Chapter 19
Processes with Independent Increments

Abstract Section 19.1 introduces the fundamental concept of infinitely divisible distributions and contains the key theorem on relationship of such processes to processes with independent homogeneous increments. Section 19.2 begins with a definition of the Wiener process based on its finite-dimensional distributions and establishes existence of a continuous modification of the process. It also derives the distribution of the maximum of the Wiener process on a finite interval. The Laws of the Iterated Logarithm for the Wiener process are established in Sect. 19.3. Section 19.4 is devoted to the Poisson processes, while Sect. 19.5 presents a characterisation of the class of processes with independent increments (the Lévy–Khintchin theorem).

19.1 General Properties

Definition 19.1.1 A process $\{\xi(t),\ t \in [a,b]\}$ given on the interval $[a,b]$ is said to be a *process with independent increments* if, for any n and $t_0 < t_1 < \cdots < t_n$, $a \le t_0$, $t_n \le b$, the random variables $\xi(t_0)$, $\xi(t_1) - \xi(t_0), \dots, \xi(t_n) - \xi(t_{n-1})$ are independent.

A process with independent increments is called *homogeneous* if the distribution of $\xi(t_1) - \xi(t_0)$ is determined by the length of the interval $t_1 - t_0$ only and is independent of t_0.

In what follows, we will everywhere assume for simplicity's sake that $a = 0$, $\xi(0) = 0$ and $b = 1$ or $b = \infty$.

Definition 19.1.2 The distribution of a random variable ξ is called *infinitely divisible* (cf. Sect. 8.8) if, for any n, the variable ξ can be represented as a sum of independent identically distributed random variables: $\xi = \xi_{1,n} + \cdots + \xi_{n,n}$. If $\varphi(\lambda)$ is the ch.f. of ξ, then this is equivalent to the property that $\varphi^{1/n}$ is a ch.f. for any n.

It is clear from the above definitions that, for a homogeneous process with independent increments, the distribution of $\xi(t)$ is infinitely divisible, because $\xi = \xi_{1,n} + \cdots + \xi_{n,n}$, where $\xi_{k,n} = \xi(kt/n) - \xi((k-1)t/n)$ are independent and distributed as $\xi(t/n)$.

A.A. Borovkov, *Probability Theory*, Universitext,
DOI 10.1007/978-1-4471-5201-9_19,

Theorem 19.1.1

(1) *Let* $\{\xi(t),\ t \ge 0\}$ *be a stochastically continuous homogeneous process with independent increments, and let* $\varphi_t(\lambda) = \mathbf{E}e^{i\lambda\xi(t)}$ *be the ch.f. of* $\xi(t)$, $\varphi(\lambda) := \varphi_1(\lambda)$. *Then*

$$\varphi_t(\lambda) = \varphi^t(\lambda), \tag{19.1.1}$$

$\varphi(\lambda) \ne 0$ *for any* λ.

(2) *Let* $\varphi(\lambda)$ *be the ch.f. of an infinitely divisible distribution. Then there exists a random process* $\{\xi(t),\ t \ge 0\}$ *satisfying the conditions of* (1) *and such that*

$$\mathbf{E}e^{i\lambda\xi(1)} = \varphi(\lambda).$$

Note that in the theorem the power $\varphi^t(\lambda)$ of the complex number $\varphi(\lambda)$ is understood as $|\varphi(\lambda)|^t e^{i\alpha(\lambda)t}$, where $\alpha(\lambda) = \arg\varphi(\lambda)$ ($\varphi(\lambda) = |\varphi(\lambda)|e^{i\alpha(\lambda)}$). But $\alpha(\lambda)$ is a multi-valued function, which is defined up to the term $2\pi k$ with integer k. Therefore, for non-integer t, the function $\varphi^t(\lambda)$ will be multi-valued as well. Since any ch.f. is continuous, after crossing the level $2\pi k$ by $\alpha(k)$ (while changing the value of λ from zero, $\alpha(0) = 0$), we are to take the "nearest" branch of $\alpha(k)$ so as to ensure continuity of the function $\varphi^t(\lambda)$. For example, for the degenerate distribution I_1 we have $\varphi(\lambda) = e^{i\lambda}$ ($\alpha(\lambda) = \lambda$), so for small $t > 0$, $\varepsilon > 0$ and for $\lambda = 2\pi + \varepsilon$ we are to set $\varphi^t(\lambda) = e^{i(2\pi+\varepsilon)t}$ rather than $\varphi^t(\lambda) = e^{i\varepsilon t}$ (although $\varphi(\lambda) = e^{i\varepsilon}$).

Denote by $\mathcal{L}$ the class of ch.f.s of all infinitely divisible distributions and by $\mathcal{L}_1$ the class of the ch.f.s of the distributions of $\xi(t)$ for stochastically continuous homogeneous processes with independent increments. Then it follows from Theorem 19.1.1 that $\mathcal{L} = \mathcal{L}_1$. The class $\mathcal{L}$ will be characterised in Sect. 19.5.

Proof (1) Let $\xi(t)$ satisfy the conditions of part (1) of the theorem. Then $\xi(t)$ can be represented as a sum of independent increments

$$\xi(t) = \sum_{j=1}^{n} \bigl[\xi(t_j) - \xi(t_{j-1})\bigr], \qquad t_0 = 0,\ t_n = t,\ t_j > t_{j-1}.$$

From this it follows, in particular, that for $t_j = j/n$, $t = 1$,

$$\varphi(\lambda) = \bigl[\varphi_{1/n}(\lambda)\bigr]^n, \qquad \varphi_{1/n}(\lambda) = \varphi^{1/n}(\lambda).$$

Raising both sides of the last equality to an integer power k, we obtain that, for any rational $r = k/n$, one has

$$\varphi_{k/n}(\lambda) = \varphi^{k/n}(\lambda),$$

which proves (19.1.1) for $t = r$. Now let t be irrational and $r_n := \lfloor tn \rfloor / n$. Since $\xi(t)$ is a stochastically continuous process, one has $\xi(r_n) \xrightarrow{p} \xi(t)$ as $n \to \infty$, and hence the corresponding ch.f.s converge: for any λ,

$$\varphi_{r_n}(\lambda) \to \varphi_t(\lambda).$$

But $\varphi_{r_n}(\lambda) = \varphi^{r_n}(\lambda) \to \varphi^t(\lambda)$. Therefore (19.1.1) necessarily holds true.

Further, by stochastic continuity of $\xi(\cdot)$, we have $\varphi_t(\lambda) = \varphi^t(\lambda) \to 1$ as $t \to 0$ for any λ. This implies that $\varphi(\lambda) \neq 0$ for any λ. This completes the proof of the first assertion of the theorem.

(2) Observe first that if $\varphi \in \mathcal{L}$ then, for any $t > 0$, φ^t is again a ch.f. Indeed,

$$\varphi^t(\lambda) = \lim_{n\to\infty} \varphi^{\lfloor tn \rfloor/n}(\lambda),$$

so that $\varphi^t(\lambda)$ is a limit of ch.f.s which is continuous at the point $\lambda = 0$. By the continuity theorem for ch.f.s, this is again a ch.f.

