# Chapter 10
# Broken Symmetries

We have discussed several problems in which the symmetry of the Hamiltonian constitutes an essential tool in the construction of the (factorized) eigenstates. In these cases, the ground state $\varphi_0$ is annihilated by the generators of the transformation associated with that symmetry. For instance, in the case of the hydrogen atom, $[\hat{H}, \hat{L}_i] = 0$ and $\hat{L}_i \varphi_0 = 0$. We say that the states carry the same symmetry as the Hamiltonian.

In the present chapter, we study situations in which this is not the case. In fact, the description with broken symmetries permeates an important fraction of today physics: ferromagnetism; superconductivity and superfluidity (condensed matter physics, nuclei, neutron stars); Hartree–Fock description (atomic and nuclear physics); molecules; quadrupole deformed nuclei; theory of electro-weak interactions and quantum chromodynamics (field theory, cosmology); etc.

The chapter is divided into two main sections. In the first one, we present the treatment of superconducting and superfluid systems by Bardeen, in collaboration with Leon N. Cooper and John R. Schrieffer. The BCS theory [66] constitutes a relevant illustration of a description based on the approximation of broken symmetries. Superconducting systems are macroscopic manifestations of quantum mechanics, and thus bear relevance in the diffuse limit between classical and quantum descriptions of nature. In addition, they display an increasing number of technical applications ranging from the transmission of electric currents without losses to the most accurate determination of the ratio $e/\hbar$.

In the second section of the chapter, we delve deeper into the concept of broken symmetry by means of a very simple mechanical example. Corrections to this approximation demand the appearance of additional degrees of freedom, the collective variables, which become compensated by the existence of constraints. Little attention, if any, is paid in quantum textbooks to the problem of quantization with constraints, an area where great progress has been made over the last 35 years [67]. This subject is not only of paramount importance in gauge field theories [68], but it also has applications in quantum mechanics, as in the description of many-body systems from moving frames of reference [69]. Moreover, the problem is conceptually significant in terms of properties of Hilbert spaces. Although we

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are here restricted to present just an outline of the BRST procedure, we hope that the reader may get some feeling of this elegant method developed by C. Becchi, A. Rouet, R. Stora and I.V. Tyutin [67] (Sects. 10.2.2–10.2.4$^\dagger$).

Quantization with constraints and superconductivity are two subjects which are seldom explained together. However, although the formalism describing superconductivity has more complexity than the one employed in Sect. 10.2, the underlying mechanism of broken symmetry is similar in both cases. Moreover, the collective sector that appears as a consequence of the BRST formalism provides a phase to the BCS solution which becomes relevant in processes involving the transference of pairs of particles (e.g. Josephson junctions).

## 10.1 The BCS Theory of Superconductivity

In many cases for which the exact eigenstates with the correct symmetry are difficult to obtain, it is a good approximation to use solutions involving a breakdown of symmetries. In this section we present superconductivity as an illustration of a broken symmetry. The formalism applies as well to nuclear superfluidity.

Heike Kamerlingh Onnes discovered in 1911 that, below a certain temperature, most metals conduct electricity without any resistance [70]. It took almost 50 years until Bardeen, Cooper and Schrieffer developed an adequate microscopic description for the quantum state of electrons in a metal, subject to attractive interactions. They were able to explain the behavior of all superconducting materials known at the time when the theory was developed [66]. Those materials are known today as low critical temperature superconductors. The magnetic response of the superconductors as well as the formal discussion of a possible microscopic theory for the superconductivity found in copper oxide superconductors (high critical temperature superconductors) is not the subject of this introduction.

### *10.1.1$^\dagger$ The Conjugate Variable to the Number of Particles*

We consider now the case of the *gauge* symmetry,[1] associated with the number of particles $N$. For all systems treated so far, the number of particles has been a conserved quantity and its conjugate variable has been ignored. In fact, it has been completely undetermined, as in (10.38). Thus, we start by looking for the conjugate variable $\theta$ to the number of particles. This variable does exist, as can be verified for the boson case within the harmonic description.

According to (3.42), the number of bosons $\hat{N} = a^+a$ is given by $\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$. The corresponding classical expression is

[1] The use of the word *gauge* is explained in footnote 5.

$$N = \frac{1}{2\hbar}\left(\frac{1}{M\omega}\,p^2 + M\omega\,x^2\right). \tag{10.1}$$

If the gauge angle is defined as

$$\theta = -\tan^{-1}\frac{p}{M\omega\,x}, \tag{10.2}$$

the (classical) Poisson bracket has the value

$$\{\theta, \hbar N\}_{\mathrm{PB}} = \hbar\left(\frac{\partial\theta}{\partial x}\frac{\partial N}{\partial p} - \frac{\partial\theta}{\partial p}\frac{\partial N}{\partial x}\right) = 1. \tag{10.3}$$

Therefore, according to Dirac's relation (2.48), the commutator

$$[\hat{\theta}, \hbar\hat{N}] = \mathrm{i}\hbar \tag{10.4}$$

holds. Indeed, the operator $\hbar\hat{N}$ plays the same role as a two-dimensional angular momentum, with the angle $\hat{\theta}$ being its conjugate variable.

In analogy to (4.7) and (5.37), we may also construct an operator generating rotations in gauge space

$$\begin{aligned}\mathcal{R}(\theta) &= \exp(\mathrm{i}\,\theta\hat{N})\\ \mathcal{R}(\theta)\,a^+\,\mathcal{R}(-\theta) &= \exp(\mathrm{i}\theta)\,a^+.\end{aligned} \tag{10.5}$$

We now turn our attention to the fermion case. From hereon $a^+$ denotes the creation of a fermion. A pair of fermions acts like a boson,[2] since changing the position of the pair does not produce a change of the wave function. Thus, to preserve (10.5) for the case of fermion pairs, we infer for the single fermion

$$\begin{aligned}\mathcal{R}(\theta) &= \exp(\mathrm{i}\,\theta\hat{N}_\pi)\\ \mathcal{R}(\theta)\,a^+\,\mathcal{R}(-\theta) &= \exp(\mathrm{i}\theta/2)\,a^+,\end{aligned} \tag{10.6}$$

where $\hat{N}_\pi$ is the operator corresponding to the number of pairs of fermions.[3]

Operators with the same number of single-fermion creation and annihilation operators behave like scalars under gauge transformations; operators with two more creation than annihilation operators behave as bosons do in (10.5)

$$\begin{aligned}\mathcal{R}(\theta)\,a_1^+a_2\,\mathcal{R}(-\theta) &= a_1^+a_2\\ \mathcal{R}(\theta)\,a_1^+a_2^+\,\mathcal{R}(-\theta) &= \exp(\mathrm{i}\theta)\,a_1^+a_2^+.\end{aligned} \tag{10.7}$$

[2]We have used already a similar substitution in (8.46).

[3]The factor $1/2$ in the r.h.s. of (10.6) plays a similar role as the same factor in the rotation of spin states (5.26).

