Chapter 24
Decay of a Discrete State into a Continuum of States: Fermi's Golden Rule

The last topics I will cover are Fermi's Golden Rule and irreversible decay. These are pretty interesting topics that involve situations in which a discrete state of a quantum system is coupled to an infinite *continuum* of quantum states. Although one starts with a Hermitian Hamiltonian, there is *irreversible* behavior in the system. This is approximately true in many cases of practical interest. For example, an atom that is prepared in an excited state decays as a result of its interaction with the continuum of vacuum field modes. A hydrogen atom placed in a high frequency optical field is photo-ionized into a continuum of (Coulomb modified) free-particle states. The reason we can get irreversible behavior in these cases is connected with the fact that the quantization volume goes to infinity. This feature in already seen in the quantum revivals of a wave packet in the infinite square square well discussed in Chap. 6. There are always quantum revivals, but the revival time goes to infinity as the well size approaches infinity. In other words, an outgoing wave packet can never be reflected back by the boundary of the quantization volume. In problems of this nature, I start by writing equations in which *all* states are discrete and then take the limit in which *some* of the states form a continuum.

24.1 Discrete State Coupled to a Continuum

The generic problem that I am interested in is illustrated in Fig. 24.1. A discrete state, labeled by 0 and having energy E_0, is coupled to a large number of states labeled by n. Eventually, I take the limit of the number of states going to infinity and the energy spacing ϵ between the states going to zero. In doing so I will replace any sums over n by an integral over energy using the prescription

$$\sum_n \rightarrow \int \rho(E) dE, \tag{24.1}$$

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Fig. 24.1 Discrete state 0 coupled to a discrete continuum of states labeled by n. The energy difference between successive levels is ϵ

where $\rho(E)$ is referred to as the (energy) *density of states*. I have already calculated the density of states for the radiation field in Chap. 1 and for matter waves in Chap. 5.

There are two general classes of problems in which levels schemes such as those shown in Fig. 24.1 are encountered. In processes such as spontaneous emission, the Hamiltonian is of the form

$$\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} + \hat{V}_{AF}, \tag{24.2}$$

where $\hat{H}_{\text{atom}}$ is the atomic Hamiltonian, $\hat{H}_{\text{field}}$ is the (quantized) Hamiltonian for the vacuum radiation field, and $\hat{V}_{AF}$ is the atom–field interaction potential. The eigenstates of $\hat{H}_{\text{field}}$ constitute the continuum levels of the problem. The atom is prepared initially in some excited state and decays to a lower energy state as a result of the interaction with the vacuum field. In this problem, all operators are bona-fide, time-independent Schrödinger operators.

In the second class of problems such as photoionization, a time-dependent classical field drives an electron in an atom from some initial bound state to a continuum of unbound energy states. Although the ionized electron still sees a residual field from the ion that is left behind, it is sometimes a good approximation to consider the final states of the electron as free-particle states. The external field is assumed to have a constant amplitude and to oscillate at frequency ω.

In addressing both types of problems, I assume that at $t = 0$ the atom is prepared in eigenket $|0\rangle$ and begins to make transitions to states $|n\rangle$. In the interaction representation, the state amplitudes evolve according to

$$\dot{c}_0(t) = -\frac{i}{\hbar}\sum_n V_{0n}(t)e^{-i\omega_{n0}t}c_n(t) \tag{24.3}$$

$$\dot{c}_n(t) = -\frac{i}{\hbar}V_{n0}(t)e^{i\omega_{n0}t}c_0(t), \tag{24.4}$$

where

$$\omega_{n0} = \left(E_n - E_0\right)/\hbar. \tag{24.5}$$

and $V_{n0}(t)$ are matrix elements of the interaction that couple the initial state to the continuum states. For the problems I consider, either $V_{n0}(t)$ is constant or it oscillates at frequency ω_L. In the latter case, I will make the RWA and replace $V_{n0}(t)$ by $V_{n0}e^{-i\omega_L t}/2$; the net effect is that of a constant perturbation with

$$\omega_0 \to \omega_0' = \omega_0 + \omega_L. \tag{24.6}$$

Thus without loss of generality, I can take the equations for the state amplitudes to be

$$\dot{c}_0(t) = -\frac{i}{\hbar}\sum_n V_{0n}e^{-i\omega_{n0}t}c_n(t) \tag{24.7a}$$

$$\dot{c}_n(t) = -\frac{i}{\hbar}V_{n0}e^{i\omega_{n0}t}c_0(t), \tag{24.7b}$$

where the coupling matrix elements are *constants* and any time dependence of an applied oscillatory field is incorporated by a redefinition of the initial state energy

$$E_0 \to E_0' = \hbar\omega_0' = \hbar\left(\omega_0 + \omega_L\right). \tag{24.8}$$

I integrate Eq. (24.7b) and substitute the result into Eq. (24.7a) to obtain

$$\dot{c}_0(t) = -\frac{1}{\hbar^2}\sum_n |V_{n0}|^2 \int_0^t dt' e^{-i\omega_{n0}(t-t')}c_0(t'). \tag{24.9}$$

The prescription (24.1) is used to convert the sum to an integral,

$$\dot{c}_0(t) = -\frac{1}{\hbar^2}\int dE\rho(E)\,|V(E)|^2 \int_0^t dt' e^{-i(E-E_0)(t-t')/\hbar}c_0(t'), \tag{24.10}$$

where I have set $V_{n0} \equiv V(E)$ to allow for the possibility that the matrix elements depend on energy. To proceed further I need to know something about the nature of $\rho(E)$ and $V(E)$. For the moment I assume that these quantities are slowly varying functions of energy compared with the exponential factor in the integrand. Since the exponential factor makes it maximum contribution at $E = E_0$, I evaluate both $\rho(E)$ and $V(E)$ at $E = E_0$ and remove them from the integral. Moreover I assume that the range of allowed energies in the integration extends from $-\infty$ to ∞. If this is the case, then

$$\begin{aligned}\dot{c}_0(t) &= -\frac{|V(E_0)|^2}{\hbar^2}\rho(E_0)\int_0^t dt' \int_{-\infty}^{\infty} dE e^{-i(E-E_0)(t-t')/\hbar}c_0(t') \\ &= -\frac{2\pi\,|V(E_0)|^2}{\hbar}\rho(E_0)\int_0^t dt'\delta\left(t-t'\right)c_0(t') = -\frac{\Gamma}{2}c_0(t),\end{aligned} \tag{24.11}$$

where

$$\Gamma = \frac{2\pi}{\hbar} |V(E_0)|^2 \rho(E_0) \tag{24.12}$$

is the decay rate of the initial state population. (In deriving this result, I used the relationship $\int_0^t dt' \delta(t - t') = 1/2$, based on the fact that $\delta(t - t')$ is a symmetric function about $t' = t$.) Equation (24.12) is known as Fermi's *Golden Rule*, although it is usually derived using a somewhat different approach.

From Eq. (24.11), we see that

$$c_0(t) = e^{-\Gamma t/2}; \qquad |c_0(t)|^2 = e^{-\Gamma t}. \tag{24.13}$$

Both the initial state amplitude and initial state probability undergo *exponential decay*. The irreversible behavior occurs because I have taken an infinite quantization volume—the emitted ionized electron or emitted photon cannot return to re-excite the quantum system undergoing decay.

