Chapter 12
Random Walks and Factorisation Identities

Abstract In this chapter, several remarkable and rather useful relations establishing interconnections between different characteristics of random walks (the so-called boundary functionals) are derived, and the arising problems are related to the simplest boundary problems of Complex Analysis. Section 12.1 introduces the concept of factorisation identity and derives two fundamental identities of that kind. Some consequences of these identities, including the trichotomy theorem on the oscillatory behaviour of random walks and a one-sided version of the Strong Law of Large Numbers are presented in Sect. 12.2. Pollaczek–Spitzer's identity and an identity for the global maximum of the random walk are derived in Sect. 12.3, followed by illustrating these results by examples from the ruin theory and the theory of queueing systems in Sect. 12.4. Sections 12.5 and 12.6 are devoted to studying the cases where factorisation components can be obtained in explicit form and so closed form expressions are available for the distributions of a number of important boundary functionals. Sections 12.7 and 12.8 employ factorisation identities to derive the asymptotic properties of the distribution of the excess of a random walk of a high level and that of the global maximum of the walk, and also to analyse the distribution of the first passage time.

In the present chapter we derive several remarkable and rather useful relations establishing interconnections between different characteristics of random walks (the so-called boundary functionals) and also relate the arising problems with the simplest boundary problems of complex analysis.

12.1 Factorisation Identities

12.1.1 Factorisation

On the plane of a complex variable λ, denote by Π the real axis $\operatorname{Im}\lambda = 0$ and by Π_+ (Π_-) the half-plane $\operatorname{Im}\lambda > 0$ ($\operatorname{Im}\lambda < 0$). Let $\mathfrak{f}(\lambda)$ be a continuous function defined on Π.

Definition 12.1.1 If there exists a representation

$$\mathfrak{f}(\lambda) = \mathfrak{f}_+(\lambda)\mathfrak{f}_-(\lambda), \qquad \lambda \in \Pi, \tag{12.1.1}$$

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where $\mathfrak{f}_\pm$ are analytic in the domains $\Pi_\pm$ and continuous on $\Pi_\pm \cup \Pi$, then we will say that the function $\mathfrak{f}$ *allows factorisation*. The functions $\mathfrak{f}_\pm$ are called *factorisation components* (*positive* and *negative*, respectively).

Further, denote by $\mathcal{K}$ the class of functions $\mathfrak{f}$ defined on Π that are continuous and such that

$$\sup_{\lambda\in\Pi} \left|\mathfrak{f}(\lambda)\right| < \infty, \qquad \inf_{\lambda\in\Pi} \left|\mathfrak{f}(\lambda)\right| > 0. \tag{12.1.2}$$

Similarly we define the classes $\mathcal{K}_\pm$ of functions analytic in $\Pi_\pm$ and continuous on $\Pi_\pm \cup \Pi$, such that

$$\sup_{\lambda\in\Pi_\pm} \left|\mathfrak{f}_\pm(\lambda)\right| < \infty, \qquad \inf_{\lambda\in\Pi_\pm} \left|\mathfrak{f}_\pm(\lambda)\right| > 0. \tag{12.1.3}$$

Definition 12.1.2 If, for an $\mathfrak{f} \in \mathcal{K}$, there exists a representation (12.1.1), where $\mathfrak{f}_\pm \in \mathcal{K}_\pm$, then we will say that the function $\mathfrak{f}$ allows *canonical factorisation*.

Representations of the form

$$\mathfrak{f}(\lambda) = \mathfrak{f}_+(\lambda)\mathfrak{f}_-(\lambda) f_0, \qquad \mathfrak{f}(\lambda) = \frac{\mathfrak{f}_+(\lambda) f_0}{\mathfrak{f}_-(\lambda)}, \quad \lambda \in \Pi,$$

where $f_0 = \mathrm{const}$ and $\mathfrak{f}_\pm \in \mathcal{K}_\pm$, are also called canonical factorisations.

