by Eq. (9.63) when the system is in an eigenstate of $\hat{L}^2$ and $\hat{L}_z$. There is no ϕ dependence in the angular probability distribution because I have chosen eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$. Had I chosen $\hat{L}^2$ and $\hat{L}_x$ the eigenfunctions would be *linear combinations* of the $Y_{\ell}^{m}(\theta,\phi)$, and the absolute square of an eigenfunction could depend on ϕ, but it would be a function only of the angle that $\mathbf{L}$ makes with the x-axis (see problems). The angular probability distribution associated with the simultaneous eigenfunctions of a spherically symmetric Hamiltonian and the operators $\hat{L}^2$ and $\hat{L}_z$ does not depend in any way on the specific form of the potential.

9.3 Summary

We have seen that angular momentum plays a critical role in problems having spherical symmetry. I obtained the eigenvalues and simultaneous eigenfunctions of the operators $\hat{L}^2$ and $\hat{L}_z$. Moreover I was able to make a correspondence with the analogous classical problem to help give you a physical interpretation to the spherical harmonics for problems involving spherical symmetry.

9.4 Appendix: Classical Angular Distribution

The classical polar angle probability distribution for central force motion is given by

$$W_\ell^m(\theta)^{\text{class}} = \frac{1}{T_{21}}\left|\frac{dt}{d\theta}\right| \tag{9.70}$$

where

$$T_{21} = \int_{r_{\min}}^{r_{\max}} \frac{dt}{dr}dr \tag{9.71}$$

and $r_{\min}$ is the minimum value of r and $r_{\max}$ the maximum value of r for the orbit. For *bound* orbits of a particle having mass m, it is easy to calculate $r_{\min}$ and $r_{\max}$ as roots of

$$E - V(r) - \frac{\hbar^2\ell(\ell+1)}{2mr^2} = 0. \tag{9.72}$$

On the other hand, for unbound orbits corresponding to a scattering problem, $r_{\max}$ approaches infinity. For *closed* orbits, T_{21} can be related to the period of the orbit.

The probability distribution (9.70) must be averaged over all possible classical trajectories consistent with Eqs. (9.59)–(9.61). A simple way to envision this is to first imagine the angular momentum in the z-direction. The radial coordinate r is a function of $(\phi - \phi_0)$, $r = r(\phi - \phi_0)$, where ϕ is the azimuthal angle and ϕ_0 is a constant that determines the orientation of the orbit in the $x-y$ plane. I must average the results over all values of ϕ_0 from 0 to 2π. For bound state orbits $\phi - \phi_0$ varies from 0 to 2π, but for unbound orbits $\phi - \phi_0$ may vary from some fixed angle β to $2\pi - \beta$. For example, in the Coulomb problem with positive energy, you can have

$$\beta = \cos^{-1}(1/\epsilon), \tag{9.73}$$

where ϵ is the eccentricity of the orbit; this value of β corresponds to an asymptote of a hyperbolic orbit. Once the average over ϕ_0 is performed, the entire result can be rotated such that the plane of the orbit is perpendicular to **L**. Following this procedure, I must calculate

$$W_{\ell m}(\theta)^{\text{class}} = \frac{1}{T_{21}} \left\langle \left| \frac{dt}{d\theta} \right| \right\rangle_{\phi_0}, \tag{9.74}$$

which is independent of the azimuthal angle of **L**.

To calculate $dt/d\theta$, I start from

$$\mathbf{L} \cdot \mathbf{r} = 0, \tag{9.75}$$

with

$$\mathbf{L} = L\left(\mathbf{u}_z \cos\alpha + \mathbf{u}_x \sin\alpha\right) \tag{9.76a}$$

$$\mathbf{r} = r\left(\mathbf{u}_z \cos\theta + \mathbf{u}_x \sin\theta\cos\phi + \mathbf{u}_y \sin\theta\sin\phi\right) \tag{9.76b}$$

