1 Fundamentals of Passive Circuit Analysis

1.1 Laplace Transform

Laplace transform "transforms" a function, f(t), in time-domain to a function, F(s), in frequency domain. We will use Laplace transform to determine the impedance of a passive element in this section.

It is defined as:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)\,\mathbf{exp}(-st)dt \tag{1.1}$$

1.2 Definitions of Passive Elements

Current-voltage relationship across a resistor, R, is defined as shown in Fig. 1.1.

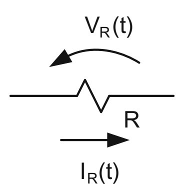

Fig. 1.1 Current-voltage relationship across a resistor, R

This relationship is called Ohm's law and mathematically expressed as follows:

$$V_R(t) = R\,I_R(t) \tag{1.2}$$

Taking Laplace transform of both sides of Eq. 1.2 yields:

$$\mathcal{L}[V_R(t)] = V_R(s) = R[I_R(t)] = R\,I_R(s)$$

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Thus, the impedance of the resistor becomes:

$$Z_R(s) = \frac{V_R(s)}{I_R(s)} = R \quad (1.3)$$

Current-voltage relationship across a capacitor, C, is defined as shown in Fig. 1.2.

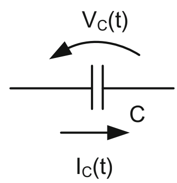


Fig. 1.2 Current-voltage relationship across a capacitor, C

This relationship originates from the Coulomb's law and mathematically expressed as follows:

$$I_C(t) = C\frac{dV_C(t)}{dt} \quad (1.4)$$

Taking Laplace transform of both sides of Eq. 1.4 yields:

$$\mathcal{L}[I_C(t)] = I_C(s) = C\mathcal{L}\left[\frac{dV_C(t)}{dt}\right] = C[sV_C(s) - V_C(0)] = sCV_C(s)$$

where $V_C(0)$ is assumed 0 V.

Thus, the impedance of the capacitor becomes:

$$Z_C(s) = \frac{V_C(s)}{I_C(s)} = \frac{1}{sC} \quad (1.5)$$

Current-voltage relationship across an inductor, L, is defined as shown in Fig. 1.3.

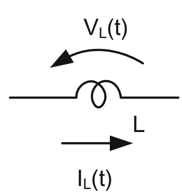


Fig. 1.3 Current-voltage relationship across an inductor, L

This relationship is mathematically expressed as follows:

$$V_L(t) = L\frac{dI_L(t)}{dt} \tag{1.6}$$

Taking Laplace transform of both sides of Eq. 1.6 yields:

$$\mathcal{L}[V_L(t)] = V_L(s) = L\mathcal{L}\left[\frac{dI_L(t)}{dt}\right] = L[sI_L(s) - I_L(0)] = sLI_L(s)$$

where $I_L(0)$ is assumed 0 A.

Thus, the impedance of the capacitor becomes:

$$Z_L(s) = \frac{V_L(s)}{I_L(s)} = sL \tag{1.7}$$

1.3 Time-Domain Analysis of First Order Passive Circuits

1.3.1 RC Circuits

In this section, we will examine the circuit behavior of simple RC circuits. A circuit composed of a series combination of a resistor and a capacitor is shown in Fig. 1.4. The switch is assumed to close at t = 0.

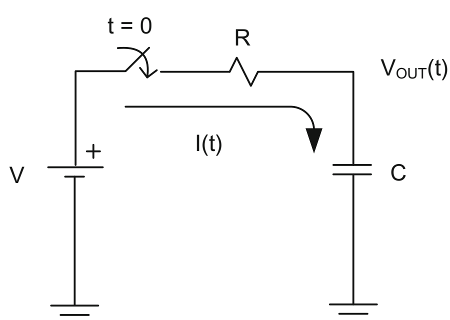


Fig. 1.4 A simple RC circuit where the switch closes at t = 0

Applying Kirchoff's voltage law to the circuit yields:

$$V = R\,I(t) + V_{OUT}(t) \tag{1.8}$$

But,

$$I(t) = C\frac{dV_{OUT}(t)}{dt} \tag{1.9}$$

In order to use Eq. 1.9 in Eq. 1.8, we need to differentiate both sides of Eq. 1.8.

$$\frac{dV}{dt} = R\frac{dI(t)}{dt} + \frac{dV_{OUT}(t)}{dt} \tag{1.10}$$

Substituting Eq. 1.9 into Eq. 1.10 yields:

$$\frac{dV}{dt} = R\frac{dI(t)}{dt} + \frac{I(t)}{C} = 0 \tag{1.11}$$

$$\frac{dI(t)}{dt} + \frac{I(t)}{RC} = 0 \tag{1.12}$$

In Eq. 1.11, the right side is equal to zero because V is a constant voltage and its derivative becomes zero.

We know I(t) is composed of a general and a particular solution for this first-order constant coefficient differential equation. Thus,

$$I(t) = I_{PART}(t) + I_{GEN}(t)$$

But, $I_{PART}(t) = 0$ since the right hand side of the differential equation in Eq. 1.12 is zero. $I_{GEN}(t)$, on the other hand, is expressed as follows:

$$I_{GEN}(t) = A\,\mathbf{exp}(st) \tag{1.13}$$

Thus,

$$I(t) = I_{PART}(t) + I_{GEN}(t) = A\,\mathbf{exp}(st) \tag{1.14}$$

To find s, we need to take the Laplace transform of both sides of the differential equation in Eq. 1.12.

Thus,

$$\mathcal{L}\left[\frac{dI(t)}{dt} + \frac{I(t)}{RC}\right] = sI(s) + \frac{1}{RC}I(s) = 0 \tag{1.15}$$

Since $I(s) \neq 0$, then Eq. 1.15 reveals $s + \frac{1}{RC} = 0$ (characteristic equation). Therefore, $s = -\frac{1}{RC}$, and I(t) becomes:

$$I(t) = A\,\mathbf{exp}\left(-\frac{t}{RC}\right) \tag{1.16}$$

To find A, we need to use the initial voltage across the capacitor. In this case, we assume $V_{OUT}(0) = 0$.

But, at t = 0, we have:

$$V = R\,I(0) + V_{OUT}(0) = R\,I(0) \tag{1.17}$$

Thus,

$$I(0) = \frac{V}{R}$$

and

$$I(t) = \frac{V}{R}\,\mathbf{exp}\left(-\frac{t}{RC}\right) \tag{1.18}$$

The current in Eq. 1.18 is plotted as a function of time in Fig. 1.5.

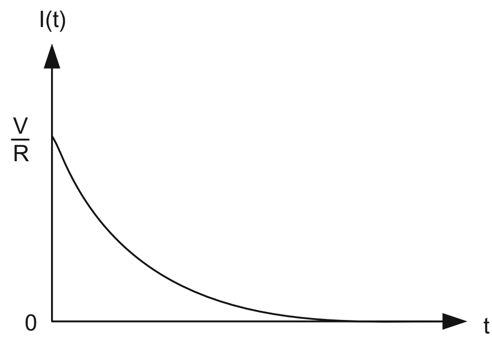


Fig. 1.5 The current waveform of the RC circuit in Fig. 1.4 as a function of time

Substituting I(t) in Eq. 1.18 into Eq. 1.8 yields:

$$V_{OUT}(t) = V - R\,I(t) = V - R\,\frac{V}{R}\,\mathbf{exp}\left(-\frac{t}{RC}\right)$$

$$V_{OUT}(t) = V - R\,I(t) = V\left[1 - \mathbf{exp}\left(-\frac{t}{RC}\right)\right] \tag{1.19}$$

The voltage in Eq. 1.19 is plotted as a function of time in Fig. 1.6.