### 10.1.2† The Monopole Pairing Operator and the Hamiltonian

Two identical fermions may be created at the same point of space if they are in a singlet-spin state. The associated operator in the $x$-representation is

$$\Gamma^{+}_{x,0} = \frac{1}{\sqrt{2}} \sum_{n,n'} \langle x|n\rangle\langle x|n'\rangle \sum_{m_s=\pm\frac{1}{2}} a^{+}_{n,m_s}\, a^{+}_{n',-m_s}, \tag{10.8}$$

see Sect. 11.1†. We consider the case for which the creation of two particles at the same point is homogeneous over all space (monopole pairing). If intermediate momentum eigenstates are used, the total operator for creating two particles at the same point is

$$\begin{aligned}\hat{P}^{+} &= \int \mathrm{d}x\, \Gamma^{+}_{x,0} = \frac{1}{\sqrt{2}} \sum_{p,p'} \int \mathrm{d}x\, \langle x|p\rangle\langle x|p'\rangle \sum_{m_s=\pm\frac{1}{2}} a^{+}_{p,m_s} a^{+}_{p',-m_s} \\ &= \frac{1}{\sqrt{2}} \sum_{p>0} \sum_{m_s=\pm\frac{1}{2}} a^{+}_{p,m_s} a^{+}_{-p,-m_s} \\ &\to \sum_{p>0} a^{+}_{p} a^{+}_{-p}. \end{aligned} \tag{10.9}$$

We have used (11.7) for the overlaps $\langle x|p\rangle$ and (11.6) for the delta function. The explicit dependence on spin has been omitted in the last line, for the sake of simplicity. The notation $p > 0$ indicates that the sum includes only a single term for each (degenerate) pair of momentum states $(p, -p)$. In the nuclear case the momentum representation may be replaced by eigenstates of the Woods–Saxon potential (7.15), which can also be paired in time-reversed states (Sect. 9.7†).

A simplified, schematic Hamiltonian may be written as

$$\begin{aligned}\hat{H} &= \hat{H}_{sp} + \hat{H}_{\rm tb} \\ \hat{H}_{sp} &= \sum_{p>0} \epsilon_p \left( a^{+}_{p} a_{p} + a^{+}_{-p} a_{-p} \right) \\ \hat{H}_{\rm tb} &= -g\, \hat{P}^{+}\, \hat{P}. \end{aligned} \tag{10.10}$$

It turns out that there is a net effective attraction between the electrons, due to the interactions between the electrons with the vibrations of the ions in the lattice. Thus the strength $g > 0$. However, this attraction is very weak, and only a tiny thermal agitation is needed to destroy it. In the nuclear case there is a natural attraction between nucleons without invoking other processes.

The two-body term $\hat{H}_{\rm tb}$ allows pairs of nearby particles to jump from one momentum state to the other, thus generating correlations in their motion. It is called a *contact* interaction because the leap only takes place if they are at the same place. Consistently, a delta force also displays fairly constant matrix elements for the jump of pairs of particles.[4] The expression (10.10) has the advantage over a delta interaction that it has a simpler expression, due to the fact that it is written in a separable form. Nevertheless, in spite of its simplifications, this Hamiltonian is still difficult to solve (but for special cases as those described in Problems 1 and 2). Therefore, we resort to approximations. However, see Ref. [132].

Note that the Hamiltonian (10.10) is a scalar under gauge transformations (10.6).

There are also normal electrons (not bound in pairs), moving around the metal in an ordinary way. We disregard this complication here.

### *10.1.3*† *The BCS Hamiltonian*

We have mentioned that a pair of fermions acts like a boson. In a system with $N >> 1$ bosons, the creation of all of them in a single state is favored over their distribution in different states, since the amplitude for creating a new pair is proportional to $\sqrt{N+1}$ (3.35). The essential feature of the superconducting phase is the presence of many bosons (i.e. pairs of bound electrons) in a single quantum state. This is called a condensate.

The mean-field approximation is a straightforward extension of the Hartree–Fock treatment (Sect. 8.6.1†). It is also obtained by using a representation in which large matrix elements of some operator become expectation values, ignoring their quantum fluctuations. In the superconducting case this operator is $P^+$, which bears some relation with the persistance of currents. Thus,

$$\hat{P}^+ \rightarrow \langle 0|P^+|0\rangle = \sum_{p>0} \langle 0|a_p^+ a_{-p}^+|0\rangle = \frac{\Delta}{g}\exp(\mathrm{i}\theta), \tag{10.11}$$

where the (real) modulus is given in units of the interaction strength. Thus, we require that $\Delta/g >> 1$.

The phase angle $\theta$ of the condensate represents the orientation of the system in gauge space. Since it is conjugate to the number of bosons in the condensate, it is completely undefined [as in (10.38)] for systems with a definite number of pairs of particles. The choice of a particular value of $\theta$ in (10.11) implies that:

- The original symmetry of the Hamiltonian is broken in the state $\varphi_0$.
- The number of pairs of particles becomes ill defined. This is consistent with the fact that the expectation value of an operator creating two particles is different from zero.

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[4]There are also cases of superconductivity for which other components of the interaction have to be introduced.

- We may request that at least the average number of pairs of particles has a prescribed value $A_\pi$. Thus, we add to the Hamiltonian (10.10) a term changing the origin of single-particle energies, which has a vanishing expectation value

$$-2\mu\,(\hat{N}_\pi - A_\pi)\,; \qquad A_\pi \equiv \langle 0|N_\pi|0\rangle. \tag{10.12}$$

The (pair-degenerate) single-particle energies $\epsilon_p$ become replaced by $\epsilon_p - \mu$. The constant $\mu$ is used to fix the average number of particles to the number $2A_\pi$, and plays a similar role as the Fermi energy in normal systems (see Sect. 7.7[†]). It is called a Lagrange multiplier.

In the following we choose $\theta = 0$. However, since this option is arbitrary,[5] physical results should not depend on it.

The monopole pairing operator can be written as

$$\hat{P}^+ = \frac{\Delta}{g} + \left(\hat{P}^+ - \frac{\Delta}{g}\right), \tag{10.13}$$

where the first term on the right hand side is supposed to be much larger than the term between parenthesis. The $BCS$ Hamiltonian is obtained by expanding $\hat{H}_{\text{tb}}$ in powers of $\Delta/g$. To leading orders,[6]

$$\begin{aligned}\hat{H} \to \hat{H}_{\text{BCS}} = {} & 2\mu A_\pi - \frac{\Delta^2}{g} \\ & + \sum_{p>0}\left[(\epsilon_p - \mu)\left(a_p^+ a_p + a_{-p}^+ a_{-p}\right) - \Delta\left(a_p^+ a_{-p}^+ + a_{-p} a_p\right)\right].\end{aligned} \tag{10.14}$$

Since this Hamiltonian is only quadratic in the fermion creation and annihilation operators, it is easy to diagonalize. We use the transformation

$$\alpha_{\pm p}^+ = U_p a_{\pm p}^+ \mp V_p a_{\mp p}. \tag{10.15}$$

The operators $\alpha_{\pm p}^+$ create excitations called quasi-particles. They carry good momentum $\pm p$, since the annihilation of an entity carrying momentum $\mp p$ implies the creation of the momentum $\pm p$. The normalization condition implies

$$U_p^2 + V_p^2 = 1. \tag{10.16}$$

[5] The word *gauge* is used in this chapter by the similarity with electromagnetic theory, where a gradient may be added to the vector potential without altering the physical results.

[6] Parts of the residual terms in $\hat{H} - \hat{H}_{\text{BRST}}$ are taken into account in Sect. 10.1.5[†].