(without loss of generality, I can take **L** in the $x - z$ plane). Clearly,

$$\cos\alpha = L_z/L. \tag{9.77}$$

From Eqs. (9.75) and (9.76), I find

$$\cos\phi = -\cot\alpha\cot\theta \tag{9.78}$$

and

$$\sin\phi = \frac{\sqrt{\sin^2\alpha - \cos^2\theta}}{\sin\alpha\sin\theta}. \tag{9.79}$$

The square of the angular momentum is given by

$$L^2 = m^2 r^4\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right). \tag{9.80}$$

Using Eq. (9.78), I obtain

$$-\sin\phi\dot{\phi} = \frac{\cot\alpha}{\sin^2\theta}\dot{\theta}, \tag{9.81}$$

allowing me to calculate

$$\dot{\phi} = -\frac{\cos\alpha}{\sin\theta\sqrt{\sin^2\alpha - \cos^2\theta}}\dot{\theta}. \tag{9.82}$$

Combining Eqs. (9.80) and (9.82), I can express the square of the angular momentum as

$$L^2 = m^2 r^4 \dot{\theta}^2 \frac{\sin^2\theta}{\sin^2\alpha - \cos^2\theta}, \tag{9.83}$$

from which it follows that

$$\frac{dt}{d\theta} = \frac{mr^2}{L} \frac{\sin\theta}{\sqrt{\sin^2\alpha - \cos^2\theta}}. \tag{9.84}$$

Finally, using Eqs. (9.74) and (9.84), I arrive at

$$W_{\ell m}(\theta)^{\text{class}} = \frac{1}{T_{21}} \left\langle \left| \frac{dt}{d\theta} \right| \right\rangle_{\phi_0} = \frac{1}{T_{21}} \frac{m \left\langle r^2 \right\rangle_{\phi_0}}{L} \frac{\sin\theta}{\left(\sin^2\alpha - \cos^2\theta\right)^{1/2}}. \tag{9.85}$$

Equation (9.85) is somewhat surprising. It seems to imply that

$$\frac{1}{T_{21}} \frac{m \left\langle r^2 \right\rangle_{\phi_0}}{L}$$

must be independent of energy for *any* central potential. I now show that this is actually the case, starting from

$$\left\langle r^2 \right\rangle_{\phi_0} = \frac{1}{2\pi} \int_0^{2\pi} r^2 (\phi - \phi_0)\, d\varphi_0 = \frac{1}{2\pi} \int_0^{2\pi} r^2(\bar{\phi}) d\bar{\phi}, \tag{9.86}$$

where $\bar{\phi} = \phi - \phi_0$. By writing

$$d\bar{\phi} = \frac{d\bar{\phi}}{dt} \frac{dt}{dr} dr = \dot{\phi} \frac{dt}{dr} dr, \tag{9.87}$$

I can obtain

$$\left\langle r^2 \right\rangle_{\phi_0} = \frac{2}{2\pi} \int_{r_{\min}}^{r_{\max}} \dot{\phi} r^2 \frac{dt}{dr} dr = \frac{L}{\pi m} \int_{r_{\min}}^{r_{\max}} \frac{dt}{dr} dr = \frac{LT_{21}}{2\pi m}, \tag{9.88}$$

where Eq. (9.71) was used. The extra factor of 2 in Eq. (9.88) arises from the fact that as $\bar{\phi}$ varies from 0 to 2π, r varies *twice* from $r_{\min}$ to $r_{\max}$. The classical probability distribution, calculated using Eqs. (9.85) and (9.88) is

$$W_{\ell m}(\theta)^{\text{class}} = \frac{1}{T_{21}} \left\langle \left| \frac{dt}{d\theta} \right| \right\rangle_{\phi_0} = \frac{1}{\pi} \frac{\sin\theta}{\left(\sin^2\alpha - \cos^2\theta\right)^{1/2}}. \tag{9.89}$$

It is easy to verify that $W_{\ell m}(\theta)^{\text{class}}$ is normalized properly,

$$\int_{\pi/2-\alpha}^{\pi/2+\alpha} W_{\ell m}(\theta)^{\text{class}}\, d\theta = 1. \tag{9.90}$$

Using the value of α defined in Eq. (9.77),

$$\alpha = \cos^{-1}(L_z/L) = \cos^{-1}[m/\sqrt{\ell(\ell+1)}], \tag{9.91}$$

I arrive at Eq. (9.64).