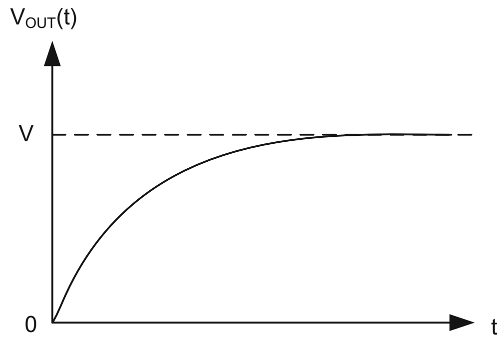


Fig. 1.6 The voltage waveform of the RC circuit in Fig. 1.4 as a function of time

1.3.2 RL Circuits

In this section, we will examine the circuit behavior of simple RL circuits. A circuit composed of a series combination of a resistor and an inductor is shown in Fig. 1.7. The switch is assumed to close at t = 0.

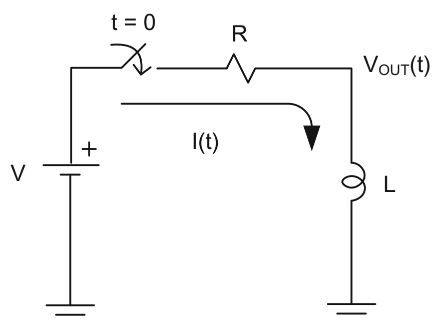


Fig. 1.7 A simple RL circuit where the switch closes at t = 0

Applying Kirchoff's voltage law to the circuit yields:

$$V = R\,I(t) + V_{OUT}(t) \tag{1.20}$$

But, the current-voltage relationship across the inductor is

$$V_{OUT}(t) = L\frac{dI(t)}{dt} \tag{1.21}$$

Substituting Eq. 1.21 into Eq. 1.20 yields:

$$V = R\,I(t) + L\frac{dI(t)}{dt} \tag{1.22}$$

Rearranging the terms in Eq. 1.22 produces:

$$\frac{dI(t)}{dt} + \frac{R}{L}I(t) = \frac{V}{L} \tag{1.23}$$

Eq. 1.23 is a constant coefficient linear differential equation. Therefore, I(t) in this equation is composed of general and particular solutions, $I_{GEN}(t)$ and $I_{PART}(t)$, respectively. Thus,

$$I(t) = I_{PART}(t) + I_{GEN}(t) \tag{1.24}$$

$I_{PART}(t) = K$ since the right hand side of the differential equation in Eq. 1.23 is a constant. $I_{GEN}(t)$, on the other hand, is expressed as follows:

$$I_{GEN}(t) = A\ \mathbf{exp}(st) \tag{1.25}$$

Combining $I_{GEN}(t)$ and $I_{PART}(t)$ yields:

$$I(t) = I_{PART}(t) + I_{GEN}(t) = K + A\ \mathbf{exp}(st) \tag{1.26}$$

To find K, we need to substitute $I_{PART}(t) = K$ into Eq. 1.23.

$$\frac{dI_{PART}(t)}{dt} + \frac{R}{L}I_{PART}(t) = \frac{V}{L}$$

$$I_{PART}(t) = K = \frac{V}{R} \tag{1.27}$$

To find s in Eq. 1.26, we need to equate the right hand side of the differential equation in Eq. 1.23 to zero, and take the Laplace transform of both sides.

Therefore:

$$\frac{dI_{GEN}(t)}{dt} + \frac{R}{L}I_{GEN}(t) = 0$$

$$\mathcal{L}\left[\frac{dI_{GEN}(t)}{dt} + \frac{R}{L}I_{GEN}(t)\right] = sI_{GEN}(s) + \frac{R}{L}I_{GEN}(s) = 0$$

Since $I_{GEN}(s) \neq 0$, then $s + \frac{R}{L} = 0$

Therefore, $s = -\frac{R}{L}$

$$I_{GEN}(t) = A\ \mathbf{exp}\left(-\frac{R}{L}t\right) \tag{1.28}$$

Thus,

$$I(t) = I_{PART}(t) + I_{GEN}(t) = \frac{V}{R} + A\ \mathbf{exp}\left(-\frac{R}{L}t\right) \tag{1.29}$$

To solve for A, we need to use the initial condition, I(0) = 0. Thus,

$$I(0) = \frac{V}{R} + A = 0$$
$$A = -\frac{V}{R} \tag{1.30}$$
$$I(t) = \frac{V}{R}\left[1 - \mathbf{exp}\left(-\frac{R}{L}t\right)\right]$$

This current is plotted as a function of time in Fig. 1.8.

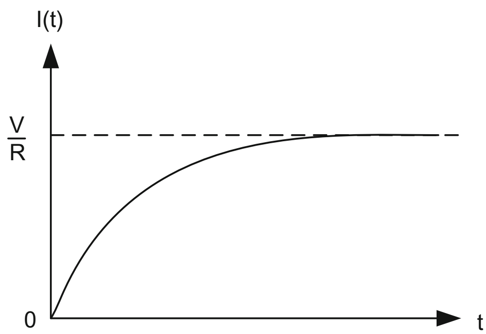


Fig. 1.8 The current waveform of the RL circuit in Fig. 1.7 as a function of time

For the output voltage, we need to use Eq. 1.20.

$$V = R\,I(t) + V_{OUT}(t)$$

$$V_{OUT}(t) = V - R\,I(t)$$

Substituting I(t) in Eq. 1.30 into $V_{OUT}(t)$ above yields:

$$V_{OUT}(t) = V\,\mathbf{exp}\left(-\frac{R}{L}t\right) \tag{1.31}$$

This voltage is plotted as a function of time in Fig. 1.9.

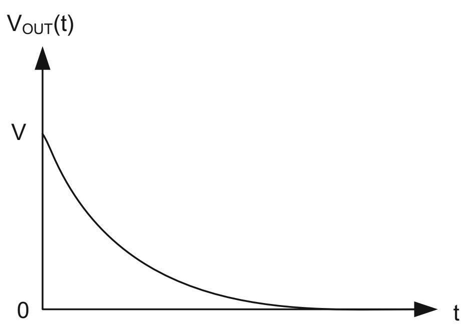


Fig. 1.9 The voltage waveform of the RL circuit in Fig. 1.7 as a function of time

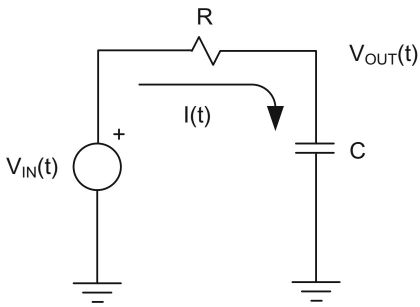


Fig. 1.10 A simple RC circuit with a pulse input

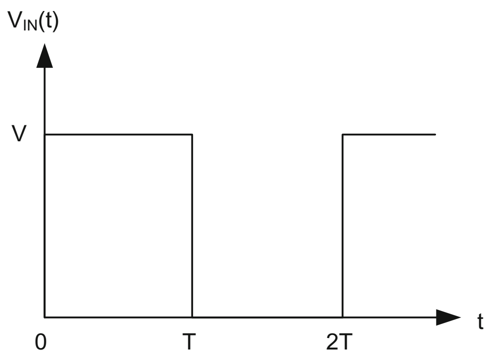


Fig. 1.11 Pulse input waveform

Example 1.1 Assume the RC circuit in Fig. 1.10 receives a pulse input as shown in Fig. 1.11. Let us derive the response of the circuit at the output node, $V_{OUT}(t)$.

