Chapter 11
Eigenvectors of the Position Operator Path Integral Formulation

In the first place, we enlarge the Hilbert space[1] by broadening the normalization procedure through the introduction of the Dirac delta function.[2] Subsequently, the notion of propagator as a probability amplitude is introduced. This concept allows us to present the formulation of quantum mechanics in terms of path integrals. This formalism, due to Feynman [72], provides a most powerful link between classical and quantum mechanics. It is widely applied in field theory, statistical mechanics, cosmology, financial mathematics, etc.

11.1† Eigenvectors of the Position Operator and the Delta Function

It is convenient to include the eigenvectors φ_x of the position operator $\hat{x}$ in the quantum formalism.

$$\hat{x}\,\varphi_x = x\,\varphi_x \ . \tag{11.1}$$

We must give a value to the scalar product between continuum eigenstates $\langle x|x'\rangle$. By analogy with (2.58), we write the unit operator as

$$I = \int \mathrm{d}x\,|x\rangle\langle x| \ . \tag{11.2}$$

Thus, either discrete or continuous states φ_n may be expressed as

$$\varphi_n = \int \mathrm{d}x'|x'\rangle\langle x'|n\rangle \ . \tag{11.3}$$

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[1] Another enlargement was accomplished in Sect. 10.2.4†.

[2] Since the mathematical level of the present chapter is somewhat higher than in the other ones, it is only reccommended for a second lecture or to physics students oriented towards mathematical formalizations.

D. Bes, *Quantum Mechanics*, Graduate Texts in Physics,
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Let us apply the bra $\langle x|$ to both sides of this equation

$$\langle x|n\rangle = \int \mathrm{d}x'\,\delta(x-x')\langle x'|n\rangle$$
$$\langle x|x'\rangle \equiv \delta(x-x'), \tag{11.4}$$

where the scalar product between position eigenstates is given by the Dirac delta function, a generalization of the Kronecker delta to the continuum case.[3] Since $\delta(x-x')$ vanishes for $x \neq x'$ and equals ∞ if $x = x'$, the limits of integration in (11.4) can be arbitrary, provided that the point x is enclosed within the interval of integration.

There exists a continuous formalism quite parallel to the discrete one used so far in this book. In fact, the wave functions $\varphi_n(x)$ introduced in connection with the Schrödinger representation may be interpreted as the probability amplitude of the system described by the state vector φ_n to be at the position x.

$$\varphi_n(x) = \langle x|n\rangle\ . \tag{11.5}$$

Upon a measurement of the position performed with a precision Δx, the state collapses into a wave packet of dimension Δx with probability $|\varphi_n(x)|^2\,\Delta x$.

There are several mathematical representations of the delta function, such as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\,\exp(\mathrm{i}kx)$$
$$= \lim_{\eta\to 0}\frac{1}{\eta\sqrt{\pi}}\exp(-x^2/\eta^2)\ . \tag{11.6}$$

The delta-normalization is not only applicable to the position eigenstates (as in (11.4)). In particular, the eigenfunctions (4.32) of the momentum operator can be written as

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}kx)$$
$$\langle k|k'\rangle = \delta(k-k')\ , \tag{11.7}$$

where (11.6) has been applied.

It is also possible to define the momentum probability amplitudes

$$\varphi_n(p) = \langle p|n\rangle = \int \mathrm{d}x\,\langle p|x\rangle\langle x|n\rangle = \frac{1}{(2\pi\hbar)^{1/2}}\int \mathrm{d}x\,\exp(-\mathrm{i}px/\hbar)\,\langle x|n\rangle\ . \tag{11.8}$$

The probability amplitudes $\langle x|n\rangle$ and $\langle p|n\rangle$ are the Fourier transforms of each other.

[3] The delta function is not a proper function, but a distribution. It is only defined within integrals such as in (11.4).

11.2† The Propagator

The propagator (or Green function) is the probability amplitude for the transition from (x_1, t_1) to (x_2, t_2).

$$\langle x_2, t_2; |x_1, t_1\rangle = \langle x_2|\mathcal{U}(t_2 - t_1)|x_1\rangle \ , \tag{11.9}$$

where we have assumed that $t_2 \geq t_1$. The propagator summarizes the quantum mechanics of the system.