Lemma 12.1.1 *The components $\mathfrak{f}_\pm$ of a canonical factorisation of a function $\mathfrak{f} \in \mathcal{K}$ are defined uniquely up to a constant factor.*

Proof Together with the canonical factorisation (12.1.1), let there exist another canonical factorisation

$$\mathfrak{f}(\lambda) = \mathfrak{g}_+(\lambda)\mathfrak{g}_-(\lambda), \quad \lambda \in \Pi.$$

Then

$$\mathfrak{f}_+(\lambda)\mathfrak{f}_-(\lambda) = \mathfrak{g}_+(\lambda)\mathfrak{g}_-(\lambda), \quad \lambda \in \Pi,$$

and, by (12.1.2), we can divide both sides of the inequality by $\mathfrak{g}_+(\lambda)\mathfrak{f}_-(\lambda)$. We get

$$\frac{\mathfrak{f}_+(\lambda)}{\mathfrak{g}_+(\lambda)} = \frac{\mathfrak{g}_-(\lambda)}{\mathfrak{f}_-(\lambda)},$$

where, by virtue of (12.1.2), the function $\frac{\mathfrak{f}_+(\lambda)}{\mathfrak{g}_+(\lambda)}$ $\left(\frac{\mathfrak{g}_-(\lambda)}{\mathfrak{f}_-(\lambda)}\right)$ belongs to the class $\mathcal{K}_+$ ($\mathcal{K}_-$). We have obtained that the function $\frac{\mathfrak{f}_+(\lambda)}{\mathfrak{g}_+(\lambda)}$, analytical in Π_+, can be analytically continued over the line Π onto the half-plane Π_- (to the function $\frac{\mathfrak{g}_-(\lambda)}{\mathfrak{f}_-(\lambda)}$). After such a continuation, in view of (12.1.3), this function remains bounded on the whole complex plane. By Liouville's theorem, bounded entire functions must be constant, i.e. there exists a constant c, such that, on the whole plane

$$\frac{\mathfrak{f}_+(\lambda)}{\mathfrak{g}_+(\lambda)} = \frac{\mathfrak{g}_-(\lambda)}{\mathfrak{f}_-(\lambda)} = c,$$

holds, so $\mathfrak{f}_+(\lambda) = c\mathfrak{g}_+(\lambda)$, $\mathfrak{f}_-(\lambda) = c^{-1}\mathfrak{g}_-(\lambda)$. The lemma is proved. □

The factorisation problem consists in finding conditions under which a given function $\mathfrak{f}$ admits a factorisation, and in finding the components of the factorisation. This problem has a number of important applications to solving integral equations and is a version of the well-known Cauchy–Riemann boundary-value problem in complex function theory. We will see later that factorisation is also an important tool for studying the so-called boundary problems in probability theory.

12.1.2 The Canonical Factorisation of the Function $\mathfrak{f}_z(\lambda) = 1 - z\varphi(\lambda)$

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space on which a sequence $\{\xi_k\}_{k=1}^{\infty}$ of independent identically distributed ($\xi_k \stackrel{d}{=} \xi$) random variables is given. Put, as before, $S_n := \sum_{k=1}^{n} \xi_k$ and $S_0 = 0$. The sequence $\{S_k\}_{k=0}^{\infty}$ forms a *random walk*.

First of all, note that the function

$$\mathfrak{f}_z(\lambda) := 1 - z\varphi(\lambda), \qquad \varphi(\lambda) := \mathbf{E}e^{i\lambda\xi}, \quad \lambda \in \Pi,$$

belongs to $\mathcal{K}$, for all z with $|z| < 1$ (here z is a complex-valued parameter). This follows from the inequalities $|\varphi(\lambda)| \le 1$ for $\lambda \in \Pi$ and $|z\varphi(\lambda)| < |z| < 1$.