This circuit can be examined in two distinct time intervals: $0 < t < T$ and $T < t < 2T$. The circuit analysis for $0 < t < T$ is the same as the analysis for the RC circuit in Fig. 1.4. Thus,

$$V_{OUT}(t) = V\left[1 - \mathbf{exp}\left(-\frac{t}{RC}\right)\right]$$

If the time duration, T, is long enough such that $T \gg RC$, the output voltage, $V_{OUT}(t)$, at $t = T$ becomes:

$$V_{OUT}(T) = V\left[1 - \mathbf{exp}\left(-\frac{T}{RC}\right)\right] \approx V \tag{1.32}$$

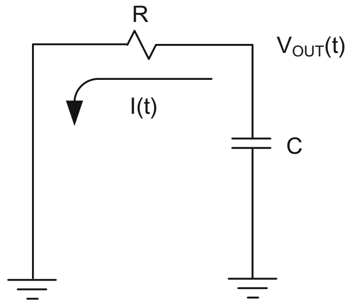


Fig. 1.12 The equivalent RC circuit in Fig. 1.10 during T < t < 2T

Therefore, $V_{OUT}(T) = V$ becomes the initial condition for the circuit for $T < t < 2T$. However, at $t = T$, the input transitions to 0 V, and the circuit in Fig. 1.10 transforms into the RC circuit shown in Fig. 1.12. Here, the current flows into the opposite direction of the current in Fig. 1.10.

Writing the Kirchoff's voltage law for this circuit yields:

$$V_R(t) + V_C(t) = 0 \tag{1.33}$$

Employing the Ohms's law in Eq. 1.33 yields:

$$R\,I(t) + V_C(t) = 0$$

Differentiating both sides of this equation, and substituting the current-voltage relationship for the capacitor produces:

$$R\,\frac{dI(t)}{dt} + \frac{dV_C(t)}{dt} = 0$$

$$R\,\frac{dI(t)}{dt} + \frac{I(t)}{C} = 0$$

$$\frac{dI(t)}{dt} + \frac{I(t)}{RC} = 0 \tag{1.34}$$

Solving the differential equation in Eq. 1.34 yields:

$$I(t) = A\,\mathbf{exp}\left(-\frac{(t-T)}{RC}\right) \tag{1.35}$$

Then the output voltage becomes:

$$V_{OUT}(t) = R\,I(t) = R\;A\;\mathbf{exp}\left(-\frac{(t-T)}{RC}\right) \tag{1.36}$$

Fetching the initial condition from Eq. 1.32, and using it in Eq. 1.36 produces:

$$R\ A = V$$

or

$$A = \frac{V}{R} \tag{1.37}$$

Substituting A in Eq. 1.37 into I(t) in Eq. 1.35 yields:

$$I(t) = \frac{V}{R}\,\mathbf{exp}\left[-\frac{(t-T)}{RC}\right] \tag{1.38}$$

Plotting $I(t) = \frac{V}{R}\mathbf{exp}\left(-\frac{t}{RC}\right)$ during $0 < t < T$, and $I(t) = -\frac{V}{R}\mathbf{exp}\left[-\frac{(t-T)}{RC}\right]$ during $T < t < 2T$ yields the following waveform in Fig. 1.13. In this figure, I(t) becomes a negative quantity during $T < t < 2T$ because the current in Fig. 1.12 flows in the opposite direction of the current in Fig. 1.10.

$V_{OUT}(t)$ becomes $V_{OUT}(t) = R\,I(t)$.

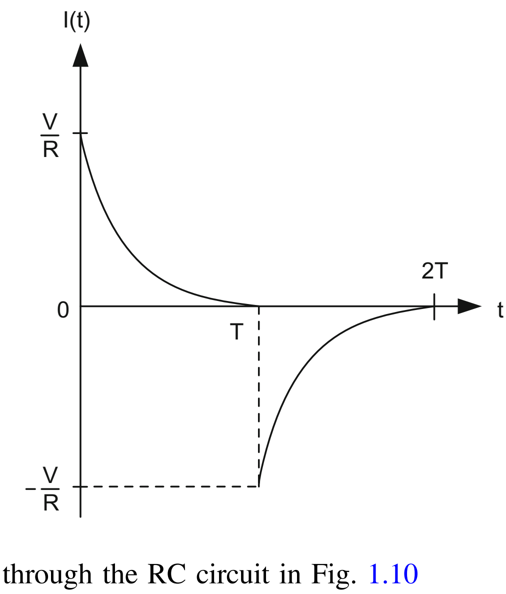


Fig. 1.13 The current through the RC circuit in Fig. 1.10

Thus,

$$V_{OUT}(t) = R\,I(t) = V\,\mathbf{exp}\left[-\frac{(t-T)}{RC}\right] \tag{1.39}$$

Plotting $V_{OUT}(t) = V\left[1 - \mathbf{exp}\left(-\frac{t}{RC}\right)\right]$ during $0 < t < T$, and $V_{OUT}(t) = V\,\mathbf{exp}\left[-\frac{(t-T)}{RC}\right]$ during $T < t < 2T$ yields the following waveform in Fig. 1.14.

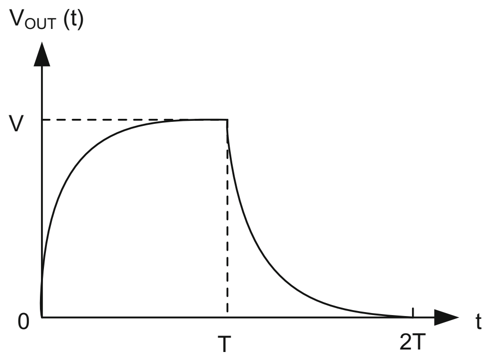


Fig. 1.14 The voltage at the output of the RC circuit in Fig. 1.10

1.4 First Order Passive Circuit Analysis Using Natural Frequencies

Natural frequencies method is an easy way to find the current through a passive element and the voltage value across it. This method only uses the initial and final conditions on the element and the natural frequencies of the circuit.

There are three steps involved to evaluate the current or the voltage value as a function of time:

Step 1: Evaluate the initial ($t = 0$) current or voltage values at a passive element
Step 2: Evaluate the final ($t \to \infty$) current or voltage values at a passive element
Step 3: Evaluate the natural frequencies of the circuit when there is no current or voltage source applied to the circuit.

Once these three steps are completed, we can determine the current and voltage values for a passive element and plot the corresponding waveforms.

To show how this method works, assume the simple RC circuit in Fig. 1.15 with a fixed voltage source (the input voltage source can also be time-dependent, however this does not alter the analysis) at its input.

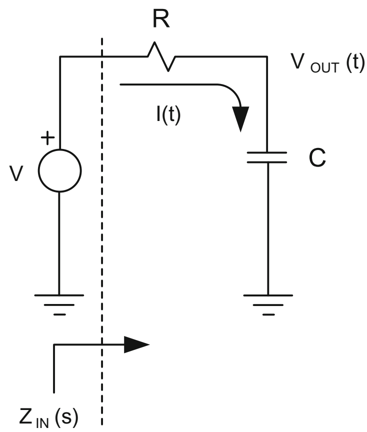


Fig. 1.15 A simple RC circuit with a fixed voltage source, V, at its input

Assume that $V_{OUT}(0) = 0$ V. The initial output voltage may have a different value; however, this does not alter the circuit analysis. The initial current, I(0), becomes:

$$I(0) = \frac{V}{R} \tag{1.40}$$

This completes step 1.

For step 2, we need to identify the value of the current through the circuit, and the voltage across the capacitor at $t \to \infty$.

We know that as a constant current flows through a capacitor, and charges it completely, the capacitor becomes an open circuit. Thus, the current through this circuit becomes 0 A, and voltage across the capacitor becomes equal to V at $t \to \infty$.