Although the behavior is irreversible, the fact that the decay is *exponential* depends on the assumptions that were made concerning the density of states and $V(E)$. If both $\rho(E)$ and $V(E)$ are constant for $-\infty < E < \infty$, then Eq. (24.11) is *exact*. This results in what is known is a *Markov process* since the effective correlation time of the continuum of states giving rise to the decay is equal to zero. The delta function appearing in the integrand of Eq. (24.11) is a signature of the fact that the process is Markovian—it has no temporal memory. At any instant of time, the decay is independent of past events—it is *always* exponential. Of course, both $\rho(E)$ and $V(E)$ usually depend on energy. Moreover the energy spectrum is usually bounded from below since negative kinetic energies of particles and negative energies of photons are not allowed. As a consequence, the decay of a discrete state into the continuum can never be *exactly* exponential. However, to a very good approximation, processes such as spontaneous decay and particle decay can be represented as Markovian in nature. I will return to this point in discussing the Zeno effect later in this chapter. I will also discuss how the decay process is modified if the continuum is bounded from above and below. For the present, however, I consider two examples, photoionization and spontaneous decay.

24.1.1 Photoionization

As a first example of Fermi's Golden Rule, I consider the photoionization of hydrogen from its ground state produced by a high frequency (X-ray) radiation field. It is assumed that the frequency of the field is much greater than the ionization energy of hydrogen divided by h. In this limit, the electron that emerges can be treated in first approximation as a free particle. Moreover, since the matrix element involves a coupling from the ground state, the integral that determines the value of the matrix element is restricted to radii on the order of the Bohr radius. As a consequence, I use the dipole approximation and assume an interaction potential of the form

$$\hat{V}(t) = -\hat{\mathbf{p}}_e \cdot \mathbf{E}_0 \cos(\omega_L t) = e\mathbf{E}_0 \cdot \hat{\mathbf{r}} \cos(\omega_L t), \tag{24.14}$$

where $\mathbf{E}_0$ is the field amplitude, $\hat{\mathbf{p}}_e$ is the dipole moment operator of the atom, and $\hat{\mathbf{r}}$ is the position operator of the atom. In the rotating wave approximation, the equations of motion for the state amplitudes are

$$\dot{c}_0(t) = -\frac{i}{\hbar}\sum_n V_{0n} e^{-i(E_n - E_0 - \hbar\omega_L)t/\hbar} c_n(t) \tag{24.15a}$$

$$\dot{c}_n(t) = -\frac{i}{\hbar} V_{n0} e^{i(E_n - E_0 - \hbar\omega_L)t/\hbar} c_0(t), \tag{24.15b}$$

where

$$V_{0n} = -\langle n|\hat{\mathbf{p}}_e \cdot \mathbf{E}_0|0\rangle/2 \tag{24.16}$$

is one-half the matrix element of the interaction operator between the ground state $|0\rangle = |n = 1, \ell = 0, m = 0\rangle$ of the electron in hydrogen and the final state $|n\rangle$ of the electron, which is approximated as a free-particle state. For the ionization problem, I have to evaluate matrix elements of $e\mathbf{E}_0 \cdot \hat{\mathbf{r}}$ between the final free particle states (quantized using periodic boundary conditions having period L in all directions),

$$\left(\psi_f\right)_n = \frac{e^{i\mathbf{k}_n\cdot\mathbf{r}}}{\sqrt{L^3}} \to \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{L^3}}, \tag{24.17}$$

and the initial state

$$\psi_0 = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}, \tag{24.18}$$

where a_0 is the Bohr radius. In the final state wave function

$$k_{nx} = \frac{2\pi n_y}{L}; \quad k_{ny} = \frac{2\pi n_y}{L}; \quad k_{nz} = \frac{2\pi n_z}{L}, \tag{24.19}$$

and the n_x, n_y, n_z are integers (positive, negative, or zero), as discussed in the Appendix of Chap. 5. To go over to continuum states I use the prescription given by Eq. (5.161), namely

$$\sum_{n_x,n_y,n_z} \to \left(\frac{L}{2\pi}\right)^3 \int d\mathbf{k} = \left(\frac{L}{2\pi}\right)^3 \int k^2 dk d\Omega_k. \tag{24.20}$$

I can then talk about a *density of states per unit solid angle* $\rho(E, \Omega_k)$ by setting

$$\left(\frac{L}{2\pi}\right)^3 \int k^2 dk = \int \rho(E, \Omega_k) dE. \tag{24.21}$$

With $k_E = \sqrt{2mE/\hbar^2}$ and m the electron mass, I find that $\rho(E, \Omega_k)$ is then given by

$$\rho(E, \Omega_k) = \frac{mL^3}{8\pi^3\hbar^2} k_E. \tag{24.22}$$

According to Eq. (24.6), I must evaluate k_E at

$$k_E = k_f = \sqrt{\frac{2m}{\hbar}(\omega_L - \omega_0)} \tag{24.23}$$

where

$$\hbar\omega_0 = 13.6 \text{ eV} = -E_0 \tag{24.24}$$

is the ionization energy of hydrogen. It is assumed that $\omega_L \gg \omega_0$, but that $(\omega_L - \omega_0) / (\omega_L + \omega_0) \ll 1$ to insure the validity of the RWA.

If I take the field along the z axis, then

$$\hat{V}(t) = e\mathcal{E}_0 \hat{z} \cos(\omega_L t) = e\mathcal{E}_0 \widehat{r \cos\theta} \cos(\omega_L t), \tag{24.25}$$

where $\mathcal{E}_0$ is the magnitude of the field amplitude. From Eq. (24.12), I can calculate the *photoionization rate per unit solid angle* as

$$\Gamma(\Omega_k) = \frac{d\Gamma}{d\Omega_k} = \frac{2\pi}{\hbar} \left|V(E_f, \Omega_k)\right|^2 \rho(E_f = \hbar^2 k_f^2/2m, \Omega_k) \tag{24.26}$$

where

$$V(E_f, \Omega_k) = \frac{1}{2} \int d\mathbf{r} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{L^3}} r \cos\theta \frac{e^{-r/a}}{\sqrt{\pi a^3}}\bigg|_{k=k_f}. \tag{24.27}$$

As you can see, the matrix element $V(E_f)$ depends on the direction of emission $\mathbf{k}$ of the electron.

To evaluate the matrix element, I use the spherical wave expansion

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell j_\ell(kr) \left[Y_\ell^m(\theta, \phi)\right]^* Y_\ell^m(\theta_k, \phi_k), \tag{24.28}$$

where (θ, ϕ) are the spherical angles of $\mathbf{r}$ and (θ_k, ϕ_k) are the spherical angles of $\mathbf{k}$. Since $\cos\theta = \sqrt{\frac{4\pi}{3}} Y_1^0(\theta, \phi)$, the integral over angles (θ, ϕ) in Eq. (24.27) is nonvanishing only if $\ell = 1$ and $m = 0$, yielding

$$\begin{aligned}\int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r \cos\theta e^{-r/a} &= 4\pi i \sqrt{\frac{3}{4\pi}} Y_1^0(\theta_k, \phi_k) \int_0^\infty dr j_1(kr) r^3 e^{-r/a} \\ &= 4\pi i \frac{8a^5 k \cos\theta_k}{(1 + k^2a^2)^3}.\end{aligned} \tag{24.29}$$

By combining Eqs. (24.26), (24.22), (24.27), and (24.29) I arrive at

$$\Gamma(\Omega_k) = \frac{64 m k_f^3 a_0^7 e^2 \mathcal{E}_0^2 \cos^2\theta_k}{\pi\hbar^3 \left(1 + k_f^2 a_0^2\right)^6}. \tag{24.30}$$

The total ionization rate is

$$\Gamma = \int d\Omega_k \Gamma(\Omega_k) = \frac{4\pi}{3} \frac{64 m k_f^3 a_0^7 e^2 \mathcal{E}_0^2}{\pi\hbar^3 \left(1 + k_f^2 a_0^2\right)^6}. \tag{24.31}$$

Note that

$$\begin{aligned}k_f a_0 &= \sqrt{\frac{2mca_0^2}{\hbar} \frac{(\omega_L - \omega_0)}{c}} \approx \sqrt{\frac{2mca_0}{\hbar} \frac{\omega_L a_0}{c}} \\ &= \sqrt{\frac{2mca_0}{\hbar} k_L a_0} \approx 17\sqrt{k_L a_0}.\end{aligned} \tag{24.32}$$

For the dipole approximation to be valid, it is necessary that $k_L a_0 \ll 1$. On the other hand, the theory is probably valid only in the limit that $k_f a_0 \gg 1$. Thus the result should not be viewed as an accurate description, except for a limited range of parameter space.