Now we will construct a random process $\xi(t)$ with independent increments by specifying its finite-dimensional distributions. Put

$$0 = t_0 < t_1 < \cdots < t_k, \qquad \Delta_j := \xi(t_j) - \xi(t_{j-1}), \qquad \delta_j := t_j - t_{j-1},$$

and observe that

$$\sum_{j=1}^k \lambda_j \xi(t_j) = \sum_{j=1}^k \lambda_j \sum_{l=1}^j \Delta_l = \sum_{j=1}^k \Delta_j \sum_{l=j}^k \lambda_l.$$

Define the ch.f. of the joint distribution of $\xi(t_1), \ldots, \xi(t_k)$ by the equality (postulating independence of Δ_j)

$$\mathbf{E}\exp\left\{i\sum_1^k \lambda_j \xi(t_j)\right\} := \mathbf{E}\exp\left\{i\sum_{j=1}^k \Delta_j \sum_{l=j}^j \lambda_l\right\} = \prod_{j=1}^k \varphi\left(\sum_{l=j}^k \lambda_l\right)^{\delta_j}.$$

Thus, we have used φ to define the finite-dimensional distributions of $\xi(t)$ in $\langle \mathbb{R}^T, \mathfrak{B}_{\mathbb{R}}^T\rangle$ with $T = [0, \infty)$ which, as one can easily see, are consistent. By Kolmogorov's theorem, there exists a distribution of a random process $\xi(t)$ in $\langle \mathbb{R}^T, \mathfrak{B}_{\mathbb{R}}^T\rangle$. That process is by definition a homogeneous processes with independent increments.

To prove stochastic continuity of $\xi(t)$, note that, as $h \to 0$,

$$\mathbf{E}e^{i\lambda(\xi(t+h)-\xi(t))} = \varphi^h(\lambda) \to \varphi_0(\lambda),$$

where

$$\varphi_0(\lambda) = \begin{cases} 1 & \text{if } \varphi(\lambda) \neq 0, \\ 0 & \text{if } \varphi(\lambda) = 0. \end{cases}$$

Thus the limiting function $\varphi_0(\lambda)$ can assume only two values: 0 and 1. But it is bound to be a ch.f. since it is continuous at the point $\lambda = 0$ ($\varphi(\lambda) \neq 0$ in a neighbourhood of the point $\lambda = 0$) and is a limit of ch.f.s. Therefore $\varphi_0(\lambda)$ is continuous, $\varphi_0(\lambda) \equiv 1$, $\varphi^h(\lambda) \to 1$, and

$$\xi(t+h) - \xi(t) \xrightarrow{p} 0 \quad \text{as } h \to 0.$$

The theorem is proved. □

Corollary 19.1.1 *Let the conditions of part* (1) *of Theorem* 19.1.1 *be met. If, for all* t, $\mathbf{E}|\xi(t)| < \infty$ *then*

$$\mathbf{E}\xi(t) = t\mathbf{E}\xi(1).$$

If $\mathbf{E}(\xi(1))^2 < \infty$ *then*

$$\operatorname{Var}\xi(t) = t \operatorname{Var}\xi(1).$$

Proof For the sake of brevity, put $a := \mathbf{E}\xi(1)$. Then, differentiating (19.1.1) in λ at the point $\lambda = 0$, we obtain

$$\begin{aligned}
\mathbf{E}\xi(t) &= -i\varphi_t'(0) = -it\varphi^{t-1}\varphi'(0) = at, \\
\mathbf{E}\xi^2(t) &= -\varphi_t''(0) = -t(t-1)\varphi^{t-2}(0)\big(\varphi'(0)\big)^2 - t\varphi^{t-1}(0)\varphi''(0) \\
&= t(t-1)a^2 + t\mathbf{E}\xi^2(1), \\
\operatorname{Var}\xi(t) &= t\big(\mathbf{E}\xi^2(1) - a^2\big) = t \operatorname{Var}\xi(1).
\end{aligned}$$

The corollary is proved. □

In the next theorem we put, as before, $T = [0, 1]$ or $T = [0, \infty)$.

Theorem 19.1.2 *Homogeneous stochastically continuous processes with independent increments* $\{\xi(t),\ t \in T\}$ *have modifications in the space* $D(T)$, *i.e. the process* $\xi(t)$ *can be given in* $\langle D(T), \mathfrak{B}_D^T\rangle$ *and hence have no discontinuities of the second type.*

Proof To simplify the argument, assume that $\mathbf{E}\xi^2(1)$ exists, or, which is the same, that the second derivative $\varphi''(\lambda)$ exists. Then

$$\begin{aligned}
\mathbf{E}\big(\xi(t) - \xi(t-h)\big)^2 &= \varphi_h''(0) = -h(h-1)\big[\varphi'(0)\big]^2 - h\varphi''(0) \le c|h|, \\
\mathbf{E}\Big(\big|\xi(t+h_2) - \xi(t)\big|^2 \big|\xi(t) - \xi(t-h_1)\big|^2\Big) &\le c^2 h_1 h_2 \le c^2(h_1 + h_2),
\end{aligned}$$

and the assertion follows from the second criterion of Theorem 18.2.3. The theorem is proved. □

In the general case, the proof is more complicated: one has to make use of criterion (18.2.4) and bounds for $\mathbf{P}(\xi(t) - \xi(t-h)| > \varepsilon)$.

Now we will consider the two most important processes with independent increments: the so-called Wiener and Poisson processes.

19.2 Wiener Processes. The Properties of Trajectories

Definition 19.2.1 The *Wiener process* is a homogeneous process with independent increments for which the distribution of $\xi(1)$ is normal.

In other words, this is a process for which

$$\varphi(\lambda) = e^{i\lambda\alpha - \sigma^2\lambda^2/2}, \qquad \varphi_t(\lambda) = \varphi^t(\lambda) = e^{i\lambda t\alpha - \sigma^2\lambda^2 t/2}$$

for some α and $\sigma^2 \ge 0$. The second equality means that the increments $\xi(t+u) - \xi(u)$ are normally distributed with parameters $(\alpha t, \sigma^2 t)$. All joint distributions of $\xi(t_1), \dots, \xi(t_n)$ are clearly also normal.

The numbers α and σ are called the *shift* and *diffusion coefficients*, respectively. Introducing the process $\xi_0(t) := (\xi(t) - \alpha t)/\sigma$ which is obtained from $\xi(t)$ by an affine transformation, we obtain that its ch.f. equals

$$\mathbf{E} e^{i\lambda\xi_0(t)} = e^{-i\lambda\alpha t/\sigma}\varphi_t(\lambda/\sigma) = e^{-\lambda^2 t/2}.$$

Such a process with parameters $(0, t)$ is often called the *standard Wiener process*. We consider it in more detail.

Theorem 19.2.1 *The Wiener process has a continuous modification.*

This means, as we know, that the Wiener process $\{\xi(t),\ t \in [0,1]\}$ can be considered as given on the measurable space $\langle C(0,1), \mathfrak{B}_C^{[0,1]}\rangle$ of continuous functions.

Proof We have $\xi(t+h) - \xi(t) \in\!\!\!= \mathbf{\Phi}_{0,h}$ and $h^{-1/2}(\xi(t+h) - \xi(t)) \in\!\!\!= \mathbf{\Phi}_{0,1}$. Therefore

$$\mathbf{E}\big(\xi(t+h) - \xi(t)\big)^4 = h^2 \mathbf{E}\xi(1)^4 = 3h^2.$$

This means that the conditions of Theorem 18.2.1 are satisfied. □

Thus we can assume that $\xi(\cdot) \in C(0,1)$. The standard Wiener process with *continuous* trajectories will be denoted by $\{w(t),\ t \in T\}$.