For step 3, the natural frequencies of the circuit must be calculated. When the input voltage source is separated from the circuit in Fig. 1.15, the equivalent input impedance of the circuit, $Z_{IN}(s)$, becomes:

$$Z_{IN}(s) = \frac{V_{IN}(s)}{I_{IN}(s)} = R + \frac{1}{sC} = \frac{1 + sRC}{sC} \tag{1.41}$$

The natural frequencies method dictates to "terminate" all independent current and voltage sources in a circuit. Therefore, when the input voltage source, V_{IN}, is terminated or forced to become 0 V, the numerator of $Z_{IN}(s)$ automatically becomes zero, revealing $(sRC + 1) = 0$.

Thus,

$$s = -\frac{1}{RC} \tag{1.42}$$

This is the natural frequency of the circuit, which is inversely proportional to t.

Therefore, the initial current starts at V/R but reduces to 0 A with a time constant $\tau = RC$ as t approaches ∞. Similarly, the output voltage starts at 0 V, and increases towards the value, V, with the same time constant, $\tau = RC$. Here, the time constant is inversely proportional to the natural frequency as mentioned previously.

Then, the equations for I(t) and $V_{OUT}(t)$ are calculated as follows:

$$I(t) = \frac{V}{R}\mathbf{exp}(st) = \frac{V}{R}\mathbf{exp}\left(-\frac{t}{RC}\right) \quad (1.43)$$

Similarly,

$$V_{OUT}(t) = V[1 - \mathbf{exp}(st)] = V\left[1 - \mathbf{exp}\left(-\frac{t}{RC}\right)\right] \quad (1.44)$$

1.5 Time-Domain Analysis of Second Order Passive Circuits

An RLC circuit composed of a series combination of a resistor, inductor and capacitor is considered a second order passive network as shown in Fig. 1.16. In this circuit, assume the switch is closed at t = 0.

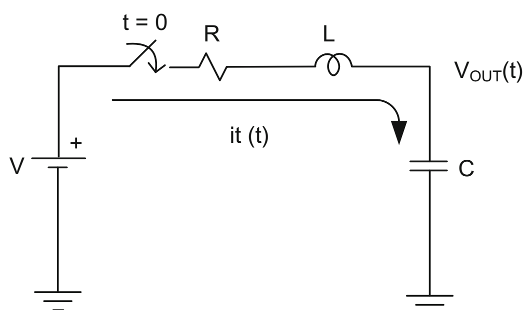


Fig. 1.16 A simple second order RLC circuit

The Kirchoff's voltage law for the circuit yields:

$$V = R\,I(t) + V_L(t) + V_{OUT}(t) \quad (1.45)$$

But,

$$V_L(t) = L\frac{dI(t)}{dt} \quad (1.46)$$

$$I(t) = C\frac{dV_{OUT}(t)}{dt} \quad (1.47)$$

To be able to substitute Eqs. 1.46 and 1.47 into Eq. 1.45, we need to differentiate both sides of Eq. 1.45. Thus,

$$0 = R\,\frac{dI(t)}{dt} + \frac{dV_L(t)}{dt} + \frac{dV_{OUT}(t)}{dt} = R\frac{dI(t)}{dt} + L\frac{d^2I(t)}{dt^2} + \frac{I(t)}{C}$$

Reorganizing the terms yields:

$$\frac{d^2I(t)}{dt^2} + \frac{R}{L}\frac{dI(t)}{dt} + \frac{I(t)}{LC} = 0 \quad (1.48)$$

But,

$$I(t) = I_{GEN}(t) + I_{PART}(t) \quad (1.49)$$

Here, $I_{PART}(t) = 0$ and $I_{GEN}(t) = A\,\mathbf{exp}(s_1t) + B\,\mathbf{exp}(s_2t)$ since the differential equation in Eq. 1.48 is a second order constant coefficient differential equation.

Thus,

$$I(t) = A\,\mathbf{exp}(s_1t) + B\,\mathbf{exp}(s_2t) \quad (1.50)$$

s_1 and s_2 are the natural frequencies of this circuit and computed by solving the characteristic equation. Taking Laplace transform of Eq. 1.48 yields:

$$\mathcal{L}\left[\frac{d^2I(t)}{dt^2} + \frac{R}{L}\frac{dI(t)}{dt} + \frac{I(t)}{LC}\right] = s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad (1.51)$$

Thus,

$$s_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \quad (1.52)$$

Let $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$ in $s_{1,2}$. Then,

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad (1.53)$$

We still need two initial conditions to solve A and B. The first initial condition, $I(0) = 0$, because the initial current stored in the inductor is assumed 0 A.

The second initial condition, $\frac{dI(0)}{dt}$, requires evaluating the RLC circuit at $t = 0$, and constructing its equivalent circuit as shown in Fig. 1.17.

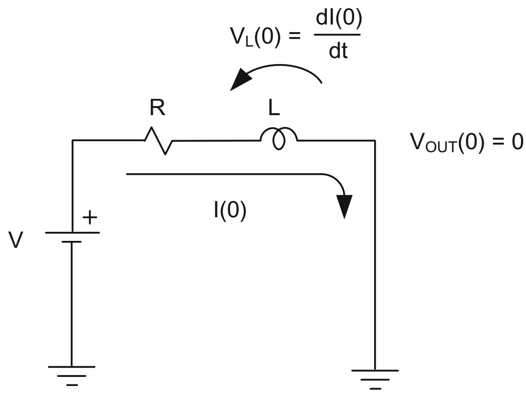


Fig. 1.17 The equivalent RLC circuit in Fig. 1.16 at t = 0

Therefore,

$$V = R\,I(0) + V_L(0) = R\,I(0) + L\frac{dI(0)}{dt} = L\frac{dI(0)}{dt}$$

Here, I(0) = 0. Then,

$$\frac{dI(0)}{dt} = \frac{V}{L} \tag{1.54}$$

Applying the first initial condition, I(0) = 0, to Eq. 1.50 yields:

$$I(0) = 0 = A + B$$

$$A = -B$$

$$I(t) = A(\mathbf{exp}(s_1 t) - \mathbf{exp}(s_2 t)) \tag{1.55}$$

Applying the second initial condition, $\frac{dI(0)}{dt} = \frac{V}{L}$, to Eq. 1.55 yields:

$$\begin{aligned} \frac{dI(0)}{dt} &= A(s_1 - s_2) = \frac{V}{L} \\ A &= \frac{V}{(s_1 - s_2)L} \end{aligned} \tag{1.56}$$

Substituting Eq. 1.56 into Eq. 1.55 produces:

$$I(t) = \frac{V}{(s_1 - s_2)L}(\mathbf{exp}(s_1 t) - \mathbf{exp}(s_2 t)) \tag{1.57}$$

where, $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$.