Theorem 12.1.1 (The first factorisation identity) *For* $|z| < 1$, *the function* $\mathfrak{f}_z(\lambda)$ *admits the canonical factorisation*

$$\mathfrak{f}_z(\lambda) = \mathfrak{f}_{z+}(\lambda) C(z) \mathfrak{f}_{z-}(\lambda), \quad \lambda \in \Pi, \tag{12.1.4}$$

where

$$\begin{aligned} \mathfrak{f}_{+}(\lambda) &= \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}\left(e^{i\lambda S_k};\ S_k > 0\right) \right\} \in \mathcal{K}_+, \\ \mathfrak{f}_{z-}(\lambda) &= \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}\left(e^{i\lambda S_k};\ S_k < 0\right) \right\} \in \mathcal{K}_-, \\ C(z) &= \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{P}(S_k = 0) \right\}. \end{aligned} \tag{12.1.5}$$

Proof Since $|z| < 1$, $\ln(1 - z\varphi(\lambda))$ exists, understood in the principal value sense. The following equalities give the desired decomposition:

$$\begin{aligned} \mathfrak{f}_z(\lambda) = e^{\ln(1 - z\varphi(\lambda))} &= \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k \varphi^k(\lambda)}{k} \right\} = \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}e^{i\lambda S_k} \right\} \\ &= \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}\left(e^{i\lambda S_k};\ S_k > 0\right) \right\} \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{P}(S_k = 0) \right\} \\ &\quad \times \exp\left\{ -\sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}\left(e^{i\lambda S_k};\ S_k < 0\right) \right\}. \end{aligned}$$

Show that $\mathfrak{f}_{z+}(\lambda) \in \mathcal{K}_+$. Indeed, the function $\mathbf{E}(e^{i\lambda S_k};\ S_k > 0)$, for every k and $\lambda \in \Pi_+ \cup \Pi$, does not exceed 1 in the absolute value, is analytic in Π_+, and is continuous on $\Pi_+ \cup \Pi$. Analyticity follows from the differentiability of this function at any point $\lambda \in \Pi_+$ (see also Property 6 of ch.f.s in Sect. 7.1). The function $\ln \mathfrak{f}_{z+}(\lambda)$ is a uniformly converging series of functions analytic in Π_+, and hence possesses the same properties together with the function $\mathfrak{f}_{z+}(\lambda)$. The same can be said about the continuity on $\Pi \cup \Pi_+$.

That $\mathfrak{f}_{z-}(\lambda) \in \mathcal{K}_-$ is established in a similar way. The theorem is proved. □

12.1.3 The Second Factorisation Identity

The second factorisation identity is associated with the so-called boundary functionals of the random walk $\{S_k\}$. On the main probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ we define, together with $\{\xi_k\}$, the random variable

$$\eta_+^0 := \min\{k \ge 1;\ S_k \ge 0\}.$$

This is the *first-passage time to zero level*. For the elementary events such that all $S_k < 0$, $k \ge 1$, we put $\eta_+^0 := \infty$. Like the random variable $\eta(0)$ in Sect. 10.1, the variable η_+^0 is a Markov time.

The random variable $\chi_+^0 := S_{\eta_+^0}$ is called the *first nonnegative sum*. It is defined on the set $\{\eta_+^0 < \infty\}$ only.

The *first passing time of zero from the right*

$$\eta_-^0 := \min\{k \ge 1;\ S_k \le 0\}$$

possesses quite similar properties, and so does the *first nonpositive sum* $\chi_-^0 := S_{\eta_-^0}$.

Studying the properties of the introduced random variables, which are called boundary functionals of the random walk $\{S_k\}$, is of significant independent interest. For instance, the variable η_+^0 is a stopping time, and understanding its nature is essential for studying stopping times in many more complex problems (see e.g. the problems of the renewal theory in Chap. 10, the problems of statistical control described in Sect. 4.4 and so on). Moreover, the variables η_+^0 and χ_+^0 will be needed to describe the extrema

$$\zeta := \sup(S_1, S_2, \ldots) \quad \text{and} \quad \gamma := \inf(S_1, S_2, \ldots),$$

which are also termed boundary functionals and play an important role in the problems of mathematical statistics, queueing theory (see Sect. 12.4), etc.