24.1.2 Spontaneous Decay

As a second example, I calculate the spontaneous emission rate at which an atom decays from an excited state to its ground state. To solve this problem it is necessary to quantize the radiation field since the continuum states that lead to decay are those of the vacuum field. You can find discussions of field quantization in most any book on quantum optics. I will not present any formal derivation of field quantization. A standard treatment involves the use of periodic boundary conditions for the field.

Once the normal modes of the field are found (in this case, plane waves subject to periodic boundary conditions), creation and annihilation operators are assigned to each field mode. The net effect is that the Hamiltonian for the quantized field can be written as

$$\hat{H}_{\text{field}} = \sum_{\mathbf{k}_n} \hbar\omega_{k_n} a^{\dagger}_{\mathbf{k}_n} a_{\mathbf{k}_n} \tag{24.33}$$

where a_{k_n} ($a^{\dagger}_{k_n}$) is a destruction (creation) operator for mode n having frequency $\omega_{k_n} = ck_n$. The summation index $\mathbf{k}_n$ is meant to imply a summation over $\{n_x, n_y, n_z\}$ with

$$\mathbf{k}_n = \frac{2\pi n_x}{L}\mathbf{u}_x + \frac{2\pi n_y}{L}\mathbf{u}_y + \frac{2\pi n_z}{L}\mathbf{u}_z. \tag{24.34}$$

The corresponding electric field is

$$\mathbf{E}(\mathbf{R}) = i\sum_{\mathbf{k}_n}\left(\frac{\hbar\omega_{k_n}}{2\epsilon_0\mathcal{V}}\right)^{1/2} \boldsymbol{\epsilon}_{\mathbf{k}_n}\left(a_{\mathbf{k}_n}e^{i\mathbf{k}_n\cdot\mathbf{R}} - a^{\dagger}_{\mathbf{k}_n}e^{-i\mathbf{k}_n\cdot\mathbf{R}}\right), \tag{24.35}$$

where $\mathcal{V} = L^3$ is the quantization volume and $\boldsymbol{\epsilon}_{\mathbf{k}_n}$ is the field polarization unit vector. There are two, independent polarizations for each $\mathbf{k}$. The creation and destruction operators for the field are similar to those for the harmonic oscillator, having the same commutation relations. Instead of creating and destroying number states of the oscillator, they create and destroy *photon states* of the field. That is,

$$a^{\dagger}_{\mathbf{k}_n}|0\rangle = |1_{\mathbf{k}_n}\rangle, \tag{24.36}$$

where $|0\rangle$ is the *vacuum state* and $|1_{\mathbf{k}_n}\rangle$ is a state with one photon in mode $\mathbf{k}_n$ of the field. These are the only states I need to discuss spontaneous decay.

Rather than use Fermi's Golden Rule to derive an expression for the transition rate, I will calculate the state vector of the system as a function of time directly. The total Hamiltonian is given by

$$\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} + \hat{V}_{AF},$$

where $\hat{H}_{\text{atom}}$ is the atomic Hamiltonian, $\hat{H}_{\text{field}}$ is the free-field Hamiltonian given by Eq. (24.33), and $\hat{V}_{AF}$ is the atom–field interaction potential given by

$$\hat{V}_{AF} = -\hat{\mathbf{p}}_e \cdot \mathbf{E}(\mathbf{R} = \mathbf{0}), \tag{24.37}$$

where $\hat{\mathbf{p}}_e$ is the dipole moment operator of the atom. The nucleus of the atom is assumed to be fixed at the origin.

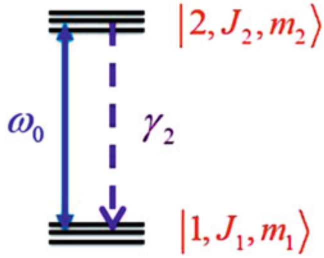

Fig. 24.2 Spontaneous emission between an excited and ground state manifold of levels

The eigenkets of $\hat{H}_{\text{atom}}$ are written as $|n, J_n, m_n\rangle$, where J_n is the total angular momentum of a manifold of levels in electronic state n and m_n a magnetic quantum number associated with J_n. In practice I consider emission from a state $|2, J_2, m_2\rangle$ which has been excited at time $t = 0$ to the manifold of ground states denoted by $|1, J_1, m_1\rangle$ (see Fig. 24.2). The transition is driven by the atom-vacuum field interaction. Of course, it is impossible to *instantaneously* excite the atom at a given time, so what I really mean is that the atom is excited in a time τ that is much less than the decay rate of the system, but much longer than the inverse of the transition frequency. It is important not to forget that it is simply a vacuum field induced decay process with which we are dealing, since the algebra can get a little messy.

Initially, the system is in the state $|2, J_2, m_2; 0\rangle$ where the 0 labels the vacuum state of the field. In the interaction representation, it follows from Eq. (24.7a) that the excited state amplitude evolves as

$$\dot{c}_{2,J_2,m_2;0}(t) = \frac{1}{i\hbar} \sum_{\mathbf{k},\boldsymbol{\epsilon}_\mathbf{k}} \sum_{m_1} e^{-i(\omega_k - \omega_0)t} \times \langle 2, J_2, m_2; 0 | \hat{V}_{AF} | 1, J_1, m_1; \mathbf{k}, \boldsymbol{\epsilon}_\mathbf{k} \rangle c_{1,J_1,m_1;\mathbf{k},\boldsymbol{\epsilon}_\mathbf{k}}(t) . \quad (24.38)$$

where $\omega_0 = \omega_{21} = (E_2 - E_1)/\hbar$ is the transition frequency, $\{\mathbf{k}, \boldsymbol{\epsilon}_\mathbf{k}\}$ labels a one-photon state of the field having propagation vector $\mathbf{k}$ and polarization $\boldsymbol{\epsilon}_\mathbf{k}$. The sum over $\boldsymbol{\epsilon}_\mathbf{k}$ refers to a sum over the two independent field polarizations for each $\mathbf{k}$ and the sum over $\mathbf{k}$ is a shorthand notation for summing over all $\mathbf{k}_n$. A formal solution for the ground state amplitude, obtained using Eq. (24.7b), is

$$c_{1,J_1,m_1;\mathbf{k},\boldsymbol{\epsilon}_\mathbf{k}}(t) = \frac{1}{i\hbar} \int_0^t e^{i(\omega_k - \omega_0)t'} \sum_{J_2', m_2'} \times \langle 1, J_1, m_1; \mathbf{k}, \boldsymbol{\epsilon}_\mathbf{k} | \hat{V}_{AF} | 2, J_2', m_2'; 0 \rangle c_{2,J_2',m_2';0}(t') , \quad (24.39)$$

which, when substituted back into Eq. (24.38), yields

$$\dot{c}_{2,J_2,m_2;0}(t) = -\frac{1}{\hbar^2}\sum_{\mathbf{k},\boldsymbol{\epsilon}_{\mathbf{k}}}\sum_{J_2',m_2'}\int_0^t e^{-i(\omega_k-\omega_0)(t-t')}c_{2,J_2',m_2';0}(t')$$
$$\times\langle 2,J_2,m_2;0|\hat{V}_{AF}|1,J_1,m_1;\mathbf{k},\boldsymbol{\epsilon}_{\mathbf{k}}\rangle$$
$$\langle 1,J_1,m_1;\mathbf{k},\boldsymbol{\epsilon}_{\mathbf{k}}|\hat{V}_{AF}|2,J_2',m_2';0\rangle\,, \tag{24.40}$$

where I have allowed for transitions back to another level J_2' in the same state 2 electronic state manifold.