From Eq. 1.45, we also have:

$$V_{OUT}(t) = V - R\,I(t) - V_L(t) = V - R\,I(t) - L\frac{dI(t)}{dt}$$

But,

$$\frac{dI(t)}{dt} = \frac{V}{(s_1 - s_2)L}(s_1\,\mathbf{exp}(s_1 t) - s_2\,\mathbf{exp}(s_2 t))$$

Thus,

$$V_{OUT}(t) = V - \frac{VR}{(s_1 - s_2)L}(\mathbf{exp}(s_1 t) - \mathbf{exp}(s_2 t)) - \frac{V}{(s_1 - s_2)}(s_1\,\mathbf{exp}(s_1 t) - s_2\,\mathbf{exp}(s_2 t))$$

$$V_{OUT}(t) = V\left[1 - \frac{\mathbf{exp}(s_1 t)}{(s_1 - s_2)}\left(\frac{R}{L} + s_1\right) + \frac{\mathbf{exp}(s_2 t)}{(s_1 - s_2)}\left(\frac{R}{L} + s_2\right)\right] \qquad (1.58)$$

If $\alpha > \omega_0$, substituting $s_1 - s_2 = -2\sqrt{\alpha^2 - \omega_0^2}$ into $V_{OUT}(t)$ in Eq. 1.58 yields:

$$V_{OUT}(t) = V\left[1 + \frac{\mathbf{exp}(-\alpha t)\mathbf{exp}\left(-\sqrt{\alpha^2 - \omega_0^2}t\right)}{2\sqrt{\alpha^2 - \omega_0^2}}\left(\frac{R}{L} - \alpha - \sqrt{\alpha^2 - \omega_0^2}\right)\right.$$
$$\left. - \frac{\mathbf{exp}(-\alpha t)\mathbf{exp}\left(\sqrt{\alpha^2 - \omega_0^2}t\right)}{2\sqrt{\alpha^2 - \omega_0^2}}\left(\frac{R}{L} - \alpha + \sqrt{\alpha^2 - \omega_0^2}\right)\right]$$

$$V_{OUT}(t) = V\left\{1 + \frac{\mathbf{exp}(-\alpha t)}{\sqrt{\alpha^2 - \omega_0^2}}\left[\left(\alpha - \frac{R}{L}\right)\sinh\sqrt{\alpha^2 - \omega_0^2}t - \sqrt{\alpha^2 - \omega_0^2}\cosh\sqrt{\alpha^2 - \omega_0^2}t\right]\right\} \qquad (1.59)$$

This equation can be plotted in Fig. 1.18.

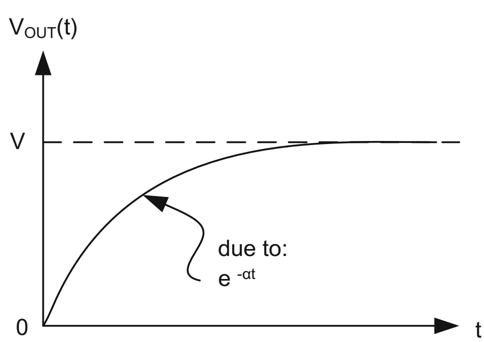


Fig. 1.18 The output voltage waveform when $\alpha > \omega_0$

However, for $\alpha < \omega_0$

$$s_1 - s_2 = -2j\sqrt{\omega_0^2 - \alpha^2}$$

Thus,

$$V_{OUT}(t) = \left\{1 - j\frac{\mathbf{exp}(-\alpha t)}{2\sqrt{\omega_0^2 - \alpha^2}}\left[\left(\frac{R}{L} - \alpha\right)\left(\mathbf{exp}\left(-j\sqrt{\omega_0^2 - \alpha^2}\right)t - \mathbf{exp}\left(j\sqrt{\omega_0^2 - \alpha^2}\right)t\right)\right]\right.$$
$$\left. -j\sqrt{\omega_0^2 - \alpha^2}\left[\mathbf{exp}\left(-j\sqrt{\omega_0^2 - \alpha^2}\right)t + \mathbf{exp}\left(j\sqrt{\omega_0^2 - \alpha^2}\right)t\right]\right\}$$

Using $\mathbf{exp}(j\theta) = \cos\theta + j\sin\theta$ (Euler's equation) in $V_{OUT}(t)$ yields:

$$V_{OUT}(t) = V\left[1 - \frac{\mathbf{exp}(-\alpha t)}{\sqrt{\omega_0^2 - \alpha^2}}\left(\alpha \sin\sqrt{\omega_0^2 - \alpha^2 t} + \sqrt{\omega_0^2 - \alpha^2}\cos\sqrt{\omega_0^2 - \alpha^2 t}\right)\right] \quad (1.60)$$

This equation produces quite a different result from Fig. 1.18, and contains the component of oscillation from sine and cosine terms in Eq. 1.60 as shown in Fig. 1.19.

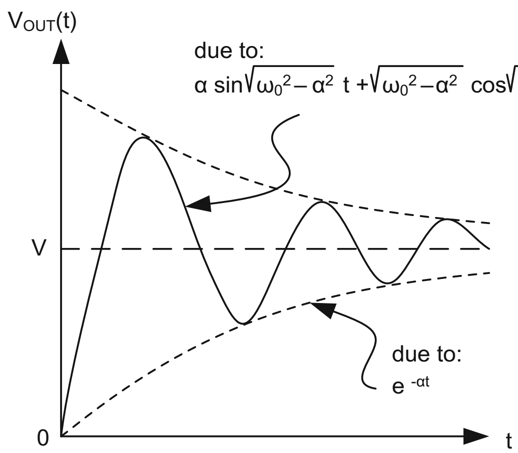


Fig. 1.19 The output voltage waveform when $\alpha < \omega_0$

When the resistor in component in this circuit is completely eliminated, then $V_{OUT}(t)$ will reduce to $V_{OUT}(t) = V(1 - \cos\omega_0 t)$ and oscillate with an angular frequency of ω_0. The resultant waveform is plotted in Fig. 1.20.

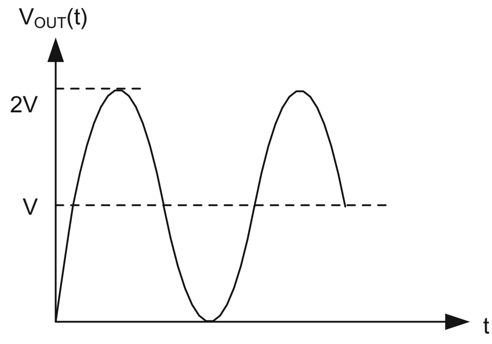


Fig. 1.20 The output voltage waveform when R = 0

1.6 Transfer Function and Circuit Stability

Transfer function, H(s), defines the relationship between the input and output of a passive circuit in frequency domain. The passive network has to be defined in terms of individual impedances of R, L and C to produce H(s) as shown in Fig. 1.21.

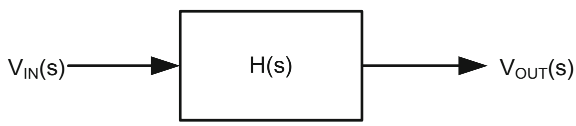


Fig. 1.21 Input-output relationship of a passive network in frequency (s) domain

$$V_{OUT}(s) = H(s)\,V_{IN}(s)$$

Thus,

$$H(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} \tag{1.61}$$

The transfer function, H(s), in Eq. 1.61 can be described as:

$$H(s) = \frac{N(s)}{D(s)}$$

In this equation, if N(s) = 0, the roots of the polynomial gives the zeros of H(s). Similarly, if D(s) = 0, the roots of the polynomial produces the poles of H(s), which describes the stability of the circuit. The poles of the circuit are the same as the natural frequencies, and follow the form in Eq. 1.62:

$$s = \alpha + j\omega \tag{1.62}$$

Here, α is the real component and ω is the imaginary component of a pole. The root(s) from D(s) = 0 can be plotted on a complex plane as shown in Fig. 1.22.

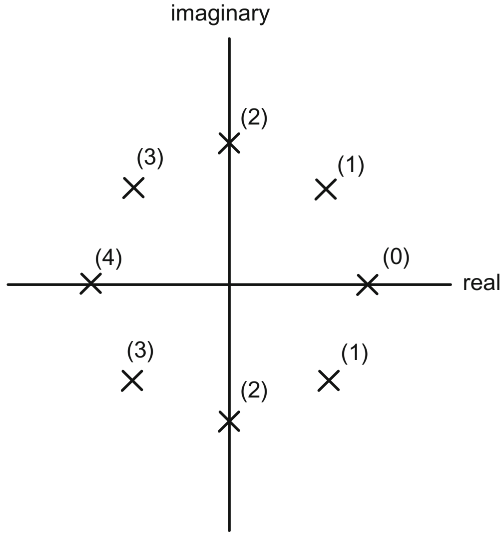


Fig. 1.22 Possible locations of poles in a complex plane

In this figure, if the pole(s) is at position (0) or in complex conjugate form at position (1), this situation creates instability in the circuit. The resulting waveform grows in amplitude as a function of time similar to Fig. 1.23.