Now note that the *trajectories of the Wiener process $w(t)$, being continuous, are not differentiable with probability* 1 *at any given point t*.

By virtue of the homogeneity of the process, it suffices to prove its nondifferentiability at the point 0. If, with a positive probability, i.e. on an event set $A \subset \Omega$ with $\mathbf{P}(A) > 0$, there existed the derivative

$$w'(0) = w'(0, \omega) = \lim_{t \to 0} \frac{w(t)}{t},$$

then, on the same event, there would exist the limit

$$\begin{aligned}\lim_{k\to\infty} \frac{w(2^{-k+1}) - w(2^{-k})}{2^{-k}} &= \lim_{k\to\infty} \frac{2w(2^{-k+1})}{2^{-k+1}} - \lim_{k\to\infty} \frac{w(2^{-k})}{2^{-k}} \\ &= 2w'(0) - w'(0) = w'(0).\end{aligned}$$

But this is impossible for the following reason. The independent differences $w(2^{-k+1}) - w(2^{-k})$ have the same distribution as $w(2^{-k})$, and with the positive probability $p = 1 - \Phi(1)$ they exceed the value $\sqrt{2^{-k}}$. That is, the independent events $B_k = \{w(2^{-k+1}) - w(2^{-k}) > \sqrt{2^{-k}}\}$ have the property $\sum_{k=1}^{\infty} \mathbf{P}(B_k) = \infty$. By the Borel–Cantelli criterion, this means that with probability 1 there occur infinitely many events B_k, so that

$$\mathbf{P}\left(\limsup_{k\to\infty} \frac{w(2^{-k+1}) - w(2^{-k})}{\sqrt{2^{-k}}} > 1\right) = 1.$$

In the same way we find that

$$\mathbf{P}\biggl(\liminf_{k\to\infty}\frac{w(2^{-k+1})-w(2^{-k})}{2^{-k}}<-1\biggr)=1.$$

This implies that, with probability 1,

$$\limsup_{k\to\infty}\frac{w(2^{-k+1})-w(2^{-k})}{2^{-k}}=\infty,\qquad \liminf_{k\to\infty}\frac{w(2^{-k+1})-w(2^{-k})}{2^{-k}}=-\infty,$$

and therefore the process $w(t)$ is nondifferentiable at any given point t with probability 1.

A stronger assertion also takes place: with probability 1 *there exists no point t at which the trajectory of the process $w(t)$ would have a derivative*. In other words, the Wiener process is *nowhere differentiable* with probability 1. The proof of this fact is much more complicated and lies beyond the scope of the book.

The reader can easily verify that $w(t)$ has, in a certain sense, a parabola property. Namely, for any $c>0$, the process $w^*(t)=c^{-1/2}w(ct)$ is again a Wiener process.

The properties of continuity of trajectories and independence of increments for the Wiener process allow us to find, in an explicit form, the distributions of

$$\overline{w}(t)=\max_{u\in[0,t]}w(u)$$

and of the time of the first passage of a given level which is defined, for a given $x>0$, by

$$\eta(x):=\inf\bigl\{t:w(t)\ge x\bigr\}=\inf\bigl\{t:w(t)=x\bigr\}.$$

Theorem 19.2.2

$$\mathbf{P}\bigl(\overline{w}(t)>x\bigr)=2\mathbf{P}\bigl(w(t)>x\bigr)=2\biggl(1-\Phi\biggl(\frac{x}{\sqrt{t}}\biggr)\biggr).\qquad(19.2.1)$$

The distribution of $\eta(1)$ is stable and has the density

$$\frac{1}{\sqrt{2\pi}\,t^{3/2}}e^{-\frac{1}{2t}},\quad t>0.\qquad(19.2.2)$$

Distribution (19.2.1) is sometimes called the *double normal tail law*, while the distribution with density (19.2.2) is called the Lévy distribution (see Sect. 8.8).

Proof Since

$$\bigl\{\eta(x)=v\bigr\}=\bigcap_{n=1}^{\infty}\bigl\{\overline{w}(v-1/n)<x,\,w(v)=x\bigr\}\in\mathfrak{F}_v:=\sigma\bigl\{w(u);\,u\le v\bigr\}$$

and $w(t)-w(v)\stackrel{d}{=}w(t-v)$ for $t>v$ does not depend on $\mathfrak{F}_v$, we have

$$\begin{aligned}\mathbf{P}\bigl(w(t)>x\bigr)&=\int_0^t\mathbf{P}\bigl(\eta(x)\in dv\bigr)\mathbf{P}\bigl(w(t-v)>0\bigr)\\&=\frac{1}{2}\int_0^t\mathbf{P}\bigl(\eta(x)\in dv\bigr)=\frac{1}{2}\mathbf{P}\bigl(\overline{w}(t)\ge x\bigr).\end{aligned}$$

This implies the first assertion of the theorem.

The same equalities imply that

$$\mathbf{P}\big(\eta(x) < v\big) = \mathbf{P}\big(\overline{w}(v) > x\big) = 2\left(1 - \Phi\left(\frac{x}{\sqrt{v}}\right)\right) = \frac{2}{\sqrt{2\pi}} \int_{x/\sqrt{v}}^{\infty} e^{-s^2/2}\, ds,$$

which yields, for the density f_η of the variable $\eta := \eta(1)$,

$$f_\eta(v) = \frac{e^{-\frac{1}{2v}}}{\sqrt{2\pi}\, v^{3/2}}.$$

In order to prove that this distribution is stable, note that

$$\eta(n) = \eta_1 + \cdots + \eta_n,$$

where η_i are distributed as η and are independent (since the path of $w(t)$ first attains level 1; then level 2, starting at a point with ordinate 1; then level 3, and so on). Using the same argument as above, we obtain that

$$\mathbf{P}\big(\eta(n) < v\big) = \mathbf{P}\big(\overline{w}(v) > n\big) = \mathbf{P}\big(\overline{w}\big(vn^{-2}\big) > 1\big) = \mathbf{P}\big(\eta < vn^{-2}\big),$$

so the distributions of η and $\eta(n)$ coincide up to a scale transformation. This implies the stability of the distribution of η (see Sect. 8.8). Since $\eta \geq 0$ and $\mathbf{P}(\eta > x) \sim \sqrt{\frac{2}{\pi x}}$ as $x \to \infty$, we obtain that it is, up to a scale transformation, the distribution $\mathbf{F}_{1/2,1}$ with parameters $\beta = 1/2$, $\rho = 1$ (cf. Sect. 8.8). The theorem is proved. □

19.3 The Laws of the Iterated Logarithm

Using an argument similar to that employed at the end of the previous section, one can establish a much stronger assertion: the trajectory of $w(t)$ in the neighbourhood of the point $t = 0$, graphically speaking, "completely shades" the interior of the domain bounded by the two curves

$$y = \pm\sqrt{2t \ln\ln\frac{1}{t}}.$$

The exterior of this domain remains untouched. This is the so-called *law of the iterated logarithm.*

Theorem 19.3.1

$$\mathbf{P}\left(\limsup_{t\to 0} \frac{w(t)}{\sqrt{2t \ln\ln \frac{1}{t}}} = 1\right) = 1,$$

$$\mathbf{P}\left(\liminf_{t\to 0} \frac{w(t)}{\sqrt{2t \ln\ln \frac{1}{t}}} = -1\right) = 1.$$

Thus, if we consider the sequence of random variables $w(t_n)$, $t_n \downarrow 0$, then, for any $\varepsilon > 0$,

$$(1 \pm \varepsilon)\sqrt{2t_n \ln\ln \frac{1}{t_n}}$$

will be upper and lower sequences, respectively, for that sequence.