- Using (9.2) and (11.4) one obtains

$$\lim_{t_2 \to t_1} \langle x_2, t_2|x_1, t_1\rangle = \langle x_2|x_1\rangle = \delta(x_2 - x_1) \ . \tag{11.10}$$

- The time evolution of a state $\varphi_n(x, t)$ is given by the space integral

$$\varphi_n(x_2, t_2) = \int \mathrm{d}x_1 \ \langle x_2, t_2|x_1, t_1\rangle \ \varphi_n(x_1, t_1) \ . \tag{11.11}$$

- The propagator is also related to more familiar concepts, as energy levels and wave functions. We can write (11.9) as

$$\begin{aligned}\langle x_2, t|x_1, 0\rangle &= \sum_n \langle x_2|n\rangle \exp(-\mathrm{i}E_n t/\hbar)\langle n|x_1\rangle \\ &= \sum_n \varphi_n(x_2)\exp(-\mathrm{i}E_n t/\hbar)\varphi_n^*(x_1) \\ \int \mathrm{d}x \ \langle x, t|x, 0\rangle &= \sum_n \exp(-\mathrm{i}E_n t/\hbar) \ ,\end{aligned} \tag{11.12}$$

  where $t = t_2 - t_1$. A Fourier transform is performed on both sides of (11.12)

$$\begin{aligned}\langle x_2, z|x_1, 0\rangle &= \sum_n \int_0^\infty \mathrm{d}t \ \varphi_n(x_2)\exp[\mathrm{i}(z - E_n)t/\hbar)]\varphi_n^*(x_1) \\ &= \mathrm{i}\hbar \sum_n \frac{\varphi_n(x_2) \ \varphi_n^*(x_1)}{z - E_n} \ ,\end{aligned} \tag{11.13}$$

  where $\mathrm{Im}(z) > 0$ has been assumed. Singularities arise at $\mathrm{Im}(z) > 0$. The poles of the energy propagator yield the energy of the states, while wave functions are given by the residues, in the case of a discrete spectrum. The propagator displays a cut for a continuous spectrum.
- An alternative interpretation of the propagator regarded as a function of x_2 is that it represents the wave function at time t_2 of a particle that was localized before at

(x_1, t_1). Thus, the propagator satisfies the time-dependent Schrödinger equation in the variables (x_2, t_2).

- The composition property of quantum mechanics allows us to divide the time interval $t_2 - t_1$ into two segments

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int \mathrm{d}x'' \langle x_2, t_2 | x'', t'' \rangle \, \langle x'', t'' | x_1, t_1 \rangle \;, \tag{11.14}$$

since the unit operator may be expressed as

$$I = \int \mathrm{d}x'' | x'', t'' \rangle \langle x'', t'' | = \exp[-\mathrm{i}\hat{H} t''/\hbar] \int \mathrm{d}x'' | x'' \rangle \langle x'' | \exp(\mathrm{i}\hat{H} t''/\hbar) \;. \tag{11.15}$$

Here $\hat{H}$ is assumed to be time independent and (11.2) has been used.

The amplitude for the transition from (x_1, t_1) to (x_2, t_2) may be expressed as the result of transition from (x_1, t_1) to all available intermediate points x'' followed by transitions from (x'', t'') to (x_2, t_2). In a two-slit experiment, the total probability amplitude for the particle being detected at the point x_2 on the screen is the sum of the amplitudes for the particle starting at the point x_1 and passing through either of the two holes. Interference plays here a fundamental role, since the probability of finding the particle at x_2 results from a double-path interference process.