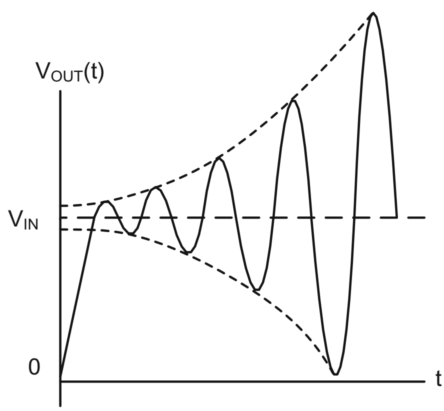


Fig. 1.23 Output of a passive network with complex poles at position (1) in Fig. 1.22

If the poles lay on the imaginary axis or in position (2) on the complex plane, the output waveform becomes oscillatory as shown in Fig. 1.24.

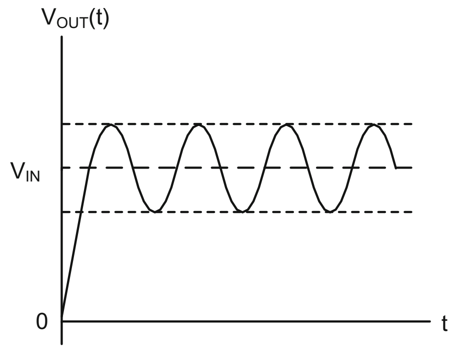


Fig. 1.24 Output of a passive network with complex poles at position (2) in Fig. 1.22

If the poles shift to the left hand side of the complex plane in Fig. 1.22 such as in position (3), the output of the circuit still becomes oscillatory but reaches to a stable voltage as shown in Fig. 1.25.

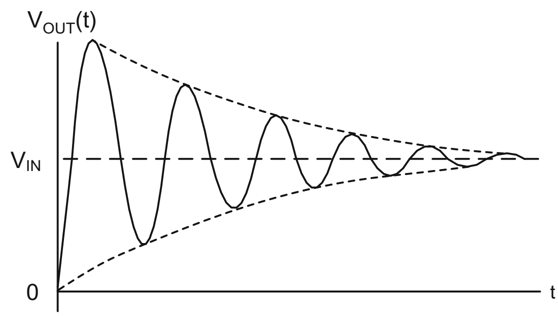


Fig. 1.25 Output of a passive network with complex poles at position (3) in Fig. 1.22

If the pole(s) lay on the negative real axis of the complex plane such as position (4), then the oscillation at the output disappears, and the waveform reaches to a stable value as shown in Fig. 1.26.

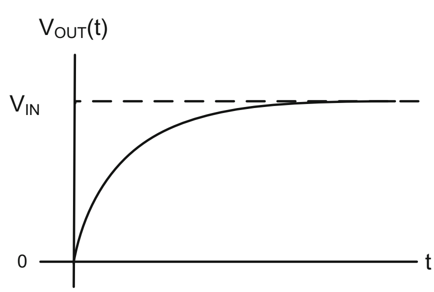


Fig. 1.26 Output of a passive network with real pole(s) at position (4) in Fig. 1.22

Example 1.2 Assume the following RC circuit in Fig. 1.27 to evaluate its stability.

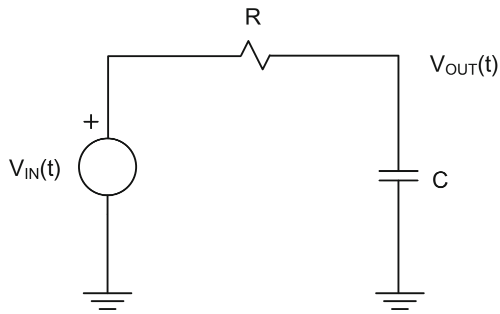


Fig. 1.27 The RC circuit for stability analysis

First, each time-domain component in the circuit is transformed to a frequency-domain component using Laplace transform.

Thus, the circuit transforms into Fig. 1.28.

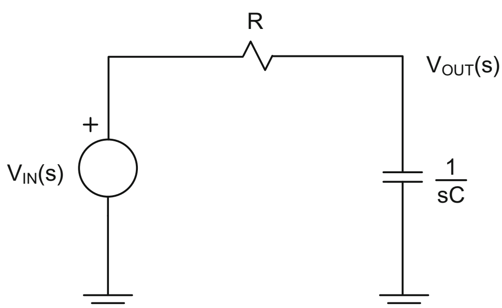


Fig. 1.28 The frequency-domain equivalent of the RC circuit in Fig. 1.27

Then the relationship between the input and output becomes:

$$H(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1} = \frac{1}{RC\left(s + \frac{1}{RC}\right)} \tag{1.63}$$

To find the pole(s), the denominator needs to be zero.

$$D(s) = RC\left(s + \frac{1}{RC}\right) = 0$$

Thus, the pole becomes:

$$s = -\frac{1}{RC} \tag{1.64}$$

Plotting this single pole on complex plane yields the following in Fig. 1.29.

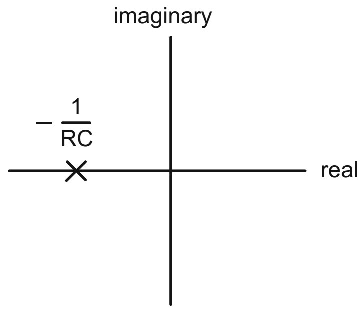


Fig. 1.29 A single pole representation of the RC circuit in Fig. 1.27

From the position of the pole, we know that the circuit is stable, and $V_{OUT}(t)$ approaches to a stable value as shown in Fig. 1.30.

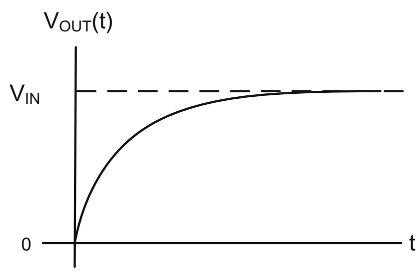


Fig. 1.30 The output response of the RC circuit in Fig. 1.27

Example 1.3 Now, let us assume the following RLC circuit in Fig. 1.31 to evaluate its stability. Again, we need to convert the time-domain components in this circuit to frequency-domain components. Taking the Laplace transform of $V_{IN}(t)$, and each passive component in the circuit transforms Fig. 1.31 to Fig. 1.32.