The sum over **k** is converted to an integral using

$$\sum_{\mathbf{k}_n}\rightarrow\frac{\mathcal{V}}{(2\pi)^3}\int d^3k=\frac{\mathcal{V}}{(2\pi)^3}\int_0^\infty k^2dk\int d\Omega_k$$
$$=\frac{\mathcal{V}}{(2\pi)^3}\int_0^\infty\frac{\omega_k^2}{c^3}d\omega_k\int d\Omega_k. \tag{24.41}$$

In this continuum limit, Eq. (24.35) is replaced by

$$\mathbf{E}(\mathbf{R}=\mathbf{0})=i\frac{1}{(2\pi)^3}\int_0^\infty\frac{\omega_k^2}{c^3}d\omega_k\int d\Omega_k\left(\frac{\hbar\omega_k\mathcal{V}}{2\epsilon_0}\right)^{1/2}\boldsymbol{\epsilon}_{\mathbf{k}}\left(a_{\mathbf{k}}-a_{\mathbf{k}}^\dagger\right), \tag{24.42}$$

By combining Eq. (24.37) with Eqs. (24.40)–(24.42), I find that the quantization volume cancels (as it must, if the results are to make any sense) and that I am faced with an integral of the form

$$\int_0^t dt'\int_0^\infty\omega_k^3d\omega_k e^{-i(\omega_k-\omega_0)(t-t')}c_{2,J_2',m_2';0}(t'). \tag{24.43}$$

Although the integral over ω_k diverges, it is not unreasonable to cut off the integral at some value of $(\omega_k-\omega_0)$ of order ω_0.[1] With such a cutoff, the major contribution to the ω_k integral comes from a region $|\omega_k-\omega_0|\sim\gamma\ll\omega_0$. This allows me to replace ω_k^3 by ω_0^3, remove it from the integral and extend the lower integration limit of the ω_k integral to $-\infty$. These two approximations constitute the so-called *Weisskopf-Wigner approximation*,[2] and are equivalent to the Markov

[1] Based on theoretical considerations, the cutoff frequency could be determined by the range of validity of the dipole approximation, $ka_0=\omega_ka_0/c\approx 1$, which gives an ω_k(cutoff) of order 10^{18} s^{-1}. Based on *experimental* considerations, the cutoff can be taken at $\omega_k(\text{cutoff})=\omega_0+(1/T)$, where $\gamma\ll(1/T)\ll\omega_0$ and T is the time it takes to excite the atom to its ground to excited state [see P. R. Berman, *Wigner-Weisskopf approximation under typical experimental conditions*, Physical Review A **72**, 025804 (2005)].

[2] V. Weisskopf and E. Wigner, *Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie*, Zeitschrift für Physik 63, pp. 54–73 (1930); a translation is available, *Calculation of the Natural Line Width on the Basis of Dirac's Theory of Light,* in W. R. Hindmarsh, *Atomic*

approximation used in arriving at Fermi's Golden Rule. In the Weisskopf–Wigner approximation,

$$\begin{aligned}&\int_0^t dt' \int_0^\infty \omega_k^3 d\omega_k e^{-i(\omega_k-\omega_0)(t-t')} c_{2,J_2',m_2';0}(t') \\ &\approx \omega_0^3 \int_0^t dt' \int_{-\infty}^\infty d\omega_k e^{-i(\omega_k-\omega_0)(t-t')} c_{2,J_2',m_2';0}(t') \\ &= 2\pi\omega_0^3 \int_0^t \delta\left(t-t'\right) c_{2,J_2',m_2';0}(t')dt' = \pi\omega_0^3 c_{2,J_2',m_2';0}(t). \end{aligned} \tag{24.44}$$

Combining this result with Eq. (24.37) with Eqs. (24.40)–(24.42), I find

$$\begin{aligned}\dot{c}_{2,J_2,m_2;0}(t) = -\frac{1}{\hbar}\frac{\pi\omega_0^3}{2\epsilon_0(2\pi)^3c^3}\sum_{J_2',m_2'}\int d\Omega_k \\ \times\sum_{\boldsymbol{\epsilon}_{\mathbf{k}}}\langle 2,J_2,m_2|\hat{\mathbf{p}}_e\cdot\boldsymbol{\epsilon}_{\mathbf{k}}|1,J_1,m_1\rangle \\ \times\langle 1,J_1 m_1|\hat{\mathbf{p}}_e\cdot\boldsymbol{\epsilon}_{\mathbf{k}}|2,J_2',m_2'\rangle c_{2,J_2',m_2';0}(t)\,. \end{aligned} \tag{24.45}$$

I now need to carry out the angular integration. To do so I must write explicit expressions for the unit polarization vectors. The unit vectors $\boldsymbol{\epsilon}_{\mathbf{k}}^{(1)}$, $\boldsymbol{\epsilon}_{\mathbf{k}}^{(2)}$, and $\hat{\mathbf{k}}$ make up a right-handed system with

$$\hat{\mathbf{k}} = \sin\theta_k\cos\phi_k\mathbf{u}_x + \sin\theta_k\sin\phi_k\mathbf{u}_y + \cos\theta_k\mathbf{u}_z\,; \tag{24.46a}$$

$$\boldsymbol{\epsilon}_{\mathbf{k}}^{(1)} = \hat{\boldsymbol{\theta}}_k = \cos\theta_k\cos\phi_k\mathbf{u}_x + \cos\theta_k\sin\phi_k\mathbf{u}_y - \sin\theta_k\mathbf{u}_z\,; \tag{24.46b}$$

$$\boldsymbol{\epsilon}_{\mathbf{k}}^{(2)} = \hat{\boldsymbol{\phi}}_k = -\sin\phi_k\mathbf{u}_x + \cos\phi_k\mathbf{u}_y\,. \tag{24.46c}$$

It is then straightforward to carry out the summations and integrations in Eq. (24.45) to obtain

$$\begin{aligned}\sum_{\lambda=1}^{2}\sum_{i,j=1}^{3} d_i\bar{d}_j\int d\Omega_k\left(\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}\right)_i\left(\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}\right)_j &= \frac{8\pi}{3}\sum_{i,j}\delta_{ij}d_i\bar{d}_j \\ &= \frac{8\pi}{3}\mathbf{d}\cdot\bar{\mathbf{d}}, \end{aligned} \tag{24.47}$$

Spectra (Pergamon Press, London, 1967), pp. 304–327. For an extensive discussion of the validity of the Weisskofp-Wigner approximation, see the article by Paul R. Berman and George W. Ford, *Spontaneous Decay, Unitarity, and the Weisskopf-Wigner Approximation*, in *Advances in Atomic, Molecular, and Optical Physics*, edited by E. Arimondo, P. R. Berman, and C. C. Lin (Elsevier-Academic Press, New York, 2010), volume **59**, pp. 175–221.

where

$$\mathbf{d} = \langle 2, J_2, m_2|\hat{\mathbf{p}}_e|1, J_1, m_1\rangle; \tag{24.48a}$$

$$\bar{\mathbf{d}} = \langle 1, J_1 m_1|\hat{\mathbf{p}}_e|2, J_2', m_2'\rangle. \tag{24.48b}$$

I have denoted the first and second matrix elements in Eq. (24.45) by **d** and $\bar{\mathbf{d}}$, respectively.