- The interval $t_2 - t_1$ may also be further subdivided into n segments

$$\langle x_{2,2} | x_1, t_1 \rangle = \oint [\mathrm{d}x] \prod_{\nu=1}^{n} \langle x^{(\nu)}, t^{(\nu)} | x^{(\nu-1)}, t^{(\nu-1)} \rangle$$

$$\oint \equiv \int\int \ldots \int \;, \qquad [\mathrm{d}x] \equiv \prod_{\nu=1}^{n-1} \mathrm{d}x^{(\nu)} \;. \tag{11.16}$$

Here $t_2 - t_1 = \sum_{\nu=1}^{n} (t^{(\nu)} - t^{(\nu-1)})$ and $x^{(n)} = x_2$, $t^{(n)} = t_2$, $x^{(0)} = x_1$, $t^{(0)} = t_1$. These last two positions are fixed points, not variables of integration.

11.2.1† The Free Particle Propagator

We calculate now the propagator of a free particle. In this case, a continuous quantum number k labels the eigenstates of the momentum (11.7)

$$\langle x | k \rangle = \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i}kx) \;, \qquad E_k = \frac{\hbar^2 k^2}{2M} \;. \tag{11.17}$$

We obtain a Gaussian integral by introducing these expressions in (11.12)

$$\langle x_2, t_2 | x_1, t_1 \rangle_{\text{free}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \exp\left[-\mathrm{i}k(x_2 - x_1) - \mathrm{i}k^2 \hbar(t_2 - t_1)/2M\right] . \tag{11.18}$$

Completing squares yields the exponential function

$$\exp\left(-\mathrm{i}\frac{\hbar(t_2 - t_1)}{2M}\left[(k + \alpha)^2 - \alpha^2\right]\right) . \tag{11.19}$$

Therefore,

$$\begin{aligned} \langle x_2, t_2 | x_1, t_1 \rangle_{\text{free}} &= \frac{1}{2\pi} \exp\left[\frac{\mathrm{i}M(x_2 - x_1)^2}{2\hbar(t_2 - t_1)}\right] \int_{-\infty}^{\infty} \mathrm{d}k' \exp[-\beta^2 k'^2] \\ &= \sqrt{\frac{M}{\mathrm{i}2\pi\hbar(t_2 - t_1)}} \exp\left[\frac{\mathrm{i}M(x_2 - x_1)^2}{2\hbar(t_2 - t_1)}\right] , \end{aligned} \tag{11.20}$$

where

$$\alpha = \frac{(x_2 - x_1)M}{(t_2 - t_1)\hbar} , \qquad \beta^2 = \frac{\mathrm{i}\hbar(t_2 - t_1)}{2M} , \qquad k' = k + \alpha . \tag{11.21}$$

The free particle propagator (11.20) displays a central role in the next section.

11.3† Path Integral Formulation of Quantum Mechanics

Consider the case of a particle moving in a one-dimensional potential $V(x)$. The particle starts at the position (x_1, t_1) and ends at (x_2, t_2). The time $t_2 - t_1$ is divided into n equal intervals of duration τ. The propagator between the positions $x^{(\nu-1)}$ and $x^{(\nu)}$ during this time interval is[4]

$$\left\langle x^{(\nu)} \left| \exp\left[-\frac{\mathrm{i}\tau}{\hbar}(T + V)\right] \right| x^{(\nu-1)} \right\rangle \tag{11.22}$$

$$= \left\langle x^{(\nu)} \left| \exp\left(-\frac{\mathrm{i}\tau}{\hbar}T\right) \exp\left(-\frac{\mathrm{i}\tau}{\hbar}V\right) \exp\left(-\frac{\tau^2}{2\hbar^2}[T, V]\right) \right| x^{(\nu-1)} \right\rangle ,$$

where T is the kinetic energy. In the limit $\tau \to 0$, the term proportional to the commutator $[T, V]$ can be neglected, since it is of higher order in $\tau/\hbar$.