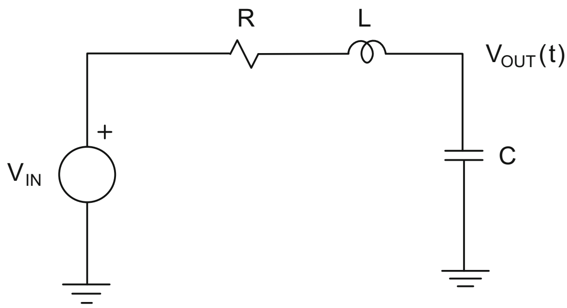


Fig. 1.31 The RLC circuit for stability analysis

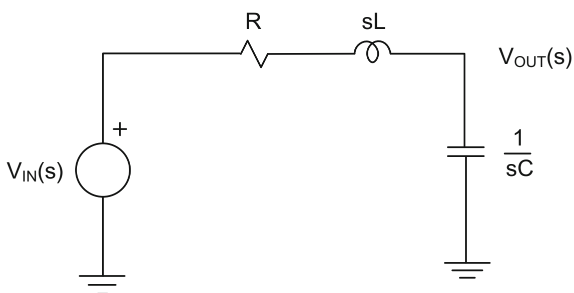


Fig. 1.32 The frequency domain equivalent of the RLC circuit in Fig. 1.31

The transfer function, H(s), is evaluated as follows:

$$H(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{s^2CL + sRC + 1} \tag{1.65}$$

To find the pole(s), the denominator of H(s) in Eq. 1.65 needs to be zero. Thus,

$$D(s) = s^2CL + sRC + 1 = 0 \tag{1.66}$$

Solving the polynomial in Eq. 1.66 reveals:

$$s_{1,2} = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2} \tag{1.67}$$

The roots, s_1 and s_2, can be simplified as follows:

$$s_1 = -\alpha_0 + \sqrt{\alpha_0^2 - \omega_0^2}$$
$$s_2 = -\alpha_0 - \sqrt{\alpha_0^2 - \omega_0^2}$$

where,

$$\alpha_0 = \frac{R}{2L}$$
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Assuming that L, R and C are all non-zero and non-negative, there are three possible cases depending on the passive component values.

Case 1: when $\alpha_0 > \omega_0$ then both poles become real.

Plotting these poles in Fig. 1.33 yields:

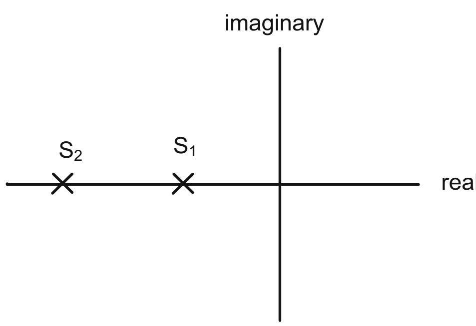


Fig. 1.33 The real poles of the RLC circuit in Fig. 1.31 when $\alpha_0 > \omega_0$

The output of the circuit, therefore, approaches to a stable value as $t \to \infty$ as shown in Fig. 1.34.

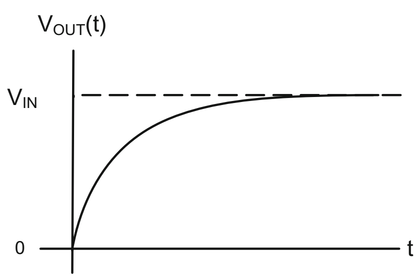


Fig. 1.34 The output of the RLC circuit in Fig. 1.31 when $\alpha_0 > \omega_0$

Case 2: when $\alpha_0 = \omega_0$, then both poles are still real but they collapse on top of each other when plotted on complex plane as shown in Fig. 1.35.

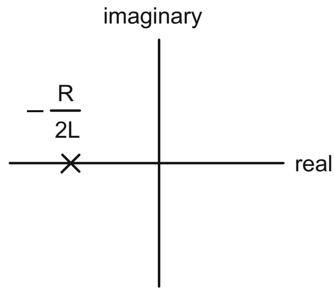


Fig. 1.35 The real poles of the RLC circuit in Fig. 1.31 when $\alpha_0 = \omega_0$

The output of the circuit still produces a stable value as $t \rightarrow \infty$ in Fig. 1.36.

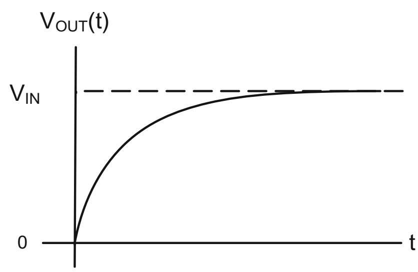


Fig. 1.36 The output of the RLC circuit in Fig. 1.31 when $\alpha_0 = \omega_0$

Case 3: when $\alpha_0 < \omega_0$, then both poles become complex conjugate as shown in Fig. 1.37.

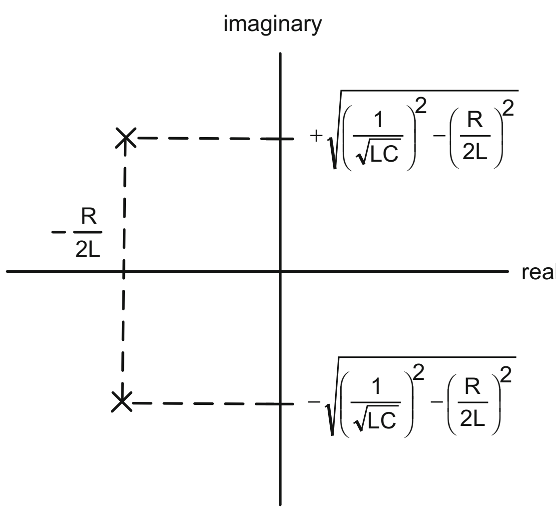


Fig. 1.37 The complex conjugate poles of the RLC circuit in Fig. 1.31 when $\alpha_0 < \omega_0$

The output of the circuit produces an oscillatory waveform but reaches to a stable value as t approaches infinity in Fig. 1.38.

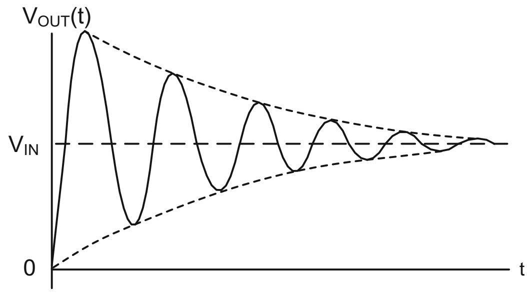
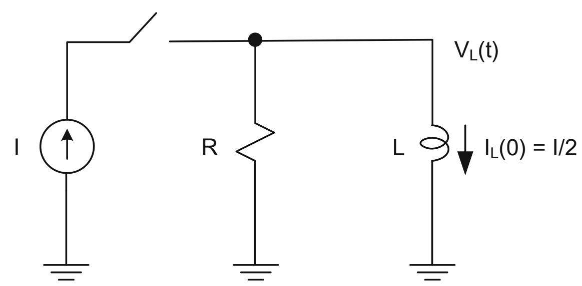


Fig. 1.38 The output of the RLC circuit in Fig. 1.31 as a result of complex conjugate poles when $\alpha_0 < \omega_0$

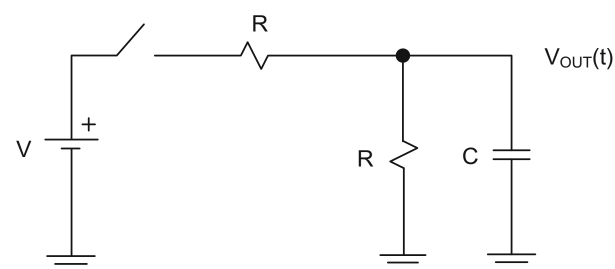
Review Questions

1. The following RL circuit is given:
In this circuit, the current, I, represents a constant current value. The current in the inductor L at t = 0 s is I/2 as shown in the schematic. At t = 0, the switch closes. Using the natural frequencies method

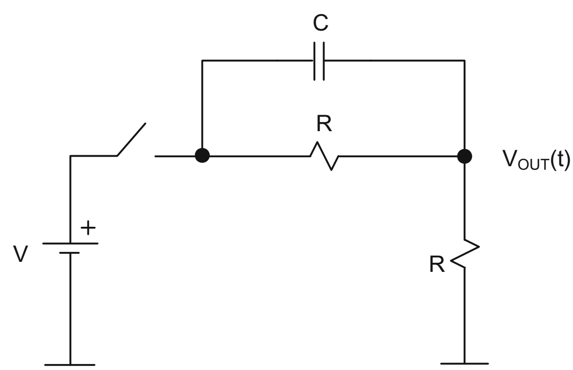
(a) Find the current through the inductor and plot it.
(b) Find the voltage across the inductor and plot it.