Put, as before, $\varphi(\lambda) := \varphi_\xi(\lambda) = \mathbf{E}e^{i\lambda\xi}$.

Theorem 12.1.2 (The second factorisation identity) *For the ch.f. of the joint distributions of the introduced random variables, for* $|z| < 1$ *and* $\operatorname{Im}\lambda = 0$, *the canonical factorisation*

$$\begin{aligned}\mathfrak{f}_z(\lambda) &:= 1 - z\varphi(\lambda)\\ &= \big[1 - \mathbf{E}\big(e^{i\lambda\chi_+^0} z^{\eta_+^0};\ \eta_+^0 < \infty\big)\big] D^{-1}(z)\big[1 - \mathbf{E}\big(e^{i\lambda\chi_-^0} z^{\eta_-^0};\ \eta_-^0 < \infty\big)\big],\end{aligned}$$

of $\mathfrak{f}_z(\lambda)$ *holds true*, *where*

$$D(z) := 1 - \mathbf{E}\big(z^{\eta_+^0};\ \chi_+^0 = 0,\ \eta_+^0 < \infty\big) = 1 - \mathbf{E}\big(z^{\eta_-^0};\ \chi_-^0 = 0,\ \eta_-^0 < \infty\big).$$

Proof Set $\zeta_n := \max\{S_1, \dots, S_n\}$. We have

$$\begin{aligned}\varphi^n(\lambda) = \mathbf{E}e^{i\lambda S_n} &= \sum_{k=1}^n \mathbf{E}\big(e^{i\lambda S_n};\ \eta_+^0 = k\big) + \mathbf{E}\big(e^{i\lambda S_n};\ \zeta_n < 0\big)\\ &= \sum_{k=1}^n \mathbf{E}\big(e^{i\lambda(S_n - S_k)} e^{i\lambda S_k}\mathrm{I}\big(\eta_+^0 = k\big)\big) + M_n, \qquad (12.1.6)\end{aligned}$$

where $M_n = \mathbf{E}(e^{i\lambda S_n};\ \zeta_n < 0)$ and $\mathrm{I}(A)$ is the indicator of the event A. For each fixed k, the random variables $S_n - S_k$ and $S_k\mathrm{I}(\eta_+^0 = k) = \chi_+^0\mathrm{I}(\eta_+^0 = k)$ are independent. Hence,

$$\varphi^n(\lambda) = \sum_{k=1}^n \varphi^{n-k}(\lambda)\mathbf{E}\big(e^{i\lambda\chi_+^0};\ \eta_+^0 = k\big) + M_n.$$

Now multiply both sides by z^n, $n = 0, 1, \dots,$ and then sum up over n. We will use the convention that, for $n = 0$,

$$\sum_{k=1}^n = 0, \qquad M_n = 1.$$

For the convolution of two sequences $c_n = \sum_{k=1}^n a_k b_{n-k}$, we have

$$\sum_{n=0}^\infty c_n z^n = \sum_{n=1}^\infty a_n z^n \sum_{n=0}^\infty b_n z^n,$$

provided that the series in this equality converges absolutely. Since $|z| < 1$ and $|\varphi(\lambda)| \le 1$ for $\mathrm{Im}\,\lambda = 0$, one has