[4]Use is made of the mathematical identity $\exp(A + B) = \exp(A)\exp(B)\exp(-\frac{1}{2}[A, B])$ (times exponential terms involving two or more commutators).

$$
\begin{aligned}
&\lim_{\tau\to 0}\left\langle x^{(\nu)}\left|\exp\left[-\frac{\mathrm{i}\tau}{\hbar}(T + V)\right]\right| x^{(\nu-1)}\right\rangle \\
&= \lim_{\tau\to 0}\langle x^{(\nu)}, t^{(\nu)}|x^{(\nu-1)}, t^{(\nu-1)}\rangle_{\text{free}} \exp\left[-\frac{\mathrm{i}\tau}{\hbar}V(x^{(\nu-1)})\right] \\
&= \lim_{\tau\to 0}\left(\frac{M}{\mathrm{i}2\pi\hbar\tau}\right)^{1/2} \exp\left[\frac{\mathrm{i}\tau}{\hbar}\left(T(\dot{x}^{(\nu)}) - V(x^{\nu})\right)\right] \\
&= \lim_{\tau\to 0}\left(\frac{M}{\mathrm{i}2\pi\hbar\tau}\right)^{1/2} \exp\left[\frac{\mathrm{i}\tau}{\hbar}\mathcal{L}(\dot{x}^{(\nu)}, x^{(\nu)})\right] ,
\end{aligned} \tag{11.23}
$$

where $\mathcal{L}$ is the classical Lagrangian. The potential term has been factored out during each interval $\tau \to 0$ because it stays constant within such interval. The result (11.20) for the kinetic energy part of the Hamiltonian has been used.

The particle travels from x_1 to x_2 through a series of intermediate steps $x^{(1)}, x^{(2)}, \ldots$, which define a "path." The total amplitude for the particle to begin at x_1 and end up at x_2 is given by the sum over all possible paths. The number of integrations become infinite as the time interval τ tends to zero. In this limit, the propagator for the total time interval $t_2 - t_1$ may be written as a path integral

$$
\lim_{\tau\to 0, n\to\infty}\langle x_2, t_2|x_1, t_1\rangle = \lim_{\tau\to 0, n\to\infty}\left(\frac{M}{\mathrm{i}2\pi\hbar\tau}\right)^{n/2} \oint [\mathrm{d}x] \exp\left[\frac{\mathrm{i}}{\hbar}\mathcal{S}(x_2, t_2|x_1, t_1)\right] , \tag{11.24}
$$

where

$$
\mathcal{S}(x_2, t_2|x_1, t_1) = \sum_{\nu=1}^{n} \tau\langle x^{(\nu)}, t^{(\nu)}|x^{(\nu-1)}, t^{(\nu-1)}\rangle = \int_{t_1}^{t_2} \mathrm{d}t\, \mathcal{L}(t) . \tag{11.25}
$$

The action S has the same dimension as $\hbar$. The symbols $\oint$ and $[\mathrm{d}x]$ have been defined in (11.16).

- All the paths contribute equally in magnitude, but the phase of their contribution is different. The phase is given by the classical action (in units of $\hbar$).
- Expression (11.24) has been derived from laws of quantum mechanics to which the reader has become familiar by now. One of them is the superposition principle, used in summing the contributions of alternative paths. The other is the composition property of the transition amplitude, which allows us to divide the total time interval into segments of vanishing duration τ. Feynman's original approach was to adopt (11.24) as an hypothesis and, subsequently, derive the time-dependent Schrödinger equation.
- The path integral relates certain quantum probability amplitudes to the classical action. Therefore, it provides a deep link between classical and quantum physics which is explained in [72] as follows: the classical approximation corresponds to the case of large $\mathcal{S}$ in relation to $\hbar$. A small change in the path induces a change

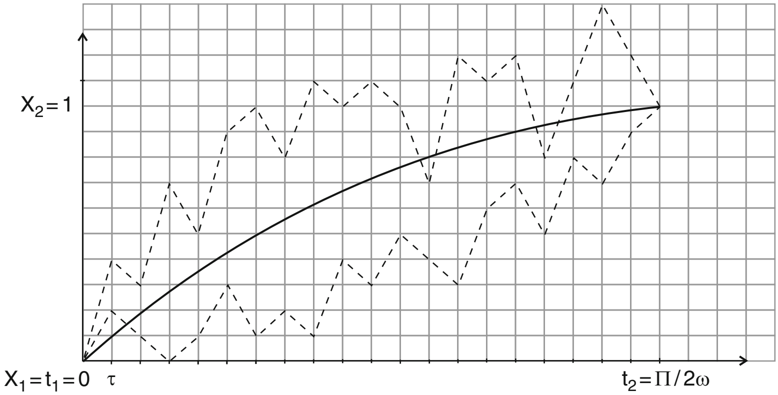

Fig. 11.1 Trajectory of a particle moving in a harmonic oscillator potential: minimum action trajectory (*full line*) and two other possible trajectories (*dashed lines*)