The diagonalization amounts to insure the vanishing of the terms creating two-quasi-particles in (10.14). Inversion of (10.15) and replacement in $\hat{H}_{\text{BCS}}$ yields the equations

$$2U_pV_p = \frac{\Delta}{E_p} \quad \text{and} \quad U_p^2 - V_p^2 = \frac{\epsilon_p - \mu}{E_p}, \tag{10.17}$$

which determine the ground state energy $W$, the quasi-particle excitation energies $E_p$, and the amplitudes $U_p, V_p$.

$$\hat{H}_{\text{BCS}} = W + \sum_{p>0} E_p \left(\alpha_p^+ \alpha_p + \alpha_{-p}^+ \alpha_{-p}\right) \tag{10.18}$$

$$W = 2\sum_{p>0} \epsilon_p V_p^2 - \frac{\Delta^2}{g}\ ; \qquad E_p = \sqrt{(\epsilon_p - \mu)^2 + \Delta^2}\ ;$$

$$U_p = \frac{1}{\sqrt{2}}\left(1 + \frac{\epsilon_p - \mu}{E_p}\right)^{1/2}\ ; \qquad V_p = \frac{1}{\sqrt{2}}\left(1 - \frac{\epsilon_p - \mu}{E_p}\right)^{1/2}.$$

The Hamiltonian $\hat{H}_{\text{BCS}}$ describes a system of independent quasi-particles. The condensate provides an energy of order $\Delta$ to each single-particle excitation energy $|\epsilon_p - \mu|$, and an extra contribution to the binding energy.

In the limit $\Delta \to 0$, $V_p = 1$ (0) and $U_p = 0$ (1) for single-particle energies $(\epsilon_p - \mu) < 0$ ($> 0$). For a normal system, $V_p$ drops abruptly from 1 to 0 at the Fermi level, while in superconducting systems there is a diffuseness of the Fermi energy extending over a range of size $\Delta$. The BCS solution is able to describe solutions ranging from a cylindrically symmetric vacuum (completely undetermined $\theta$) to a vacuum which does not display this symmetry.

There are still two parameters in this solution, $\Delta$ and $\mu$, which must be determined. This is accomplished by making use of the following two requirements:

- The expectation value (10.11) and the first of requirements (10.17) yield the self-consistent condition

$$\frac{2}{g} = \sum_{p>0} \frac{1}{E_p}. \tag{10.19}$$

  A solution with a non-vanishing $\Delta$ is always obtained for sufficiently large values of $g$.
- The requirement that the average number of the pair of particles corresponds to a certain prefixed value $A_\pi$ is written as

$$\frac{1}{2}\left\langle 0 \right| \sum_p a_p^+ a_p \left| 0 \right\rangle = \sum_{p>0} V_p^2 = A_\pi. \tag{10.20}$$

### 10.1.4† The Ground State

The ground state $\varphi_0$ represents the vacuum of quasi-particles ($\alpha_p \varphi_0 = 0$). In terms of particles, it is given by

$$\varphi_0 = \prod_{p>0} \left( U_p + V_p a_p^+ a_{-p}^+ \right) |\rangle, \tag{10.21}$$

where $|\rangle$ is the true vacuum of particles.

The matrix elements for the creation and the destruction of a particle between the ground state and the states $\alpha_p^+ \varphi_0$ are given by

$$\langle \alpha_p^+ | a_p^+ | 0 \rangle = U_p, \qquad \langle \alpha_p^+ | a_p | 0 \rangle = -V_p. \tag{10.22}$$

Therefore, $U_p^2$ ($V_p^2$) in (10.18) is the probability that the particle state $p$ is empty (occupied).

The expectation value of the operator creating a pair of particles is

$$\langle 0 | a_p^+ a_{-p}^+ | 0 \rangle = U_p V_p = \frac{\Delta}{2E_p} \leq \frac{1}{2}. \tag{10.23}$$

This expectation number is larger for states within the gap ($|\epsilon_p - \mu| << \Delta$) and decreases for distant states ($|\epsilon_p - \mu| >> \Delta$). Replacement of (10.23) in (10.19) yields the ratio $\Delta/g$. It is a large number (as required), since all contributions to the sum (10.11) have the same sign.

As an illustration, let us consider the case of 19 pairs of particles allowed to move in 38 single-particle equidistant levels. The distance between consecutive levels will be the unit of energy. For $g = 0.456$, (10.19) and (10.20) yield the parameters $\Delta = 4.50$ and $\mu = 19.50$. The quasi-particle energies $E_p$ are compared with the particle excitation energies $|\epsilon_p - \mu|$ in Fig. 10.1c. The occupation probabilities $V_p^2$ are represented in Fig. 10.1a and the amplitudes $U_p V_p$ in Fig. 10.1b.

The particles constituting a pair are not very close in real space, in spite of the fact that a contact interaction has been used: the mean value of this distance is somewhat larger than the average distance between different pairs of particles. This is due to the fact that, in the condensate, there are many pairs of particles occupying the same state.

### 10.1.5† The Excitation Spectrum

The ground state (10.21) displays an even number of particles. In fact, although the transformation (10.15) to quasi-particles does not conserve the number of particles, it preserves the parity in the number of particles. Systems with an even number of particles are represented by states with an even number of quasi-particles, including

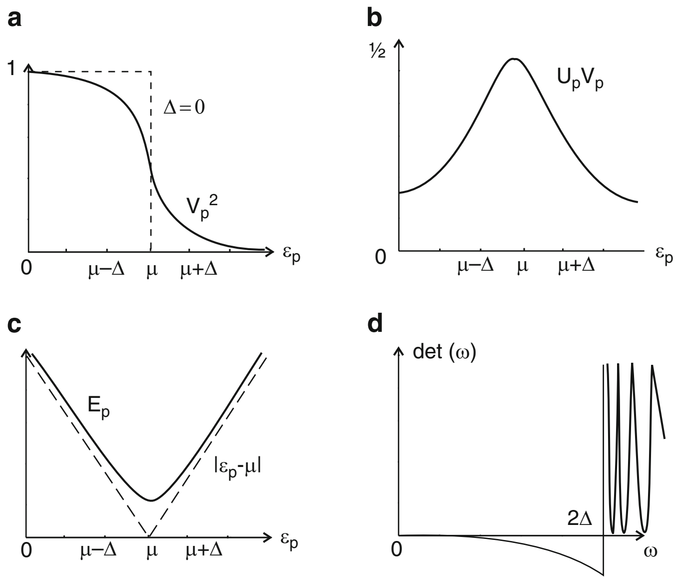


**Fig. 10.1** Occupation probabilities $V_p^2$ (**a**); amplitudes $U_p V_p$ (**b**); and quasi-particle energies $E_p$ (**c**), as functions of the particle energies $\epsilon_p$. The determinant $\det(\omega)$ is represented as a function of $\omega$ in (**d**) [see (10.29)]

the vacuum state $\varphi_0$. Odd systems are in correspondence with states with an odd number of quasi-particles, like $\alpha_p^+ \varphi_0$.

Therefore, the lowest excited states of an even system consists of two quasi-particle states $\alpha_{p_1}^+ \alpha_{p_2}^+ \varphi_0$, with energies $E_{p_1} + E_{p_2} \geq 2\Delta$. Thus, the spectrum displays a gap in the vicinity of the ground state.

In a normal conductor, resistance originates from the heat produced by collisions of the moving electrons with the ionic lattice. In a superconductor, the current is due to pairs of electrons moving together, each pair with a non-vanishing center of mass momentum. These pairs form a highly collective quantum condensate: breaking a pair requires a change of energy of all other pairs, which is of order of the gap ($\approx 2.7$ meV for Nb). Thus, resistance is suppressed.

However, since there is no restoring force in the $\theta$ angular direction, we should expect the existence of a zero frequency boson, which appears to be precluded by the existence of a gap. Nevertheless, we have seen in Sect. 8.6.2$^\dagger$ that low energy bosons may appear as RPA excitations of the ground state in fermion systems. In the following, we search for this type of excitations in the superconducting case.

We again take advantage of the fact that a pair of fermion operators acts in many respects as a boson, and make the replacement

$$\alpha_p^+ \alpha_{-p}^+ \to \gamma_p^+ \ ; \quad [\gamma_p, \gamma_{p'}^+] = \delta_{pp'}. \tag{10.24}$$

We now focus our attention on states of the form $\alpha_p^+\alpha_{-p}^+\varphi_0 \rightarrow \gamma_p^+\varphi_0$, with energies $2E_p$.