To proceed further, I write the components of the matrix elements **d** and $\bar{\mathbf{d}}$, as well as those of the dipole moment operator, in spherical form as

$$\hat{p}_{e1} = -\frac{\hat{p}_{ex} + i\hat{p}_{ey}}{\sqrt{2}}, \quad \hat{p}_{e,-1} = \frac{\hat{p}_{ex} - i\hat{p}_{ey}}{\sqrt{2}}, \quad \hat{p}_{e0} = \hat{p}_{ez}\,, \tag{24.49a}$$

$$d_1 = -\frac{d_x + id_y}{\sqrt{2}}, \quad d_{-1} = \frac{d_x - id_y}{\sqrt{2}}, \quad d_0 = d_z, \tag{24.49b}$$

such that

$$\mathbf{d} \cdot \bar{\mathbf{d}} = \sum_q (-1)^q d_q \bar{d}_{-q}\,. \tag{24.50}$$

I can now use the Wigner-Eckart theorem and the relationship

$$\left(\hat{p}_{e,q}\right)^\dagger = (-1)^q \hat{p}_{e,-q} \tag{24.51}$$

to evaluate

$$\begin{aligned} d_q &= \langle 2, J_2, m_2|\hat{p}_{e,q}|1, J_1, m_1\rangle \\ &= \frac{1}{\sqrt{2J_2+1}} \begin{bmatrix} J_1 & 1 & J_2 \\ m_1 & q & m_2 \end{bmatrix} \langle 2, J_2\|p^{(1)}\|1, J_1\rangle\,, \end{aligned} \tag{24.52}$$

and

$$\begin{aligned} \bar{d}_{-q} &= \langle 1, J_1 m_1|\hat{p}_{e,-q}|2, J_2', m_2'\rangle \\ &= \left(\langle 2, J_2', m_2'|\left(\hat{p}_{e,-q}\right)^\dagger|1, J_1, m_1\rangle\right)^* \\ &= (-1)^q \left(\langle 2, J_2' m_2'|\hat{p}_{e,q}|1, J_1, m_1\rangle\right)^* \\ &= \frac{1}{\sqrt{2J_2+1}} (-1)^q \begin{bmatrix} J_1 & 1 & J_2 \\ m_1 & q & m_2' \end{bmatrix} \left(\langle 2, J_2\|p_e^{(1)}\|1, J_1\rangle\right)^*\,. \end{aligned} \tag{24.53}$$

It then follows that

$$\sum_{q,m_1}(-1)^q d_q \bar{d}_{-q} = \frac{1}{2J_2+1}|\langle 2, J_2\|p_e^{(1)}\|1, J_1\rangle|^2 \times \sum_{q,m_1}\begin{bmatrix} J_1 & 1 & J_2 \\ m_1 & q & m_2 \end{bmatrix}\begin{bmatrix} J_1 & 1 & J_2' \\ m_1 & q & m_2' \end{bmatrix} = \frac{1}{2J_2+1}|\langle 2, J_2\|p_e^{(1)}\|1, J_1\rangle|^2 \delta_{J_2,J_2'}\delta_{m_2,m_2'}, \tag{24.54}$$

where the orthogonality property of the Clebsch-Gordan coefficients has been used. The sum is proportional to $\delta_{J_2,J_2'}\delta_{m_2,m_2'}$. Incorporating Eqs. (24.47) and (24.54) into Eq. (24.45), I find the upper state amplitude decays as

$$\dot{c}_{2,J_2,m_2;0}(t) = -\gamma c_{2,J_2,m_2;0}(t), \tag{24.55}$$

which has as solution [given $c_{2,J_2,m_2;0}(0) = 1$],

$$c_{2,J_2,m_2;0}(t) = e^{-\gamma t}, \tag{24.56}$$

where

$$\gamma = \gamma_{2,J_2;1,J_1}/2 = \frac{2}{3}\frac{\alpha_{FS}}{2J_2+1}|\langle 2, J_2\|r^{(1)}\|1, J_1\rangle|^2 \frac{\omega_{21}^3}{c^2}, \tag{24.57}$$

α_{FS} is the fine structure constant, $\langle 2, J_2\|r^{(1)}\|1, J_1\rangle$ is the reduced matrix element for the position operator, and $\gamma_{2,J_2;1,J_1}$ is the *decay rate* from state $|2, J_2\rangle$ to $|1, J_1\rangle$. For dipole allowed optical transitions from an excited state to the ground state, $\gamma_{2,J_2;1,J_1}/\omega_{21}$ is of order 10^{-7}; that is, decay rates from the first excited states of atoms back to the ground state are typically in the nanosecond to tens of nanoseconds range.

Using the quantized vacuum field, I have derived a fundamental and important equation for the spontaneous emission rate. The fact that $\dot{c}_{2,J_2,m_2;0}$ is not coupled to $c_{2,J_2',m_2';0}$ for $J_2' \neq J_2$ or $m_2' \neq m_2$ can be understood simply in terms of conservation of angular momentum. States in the excited state manifold differing in total angular momentum or the z-component of angular momentum cannot be coupled by "emitting" and "absorbing" the *same* photon. Equation (24.57) also contains the important result that the decay rate is proportional to the cube of the transition frequency.[3] I had to invoke the Weisskopf-Wigner approximation to carry out the calculation. When used consistently, the Weisskopf-Wigner approximation

[3]This assumes that the reduced matrix element is independent of frequency as it is for atoms. For an oscillator, however, the square of the reduced matrix element varies inversely with the transition frequency such that the decay rate depends on the square of the transition frequency.

results in overall conservation of probability for the atom-field system. You can use Eqs. (24.39) and (24.56) to evaluate $\sum_{m_1} \left|c_{1,J_1,m_1;\mathbf{k},\boldsymbol{\epsilon}_\mathbf{k}}(\infty)\right|^2$, which provides a measure of the spectral and directional properties of the emitted radiation (see problems).

24.2 Bounded Continuum

In many situations, the continuum is *bounded*. For example, in spontaneous decay, you cannot have emission at negative frequencies—the continuum is bounded from below at $E = 0$. When the continuum is bounded, there can never be purely exponential decay and the exact solutions are very complicated. However, there are two limiting cases that can be solved approximately. If the energy of the continuum encompasses the discrete state (as it does in spontaneous emission since the discrete state has energy $\hbar\omega_0$), and the width of the continuum is much larger than the decay rate (as in spontaneous emission), then the decay is exponential to a very good approximation. In this limit, the infinite continuum model provides a good approximation to the exact result.

The second case is very different. Imagine in the photoelectric effect that you send in a field having frequency ω_L for which $\hbar\omega_L$ is *less* than the work function $E_a = \hbar\omega_a$. In that case, you might think that the initial state amplitude remains equal to unity and this is *approximately* true. But some of the initial state amplitude is lost. The situation is similar to that in off-resonant Rayleigh scattering, where there is a small, but non-vanishing, excited state probability amplitude. To model this effect, I take the zero of energy such that $E_0 = 0$ and assume that the continuum extends from energy $E_a = \hbar\omega_a$ to $E_b = \hbar\omega_b$, with $E_b > E_a > \hbar\omega_L$. In other words, the field frequency is not sufficiently high to effectively drive transitions from the initial state into the continuum.