in $\mathcal{S}$ that is small in the scale of $\mathcal{S}$ but not negligible in the scale of $\hbar$. Thus, small changes in the path produce finite changes in the phase, and the contributions of the different trajectories mutually cancel because they do not add coherently in phase with one another. However, there is the special path for which $\mathcal{S}$ is an extreme (Fig. 11.1). In this case, there is no change in $\mathcal{S}$, at least to first order. In this way the classical laws of motion arise from the quantum laws. From the extreme condition one derives the Euler's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \,. \tag{11.26}$$

Classical physics takes into account only the trajectory $x_{\rm cl}(t)$ satisfying (11.26).

- In the quantum case, $\mathcal{S}$ may be comparable with $\hbar$, and all the trajectories must be added in detail.
- Path integrals whose exponents are quadratic in x and $\dot{x}$ may be calculated exactly. Thus, it is useful to find first the classical path $x_{\rm cl}$ using (11.26) and, subsequently, to replace $x, \dot{x}$ by $y = x - x_{\rm cl}$, $\dot{y} = \dot{x} - \dot{x}_{\rm cl}$ in the Lagrangian. Exact quantum expressions for the quadratic fluctuations around the classical path can be obtained (see the example in Sect. 11.3.1†). Higher than quadratic terms in the potential may be calculated by means of perturbation expansions.
- "Since its inception in Richard Feynman's 1942 doctoral thesis, the path integral has been a physicist's dream and a mathematician's nightmare. To a physicist, the path integral provides a powerful and intuitive way to understand quantum mechanics, building on the simple idea that quantum physics is fundamentally a theory of superposition and interference of probability amplitudes... To a mathematician, the path integral is at best an ill-defined formal expression. It is some sort of vaguely integral-like object involving a sum over badly specified collection of functions, having an undefined measure, and whose value

is apparently determined by a group of unclear and perhaps incompatible limits that may or may not yield finite answers”[5] [74].

- Feynman’s path integral is not a too convenient tool for solving problems in non-relativistic quantum mechanics, such as those tackled in this text. On the contrary, methods based on path integrals are very powerful in other branches of modern physics, such as quantum field theory and statistical mechanics. The Lagrange formulation of problems involving quantum chromodynamics (QCD), the theory of strong interactions, is amenable to quantification via path integrals and to subsequent perturbation expansion. Expressions are obtained in a Lorentz-covariant form. The relation with statistical mechanics stems from the third line in (11.12), which resembles the “sum over states” associated with the partition function

$$\mathcal{Z} = \sum_n \exp(-\beta E_n) \ . \tag{11.27}$$

In fact, one obtains the partition function from (11.12) by analytical continuation of t into the purely imaginary axis, with $\beta = \mathrm{i}t/\hbar$, real and positive.

11.3.1† *The Harmonic Oscillator Re-revisited. The Path Integral Calculation*

Let us exemplify once more with the harmonic oscillator. The Lagrangian reads

$$\mathcal{L} = \frac{M}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \ . \tag{11.28}$$

Therefore, the classical equation of motion (11.26) yields the equation $\ddot{x} + \omega^2 x = 0$. The solution is

$$\begin{aligned} x_{\rm cl} &= A \sin\omega t + B\cos\omega t \\ A &= \frac{x_2\cos\omega t_1 - x_1\cos\omega t_2}{\sin\omega(t_2-t_1)} \ , \quad B = \frac{x_1\sin\omega t_2 - x_2\sin\omega t_1}{\sin\omega(t_2-t_1)} \ . \end{aligned} \tag{11.29}$$

We make now a transformation of variables

$$y = x - x_{\rm cl} \ , \tag{11.30}$$

and thus, the action reads

$$\mathcal{S} = \mathcal{S}_{\rm cl} + \frac{M}{2}\left(\dot{y}^2 - \omega^2 y^2\right) \ , \quad y(t_1) = y(t_2) = 0 \ . \tag{11.31}$$

[5] An up-to-date attempt to put path integral methods on a sound mathematical footing can be found in [73].