In addition to the BCS Hamiltonian (10.14), there are residual interactions that are quartic products of quasi-particle operators. We select those terms which allow for the replacement of the two pairs of fermions, $\alpha_p^+\alpha_{-p}^+$ and $\alpha_{-p}\alpha_p$, by the bosons (10.24), and thus, become quadratic in these boson operators.[7] We also obtain the operator associated with the number of pairs of particles

$$\hat{H}_b = W + \sum_{p>0} 2E_p\gamma_p^+\gamma_p - g\sum_{p,p'>0}(U_p^2\gamma_p^+ - V_p^2\gamma_p)(-V_{p'}^2\gamma_{p'}^+ + U_{p'}^2\gamma_{p'})$$

$$(\hat{N}_\pi)_b = A_\pi + \sum_{p>0} U_pV_p(\gamma_p^+ + \gamma_p). \tag{10.25}$$

We carry below the uncoupling of the bosons $\gamma_p^+, \gamma_p$ in $\hat{H}_b$, and we find that a zero frequency root $\omega_0 = 0$ is always present (Fig. 10.1d). The associated creation operator $\Gamma_0^+$ $(= \Gamma_0)$ is proportional to the boson term of the operator[8] $\hat{N}_\pi$ in the last line of (10.25).

#### The Uncoupling of the Hamiltonian (10.25) and the Presence of the Zero-Frequency Excitation

The procedure is completely similar to the one applied in the RPA (Sect. 8.6.2$^\dagger$). We perform the linear transformation

$$\Gamma_\nu^+ = \sum_{p>0}(\lambda_{\nu p}\gamma_p^+ - \mu_{\nu p}\gamma_p). \tag{10.26}$$

Use is made of the commutation

$$[\hat{H}_b, \Gamma_\nu^+] = \sum_{p>0}\left(\lambda_{\nu p}2E_p - \Lambda_{\nu 1}V_p^2 - \Lambda_{\nu 2}U_p^2\right)\gamma_p^+ + \sum_{p>0}\left(\mu_{\nu p}2E_p + \Lambda_{\nu 1}U_p^2 + \Lambda_{\nu 2}V_p^2\right)\gamma_p \tag{10.27}$$

$$\Lambda_{\nu 1} \equiv g\sum_{p>0}(\lambda_{\nu p}V_p^2 - \mu_{\nu p}U_p^2), \qquad \Lambda_{\nu 2} \equiv g\sum_{p>0}(\lambda_{\nu p}U_p^2 - \mu_{\nu p}V_p^2).$$

[7] It can be shown that the remaining terms only yield higher order contributions in perturbation theory [69].

[8] The applicability of the BRST procedure (Sect. 10.2.1) also requires the existence of a zero frequency boson. The generator $\hat{N}_\pi$, together with the angle $\theta$, are incorporated into the unphysical sector, as in Sect. 10.2.4$^\dagger$.

The amplitudes $\lambda_{\nu p}, \mu_{\nu p}$ are obtained from the condition that the coefficients of the operators $\gamma_p^+$ and $\gamma_p$ vanish in the harmonic equation $[\hat{H}_b, \Gamma_\nu^+] - \omega_\nu \Gamma_\nu^+ = 0$. One obtains

$$\lambda_{\nu p} = \frac{\Lambda_{\nu 1} V_p^2 + \Lambda_{\nu 2} U_p^2}{2E_p - \hbar\omega_\nu}, \qquad \mu_{\nu p} = -\frac{\Lambda_{\nu 1} U_p^2 + \Lambda_{\nu 2} V_p^2}{2E_p + \hbar\omega_\nu}. \tag{10.28}$$

Introduction of (10.28) into the definitions on the last line of (10.27) yields two linear, homogeneous, coupled equations, which should have a null determinant

$$0 = \Lambda_{\nu 1} \left( A - \frac{1}{g} \right) + \Lambda_{\nu 2}\, C\ ; \qquad 0 = \Lambda_{n1}\, C + \Lambda_{n2} \left( B - \frac{1}{g} \right)$$
$$\det(\omega) = \left( A - \frac{1}{g} \right) \left( B - \frac{1}{g} \right) - C^2, \tag{10.29}$$

where

$$A = \sum_{p>0} \left( \frac{V_p^4}{2E_p - \hbar\omega} + \frac{U_p^4}{2E_p + \hbar\omega} \right); \qquad B = \sum_{p>0} \left( \frac{U_p^4}{2E_p - \hbar\omega} + \frac{V_p^4}{2E_p + \hbar\omega} \right)$$
$$C = 4 \sum_{p>0} \frac{U_p^2 V_p^2 E_p}{4E_p^2 - \hbar^2\omega^2}. \tag{10.30}$$

The Hamiltonian $\hat{H}_b$ may be expressed in terms of the bosons (10.26)

$$\hat{H}_b = W - g \sum_{p>0} V_p^4 + \sum_\nu \hbar\omega_\nu \Gamma_\nu^+ \Gamma_\nu, \tag{10.31}$$

where the frequencies $\omega_\nu$ are given by the roots of the equation $\det(\omega) = 0$. The determinant is represented in Fig. 10.1d as a function of $\omega$, using the same parameters as in Figs. 10.1a–c. Poles appear at each unperturbed excitation value $2E_p$: there are none within the gap $2\Delta$ and become compressed just above the gap; they are separated by twice the distance between consecutive levels for higher states.

The roots $\omega_\nu$ display a density similar to that of the poles (they are completely intermixed), but for the root $\omega_0 = 0$, which is present at the origin. Using the normalization condition (10.16) and the self-consistency requirement (10.19), it is simple to show that there is always a zero frequency root, independently of the single-particle spectrum and of the number of pairs of particles. Moreover, the relative amplitudes of the (coupled) bosons $\gamma_p^+ + \gamma_p$ for this root are proportional to $U_p V_p$, and thus $\Gamma_0^+ + \Gamma_0$ is proportional to the boson term in $\hat{N}_\pi$ (10.25).

### *10.1.6*† *Collective sector. Rotational Bands. Josephson Junctions*

The breaking of rotational invariance is always associated with the occurrence of rotational degrees of freedom. The simplest example is given by the diatomic molecule (see Sect. 8.4.3).

The intrinsic motion is described relative to the body-fixed coordinate frame $\theta = 0$. The orientation of the body-fixed frame relative to the laboratory frame is determined by the collective azimuthal angle $\phi$. The separation of motion into intrinsic and rotational components yields the product state

$$\Psi_{n_\nu, A_\pi} = \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i} A_\pi \phi) \prod_{\nu \neq 0} \frac{1}{\sqrt{n_\nu !}} \left(\Gamma_\nu^+\right)^{n_\nu} \varphi_0, \tag{10.32}$$

where $A_\pi$ is the number of pairs of particles. In the present subsection, the relevance of the presence of the collective eigenvector $\exp(\mathrm{i} A_\pi \phi / \sqrt{2\pi})$ (5.60) in the description of a superconductor is emphasized.

The product states (10.32) constitute an approximation, since there appear more degrees of freedom than the original ones. One consequence is that overcompleteness and Pauli violations are included in (10.32). However, a legitimation of these states is presented in Sect. 10.2.

States with the same number $n_\nu$ of finite frequency bosons can be grouped into "rotational" bands made up from systems with different numbers of pairs of particles. In particular, the set of ground states of even systems constitutes a rotational band. The associated rotational energies can be obtained as in (10.65).