$$\begin{aligned}\sum_{n=0}^\infty z^n\varphi^n(\lambda) = \frac{1}{1 - z\varphi(\lambda)} &= \sum_{n=0}^\infty z^n \sum_{k=1}^n \varphi^{n-k}(\lambda)\mathbf{E}\big(e^{i\lambda\chi_+^0};\ \eta_+^0 = k\big) + \sum_{n=0}^\infty z^n M_n\\ &= \sum_{k=1}^\infty z^k\mathbf{E}\big(e^{i\lambda\chi_+^0};\ \eta_+^0 = k\big)\sum_{n=0}^\infty z^n\varphi^n(\lambda) + \sum_{n=0}^\infty z^n M_n\\ &= \frac{1}{1 - z\varphi(\lambda)}\mathbf{E}\big(e^{i\lambda\chi_+^0} z^{\eta_+^0};\ \eta_+^0 < \infty\big) + \sum_{n=0}^\infty z^n M_n,\end{aligned}$$

or, which is the same,

$$\mathfrak{f}_z(\lambda) = 1 - z\varphi(\lambda) = \frac{1 - \mathbf{E}(e^{i\lambda\chi_+^0} z^{\eta_+^0};\ \eta_+^0 < \infty)}{\sum_{n=0}^\infty z^n\mathbf{E}(e^{i\lambda S_n};\ \zeta_n < 0)} = \frac{\mathfrak{a}_{z+}(\lambda)}{\mathfrak{a}_{z-}(\lambda)}, \qquad (12.1.7)$$

where $\mathfrak{a}_{z\pm}(\lambda)$ denote the numerator and denominator of the ratio obtained for $\mathfrak{f}_z(\lambda)$.

It is easy to see that, if we put

$$\gamma_n := \min(S_1, \dots, S_n)$$

then, repeating the above arguments, we will arrive at the equality

$$\mathfrak{f}_z(\lambda) = \frac{1 - \mathbf{E}(e^{i\lambda\chi_-^0} z^{\eta_-^0};\ \eta_-^0 < \infty)}{\sum_{n=0}^{\infty} z^n \mathbf{E}(e^{i\lambda S_n};\ \gamma_n > 0)} = \frac{\mathfrak{b}_{z-}(\lambda)}{\mathfrak{b}_{z+}(\lambda)}, \tag{12.1.8}$$

where, similarly to the above, $\mathfrak{b}_{z\mp}(\lambda)$, respectively, denote the numerator and denominator in relation (12.1.8).

Now we show that $\mathfrak{a}_{z\pm}(\lambda) \in \mathcal{K}$ and $\mathfrak{b}_{z\pm}(\lambda) \in \mathcal{K}$ for $|z| < 1$. Indeed, for $|z| < 1$ and $\operatorname{Im} \lambda = 0$,

$$\left|\mathbf{E}\big(e^{i\lambda\chi_+^0} z^{\eta_+^0};\ \eta_+^0 < \infty\big)\right| \le \mathbf{E}\big(|z|^{\eta_+^0};\ \eta_+^0 < \infty\big) < 1$$

and therefore

$$\sup_{\lambda \in \varPi} \left|\mathfrak{a}_{z+}(\lambda)\right| < \infty, \qquad \inf_{\lambda \in \varPi} \left|\mathfrak{a}_{z+}(\lambda)\right| > 0.$$

Since $\mathfrak{f}_z(\lambda) \in \mathcal{K}$, this also implies that $\mathfrak{a}_{z-}(\lambda) \in \mathcal{K}$. In the same way we obtain that $\mathfrak{b}_{z\mp}(\lambda) \in \mathcal{K}$. By equating the right-hand sides of (12.1.7) and (12.1.8) and multiplying them by $\mathfrak{a}_{z-}(\lambda)\mathfrak{b}_{z+}(\lambda)$, we get

$$\mathfrak{a}_{z+}(\lambda)\mathfrak{b}_{z+}(\lambda) = \mathfrak{a}_{z-}(\lambda)\mathfrak{b}_{z-}(\lambda), \quad \lambda \in \varPi. \tag{12.1.9}$$