This expression is exact, since in this case there are no higher order terms in y. Therefore, the path integral has been factorized into

$$\langle x_2, t_2 | x_1, t_1 \rangle = \langle 0, t_2 | 0, t_1 \rangle \exp(\mathrm{i}\mathcal{S}_{\mathrm{cl}}) \ . \tag{11.32}$$

The two factors are calculated in Sect. 11.3.2*. The results are

$$\langle 0, t_2 | 0, t_1 \rangle = \sqrt{\frac{M\omega}{\mathrm{i}2\pi\hbar \sin\omega(t_2 - t_1)}} \tag{11.33}$$

$$\mathcal{S}_{\mathrm{cl}} = \frac{M\omega}{2\sin\omega(t_2 - t_1)}[(x_1^2 + x_2^2)\cos\omega(t_2 - t_1) \ - \ 2x_1x_2] \ .$$

The dependence on the spatial coordinates x_1 and x_2 is determined only by the classical action.

Let us obtain once more the energy levels of the harmonic oscillator. We use the trace of the propagator (11.32)

$$\begin{aligned}\int \mathrm{d}x \, \langle x, t | x, 0 \rangle &= \left(\frac{M\omega}{\mathrm{i}2\pi\hbar \sin\omega t}\right)^{1/2} \int \mathrm{d}x \exp\left(-\frac{\mathrm{i}2M\omega \sin^2(\omega t/2)}{\hbar \sin\omega t} x^2\right) \\ &= \frac{1}{\mathrm{i}2\sin\omega t/2} \\ &= \exp\left(-\mathrm{i}\frac{1}{2}\omega t\right) \sum_{n=0}^{\infty} \exp(-\mathrm{i}n\omega t) \ .\end{aligned} \tag{11.34}$$

The sought energies are obtained by comparing the last line of (11.34) and of (11.12).

11.3.2 The Classical Action and the Quantum Correction to the Harmonic Oscillator*

In the first place we compute the classical action

$$\begin{aligned}S_{\mathrm{cl}} &= \frac{M}{2} \int_{t_1}^{t_2} \mathrm{d}t \, (\dot{x}_{\mathrm{cl}}^2 \ - \ \omega^2 x_{\mathrm{cl}}^2) \\ &= \frac{M\omega}{4} \left[(A^2 - B^2)\cos 2\omega t \ - \ 2AB \sin 2\omega t\right]_{t_1}^{t_2} \ ,\end{aligned} \tag{11.35}$$

which yields (11.34), upon replacement of the values A, B given in (11.29).

Let us now evaluate the propagator $\langle 0, t_2|0, t_1\rangle$ for the harmonic oscillator.[6] According to (11.23)

$$\begin{aligned}
&\langle 0, t_2|0, t_1\rangle \\
&= \lim_{n\to\infty}\left(\frac{M}{\mathrm{i}2\pi\hbar\tau}\right)^{n/2}\oint[\mathrm{d}y]\exp\left\{\frac{\mathrm{i}M}{2\hbar}\sum_{\nu=0}^{n-1}\left[\frac{(y_{\nu+1}-y_\nu)^2}{\tau}-\tau\omega^2 y_\nu^2\right]\right\} \\
&= \lim_{n\to\infty}\left(\frac{M}{\mathrm{i}2\pi\hbar\tau}\right)^{n/2}\oint[\mathrm{d}\eta]\exp\left(-\frac{M}{\mathrm{i}2\tau\hbar}\,\eta^T\sigma\eta\right)\ ,
\end{aligned} \tag{11.36}$$

with

$$y_0 = y_n = 0\ ;\quad \tau = (t_2 - t_1)/n\ ;\quad [\mathrm{d}y] = \prod_{\nu=1}^{n-1}\mathrm{d}y_\nu\ . \tag{11.37}$$