9.5 Problems

1. How does quantum angular momentum differ from classical angular momentum? Why is a quantum state with $\ell = 0$ spherically symmetric, whereas a classical state with $L = 0$ has a straight line trajectory through the origin? Why is it customary to choose $\hat{L}^2$ and $\hat{L}_z$ as the commuting operators for which to find simultaneous eigenfunctions?

2. Prove that $\hat{\mathbf{L}}$ is Hermitian, that $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$, that $\left[\hat{L}_x, \hat{p}_y\right] = i\hbar\hat{p}_z$, and that $\left[\hat{L}_x, \hat{p}^2\right] = 0$.

3. Prove that $\left[\hat{L}^2, \hat{\mathbf{L}}\right] = 0$, that $\left[\hat{L}^2, V(r)\right] = 0$, and that $\left[\hat{L}^2, \hat{p}^2\right] = 0$.

4. Prove that $\hat{L}^2$ and $\hat{\mathbf{L}}$ commute with the parity operator and that the parity of $Y_\ell^m(\theta,\phi)$ is $(-1)^\ell$.

5. By making the substitutions $x = \cos\theta$, $G_{\ell m}(\theta) \to P_\ell^m(x)$, prove that the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dG_{\ell m}(\theta)}{d\theta} - \frac{m^2}{\sin^2\theta}G_{\ell m}(\theta) = -\ell(\ell+1)\,G_{\ell m}(\theta)$$

can be transformed into

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_\ell^m(x)}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m(x) = 0.$$

6–7. Classically, if the magnitude of the angular momentum is $100\hbar$ and the z-component of angular momentum is $50\hbar$, by what angle can the motion deviate from the xy plane. Plot $W_{\ell m}(\theta) = 2\pi\sin\theta\left|Y_\ell^m(\theta,\phi)\right|^2$ as a function of θ for $\ell = 100$ and $m = 50$ to see if the quantum result corresponds to the classical one. Repeat the plot for $\ell = 100$ and $m = 100$ and for $\ell = 100$ and $m = 0$ and interpret your results.

8. A rigid rotator of mass m has a Hamiltonian given by

$$\hat{H} = \frac{\hat{L}^2}{2ma^2},$$

where a is a constant. Find the eigenfunctions and eigenenergies of the rigid rotator. Are these *rotational* energy levels equally spaced? For an H_2O molecule at room temperature, estimate the number of energy levels that are occupied and the frequency spacing of the lowest rotational transition. Show that your result implies that heating in a microwave oven, which uses a frequency of about 2.4 GHz, does *not* occur by resonant absorption by the water molecules. Of course, the bond lengths in molecules are *not* rigid, giving rise to vibrations that modify the energy levels of the "rigid" rotator.

9–10. In general, what can you say about the simultaneous eigenfunctions of $\hat{L}^2$ and $\hat{L}_x$? Specifically show that the eigenfunctions of $\hat{L}_x$ for $\ell = 1$ are

$$\Phi_{\ell=1,\ell_x}(\theta,\phi) = \begin{cases} \frac{1}{2}\left[Y_1^1(\theta,\phi) + \sqrt{2}Y_1^0(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right] \\ \frac{1}{\sqrt{2}}\left[Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)\right] \\ \frac{1}{2}\left[Y_1^1(\theta,\phi) - \sqrt{2}Y_1^0(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right] \end{cases}$$

and find the eigenvalues associated with these states (what must they be?). Note that $|\Phi_{\ell=1,\ell_x}(\theta,\phi)|^2$ now depends on ϕ in a non-trivial way. Prove, however, that $|\Phi_{\ell=1,\ell_x}(\theta,\phi)|^2$ depends only on the angle between the position vector and the x-axis.