Since $\varphi_0$ is a valid description of the ground state in the intrinsic system, any operator must be transformed to this system before operating within states (10.32). For instance, using (10.7)

$$\begin{aligned}
&\langle n_\nu = 0, A'_\pi | \mathcal{R}^{-1}(\phi)\, P^+\, \mathcal{R}(\phi) | n_\nu = 0, A_\pi \rangle \\
&= \frac{\langle 0 | P^+ | 0 \rangle}{2\pi} \int_0^{2\pi} d\phi \exp[\mathrm{i}(-A'_\pi + 1 + A_\pi)] \\
&= \frac{\Delta}{g}\, \delta_{A'_\pi, A_\pi + 1}.
\end{aligned} \tag{10.33}$$

The presence of the collective sector in (10.32) ensures the conservation of the number of particles, which the intrinsic state $\varphi_0$ by itself does not. Moreover, (10.33) is the large matrix element of the transfer operator connecting consecutive members of the ground state band.

A Josephson junction [71] is made up from two superconductors separated by a thin layer of insulating material. We assume, for simplicity, that the two superconductors are made of the same material and that the junction is symmetrical.

A potential difference $V$ may exist between the two sides of the insulating barrier. Therefore, there appears a difference between the Fermi energies of the two superconductors $\mu_1 - \mu_2 = eV$, where $e$ is the electron charge. The Lagrange multipliers are related to the angular frequencies by the canonical equation through the term (10.12) included in $\hat{H}_{\rm BCS}$,

$$\dot{\phi} = \frac{\partial H_{\rm BCS}}{\hbar\, \partial A_\pi} = \frac{2}{\hbar}\mu. \tag{10.34}$$

Thus, one obtains

$$\begin{aligned} \dot{\phi}_1 - \dot{\phi}_2 &= \frac{2}{\hbar}\, eV \\ \phi_1 - \phi_2 &= \frac{2}{\hbar}\, eV\,(t - t_0) + \delta_0 \end{aligned} \tag{10.35}$$

Our aim is to find the probability amplitude for the electrons to jump across the junction. This tunneling can be represented by a symmetrical coupling that destroys a pair on one side of the barrier and creates another on the other side ([38], Chap. 6)

$$\hat{H}_{\rm coup} = K\, \cos(\phi_1 - \phi_2), \tag{10.36}$$

where $K$ is a constant determining the intensity of the tunneling. The presence of (10.36) prevents the separate conservation of $(A_\pi)_1$ and $(A_\pi)_2$. In fact, the canonical equations $(\dot{A}_\pi)_i = -\frac{\partial H}{\partial \phi_i}$ yield the current $J$ from superconductor 1 to 2

$$\begin{aligned} J \propto (\dot{A}_\pi)_1 &= -(\dot{A}_\pi)_2 = -\frac{\partial H_{\rm coup}}{\hbar \partial \phi_1} \\ &= J_0\, \sin(\phi_1 - \phi_2) = J_0\, \sin\left(\frac{2eV}{\hbar}(t - t_0) + \delta_0\right) \end{aligned} \tag{10.37}$$

The current $J_0$ is the maximum current that can be passed by the junction. It is proportional to the coupling strength $K$.

- dc Josephson current: with no applied dc voltage, a dc current flows across the junction, with a value between $J_0$ and $-J_0$ depending on the phase shift $\delta_0$
- ac Josephson current: with a dc voltage applied across the junction, an ac current oscillates with frequency $\omega = 2eV/\hbar$. Thus, a photon of energy $\hbar\omega = 2eV$ is emitted or absorbed each time that a pair crosses the junction. Note the factor 2, which reflects the fact that it is a pair of electrons that crosses. A very precise measure of the ratio $e/\hbar$ is obtained by measuring voltage and frequency.

## 10.2 Quantization with Constraints

### *10.2.1 Constraints*

Let us consider a very simple system of one particle constrained to move along a circumference of radius $r_0$, in real space. Thus, as in the BCS case, there is an initial circular symmetry. The fact that one takes place in gauge space, and the other in real space, is immaterial. On the other hand, this toy model is much simpler to handle, because any other degree of freedom is gone, but the rotation along the angular coordinate $\theta$.

The Hamiltonian and eigenstates are given by

$$\hat{H} = \frac{\hat{L}^2}{2Mr_0^2}, \qquad \varphi_m(\theta) = \frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}m\theta), \tag{10.38}$$

where $\hbar m$ is the eigenvalue of the angular momentum $\hat{L}$ and the conjugate angle $\theta$ is completely undetermined ($0 \leq \theta \leq 2\pi$). In particular, the ground state satisfies the relation $\hat{L}\varphi_0 = 0$.

However, other descriptions are also possible. For instance, from a rotating frame of reference. In this case, states $\varphi(\theta)$ carry less symmetry than the circular symmetry associated both with the Hamiltonian and with the eigenstates (10.38). We say that there has been a breakdown of symmetry.

Moreover, this description requires the inclusion of the angle $\phi$ specifying the orientation of the moving frame relative to the laboratory. Hence, we have an overcomplete set of degrees of freedom, namely the two angles $\theta$ and $\phi$.

We call *intrinsic*, the coordinates of a system that are referred to a rotating frame of reference. The motion of the moving frame relative to the laboratory is described by means of *collective* coordinates. Therefore, in this problem:

- The rotations of the system are generated by the intrinsic angular momentum $\hat{L}$ (5.37). There is also a collective angular momentum $\hat{I}$, the generator of rotations of the moving frame.
- The classical set of equations defining the momenta in terms of partial derivatives of the Lagrange function $\mathcal{L}$ cannot be solved in this case. This failure is due to the fact that this function does not contain information about the frame itself. For instance, in the case of one particle allowed to move on a circumference of radius $r_0$, the Lagrange function may be expressed in terms of the angular velocities $\dot{\theta}$ and $\dot{\phi}$ (Fig. 10.2):

$$\mathcal{L} = \frac{\mathcal{J}}{2}\left(\dot{\theta} + \dot{\phi}\right)^2. \tag{10.39}$$

Here $\theta = \tan^{-1}(y/x)$ and $\mathcal{J} = Mr_0^2$ is the moment of inertia. From the equations

$$L = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} \qquad \text{and} \qquad I = \frac{\partial\mathcal{L}}{\partial\dot{\phi}},$$

one obtains the orbital angular momentum $L$ and the constraint $L = I$:

$$L = \mathcal{J}\left(\dot{\theta} + \dot{\phi}\right), \tag{10.40}$$

$$f \equiv L - I = 0. \tag{10.41}$$

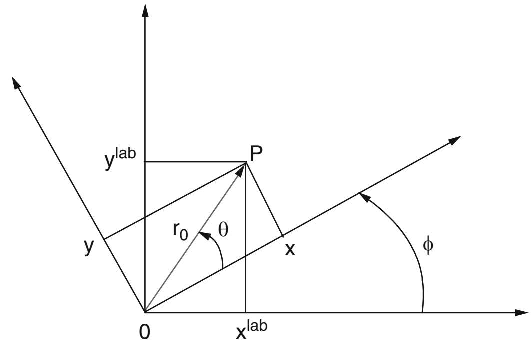


**Fig. 10.2** Intrinsic $(x, y)$ and laboratory $(x^{\text{lab}}, y^{\text{lab}})$ coordinates of a generic point $P$. The two sets of coordinates are related by a rotation. Reproduced from Fig. 1 of the second of Refs. [69], with authorization from A.A.P.T

Equation (10.41) expresses the obvious fact that if the particle is rotated through an angle relative to the moving frame, the corresponding description should be completely equivalent to the one obtained by rotating the moving frame in the opposite direction. This constitutes a mechanical analogue of a gauge invariance. The quantity $f$ is the classical generator of transformations within the gauge space associated with this simple model. It transforms a given physical trajectory into an equivalent one described from another frame. To choose a gauge means to select only one of these equivalent trajectories.