For processes, we could introduce in a natural way the notions of *upper* and *lower functions*. If, for example, a process $\xi(t)$ belongs to $C(0,\infty)$ or $D(0,\infty)$ (or is separable on $(0,\infty)$), then the respective definition for the case $t \to \infty$ has the following form.

Definition 19.3.1 A function $a(t)$ is said to be *upper* (*lower*) for the process $\xi(t)$ if, for some sequence $t_n \uparrow \infty$, the events $A_n = \{\sup_{u\ge t_n}(\xi(t) - a(t)) > 0\}$ occur finitely (infinitely) often with probability 1.

Along with Theorem 19.3.1, we will obtain here the conventional law of the iterated logarithm. The proofs of the both assertions are essentially identical. We will prove the latter and derive the former as a consequence.

Theorem 19.3.2 (The Law of the Iterated Logarithm)

$$\mathbf{P}\biggl(\limsup_{t\to\infty} \frac{w(t)}{\sqrt{2t\ln\ln t}} = 1\biggr) = 1,$$
$$\mathbf{P}\biggl(\liminf_{t\to\infty} \frac{w(t)}{\sqrt{2t\ln\ln t}} = -1\biggr) = 1.$$

Thus, for any $\varepsilon > 0$, the functions $(1 \pm \varepsilon)\sqrt{2t\ln\ln t}$ are, respectively, upper and lower for $w(t)$ as $t \to \infty$.

Proof of Theorem 19.3.2 First observe that, by L'Hospital's rule,

$$\begin{aligned}\mathbf{P}\bigl(w(t) > x\bigr) &= \frac{1}{\sqrt{2\pi t}} \int_x^\infty e^{-u^2/2t}\,du \\ &= \frac{1}{\sqrt{2\pi t}} \int_{x/\sqrt{t}}^\infty e^{-u^2/2t}\,du \sim \frac{\sqrt{t}}{\sqrt{2\pi}x} e^{-x^2/2t}\end{aligned} \tag{19.3.1}$$

as $x/\sqrt{t} \to \infty$.

Let $a > 1$ and $x_k := \sqrt{2a^k \ln\ln a^k}$. We have to show that, for any $\varepsilon > 0$,

$$\mathbf{P}\biggl(\limsup_{t\to\infty} \frac{w(t)}{\sqrt{2t\ln\ln t}} < 1 + \varepsilon\biggr) = 1, \tag{19.3.2}$$

i.e. that, with probability 1, for all sufficiently large t,

$$w(t) < (1 + \varepsilon)\sqrt{2t\ln\ln t}.$$

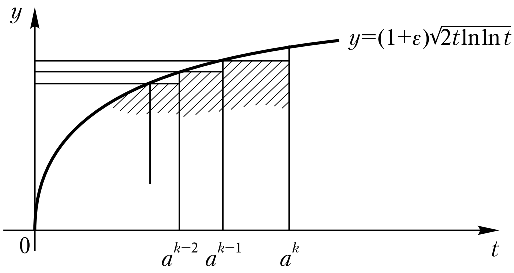

Fig. 19.1 Illustration to the proof of Theorem 19.3.2: replacing the curvilinear boundary with a step function

To this end it suffices to establish that, with probability 1, there occur only finitely many events

$$B_k := \Big\{ \sup_{a^{k-1} < u \le a^k} w(u) > (1+\varepsilon)x_{k-1} \Big\}.$$

Consider the events

$$A_k = \Big\{ \sup_{u \le a^k} w(u) > (1+\varepsilon)x_{k-1} \Big\} \supset B_k$$

(see Fig. 19.1). Because $x_k/\sqrt{a^k} \to \infty$ as $k \to \infty$, by Theorem 19.2.2 one has

$$\begin{aligned}
\mathbf{P}(A_k) &= 2\mathbf{P}\big(w\big(a^k\big) > (1+\varepsilon)x_{k-1}\big) \\
&\sim \sqrt{\frac{2}{\pi}} \frac{\sqrt{a^k}}{(1+\varepsilon)x_{k-1}} \exp\bigg\{ -\frac{2(1+\varepsilon)a^{k-1} \ln\ln a^{k-1}}{2a^k} \bigg\} \\
&= \frac{1}{(1+\varepsilon)} \sqrt{\frac{1}{\pi a \ln\ln a^{k-1}}} \frac{1}{(\ln a^{k-1})^{(1+\varepsilon)^2/a}} \\
&= c(a,\varepsilon) \frac{1}{\sqrt{(\ln(k-1) + \ln\ln a)}(k-1)^{(1+\varepsilon)^2/a}}.
\end{aligned}$$

Put $a := 1+\varepsilon > 1$. Then clearly

$$\mathbf{P}(A_k) \sim \frac{c(\varepsilon)}{k^{1+\varepsilon}\sqrt{\ln k}}$$

as $k \to \infty$.

In the above formulas, $c(a,\varepsilon)$ and $c(\varepsilon)$ are some constants depending on the indicated parameters. The obtained relation implies that $\sum_{k=1}^{\infty} \mathbf{P}(A_k) < \infty$ and hence $\sum_{k=1}^{\infty} \mathbf{P}(B_k) < \infty$ (for $B_k \subset A_k$), so that by the Borel–Cantelli criterion (Theorem 11.1.1) with probability 1 the events B_k occur only finitely often.

We now prove that, for an arbitrary $\varepsilon > 0$,

$$\mathbf{P}\bigg(\limsup_{t\to\infty} \frac{w(t)}{\sqrt{2t \ln\ln t}} > 1 - \varepsilon \bigg) = 1. \tag{19.3.3}$$

It is evident that, together with (19.3.2), this will mean that the first assertion of the theorem is true.

Consider for $a > 1$ independent increments $w(a^k) - w(a^{k-1})$ and denote by B_k the event

$$B_k := \{w(a^k) - w(a^{k-1}) > (1-\varepsilon/2)rx_k\}.$$

Since $w(a^k) - w(a^{k-1})$ is distributed as $w(a^k(1-a^{-1}))$, by virtue of (19.3.1) we find, as before, that

$$\begin{aligned}\mathbf{P}(B_k) &\sim \frac{\sqrt{a^k(1-a^{-1})}}{\sqrt{2\pi}(1-\varepsilon/2)x_k}\exp\left\{-\frac{(1-\varepsilon/2)^2 2a^k \ln\ln a^{-k}}{2a^k(1-a^{-1})}\right\}\\ &\sim \frac{c_1(a,\varepsilon)}{\sqrt{\ln k}}k^{-(1-\varepsilon/2)^2/(1-a^{-1})}.\end{aligned}$$

This implies that, for $a \ge 2/\varepsilon$, the series $\sum_{k=1}^{\infty}\mathbf{P}(B_k)$ diverges, and hence by the Borel–Cantelli criterion the events B_k occur infinitely often, with probability 1.