In the second line of (11.36) we have used a vector (η_ν) and a matrix $(\langle\nu|\sigma|\omega\rangle)$ which are defined as

$$(\eta_\nu) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1}\end{pmatrix}\ ,\quad (\langle\nu|\sigma|\omega\rangle) = \begin{pmatrix} 2-\tau^2\omega^2 & -1 & 0 & \ldots & 0 \\ -1 & 2-\tau^2\omega^2 & -1 & \ldots & 0 \\ 0 & -1 & 2-\tau^2\omega^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2-\tau^2\omega^2\end{pmatrix}. \tag{11.38}$$

Diagonalization of the matrix σ yields the matrix ϵ with eigenvalues ϵ_a $(1 \le a \le n-1)$

$$\sigma = \mathcal{U}^+\epsilon\,\mathcal{U}\ ,\qquad \zeta = \mathcal{U}\,\eta \tag{11.39}$$

Here $\mathcal{U}$ is unitary and real. The vector ζ is also real and constitutes a $(n-1)$ dimensional vector space. The Jacobian going from $[\mathrm{d}\eta]$ to $[\mathrm{d}\zeta]$ is also 1, since $|\det(\mathcal{U})| = 1$. Therefore, the integral in the second line of (11.36) has the value

$$\begin{aligned}
\oint[\mathrm{d}\eta]\exp\left(-\frac{M}{\mathrm{i}2\tau\hbar}\,\eta^T\sigma\eta\right) &= \oint[\mathrm{d}\zeta]\exp\left(-\frac{M}{\mathrm{i}2\tau\hbar}\,\zeta^T\epsilon\zeta\right) \\
&= \prod_{a=1}^{n-1}\left(\frac{\mathrm{i}2\pi\tau\hbar}{M\epsilon_a}\right)^{1/2} = \left(\frac{\mathrm{i}2\pi\tau\hbar}{M}\right)^{(n-1)/2}\frac{1}{\sqrt{\det(\sigma)}}\ .
\end{aligned} \tag{11.40}$$

The determinant $\det(\sigma)$ is derived as follows: if s_j denotes the value of the determinant of the truncated matrix consisting of the first j rows and columns of (11.38), the value of s_{j+1} is given by means of an expansion in minors

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[6] We follow the procedure used in [75].

$$s_{j+1} = (2 - \tau^2\omega^2)s_j - s_{j-1} \ , \tag{11.41}$$

with $s_1 = 2 - \tau^2\omega^2$, and s_0 is defined as 1. Equation (11.41) may be rewritten in the form

$$\frac{s_{j+1} - 2s_j + s_{j-1}}{\tau^2} = -\omega^2 \, s_j \ . \tag{11.42}$$

In the limit $\tau \to 0$ this relation yields a differential equation for the function $s(t)$ of the time variable $t = t_1 + j\tau$. This equation applies as well to the function $r(t) = \tau s(t)$, which has simpler initial conditions

$$r_{t=t_1} = 0 \ ; \qquad \left.\frac{\mathrm{d}\, r}{\mathrm{d}t}\right|_{t=t_1} = \tau \left(\frac{s_1 - s_0}{\tau}\right) = 2 - \omega^2\tau^2 - 1 = 1. \tag{11.43}$$

The solution of the equation $\frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2} = -\omega^2 \, r(t)$ is

$$\begin{aligned} r(t) &= \frac{\sin\omega(t - t_1)}{\omega} \\ \det(\sigma) &= \frac{r(t_2)}{\tau} = \frac{\sin\omega(t_2 - t_1)}{\omega\tau} \ . \end{aligned} \tag{11.44}$$

Replacement of (11.40) and (11.44) in (11.36) yields the value of the propagator $\langle 0, t_2 | 0, t_1 \rangle$ appearing in (11.33).