The coupling matrix elements are taken as $V_{E0}(t) = V_{E0}\cos(\omega_L t)$. I assume the effect of the field is sufficiently weak to allow me to solve Eq. (24.7b) with $c_0(t) \approx 1$. In that case, I find

$$c_E(t) \approx -\frac{i}{\hbar}V_{E0}\int_0^t dt' e^{i\omega_E t'}\cos\left(\omega_L t'\right) \approx \frac{V_{E0}}{2\hbar}\frac{\left(1 - e^{i(\omega_E-\omega_L)t}\right)}{\omega_E - \omega_L}, \tag{24.58}$$

where $\omega_E = E/\hbar$ and I have assumed that $\omega_E - \omega_L \ll \omega_E + \omega_L$. The initial state probability is then given by

$$|c_0(t)|^2 \approx 1 - \int_{-\infty}^{\infty} dE \left|\frac{V_{E0}}{\hbar}\right|^2 \rho(E)\frac{\sin^2\left(\frac{(\omega_E-\omega_L)t}{2}\right)}{(\omega_E - \omega_L)^2}$$

$$= 1 - \left|\frac{V}{\hbar}\right|^2 \rho \int_{E_a}^{E_b} dE \frac{\sin^2\left(\frac{(\omega_E - \omega_L)t}{2}\right)}{(\omega_E - \omega_L)^2}, \tag{24.59}$$

where, for simplicity, I assumed that both the matrix elements and density of states are constant over the energy range of the continuum. This can be rewritten as

$$\begin{aligned} |c_0(t)|^2 &\approx 1 - \frac{|V|^2 \rho}{\hbar} \int_{\omega_{aL}}^{\omega_{bL}} d\omega \frac{\sin^2\left(\frac{\omega t}{2}\right)}{\omega^2} \\ &= 1 - \frac{|V|^2 \rho}{\hbar \omega_{aL}} \int_1^{\omega_{bL}/\omega_{aL}} dx \frac{\sin^2\left(\frac{\omega_{aL} t}{2} x\right)}{x^2}, \end{aligned} \tag{24.60}$$

where

$$\omega_{aL} = \omega_a - \omega_L; \qquad \omega_{bL} = \omega_b - \omega_L. \tag{24.61}$$

This whole treatment is valid only if $|c_0(t)|^2 \approx 1$, so the correction term must be small. It *is* small, provided

$$\beta = \frac{|V|^2 \rho}{2\hbar\omega_{aL}} \ll 1, \tag{24.62}$$

a condition that is obtained by replacing $\sin^2(\omega t/2)$ by $1/2$ in the integrand, but β is *not* equal to zero. Let me take $\omega_b = \infty$. The integral is a tabulated function, but not particularly simple. The behavior of $|c_0(t)|^2$ *is* simple, however. It starts from a value of unity, oscillates and decays to a final value equal to

$$P_0(\infty) = |c_0(\infty)|^2 \approx 1 - \frac{1}{2\hbar\omega_{aL}} |V|^2 \rho \int_1^\infty dx \frac{1}{x^2} = 1 - \beta. \tag{24.63}$$

A graph of $P_0(t) = |c_0(t)|^2$, with $|c_0(t)|^2$ given by Eq. (24.60), is shown in Fig. 24.3 for $\beta = 0.1$ as a function of $\omega_{aL} t$. The oscillations appear because the field is turned on suddenly at $t = 0$. If, instead, the matrix coupling element is of the form $V_{E0}(t) = V(t)\cos(\omega_L t)$, where $V(t)$ increases from an initial value of zero to a final value of V in a time that is long compared with ω_{aL}^{-1}, $P_0(t)$ would not be oscillatory; it would decay smoothly as $1 - |V(t)|^2 \rho/(4\hbar\omega_{aL})$, assuming a final value of $1 - \beta/2$, instead of $1 - \beta$ (see problems).

If we apply this idea to photoionization with a field frequency below the ionization frequency, it looks like we violate energy conservation if an electron is ionized. Actually, the "free" electron amplitude oscillates rapidly between zero and a small value for *each* possible final energy state of the ionized electron. Thus the "free" electron cannot get very far away from the atom since each excited state returns to the initial state periodically. To measure the free electron you would have to make a measurement on a time scale less than the inverse of the oscillation

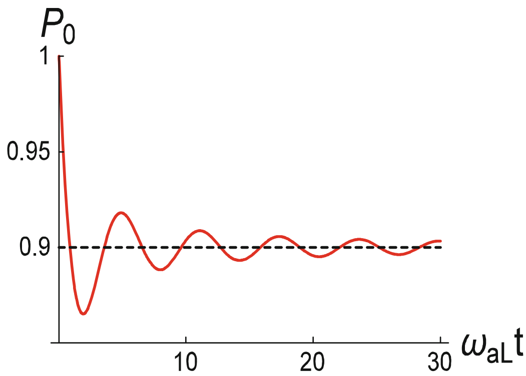

Fig. 24.3 Initial state probability P_0 as a function of $\omega_{aL}t$ for $\beta = 0.1$. The asymptotic value $P_0 \sim 0.9$ is indicated by the dashed line

frequency which would introduce Fourier components to compensate for the energy mismatch. The situation is the same for the anti-resonant component of the vacuum field interacting with a ground state atom. The "emitted" field never gets very far from the atom. On the other hand, such fields, which are of quantum origin, can be viewed as being responsible for the van der Waals interaction. Two ground state atoms separated by a distance less than the wavelength associated with a ground to excited state transition in the atoms "emit" fields that lead to an interaction energy between the atoms.

24.3 Zeno Paradox and Zeno Effect

It has become fashionable for introductory and graduate quantum mechanics texts to mention the *quantum Zeno paradox*.[4] The original Zeno paradoxes consist of a number of scenarios in which it appears that motion is impossible. For example, in the stadium paradox, the question is raised as to how long it will take to cross a stadium. To cross the stadium, you must first reach the midway point. But to reach the midway point, you must first reach the quarter way point, and so on. Since you have to reach an infinite number of points to cross the stadium, it will take you an infinite time to cross the stadium. The "paradox" is that you know it takes a finite time to cross the stadium. Although it is obvious to us that we can traverse an infinite number of points in a finite time, Zeno's paradoxes were troubling to the ancients—Aristotle spent considerable time trying to refute them.

[4]See, for example, David Griffiths, *Introduction to Quantum Mechanics*, Second Edition (Pearson Prentiss Hall, Upper Saddle River, N.J., 2005) (Sect. 12.5) and Eugen Merzbacher, *Quantum Mechanics,* Third Edition (John Wiley and Sons, New York, 1998), Sect. 19.8.

Just like the ancients were confused by Zeno's paradoxes, it appears that the quantum Zeno paradox is confusing to contemporary scientists. Part of the reason for this is related to semantics. The quantum Zeno paradox was originally formulated by Misra and Sudarshan in terms of a bubble chamber experiment.[5] Imagine that an unstable particle enters a bubble chamber and leaves a track of bubbles. Each bubble is an indication that the particle has not yet decayed. In some sense, therefore, the bubbles constitute measurements in which the particle is projected into its original state. By increasing the bubble density, it would seem that the particle is continuously projected into its initial state—it does not decay at all! The paradox is that, in the experiment, the particle decays with its normal decay rate, independent of any bubbles. What is going on?

I have shown that a Markovian decay process has no memory—decay is always exponential. For a Markovian process, you cannot change the decay rate, period. There is no Zeno paradox in this limit. The reason is simple. To modify the decay rate, you must act on the quantum system many times within the *correlation time* of the bath. For a Markovian process the correlation time of the bath giving rise to the relaxation is zero, so it is impossible to inhibit decay. Of course, no decay process is strictly Markovian. However, processes such as spontaneous emission and particle decay are close enough to Markovian to render any attempts to inhibit decay useless.