Further, by the symmetry of the process $w(t)$, it follows from relation (19.3.2) that, for all k large enough and any $\delta > 0$,

$$w(a^k) > -(1+\delta)x_k.$$

Together with the preceding argument this shows that the event

$$w(a^{k-1}) + [w(a^k) - w(a^{k-1})] = w(a^k) > -(1+\delta)x_{k-1} + (1-\varepsilon/2)x_k$$

will occur infinitely often. But the right hand-side of the above inequality can be made greater than $(1-\varepsilon)x_k$ by choosing an appropriate a. Indeed,

$$-(1+\delta)x_{k-1} + \frac{\varepsilon}{2}x_k > 0$$

once

$$(1+\delta)\sqrt{\frac{\ln\ln a^{k-1}}{a\ln\ln a^k}} < \frac{\varepsilon}{2},$$

which, in turn, can easily be achieved by taking a large enough. Thus relation (19.3.3) is proved.

The second assertion of the theorem clearly follows from the first by virtue of the symmetry of the distribution of $w(t)$. □

Now we can obtain as a consequence the local law of the iterated logarithm for the case where $t \to 0$.

Proof of Theorem 19.3.1 Consider the process $\{W(u) := uw(1/u),\ u \ge 0\}$, where we put $W(0) := 0$. The remarkable fact is that the *process* $\{W(u),\ u \ge 0\}$ *is also the standard Wiener process*. Indeed, for $t > u$,

$$\begin{aligned}\mathbf{E}\exp\{i\lambda(W(t) - W(u))\} &= \mathbf{E}\exp\left\{i\lambda\left[tw\left(\frac{1}{t}\right) - uw\left(\frac{1}{u}\right)\right]\right\}\\ &= \mathbf{E}\exp\left(i\lambda\left[w\left(\frac{1}{t}\right)(t-u) - u\left(w\left(\frac{1}{u}\right) - w\left(\frac{1}{t}\right)\right)\right]\right)\end{aligned}$$

$$= \exp\left\{-\frac{\lambda^2}{2}(t-u)^2\frac{1}{t} - \frac{\lambda^2 u^2}{2}\left(\frac{1}{u} - \frac{1}{t}\right)\right\}$$
$$= \exp\left\{-\frac{\lambda^2}{2}(t-u)\right\}.$$

The independence of increments is easiest to prove by establishing their noncorrelatedness. Indeed,

$$\mathbf{E}\big[W(u)\big(W(t)-W(u)\big)\big] = \mathbf{E}\left[uw\left(\frac{1}{u}\right)\left(tw\left(\frac{1}{t}\right) - uw\left(\frac{1}{u}\right)\right)\right]$$
$$= \mathbf{E}\left[uw\left(\frac{1}{t}\right)tw\left(\frac{1}{t}\right) - u^2w^2\left(\frac{1}{u}\right)\right] = u - u = 0.$$

To complete the proof of the theorem, it remains to observe that

$$\limsup_{t\to\infty}\frac{w(t)}{\sqrt{2t\ln\ln t}} = \limsup_{u\to 0}\frac{uw(1/u)}{\sqrt{2u\ln\ln\frac{1}{u}}} = \limsup_{u\to 0}\frac{W(u)}{\sqrt{2u\ln\ln\frac{1}{u}}}.$$

The theorem is proved. □

We could also prove the theorem by repeating the argument from the proof of Theorem 19.3.2 with $a < 1$.

In conclusion we note that Wiener processes play an important role in many theoretical probabilistic considerations and serve as models for describing various real-life processes. For example, they provide a good model for the movement of a diffusing particle. In this connection, the Wiener processes are also often called *Brownian motion processes*.

Wiener processes prove to be, in a certain sense, the limiting processes for random polygons constructed on the vertices $(k/n, S_k/\sqrt{n})$, where S_k are sums of random variables ξ_j with $\mathbf{E}\xi_j = 0$ and $\operatorname{Var}(\xi_j) = 1$. We will discuss this in more detail in Chap. 20. The concept of the stochastic integral and many other constructions and results are also closely related to the Wiener process.

19.4 The Poisson Process

Definition 19.4.1 A homogeneous process $\xi(t)$ with independent increments is said to be the *Poisson process* if $\xi(t) - \xi(0)$ has the Poisson distribution.

For simplicity's sake put $\xi(0) = 0$. If $\xi(1) \subset\!\!\!= \Pi_\mu$, then

$$\varphi(\lambda) := \mathbf{E}e^{i\lambda\xi(1)} = \exp\big\{\mu\big(e^{i\lambda} - 1\big)\big\}$$

and, as we know,

$$\varphi_t(\lambda) = \mathbf{E}e^{i\lambda\xi(t)} = \varphi^t(\lambda) = \exp\big\{\mu t\big(e^{i\lambda} - 1\big)\big\},$$

so that $\xi(t) \in \Pi_{\mu t}$. We consider the properties of the Poisson process. First of all, for each t, $\xi(t)$ takes only integer values $0, 1, 2, \ldots$. Divide the interval $[0, t)$ into segments $[0, t_1), [t_1, t_2), \ldots, [t_{n-1}, t_n)$ of lengths $\Delta_i = t_i - t_{i-1}$, $i = 1, \ldots, n$. For small Δ_i the distributions of the increments $\xi(t_i) - \xi(t_{i-1})$ will have the property that

$$\begin{aligned}
&\mathbf{P}\big(\xi(t) - \xi(t_{i-1}) = 0\big) = \mathbf{P}\big(\xi(\Delta_i) = 0\big) = e^{-mu\Delta_i} = 1 - \mu\Delta_i + O\big(\Delta_i^2\big), \\
&\mathbf{P}\big(\xi(t) - \xi(t_{i-1}) = 1\big) = \mu\Delta_i e^{-\mu\Delta_i} = \mu\Delta_i + O\big(\Delta_i^2\big), \\
&\mathbf{P}\big(\xi(t) - \xi(t_{i-1}) \ge 2\big) = O\big(\Delta_i^2\big).
\end{aligned} \tag{19.4.1}$$

Consider "embedded" rational partitions $\mathcal{R}(n) = \{t_1, \ldots, t_n\}$ of the interval $[0, t]$ such that $\mathcal{R}(n) \subset \mathcal{R}(n+1)$ and $\bigcup \mathcal{R}(n) = \mathcal{R}_1$ is the set of all rationals in $[0, t]$.

Note the following three properties.

(1) Let $\nu(n)$ be the number of intervals in the partition $\mathcal{R}(n)$ on which the increments of the process ξ are non-zero. For each ω, $\nu(n)$ is non-decreasing as $n \to \infty$. Furthermore, the number $\nu(n)$ can be represented as a sum of independent random variables which are equal to 1 if there is an increment on the i-th interval and 0 otherwise. Therefore, by (19.4.1)

$$\begin{aligned}
\mathbf{P}\big(\nu(n) \ne \xi(t)\big) &= \mathbf{P}\bigg(\bigcup_{t_i \in \mathcal{R}(n)} \big\{\xi(t_i) - \xi(t_{i-1}) \ge 2\big\} \cup \big\{\xi(t) - \xi(t_n) \ge 1\big\}\bigg) \\
&= O\Bigg(\sum_{j=1}^{n} \Delta_j^2 + (t - t_n)\Bigg),
\end{aligned}$$

where $\sum_{j=1}^{n} \Delta_j^2 \le t \max \Delta_j \to 0$ as $n \to \infty$, so that a.s.

$$\nu(n) \uparrow \xi(t) \in \Pi_{\mu t}$$

as the partitions refine.