- Our aim is to quantize this classical model. The following commutation relations hold:

$$[\hat{\theta}, \hat{L}] = [\hat{\phi}, \hat{I}] = \mathrm{i}\hbar. \tag{10.42}$$

Since we have artificially enlarged the vector space, we must expect the presence of unphysical states and operators, in addition to physical ones. The constraint (10.41) is equivalent to the quantum mechanical conditions

$$\begin{aligned} \hat{f}\varphi_{\text{ph}} &= 0, \qquad \hat{f}\varphi_{\text{unph}} \neq 0, \\ [\hat{f}, \hat{O}_{\text{ph}}] &= 0, \qquad [\hat{f}, \hat{O}_{\text{unph}}] \neq 0, \end{aligned} \tag{10.43}$$

where the labels “ph” and “unph” indicate physical and unphysical states or operators. Except in simple cases, this separation is by no means a trivial operation.

- Since the problem displays circular symmetry, there is no restoring force in the intrinsic angular direction. Therefore, we expect a zero energy state created by an operator proportional to the generator $\hat{L}$. Consequently, “infrared divergences” may prevent the applicability of perturbation theory.

### 10.2.2 Outline of the BRST Solution

The most natural thing to do would be to use the constraint (10.41) to reduce the number of variables to the initial number. However, progress has been made in the opposite direction, i.e. by enlarging the number of variables and introducing a more powerful symmetry.

The collective subspace is given by the eigenfunctions of the orbital angular momentum in two dimensions (5.60):

$$\varphi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i} m\phi). \tag{10.44}$$

The collective coordinate $\phi$, which was introduced in Sect. 10.2.1 as an artifact associated with the existence of the moving frame, has been raised to the status of a real degree of freedom.

Since this problem has only one real degree of freedom, and since this role is taken by the collective angle, all others are unphysical. There must therefore be a trade-off: the intrinsic coordinate $\theta$ has to be transferred to the unphysical subspace.

In the BRST procedure, this unphysical subspace is also integrated with auxiliary fields [67]. All effects of the unphysical degrees of freedom on any physical observable must cancel out. Moreover, the degree of freedom $(\hat{\theta}, \hat{L})$ acquires a finite frequency through its mixture with the other spurious fields, and perturbation theory becomes feasible.

Quite generally, the total zero-order state may be factorized into three terms:

- A collective term carrying a representation of the degrees of freedom corresponding to the symmetry that is being restored: $\frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}m\phi)$ for two dimensional rotations (10.44); $\sqrt{\frac{2I+1}{8\pi^2}}D^I_{MK}$ for three dimensional rotations (5.43); $\frac{1}{\sqrt{V}}\exp(\mathrm{i}\mathbf{p}\cdot\mathbf{r}/\hbar)$ for translations (7.17); etc..
- A term representing the physical intrinsic degrees of freedom. For instance, had a radial potential $\frac{1}{2}M\omega^2(r - r_0)^2$ been added to the Hamiltonian, a harmonic oscillator would describe the (real) motion along the radial direction.
- An unphysical term including the degrees of freedom associated with the broken symmetries in the intrinsic system, plus the auxiliary fields that have been introduced.

The reader may believe these conclusions and proceed to Sect. 10.3 if he or she does not wish to get involved with the somewhat abstract BRST manipulations.

Note that the BRST procedure can be applied in particular to the BCS solution (Sect. 10.1), which displays all the necessary features:

1. There is an initial circular symmetry (in gauge space).
2. This symmetry is destroyed in the chosen, arbitrary frame ($\theta = 0$).
3. There is a large parameter measuring the deformation in gauge space ($\Delta/g$). It plays the same role as $r_0$ in the toy model. Eventually, a perturbation expansion in inverse powers of this parameter becomes possible.
4. There is a zero frequency boson associated with the generator of rotations in gauge space ($\hat{N}_\pi$).

Thus, (10.32) becomes completely justified.

### 10.2.3† A Presentation of the BRST Symmetry for the Abelian Case

An elementary presentation of the quantum mechanical BRST method is given in the second of references [69]. In the following, we follow this presentation.

- A new symmetry requires additional degrees of freedom. Thus, the gauge symmetry implied the existence of both intrinsic and collective coordinates (10.40). The BRST enlarged space is obtained through further inclusion of a new boson degree of freedom,[9] with an associated constraint,

$$[\hat{\mu}, \hat{B}] = \mathrm{i}, \qquad \hat{B}\,\varphi_{\mathrm{ph}} = 0, \tag{10.45}$$

  and of two new fermion variables $\eta$ and $\bar{\eta}$, called ghosts, with their conjugate partners $\pi$ and $\bar{\pi}$

$$\{\eta, \pi\} = \{\bar{\eta}, \bar{\pi}\} = 1. \tag{10.46}$$

  All other anticommutators vanish. The ghosts carry zero angular momentum.
- The generator $\hat{\Omega}$ of BRST transformations is a linear function of the two constraints (10.43) and (10.45)

$$\hat{\Omega} = -\eta \hat{f} + \bar{\pi}\hat{B}. \tag{10.47}$$

  It is a nilpotent ($\hat{\Omega}^2 = 0$) and Hermitian ($\hat{\Omega}^+ = \hat{\Omega}$) operator, which annihilates physical states and commutes with physical operators [see (10.43)]

$$\hat{\Omega}\,\varphi_{\mathrm{ph}} = 0, \qquad [\hat{\Omega}, \hat{Q}_{\mathrm{ph}}] = 0. \tag{10.48}$$

- However, there is a set of unphysical states and operators satisfying similar properties, namely

$$\varphi_\chi = \hat{\Omega}\,\varphi_{\mathrm{unph}}, \qquad \hat{Q}_\chi = \{\hat{\Omega}, \hat{Q}_{\mathrm{unph}}\}. \tag{10.49}$$

  Therefore, we must act within the composite subspace $\varphi_{\mathrm{ph}} + \varphi_\chi$ with the set of operators $\hat{Q}_{\mathrm{ph}} + \hat{Q}_\chi$. States $\varphi_\chi$ have zero norm. Fortunately, the enlargement of space and of the set of operators does not change the values of the matrix elements, since

$$(\langle \mathrm{ph}| + \langle \chi|)(Q_{\mathrm{ph}} + Q_\chi)(|\mathrm{ph}\rangle + |\chi\rangle) = \langle \mathrm{ph}|Q_{\mathrm{ph}}|\mathrm{ph}\rangle. \tag{10.50}$$

[9] The boson term $\hat{\mu}$ may include a constant value that plays the role of the Lagrange multiplier (10.12) in the previous section. We use $\hbar = 1$ in Sects. 10.2.3 and 10.2.4.