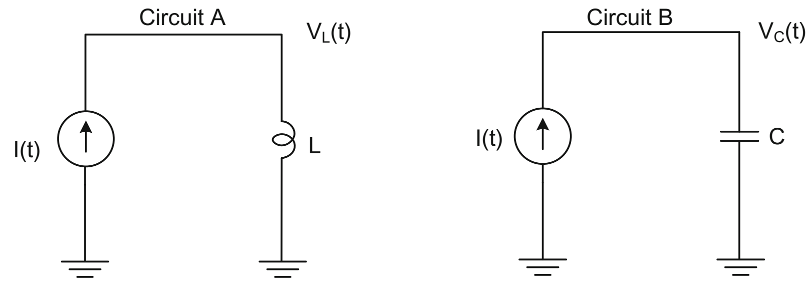
2. The following RC circuit is given.



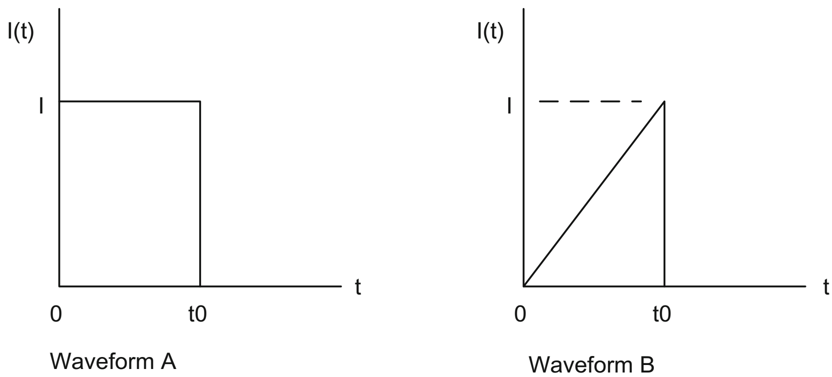
At t = 0, the switch closes. V is a constant voltage and $V_{OUT}(0) = 0$ V.
Using natural frequencies method, find the output voltage, $V_{OUT}(t)$ and plot it.
Now, change the location of the capacitor, C, as shown below. Assume the voltage across the capacitor is 0 V at t = 0. How does $V_{OUT}(t)$ change? Plot the waveform for $V_{OUT}(t)$.



3. The following circuits are given:

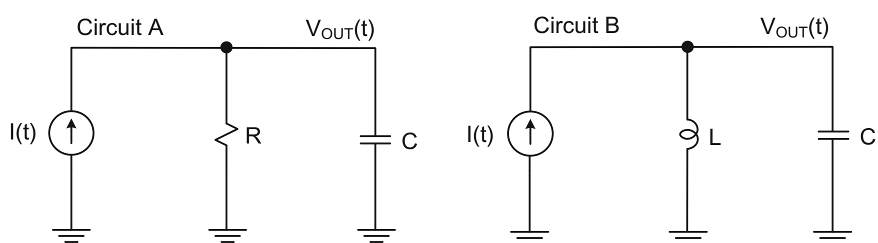


Apply the following waveforms to circuits A and B.



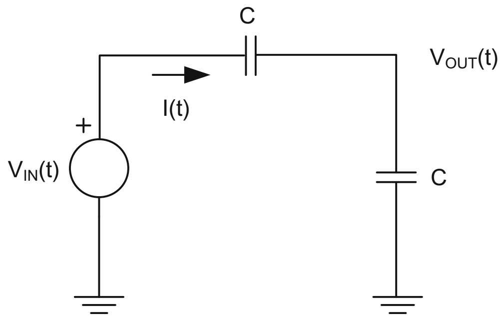
(a) Find the expression for $V_L(t)$ in circuit A, and plot it for both input waveforms A and B.
(b) Find the expression for $V_C(t)$ in circuit B, and plot it for both input waveforms A and B.

4. The following circuits are given:

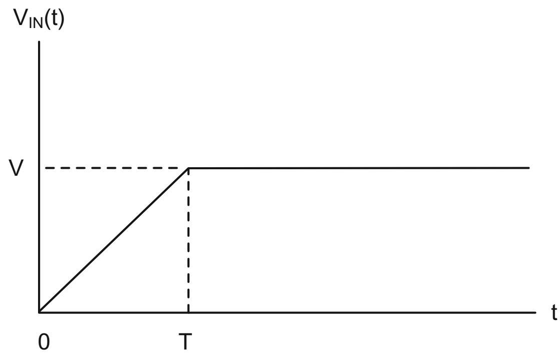


(a) Find the transfer functions, $H(s) = \frac{V_{OUT}(s)}{I(s)}$, of circuits A and B.
(b) Draw the pole-zero diagrams and find out the stability of each circuit. From stability criteria, how does $V_{OUT}(t)$ change with time?

5. The following capacitive circuit is given.

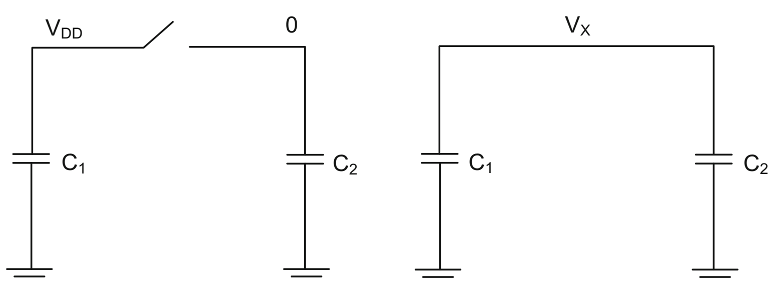


Apply the waveform below to the input of this circuit.

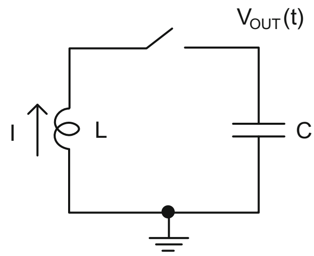


Solve I(t) and $V_{OUT}(t)$ using time-domain analysis and plot them as a function of time.

6. The capacitors, C_1 and C_2, are connected in parallel with a switch as shown below. The capacitor, C_1, has an initial voltage of V_{DD}, and capacitor, C_2, has 0 V. When the switch closes at t = 0, what will be the resultant voltage, V_X?

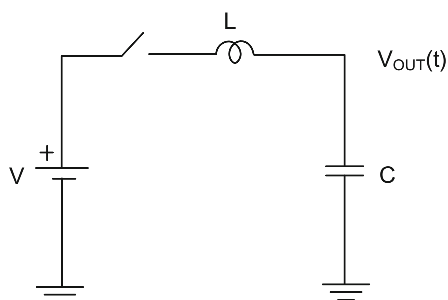


7. An LC circuit is given below:



The initial current stored in the inductor is I in the direction indicated in the circuit diagram, and the voltage across the capacitor is 0 V at t = 0 s before the switch is closed. Find the expression for $V_{OUT}(t)$ and draw its waveform using time-domain analysis.

8. A constant voltage source, V, is applied to the circuit below. If the stored current in the inductor is 0 A, and the stored voltage across the capacitor is 0 V before the switch is closed, calculate the output voltage, $V_{OUT}(t)$, for $t > 0$.



9. Assume the LC circuit below. The initial voltage stored across the capacitor is V. When the switch closes at t = 0, the current flows from the capacitor into the inductor and back into the capacitor. Assuming the stored current in the inductor is 0 A, find the current through the circuit, I(t), the voltage across the inductor, $V_{OUT}(t)$. Plot I(t) and $V_{OUT}(t)$ as a function of time.