(2) Because the maximum length of the intervals Δ_j tends to 0 as $n \to \infty$, the total length of the intervals containing jumps converges to 0.

Therefore, taking the unions of the remaining adjacent intervals Δ_j (i.e. where there are no increments of ξ), for each ω we obtain in the limit, as $n \to \infty$, $\xi(t) + 1$ intervals $(0, T_1), (T_1, T_2), \ldots, (T_\nu, t)$ on which the increments of ξ are null.

(3) Finally, by (19.4.1) the probability that at least one of the increments on the intervals Δ_j exceeds one is $\sum_j O(\Delta_j^2) = o(1)$ as $n \to \infty$, so that, with probability 1, the jumps at the points T_k are equal to 1.

Thus we have shown that, on the segment $[0, t]$, for each ω there exists a finite number $\xi(t)$ of points $T_1, \ldots, T_{\xi(t)}$ such that $\xi(u)$ takes at the rational points of the intervals (T_k, T_{k+1}) one and the same constant value equal to k. This means that one can extend the trajectories of the process $\xi(u)$, say, by continuity from the right so that $\xi(u) = k$ for all $u \in [T_k, T_{k+1})$.

Thus, for the original process $\xi(t)$ we have constructed a *modification* $\overline{\xi}(t)$ *with trajectories in* $D_+(T)$. The equivalence of $\overline{\xi}$ and ξ follows from the very construction since, by virtue of (1),

$$\mathbf{P}\big(\overline{\xi}(t) = \xi(t)\big) = \mathbf{P}\Big(\lim_{n\to\infty} \nu(n) = \xi(t)\Big) = 1.$$

One usually considers just such right (or left) continuous modifications of the Poisson process. We have already dealt with processes of this kind in Chap. 10 where more general objects—renewal processes—were defined from scratch using trajectories. That the Poisson process is a renewal process is seen from the following considerations. It is easy to establish from relations (19.4.1) that the distributions of the random variables $T_1, T_2 - T_1, T_3 - T_2, \dots$ coincide and that these variables are independent. Indeed, the difference $T_j - T_{j-1}$, $j \ge 1$, $T_0 = 0$, can be approximated by the sum $(\gamma_j - \gamma_{j-1})\Delta$ of the lengths of identical intervals of size $\Delta_i = \Delta$, where γ_j is the number of the interval in which the j-th non-zero increment of ξ occurred. Since the process $\xi(t)$ is homogeneous with independent increments, we have

$$\mathbf{P}\big((\gamma_j - \gamma_{j-1})\Delta > u\big) = \mathbf{P}\left(\gamma_1 > \frac{u}{\Delta}\right) = \big(e^{-\mu\Delta}\big)^{\lfloor u/\Delta \rfloor} \to e^{-\mu u},$$
$$\mathbf{P}\big((\gamma_j - \gamma_{j-1})\Delta > u\big) \to \mathbf{P}(T_j - T_{j-1} > u)$$

as $\Delta \to 0$. Hence the variables $\tau_j := T_j - T_{j-1}$, $j = 1, 2, 3, \dots$, have the exponential distribution, and the value $\xi(t) + 1$ can be considered as the first crossing time of the level t by the sums T_j:

$$\xi(t) = \max\{k : T_k \le t\}, \qquad \xi(t) + 1 = \min\{k : T_k > t\}.$$

Thus we obtain that the Poisson process $\xi(t)$ coincides with the renewal process $\eta(t)$ (see Chap. 10) for exponentially distributed variables $\tau_1, \tau_2, \dots$ with $\mathbf{P}(\tau_j > u) = e^{-\mu u}$.

The above and the properties of the Poisson process also imply the following remarkable property of exponentially distributed random variables. The numbers of jump points (i.e. sums T_k) which fall into disjoint time intervals δ_j are independent, these numbers being distributed according to the Poisson laws with parameters $\mu\delta_j$.

Using the last fact, one can construct a more general model of a pure jump homogeneous process with independent increments. Consider an arbitrary sequence of independent identically distributed random variables $\zeta_1, \zeta_2, \dots$ that have a ch.f. $\beta(\lambda)$ and are independent of the σ-algebra generated by the process $\xi(t)$. Construct now a new process $\zeta(t)$ as follows. To each ω we put into correspondence a new trajectory obtained from the trajectory $\xi(t)$ by replacing the first unit jump with the variable ζ_1, the second one with the variable ζ_2, and so on. It is easy to see that $\zeta(t)$ will also be a process with independent increments. The value $\zeta(t)$ will be equal to the sum

$$\zeta(t) = \zeta_1 + \cdots + \zeta_{\xi(t)} \tag{19.4.2}$$

of the random number $\xi(t)$ of random variables $\zeta_1, \zeta_2, \dots$, where $\xi(t)$ is independent of $\{\zeta_k\}$ by construction.

Hence, by the total probability formula,

$$\mathbf{E}e^{i\lambda\zeta(t)} = \sum_{k=0}^{\infty} \mathbf{P}\big(\xi(t) = k\big)\mathbf{E}e^{i\lambda(\zeta_1 + \cdots + \zeta_k)}$$
$$= \sum_{k=0}^{\infty} \frac{(\mu t)^k}{k!} e^{-\mu t}\big(\beta(\lambda)\big)^k = e^{-\mu t + \mu t\beta(\lambda)} = e^{\mu t(\beta(\lambda) - 1)}. \tag{19.4.3}$$

Definition 19.4.2 The process $\zeta(t)$ defined by formula (6) or ch.f. (7) is called a *compound Poisson process*. It is evidently a special case of the generalised renewal process (see Sect. 10.6).

As we have already noted, it is again a homogeneous process with independent increments. In formula (19.4.3), the parameter μ determines the jumps' intensity in the process $\zeta(t)$, while the ch.f. $\beta(\lambda)$ specifies their distribution. If we add a constant "drift" qt to the process $\zeta(t)$, then $\widetilde{\zeta}(t) = \zeta(t) + qt$ will clearly also be a homogeneous process with independent increments having the ch.f. $\mathbf{E}e^{i\lambda\widetilde{\zeta}(t)} = e^{t(i\lambda q+\mu(\beta(\lambda)-1))}$.

Finally, if a Wiener process $w(t)$ with zero drift and diffusion coefficient σ is given on the same probability space and is independent of $\widetilde{\zeta}(t)$, and to each ω we put into correspondence a trajectory of $\widetilde{\zeta}(t) + w(t)$, we again obtain a process with independent increments, with ch.f. $\exp\{t(i\lambda q + \mu(\beta(\lambda) - 1) - \lambda^2\sigma^2/2)\}$.

One should note, however, that these constructions by no means exhaust the whole class of processes with independent increments (and therefore the class of infinitely divisible distributions).

A description of the entire class will be given in the next section.

The Poisson processes, as well as Wiener processes, are often used as mathematical models in various applications. For example, the process of counts of cosmic particles of certain energy registered by a sensor in a given volume, or of collisions of elementary particles in an accelerator are described by the Poisson process. The same is true of the process of incoming telephone calls at a switchboard and many other processes.