Why, then, is there all this interest in the Zeno effect. I like to distinguish the *Zeno effect* from the *Zeno paradox*. I have just resolved the Zeno paradox—there can be no change in spontaneous decay rates produced by measurements on the system owing to the Markovian nature of the decay. On the other hand, there can be a Zeno *effect* for coherently driven transitions. I like to illustrate this by asking someone to explain something to me. As soon as she starts talking, I begin to yell gibberish so she must stop and begin the explanation anew. While not really an example of the quantum Zeno effect, it provides the central idea behind the effect. To cause a transition from one state to another, an interaction must build up a phase. We have already seen this in the optical Bloch equations where the application of a pulse having area equal to π leads to an inversion of a two level quantum system. If, during the pulse, you apply some *other* pulses, it is possible to inhibit the coherent build-up of the phase responsible for level inversion.[6] There is nothing magical about this—in fact you often try to *avoid* processes that lead to phase decoherence. In certain quantum information protocols, you can reverse the effects of phase decoherence using the quantum Zeno effect, but this is not phase

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[5]B. Misra and E. C. G. Sudarshan, *The Zeno's paradox in quantum theory,* Journal of Mathematical Physics **18**, 756–763 (1977).

[6]In fact, a revival of interest in the Zeno effect was generated by an article by W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, *Quantum Zeno effect*, Physical Review **41**, 2295–2300 (1990). In that article, they showed that excitation of a long-lived state from the ground state using by a π pulse could be totally inhibited if an auxiliary field is used to drive a coupled ground to dipole-allowed optical transition. The observation of the spontaneously emitted radiation (or lack therof) on the dipole-allowed transition constituted the "measurements" on the atom.

decoherence caused by Markovian processes. Rather it is phase decoherence caused by effects such as fluctuating magnetic or electric fields whose coherence time is finite.

Misra and Sudarshan proved that, *under certain conditions*, any quantum system can have its decay inhibited by continuously measuring the system. To understand their argument and to see why it fails for Markovian processes, imagine that there is a Hamiltonian $\hat{H}$ that characterizes a quantum system in which some initial discrete state $|\psi_0\rangle$ is coupled to a continuum. We can ask for the probability that we measure the quantum system in the state $|\psi_0\rangle$ at time t, predicated on the fact that we measured in state $|\psi_0\rangle$ at times $t_1, t_2 \ldots t_n$. This probability is given by

$$P(t) = \left|\langle\psi_0| e^{-i\hat{H}t_1/\hbar} |\psi_0\rangle\right|^2 \left|\langle\psi_0| e^{-i\hat{H}(t_2-t_1)/\hbar} |\psi_0\rangle\right|^2 \cdots \times \cdots \left|\langle\psi_0| e^{-i\hat{H}(t-t_n)/\hbar} |\psi_0\rangle\right|^2 . \tag{24.64}$$

Misra and Sudarshan showed rigorously that $P \sim 1$ as $n \sim \infty$, provided $\hat{H}$ is a Hermitian and semi-bounded operator (e.g., $\hat{H}$ is semi-bounded if all its eigenenergies are greater than or equal to some energy E_0). In other words, in the limit of *continuous* measurement on the system, the initial state never decays! They then went on to say that a bubble chamber does not really constitute *continuous* measurement of the undecayed particle, which explains why the decay is not inhibited.

For early times when $\omega_{n0}t \ll 1$, it follows from Eqs. (24.7) that

$$c_n(t) \approx -\frac{i}{\hbar} V_{n0} t; \tag{24.65a}$$

$$c_0(t) \approx 1 - \sum_n \frac{|V_{0n}|^2 t^2}{2\hbar^2}. \tag{24.65b}$$

If the sum is replaced by an integral over continuum states, it follows that

$$|c_0(t)|^2 \approx 1 - t^2/t_c^2, \tag{24.66}$$

where

$$t_c = \frac{\sqrt{2}}{\left[\int_{-\infty}^{\infty} dE \left|\frac{V(E)}{\hbar}\right|^2 \rho(E)\right]^{1/2}} \tag{24.67}$$

can be viewed as the correlation time of the bath. If t_c is *finite*, Eq. (24.64) reduces to

$$P(t) = \left|1 - t_1^2/t_c^2\right| \left|1 - (t_2 - t_1)^2/t_c^2\right| \cdots \left|1 - (t - t_n)^2/t_c^2\right|. \tag{24.68}$$

Taking equal time intervals $\left(t_{j+1} - t_j\right) = t/n$ and letting $n \sim \infty$, I find

$$P(t) = \lim_{n\sim\infty} \left|1 - t^2/n^2 t_c^2\right|^n = \lim_{n\sim\infty} e^{-t^2/nt_c^2} = 1. \qquad (24.69)$$

As long as t_c is finite, continuous measurement produces a quantum Zeno effect. For a Markovian process, however, $\rho(E)$ and $V(E)$ are constant for $-\infty < E < \infty$, implying that $t_c = 0$. In that limit, I found that, for $\Gamma t \ll 1$

$$|c_0(t)|^2 \approx 1 - \Gamma t, \qquad (24.70)$$

which implies that, for a Markovian process,

$$P(t) = \lim_{n\sim\infty} |1 - \Gamma t/n|^n = e^{-\Gamma t}. \qquad (24.71)$$

There is no quantum Zeno effect for a Markovian process. Spontaneous emission is very close to a Markovian process, for which t_c is of order of an inverse optical frequency, making it all but impossible to make "continuous" measurements on such a system.

24.4 Summary

The important problem of transitions from a discrete state to a continuum of states has been studied. In such cases, irreversible behavior can occur, such as that observed in photoemission and spontaneous decay. Fermi's Golden Rule can be used to evaluate the decay rate, although a better picture of the decay processes can be obtained by solving the equations of motion for the state amplitudes and forming the appropriate probability distributions for the final and initial states. Although it might appear to be possible to inhibit such decay by constant measurements on a quantum system, I have shown that this is not possible for Markovian processes.

24.5 Problems

1–2. Estimate the $2P{-}1S$ and the $2S{-}2P$ decay rates in hydrogen using Eq. (24.57). You will need to use the Wigner-Eckart theorem and look up the wave functions and frequency spacings of the levels.

3. For two fine structure levels within the same electronic state manifold, the ratio of the decay rates to some lower level is approximately equal to the ratio of cube of the transition frequencies of the two transitions. Estimate the ratio of the decay

rates for the D2 ($\lambda = 588.995$ nm) and D1 ($\lambda = 589.522$ nm) transitions in Na. The experimental ratio is approximately equal to 1.0028.[7]

4. Prove that

$$\sum_{\lambda=1}^{2} \int d\Omega_k \sum_i \left(\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}\right)_i d_i \sum_j \left(\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}\right)_j \bar{d}_j = \frac{8\pi}{3} \sum_i d_i \bar{d}_i \, .$$

5–6. The frequency spectrum $P(\omega_k)\, d\omega_k$ of the radiation emitted in spontaneous decay is given by the probability that a photon is emitted into a mode of the radiation field having that frequency. In terms of continuous frequency variables,

$$P(\omega_k)\, d\omega_k = \frac{\mathcal{V}}{(2\pi)^3} \frac{\omega_k^2}{c^3} \sum_{m_1=-J_1}^{J_1} \sum_{\lambda=1}^{2} \int d\Omega_k \left| c_{1,J_1,m_1;\mathbf{k},\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}}(\infty) \right|^2 d\omega_k,$$

where $\mathcal{V}$ is the quantization volume. Show that $P(\omega_k)$ is proportional to ω_k^3 times a Lorentzian function of ω_k. In the Weisskopf-Wigner approximation prove that all the initial energy in the atom is converted into the energy of the radiated field,

$$\int_0^\infty \hbar\omega_k P(\omega_k)\, d\omega_k = \hbar\omega_0 n_2(0),$$

where $n_2(0)$ is the initial excited state probability in the atom. Thus, when applied in a consistent manner, the Weisskopf-Wigner approximation leads to conservation of energy, although there was no guarantee that this would be the case.