This statement may be verified term by term. For instance,

$$\langle \chi | Q_\chi | \text{ph} \rangle = \langle \text{unph} | \Omega^2 \, Q_{\text{unph}} | \text{ph} \rangle \; + \; \langle \chi | Q_{\text{unph}} \, \Omega | \text{ph} \rangle = 0 + 0. \tag{10.51}$$

- We construct the BRST Hamiltonian by adding to the Hamiltonian $\hat{H}$ a $\hat{Q}_\chi$ operator

$$\hat{H}_{\text{BRST}} = \hat{H} \; + \; \{\hat{\rho}, \hat{\Omega}\} \tag{10.52}$$

For any choice of the operator $\hat{\rho}$, $\hat{H}_{\text{BRST}}$ yields the same physical eigenvalues as the original $\hat{H}$. The selection of $\hat{\rho}$ is equivalent to the selection of a gauge. One possible choice is motivated by an analogy with the covariant gauge in Yang–Mills theory

$$\hat{\rho} = \pi \hat{\mu} \; + \; \bar{\eta} \left( \hat{\theta} - \frac{1}{2\mathcal{I}} \hat{B} \right)$$
$$\hat{H}_{\text{BRST}} = \hat{H} \; - \; \hat{\mu} \hat{f} \; + \; \mathrm{i} \pi \bar{\pi} \; + \; \hat{B} \hat{\theta} - \frac{1}{2\mathcal{I}} \hat{B}^2 \; + \; \eta \bar{\eta} \, [\hat{\theta}, \hat{L}], \tag{10.53}$$

where $\mathcal{I} = M r_0^2$ is the moment of inertia and $\hat{\theta}$ is a function of the intrinsic coordinates which does not commute with $\hat{L}$. It may well be the conjugate angle, but this is not necessary. Since $\hat{H}_{\text{BRST}}$ does not commute with $\hat{L}$ (unlike $\hat{H}$), the microscopic circular symmetry is lost. Microscopic invariance is replaced by a macroscopic collective invariance.

### 10.2.4† Application of the BRST Formalism to the Abelian Toy Model

The previous subsection displays a quite general presentation of the BRST formalism as applied to Abelian transformations. We return now to the toy model described in Sect. 10.2.1. We may choose the classical, "deformed" solution $x = r_0$, $y = 0$ as the starting point for the motion in the intrinsic system. The radius $r_0$ constitutes the large distance of the problem, the *order parameter*. Thus, the leading contribution to the particle angular momentum is $\hat{L}^{(0)} = r_0 \, \hat{p}_y$. It is convenient to choose $\hat{\theta}$ as the conjugate variable to this leading order term: $\hat{\theta} = \hat{y}/r_0$. This choice fixes the particle to the moving $x$-axis. Thus, $\hat{H}_{\text{BRST}}$ (10.53) may be written as

$$\hat{H}_{\text{BRST}} = \hat{H}_b \; + \; \hat{H}_g \; + \; \hat{H}_c \; + \; \hat{H}_x$$
$$\hat{H}_b = \frac{1}{2M} \hat{p}_y^2 \; - \; r_0 \hat{\mu} \hat{p}_y \; + \; \frac{1}{r_0} \hat{B} \, \hat{y} \; - \; \frac{1}{2\mathcal{I}} \hat{B}^2$$
$$\hat{H}_g = \mathrm{i} \pi \bar{\pi} \; + \; \mathrm{i} \eta \bar{\eta}$$

$$\hat{H}_c = \hat{\mu}\,\hat{I}$$
$$\hat{H}_x = \frac{1}{2M}\hat{p}_x^2 + \frac{1}{r_0}(\hat{x} - r_0)\,\eta\,\bar{\eta} + \hat{\mu}\,[(\hat{x} - r_0)\,\hat{p}_y - \hat{y}\,\hat{p}_x]. \tag{10.54}$$

As usual in field theory, we proceed to diagonalize the quadratic Hamiltonian to define a basis of independent bosons and fermions. By completing squares, we obtain for $\hat{H}_b$

$$\hat{H}_b = \frac{1}{2M}(\hat{p}_y - M r_0 \hat{\mu})^2 + \frac{M}{2}\hat{y}^2 - \frac{1}{2\mathcal{I}}\left(\hat{B} - M r_0 y\right)^2 - \frac{\mathcal{I}}{2}\hat{\mu}^2$$
$$= \Gamma_1^+ \Gamma_1 - \gamma_0^+ \gamma_0, \tag{10.55}$$

where

$$\Gamma_1^+ = \frac{1}{\sqrt{2M}}(\hat{p}_y - M r_0 \hat{\mu}) + \mathrm{i}\frac{M}{2}\hat{y}$$
$$\gamma_0^+ = -\mathrm{i}\frac{1}{\sqrt{2\mathcal{I}}}(\hat{B} - M r_0 \hat{y}) + \sqrt{\frac{\mathcal{I}}{2}}\hat{\mu}$$
$$[\Gamma_1, \Gamma_1^+] = [\gamma_0, \gamma_0^+] = 1. \tag{10.56}$$

Thus, $\hat{H}_b$ has been written in terms of two uncoupled oscillators, with frequencies $\pm 1$. We can overcome this last inconvenience through the replacement $\gamma_0^+ \to \Gamma_0$, $\gamma_0 \to \Gamma_0^+$. Therefore,

$$\hat{H}_b = \Gamma_1^+ \Gamma_1 - \Gamma_0^+ \Gamma_0 + 1 \tag{10.57}$$
$$[\Gamma_0^+, \Gamma_0] = 1. \tag{10.58}$$

If the new vacuum state is annihilated by $\Gamma_0$, all excitations of $\hat{H}_b$ become positive, at the expense of working with the anomalous metric (10.58).[10]

The ghost sector may be written as

$$\hat{H}_g = \bar{a}a - \bar{b}b - 1$$
$$a = \mathrm{i}\bar{b}^+ = \frac{1}{\sqrt{2}}(\bar{\pi} - \mathrm{i}\eta), \quad b = -\mathrm{i}\bar{a}^+ = \frac{1}{\sqrt{2}}(\pi + \mathrm{i}\bar{\eta}). \tag{10.59}$$

Note that

$$\{\bar{a}, a\} = \{\bar{b}, b\} = 1, \text{ but } \bar{a} \neq a^+ \;;\; \bar{b} \neq b^+. \tag{10.60}$$

[10]This metric has been also employed for the boson associated with the Lagrange multiplier, for instance in QED.

Therefore, the quadratic Hamiltonian of the unphysical sector,

$$\hat{H}^{(2)}_{\text{unph}} = \Gamma_1^+ \Gamma_1 - \Gamma_0^+ \Gamma_0 + \bar{a}a + \bar{b}b, \tag{10.61}$$

is a supersymmetric Hamiltonian with a characteristic energy equal to unit. Some interesting features of this Hamiltonian are:

- The eigenvectors define the subspace

$$\varphi_{n_1,n_0,n_a,n_b} = \frac{1}{\sqrt{n_1!\, n_0!}} (\Gamma_1^+)^{n_1} (\Gamma_0^+)^{n_0} (\bar{a})^{n_a} (\bar{b})^{n_b} \varphi_0, \tag{10.62}$$

  with $n_1, n_0 = 0, 1, 2, ...$ and $n_a, n_b = 0, 1$.
- The vacuum state is annihilated by the operators $\Gamma_1$, $\Gamma_0$, $a$, $b$ and thus, by the quadratic term in the $BRST$ generator (10.47)

$$\hat{\Omega}^{(2)} = -\mathrm{i}(\Gamma_1^+ + \Gamma_0^+)a - (\Gamma_1 + \Gamma_0)\bar{b}. \tag{10.63}$$

  Therefore, this vacuum is a physical state, according to (10.43). In fact, it is the only one among the whole set of states (10.62). Moreover, the system displays the BRST symmetry ($[\hat{\Omega}, \hat{H}_{\text{BRST}}] = 0\,; \quad \hat{\Omega}\,\varphi_0 = 0$).
- The cancelation of unphysical effects would not be possible if all states (10.62) were ordinary states in Hilbert space. However, the unusual relations (10.58) and (10.60) are well defined and may be used without problems.
- As a consequence of the circular symmetry, the Hamiltonian $\hat{H}$ does not display a restoring force in the $y$-direction. However, this symmetry is lost for $\hat{H}_{\text{BRST}}$ in the intrinsic frame and, as a consequence, all unphysical degrees of freedom have acquired a finite frequency [see (10.61)]. Perturbation theory becomes feasible.
- The coupling term $\hat{H}_c$ in (10.54) can be obtained from the second equation (10.56)

$$\hat{H}_c = \frac{\hat{I}}{\sqrt{2\mathcal{I}}}(\Gamma_0^+ + \Gamma_0) \tag{10.64}$$

  Since this term is small [$\mathcal{O}\,(1/r_0)$], it can be treated in perturbation theory. The second-order contribution has the value

$$\Delta E^{(2)} = -\hat{I}^2\,{}_{\text{unph}}\langle 0|\mu|n_0 = 1\rangle\langle n_0 = 1|\mu|0\rangle_{\text{unph}} = \frac{\hat{I}^2}{2\mathcal{I}}, \tag{10.65}$$

  where (10.56) and (10.58) have been used. Second-order perturbation theory yields a positive contribution to the energy of the ground state due to the unusual metric (10.58). It agrees with the exact result in the present case. The associated eigenfunctions are given by (10.44). The spectrum displays a physical collective rotation given by (10.44) and (10.65).