Due to representation (19.4.2), the study of compound Poisson processes reduces, in many aspects, to the study of the properties of sums of independent random variables.

19.5 Description of the Class of Processes with Independent Increments

We saw in Theorem 19.1.1 that, to describe the class of distributions of stochastically continuous processes with independent increments, it suffices to describe the class of all infinitely divisible distributions. Let, as before, $\mathfrak{L}$ be the class of the ch.f.s of infinitely divisible distributions.

Lemma 19.5.1 *The class $\mathfrak{L}$ is closed with respect to the operations of multiplication and passage to the limit* (*when the limit is again a ch.f.*).

Proof (1) Let $\varphi_1 \in \mathfrak{L}$ and $\varphi_2 \in \mathfrak{L}$. Then $\varphi_1\varphi_2 = (\varphi_1^{1/n} \cdot \varphi_2^{1/n})^n$, where $\varphi_1^{1/n} \cdot \varphi_2^{1/n}$ is a ch.f.

(2) Let $\varphi_n \in \mathfrak{L}$, $\varphi_n \to \varphi$, and φ be a ch.f. Then, for any m, $\varphi_n^{1/m} \to \varphi^{1/m}$ as $n \to \infty$, where $\varphi^{1/m}$ is continuous at zero and hence is a ch.f. The lemma is proved. □

Denote by $\mathcal{L}_\Pi \subset \mathcal{L}$ the class of ch.f.s whose logarithms have the form

$$\ln \varphi(\lambda) = i\lambda q + \sum_k c_k \bigl(e^{i\lambda b_k} - 1\bigr), \quad c_k \ge 0, \quad \sum_k c_k < \infty.$$

We will call this the *Poisson class*. We already know that it corresponds to compound Poisson processes with drift q and intensities c_k of jumps of size b_k (note that $\sum_k c_k(e^{i\lambda b_k} - 1) = (\sum_k c_k)\mathbf{E}(e^{i\lambda\xi} - 1)$, where ζ assumes the values b_k with probabilities $c_k/\sum_j c_j$).

Lemma 19.5.2 *A ch.f. φ belongs to $\mathcal{L}$ if and only if $\varphi = \lim_{n\to\infty} \varphi_n$, $\varphi_n \in \mathcal{L}_\Pi$.*

Proof Sufficiency. Let

$$\ln \varphi_n = \sum_k \bigl(i\lambda q_{k,n} + c_{k,n}\bigl(e^{i\lambda b_{k,n}} - 1\bigr)\bigr),$$

and $\varphi = \lim \varphi_n$ be a ch.f. It is evident that $\varphi_n^{1/m} \in \mathcal{L}_\Pi \subset \mathcal{L}$ and $\varphi_n^{1/m} \to \varphi^{1/m}$. Therefore $\varphi^{1/m}$, being a limit of a sequence of ch.f.s which is continuous at zero, is a ch.f. itself, so that $\varphi \in \mathcal{L}$.

Necessity. Let $\varphi \in \mathcal{L}$. Then $\varphi(\lambda) \ne 0$ and there exists $\beta := \ln \varphi$ with $n(\varphi^{1/n} - 1) \to \beta$, and

$$\varphi^{1/n} - 1 = \int \bigl(e^{i\lambda x} - 1\bigr)\, dF_n(x).$$

The integral of the continuous function on the right-hand side can be viewed as a Riemann–Stieltjes integral. This means that for F_n there exists a partition of the real axis into intervals Δ_{nk} such that, for $x_{nk} \in \Delta_{nk}$ and $r_n < cn^{-2}$,

$$\int \bigl(e^{i\lambda x} - 1\bigr)\, dF_n(x) = \sum_k \int \bigl(e^{i\lambda x} - 1\bigr) P_n(\Delta_{nk}) + r_n$$

($P_n(\Delta)$ is the probability of hitting the interval Δ corresponding to F_n). We obtain

$$\beta = \lim n\bigl(\varphi^{1/n} - 1\bigr) = \lim_{n\to\infty} \Bigl[n \sum_k \bigl(e^{i\lambda x_{nk}} - 1\bigr) P_n(\Delta_{nk}) \Bigr].$$

The lemma is proved. □

Theorem 19.5.1 (Lévy–Khintchin) *A ch.f. φ belongs to $\mathcal{L}$ if and only if the function $\beta := \ln \varphi$ admits a representation of the form*

$$\beta = \beta(\lambda; a, \Psi) = i\lambda q + \int \left(e^{i\lambda x} - 1 - \frac{i\lambda x}{1 + x^2} \right) \frac{1 + x^2}{x^2}\, d\Psi(x), \tag{19.5.1}$$

where Ψ is a non-decreasing function of bounded variation (*i.e., a distribution function up to a constant factor*), *the integrand being assumed equal to $-\lambda^2/2$ at the point $x = 0$* (*by continuity*).

Proof Assume that β has the form (19.5.1). Then $\beta(\lambda)$ is a continuous function, since it is (up to a continuous additive term $i\lambda a$) a uniformly convergent integral of a continuous bounded function. Further, let $x_{nk} \neq 0$, $k = 1, \dots, n$, be points of refining partitions of intervals $[-\sqrt{n}, \sqrt{n})$. Then $\beta^0(\lambda) = \beta(\lambda) - i\lambda q$ can be represented as $\beta^0 = \lim \beta_n$ with

$$\beta_n(\lambda) := \sum_{k=1}^{n} \bigl[i\lambda q_{kn} + c_{kn}\bigl(e^{i\lambda b_{kn}} - 1\bigr)\bigr] \in \mathcal{L}_\Pi,$$

where, under a natural notational convention, one should put

$$c_{kn} = \frac{1 + x_{kn}^2}{x_{kn}^2} \Psi\bigl([x_{kn}, x_{k+1,n})\bigr), \qquad q_{kn} = \frac{1}{x_{kn}} \Psi\bigl([x_{kn}, x_{k+1,n})\bigr), \qquad b_{kn} = x_{kn},$$

Ψ being used to denote the measure $\Psi(A) = \int_A d\Psi(x)$. We obtain that φ is a limit of the sequence of ch.f.s $\varphi_n \in \mathcal{L}_\Pi$. It remains to make use of Lemma 19.5.2.

Now let $\varphi \in \mathcal{L}$. Then

$$\begin{aligned} \beta &= \lim n\bigl(\varphi^{1/n} - 1\bigr) = \lim \int \bigl(e^{i\lambda x} - 1\bigr) n\, dF_n(x) \\ &= \lim \biggl[i\lambda \int \frac{nx}{1+x^2}\, dF_n(x) \\ &\quad + \int \biggl(e^{i\lambda x} - 1 - \frac{i\lambda x}{1+x^2}\biggr) \frac{1+x^2}{x^2} \frac{nx^2}{1+x^2}\, dF_n(x) \biggr]. \end{aligned} \tag{19.5.2}$$

If we put

$$q_n := \int \frac{nx}{1+x^2}\, dF_n(x), \qquad \Psi_n(x) := \frac{nx^2}{1+x^2}\, dF_n(x), \tag{19.5.3}$$

then on the right-hand side of (19.5.2) we will have $\lim \beta_n$, $\beta_n = \beta(\lambda; q_n, \Psi_n)$.