7–8. Complementary to the frequency spectrum of spontaneous decay is the radiation pattern $I_{\mathbf{k},\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}} d\Omega_k$ (direction and polarization of radiation emitted into an element of solid angle) given by

$$I_{\mathbf{k},\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}} d\Omega_k = \frac{\mathcal{V}}{(2\pi)^3} \frac{1}{c^3} \sum_{m_1=-J_1}^{J_1} \int_0^\infty \omega_k^2 d\omega_k \left| c_{1,J_1,m_1;\mathbf{k},\boldsymbol{\epsilon}_{\mathbf{k}}^{(\lambda)}}(\infty) \right|^2 d\Omega_k.$$

In the Weisskopf-Wigner approximation, show that if an atom is prepared in the $m = 0$ sublevel of a $J_2 = 1$ excited state and decays to a $J_1 = 0$ ground state, the radiation pattern is the same as that emitted by a classical dipole oscillator whose dipole moment is in the z-direction.

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[7] U. Volz, M. Majerus, H. Liebel, A. Schmitt, and H. Schmoranzer, *Precision Lifetime Measurements on NaI $3p^2P_{1/2}$ and $3p^2P_{3/2}$ by Beam-Gas-Laser Spectroscopy,* Physical Review Letters **76**, 2862 (1996)].

9. An atom undergoes spontaneous decay with a central wavelength of 600 nm and a lifetime of 100 ns. Estimate the Rabi frequency if the field from this atom interacts with a similar atom one cm away. If, instead, the single photon pulse can be focused to an area of λ^2 when it strikes the atom, show that the Rabi frequency is of the order of the decay rate and the pulse area is of order unity; that is, a single photon pulse focused to a wavelength can fully excite an atom.

10–11. A particle having mass m moves in a one-dimensional potential

$$V(x) = \begin{cases} -V_0 & x \leq |a|/2 \\ 0 & \text{otherwise} \end{cases},$$

where V_0 is a positive constant.

Assume that the particle is in the ground state of the potential and, in addition, there is a perturbative contribution to the Hamiltonian of the form

$$\hat{V}(x,t) = \alpha\hat{x}\cos(\omega t),$$

where $\hbar\omega$ is much greater than the magnitude of the ground state energy and α is a constant. Calculate the rate at which the particle is "ionized" (escapes from the potential into unbound states). You can leave your answer in terms of the ground state energy E_1 and an integral involving the ground state eigenfunction $\psi_{E_1}(x)$. You need not evaluate the integral. [Hint: To solve this problem, assume that the final states can be approximated as free particle states. You need to calculate the density of states for these free particle states in one-dimension.]

12–13. In the calculation of Sect. 24.3 for a bounded continuum, I assumed that the field was turned on instantaneously at $t = 0$. Suppose, instead, that the coupling matrix element is $V_{E0}(t) = V(t)\cos(\omega_L t)$, where $V(t)$ is a smooth function that increases from an initial value of zero at $t = 0$ to a final value of V in some characteristic time T. Repeat the calculation of Sect. 24.3 with $\omega_b = \infty$ and show that

$$P_0(t) \approx 1 - \frac{|V(t)|^2 \rho}{4\hbar\omega_{aL}}, \tag{24.72}$$

provided $\omega_{aL}T \gg 1$. In other words, the approach to the steady-state value occurs without oscillations if $V(t)$ is turned on adiabatically with respect to the frequency ω_{aL}.

Now derive an expression for $P_0(t)$ by solving Eq. (24.7b) with $c_0(t) \approx 1$ and

$$V(t) = \begin{cases} 0 & t < 0 \\ V\left(1 - e^{-t/T}\right) & t \geq 0 \end{cases}.$$

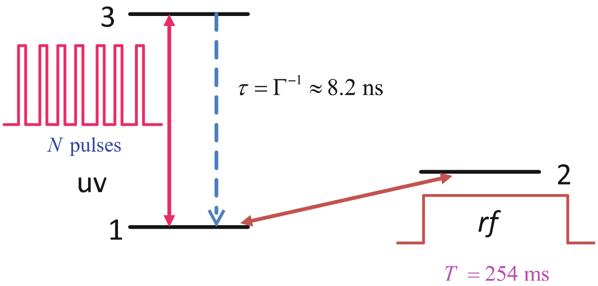

Fig. 24.4 Level scheme for Problem 24.14. Level 3 is the $2^2P_{3/2}$ that decays back to level 1. The parameters shown refer to the article of Itano et al. In their experiment the uv pulse width was 2.4 ms and up to $N = 64$ pulses were used

Plot P_0 as a function of $\omega_{aL}t$ for $\beta = |V|^2 \rho/(2\hbar\omega_{aL}) = 0.1$ and $1/\omega_{aL}T = 0.1$ and compare the result with the prediction of Eq. (24.72). There are still some very small oscillations owing to the fact that the derivative of $V(t)$ is not continuous at $t = 0$. Also plot P_0 as a function of $\omega_{aL}t$ for $\beta = 0.1$ and $1/\omega_{aL}T = 40$ (sudden turn-on of field) and compare the result with that shown in Fig. 24.3. Note that the asymptotic value for $P_0(\infty)$ is $1 - \beta$ for a sudden turn-on of the field and $1 - \beta/2$ for an adiabatic turn on of the field. The parameter

14. In the experiment of Itano et al.,[8] a transition between two, $2^2S_{1/2}$, ground state hyperfine levels of a Be ion is driven by a radio-frequency (rf) field. Let the initial state be denoted by 1 and the final state by 2. A π pulse having duration $T = 256$ ms transfers the population from state 1 to 2, assuming no other interactions played a role. To demonstrate a Zeno effect, Itano et al. added a number of ultraviolet pulses that could excite state 1 to a $2^2P_{3/2}$ level that undergoes rapid spontaneous decay back to level 1 on a time scale of order 10 ns. (see Fig. 24.4). The optical pulses are sufficiently long (but still have duration $\ll T$) and intense to insure that several spontaneously emitted photons can be observed, provided the $2^2P_{3/2}$ level is excited during the pulse. The emission of the spontaneous radiation (or lack thereof) is said to effect a "measurement" on the atom. The ion "collapses" into state 1 if the emission occurs and into state 2 if no emission occurs (no emission implies the ion must have been in state 2 when the uv pulse was applied). If N such equally spaced pulses are applied, calculate the population of the initial $2^2S_{1/2}$ at time $t = T$. Show that in the limit $N \to \infty$, this population approaches zero—there is a Zeno effect produced by the measurement pulses. To carry out this calculation, you can assume that at $t = 0$ the Bloch vector is $(u(0), v(0), w(0)) = (0, 0, -1)$. Following the first measurement pulse at time t_1, the Bloch vector "collapses" into $(u(t_1), v(t_1), w(t_1)) = (0, 0, w(t_1))$, since there is a probability $\rho_{11}(t_1)$ that the

[8] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, *Quantum Zeno effect*, Physical Review **41**, 2295–2300 (1990).

ion is in state 1 and a probability $\rho_{22}(t_1)$ that the ion is in state 2 at the time of the measurement. The ion then evolves freely under the influence of the rf pulse until the second measurement pulse at time t_2 "collapses" the Bloch vector into $(u(t_2), v(t_2), w(t_2)) = (0, 0, w(t_2))$; and so on.[9]

[9] As you know I am not a fan of the "collapse" picture. It is not necessary to invoke the collapse picture to arrive at the final result, a simple density matrix calculation gives the same result. See Ellen Block and P. R. Berman, *Quantum Zeno effect and quantum Zeno paradox in atomic physics*, Physical Review A **44**, 1466–1472 (1991).