- The presence of $\hat{H}_x$ allows for radial motion (or motion along the $x$-axis in the intrinsic frame). The spectrum also displays physical, intrinsic, finite-frequency modes.
- There are unphysical excitations described by the excited states of (10.62).
- The BRST treatment is something of an overkill for the Abelian model. In fact, the ghosts are uncoupled from the start. Their role is much more significative for non-Abelian transformations. "Nevertheless, through the Abelian calculation, we have been able to discuss properties of the BRST procedure that continue to be present in the non-Abelian case, such as the BRST symmetry, the existence of the zero-norm subspace, the construction of the unphysical subspace, and the feasibility of the perturbation expansion" [69]. Moreover, statements in Sect. 10.2.2 have been substantiated.

## 10.3 Generalizations

Symmetry breaking is common in physics.

The treatment of the two examples presented in this chapter has in common the fact that, although the generators of the transformations commute with the Hamiltonian ($[\hat{L}, \hat{H}] = [\hat{N}, \hat{H}] = 0$), the ground states are represented by states which are not annihilated by the generators. As a consequence, the states $\hat{L}\varphi_0$ and $\hat{N}\varphi_0$ have the same energy as the ground state configuration $\varphi_0$. There are many degenerate ground states. We speak of a *hidden symmetry*.

In finite systems one of these configurations may tunnel through to other configurations, so the true ground state becomes a superposition of degenerate states. One consequence is that the lost symmetry is retrieved at the collective level in the form of rotational bands, as indicated after (10.32).

For instance, in many nuclear species it is convenient to extend the spherical shell-model (Sect. 7.3.2) by including a quadrupole term in the single-particle potential $V_2 \propto r^2 Y_{20}(\theta)$ [38]. In this case, the large number measuring the deformation is the expectation value of the quadrupole operator $\langle 0| \sum_h r_h^2 Y_{20}(\theta_h) |0\rangle$. As a consequence, $\varphi_0$ is not longer an eigenstate of the angular momentum $\hat{I}$ with null eigenvalue. The associated degeneracy manifests itself in the form of low-lying rotational bands,[11] with characteristic energies $(\hbar^2/2\mathcal{I})I(I+1)$ and eigenstates $D^I_{MK}\,\varphi_\nu$ (Sect. 5.3.2$^\dagger$).

One also finds zero-energy bosons in infinite deformed systems, which are related to the one found in subsection 10.1.5$^\dagger$. They are called Goldstone bosons. Moreover, they may materialize as very light particles, like the pion. However, the procedure based on collective variables is not applied to those cases, since it may take an infinite amount of time to get across all configurations.

[11] Similar effects appear in the case of the (non-spherical) diatomic molecule (Sect. 8.4.2).

A second generalization concerns the generation of energy, a consequence similar to the increase of the single-particle energy $|\epsilon_p - \mu|$ by an amount of $\mathcal{O}(\Delta)$. Most of the mass of hadrons, such as the proton, arises not from the masses of their constituent quarks, but from the quarks' kinetic energy and the energy stored in the gluon field, through processes involving the breakdown of "gauge symmetries". For instance, in the Nambu–Jona Lasinio model of nucleons, quarks are subject to contact forces, and their condensate becomes responsible for the generation of the nucleon mass $M$: the solution of an equation similar to (10.19) yields the nucleon (relativistic) energies $E_p = (p^2 + M^2c^4)^{1/2}$, having the same form as in (10.18).

## Problems

**Problem 1.** Consider two fermions moving in a $j$-shell:

1. Calculate the size of the Hamiltonian matrix if we assume two-particle states of the form
$$\phi_{mm'} = \frac{1}{\sqrt{2}}\left[\varphi_{jm}(1)\varphi_{jm'}(2) - \varphi_{jm}(2)\varphi_{jm'}(1)\right] ;$$
2. Approximate the matrix elements of the Hamiltonian by the expression $\langle mm'|H|m''m'''\rangle = -g\,\delta_{m(-m')}\delta_{m''(-m''')}$. Calculate the size of the matrix to be diagonalized;
3. Find the eigenvectors and eigenvalues. Hint: Try a solution of the form $\Psi = \sum_m c_m\phi_{m(-m)}$: (a) with amplitudes $c_m$ = constant, (b) with amplitudes such that $\sum_m c_m = 0$.

The particles in the resulting extra-bound state $[E_0 = -g(j + \frac{1}{2})]$ are said to form a Cooper pair. This extra binding is the basis for the explanation of the phenomenon of superconductivity.

**Problem 2.** Let us consider $A_\pi$ pairs of particles which are allowed to move in degenerate, $\Omega$ pairs of time-reversed states. The particles are coupled by the pairing interaction (10.10). Calculate:

1. The exact ground state energy for $A_\pi = \Omega/2$. Hint: verify that the operators $\hbar\hat{P}^+$, $\hbar\hat{P}$ and $\hbar(\frac{1}{2}\Omega - \hat{N}_\pi)$ satisfy the same commutation relations as the three components of the angular momentum operator. Find the maximum value of $J$.
2. The moment of inertia $\mathcal{I}$ of the "rotational" band associated with the ground states of even systems. Hint: $\hat{H}_{\text{rot}} = \hbar^2(\frac{1}{2}\Omega - \hat{N}_\pi)^2/2\mathcal{I}$.
3. The lowest excitation energy for $A_\pi = \frac{1}{2}\Omega$. Hint: Use $J' = J - 1$ for the excited state.

**Problem 3.** Solve the BCS equations for the system considered in the previous problem

1. Write $E_p$, $V_p^2$, $\Delta$, $\mu$.

2. Calculate the three results obtained in Problem 2 within the BCS approximation.
3. Compare the results of the previous item with the exact results obtained in Problem 2, and explain the origin of the discrepancies.

**Problem 4.** Write the operator $\hat{P} - \frac{\Delta}{g}$ in terms of quasi-particles.

**Problem 5.** Consider a symmetric system of levels around the Fermi surface:

1. Verify that there is always one boson root at zero energy.
2. Calculate the energy of the first excited root.

**Problem 6.** Verify all terms in (10.50).

**Problem 7.** Show that all four terms in $\hat{H}^{(2)}_{\text{unph}}$ yield a positive contribution if applied to the ground state eigenvector $\varphi_0$.

**Problem 8.** 1. Obtain an expression for the operator $\hat{L}^{(0)}$ in terms of the boson degrees of freedom $\Gamma_1^+, \Gamma_0^+$.
2. Calculate the matrix element $\langle n_1 = n_0 = 1 | L^{(0)} | 0 \rangle$.