Now assume for a moment that the following continuity theorem holds for functions from $\mathcal{L}$.

Lemma 19.5.3 *If $\beta_n = \beta(\lambda; q_n, \Psi_n) \to \beta$ and β is continuous at the point $\lambda = 0$, then $\beta(\lambda)$ has the form $\beta(\lambda; q, \Psi)$, $q_n \to q$ and $\Psi_n \Rightarrow \Psi$.*

The symbol $\Rightarrow$ in the lemma means convergence at the points of continuity of the limiting function (as in the case of distribution functions) and that $\Psi_n(\pm\infty) \to \Psi(\pm\infty)$.

If the lemma is true, the required assertion of the theorem will follow in an obvious way from (19.5.2) and (19.5.3). It remains to prove the lemma.

Proof of Lemma 19.5.3 Observe first that the correspondence $\beta(\lambda; q, \Psi) \leftrightarrow (q, \Psi)$ is one-to-one. Since in one direction it is obvious, we only have to verify that β uniquely determines q and Ψ. To each β we put into correspondence the function

$$\gamma(\lambda) = \int_0^1 \left[\beta(\lambda) - \frac{1}{2}\big(\beta(\lambda+h) - \beta(\lambda-h)\big)\right] dh$$
$$= \int_0^1 \int \left(e^{i\lambda x} - \frac{1}{2}\big(e^{i(\lambda+h)x} - e^{i(\lambda-h)x}\big)\right) \frac{1+x^2}{x^2}\, d\Psi(x)\, dh,$$

where

$$\frac{1}{2}\big(e^{i(\lambda+h)x} - e^{i(\lambda-h)x}\big) = e^{i\lambda x}\cos hx,$$
$$\int_0^1 e^{i\lambda x}(1-\cos hx)\, dh = e^{i\lambda x}\left(1 - \frac{\sin x}{x}\right),$$
$$0 < c_1 < \left(1 - \frac{\sin x}{x}\right)\frac{1+x^2}{x^2} < c_2 < \infty.$$

Therefore

$$\gamma(\lambda) = \int e^{i\lambda x}\, d\Gamma(x),$$

where

$$\Gamma(x) = \int_{-\infty}^{x} \left(1 - \frac{\sin u}{u}\right)\frac{1+u^2}{u^2}\, d\Psi(u)$$

is (up to a constant multiplier) a distribution function, for which $\gamma(\lambda)$ plays the role of its ch.f. Clearly,

$$\Psi(x) = \int_{-\infty}^{x} \frac{1+u^2}{u^2}\left(1 - \frac{\sin u}{u}\right)^{-1} d\Gamma(u),$$

so that we obtained a chain of univalent correspondences $\beta \to \gamma \to \Gamma \to \Psi$ which proves the assertion.

We return to the proof of Lemma 19.5.3. Because $e^{\beta_n} \to e^{\beta}$, e^{β_n} is a ch.f., and e^{β} is continuous at the point $\lambda = 0$, we see that e^{β} is a ch.f. and hence a continuous function. This means that the convergence $\varphi_n \to \varphi$ is uniform on any interval,

$$\gamma_n(\lambda) = \int_0^1 \left[\beta_n(\lambda) - \frac{1}{2}\big(\beta_n(\lambda+h) + \beta_n(\lambda-h)\big)\right] dh$$
$$\to \int_0^1 \left[\beta(\lambda) - \frac{1}{1}\big(\beta(\lambda+h) + \beta(\lambda-h)\big)\right] dh =: \gamma(\lambda),$$

and the function $\gamma(u)$ is continuous. By the continuity theorem for ch.f.s, this means that $\gamma(u)$ is a ch.f. (of a finite measure Γ), $\Gamma_n \Rightarrow \Gamma$ (where Γ_n is the preimage of γ_n), $\Psi_n \Rightarrow \Psi$, and $q_n \to q$. Thus we establish that

$$\beta = \lim \beta_n = \lim\left[i\lambda q_n + \int \left(e^{i\lambda x} - 1 - \frac{i\lambda x}{1+x^2}\right) d\Psi_n(x)\right]$$
$$= i\lambda q + \int \left(e^{i\lambda x} - 1 - \frac{i\lambda x}{1+x^2}\right) d\Psi(x) = \beta(\lambda; q, \Psi).$$

Lemma 19.5.3 is proved. □

Theorem 19.5.1 is proved. □

Now we will make several remarks in regard to the structure of the process $\xi(t)$ and its relationship to representation (19.5.1). The function Ψ in (19.5.1) corresponds to the so-called *spectral measure of the process* $\xi(t)$ (recall that we agreed to use the same symbol Ψ for the measure itself: $\Psi(A) = \int_A d\Psi(x)$). It can be represented in the form $\mu\Psi_1(x)$, where $\mu = \Psi(\infty) - \Psi(-\infty)$ and $\Psi_1(x)$ is a distribution function.

(1) The spectral measure of the Wiener process is concentrated at the point 0. If $\Psi(\{0\}) = \sigma^2$, then $\xi(1) \in \mathbf{\Phi}_{q,\sigma^2}$.

(2) The spectral measure Ψ of a compound Poisson process has the property

$$\int |x|^{-2}\, d\Psi(x) < \infty.$$

In that case

$$G(x) = \int_{-\infty}^{x} \frac{1+u^2}{u^2}\, d\Psi(u)$$

possesses the properties of a distribution function, and $\psi(\lambda; q, \Psi)$ may be written in the form

$$i\lambda q_1 + \int \bigl(e^{i\lambda x} - 1\bigr)\, dG(x),$$

where

$$q_1 = q - \int x^{-1}\, d\Psi(x).$$

(3) Consider now the general case, but under the condition that $\Psi(\{0\}) = 0$. As we know, the function ψ can be approximated for small Δ by expressions of the form (we put $\Delta_k = [(k-1)\Delta, k\Delta)$)

$$i\lambda q + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[-\frac{i\lambda}{k\Delta}\Psi(\Delta_k) + \bigl(e^{i\lambda k\Delta} - 1\bigr)\frac{1+(k\Delta)^2}{(k\Delta)^2}\Psi(\Delta_k) \right],$$

which corresponds to the sum of Poisson processes with jumps of sizes $k\Delta$ of the respective intensities

$$\frac{1+(k\Delta)^2}{(k\Delta)^2}\Psi(\Delta_k).$$

If, say,

$$\int_{+0}^{\infty} \frac{d\Psi(x)}{x^2} = \infty,$$

then for any $\varepsilon > 0$ the total intensity of these processes with jumps from the interval $(0, \varepsilon)$ will increase to infinity as $\Delta \to 0$. This means that, with probability 1, on any time interval δ there will be at least one jump of size smaller than any given $\varepsilon > 0$, so that the trajectories of $\xi(t)$ will be everywhere discontinuous. To "compensate" these jumps, a drift of size $\Psi(\Delta_k)/k\Delta$ is added, the "total value" of such drifts being possibly unbounded (if $\int_{+0}^{\infty} x^{-1} d\Psi(x) = \infty$).

(4) For stable processes (see Sect. 8.8) the functions $\Psi(x)$ have power "branches", smooth on the half-axes, possessing the property $c_1\Psi'(x) = \Psi'(c_2 x)$ for appropriate c_1 and c_2.