Further, the functions $\mathfrak{a}_{z+}(\lambda)$ and $\mathfrak{b}_{z+}(\lambda)$ are bounded and analytic in $\varPi_+$ for the same reasons as the function $\mathfrak{f}_{z+}(\lambda)$ (see the proof of Theorem 12.1.1). Similarly, $\mathfrak{a}_{z-}(\lambda)$ and $\mathfrak{b}_{z-}(\lambda)$ are bounded and analytic in $\varPi_-$. We obtain that the function $\mathfrak{a}_{z+}(\lambda)\mathfrak{b}_{z+}(\lambda)$ is bounded and analytic in $\varPi_+$ and, by (12.1.9), has an entire bounded analytic continuation over the boundary $\varPi$ to the whole complex plane. This means that this function necessarily equals a constant c, and $\mathfrak{b}_{z+}(\lambda) = c\mathfrak{a}_{z+}^{-1}(\lambda) \in \mathcal{K}_+$, $\mathfrak{a}_{z-}(\lambda) = cb_{z-}^{-1}(\lambda) \in \mathcal{K}_-$, so relations (12.1.7) and (12.1.8) deliver a canonical factorisation of $\mathfrak{f}_z(\lambda)$.

Further, $e^{i\lambda x} \to 0$ as $\operatorname{Im} \lambda \to -\infty$, $x < 0$, and therefore

$$\mathfrak{b}_{z-}(-i\infty) = 1 - \mathbf{E}\big(z^{\eta_-^0};\ \chi_-^0 = 0,\ \eta_-^0 < \infty\big), \qquad \mathfrak{a}_{z-}(-i\infty) = 1,$$

$$\mathfrak{a}_{z-}(\lambda)\mathfrak{b}_{z-}(\lambda) = \mathfrak{a}_{z-}(-i\infty)\mathfrak{b}_{z-}(-i\infty) = 1 - \mathbf{E}\big(z^{\eta_-^0};\ \chi_-^0 = 0,\ \eta_-^0 < \infty\big) = D(z).$$

Substituting into (12.1.7) the value $\mathfrak{a}_{z-}(\lambda) = D(z)/\mathfrak{b}_{z-}(\lambda)$ derived from this equality, we obtain the assertion of the theorem. The second relation for $D(z)$ follows from the equality $D(z) = \mathfrak{a}_{z+}(i\infty)\mathfrak{b}_{z+}(i\infty)$. The theorem is proved. □

In the proof of Theorem 12.1.2 we used, in formula (12.1.6), a decomposition of $\mathbf{E}e^{i\lambda S_n}$ into summands corresponding to the disjoint events

$$\left\{\bigcup_{k=1}^{n} \{\eta_+^0 = k\}\right\} = \{\zeta_n \ge 0\} \quad \text{and} \quad \{\zeta_n < 0\}.$$

But the scheme of the proof will still work if we consider the partition of Ω into the events $\{\zeta_n > 0\}$ and $\{\zeta_n \le 0\}$. In order to do this, we introduce the random variables

$$\eta_+ := \min\{k : S_k > 0\}$$

($\eta_+ = \infty$ if $\zeta \le 0$; note that $\eta_+ = \eta(0)$ in the notation of Sect. 10.1),

$$\begin{aligned} \chi_+ &:= S_{\eta_+}, \\ \eta_- &:= \min\{k : S_k < 0\} \quad (\eta_- = \infty \text{ if } \gamma \ge 0), \\ \chi_- &:= S_{\eta_-}. \end{aligned}$$

The variable η_+ (η_-) is called the *time of the first positive* (*negative*) *sum* χ_+ (χ_-). Now we can write, together with equalities (12.1.7) and (12.1.8), the relations

$$\begin{aligned} \mathfrak{f}_z(\lambda) = 1 - z\varphi(\lambda) &= \frac{1 - \mathbf{E}(e^{i\lambda\chi_+} z^{\eta_+};\ \eta_+ < \infty)}{\sum_{n=0}^{\infty} z^n \mathbf{E}(e^{i\lambda S_n};\ \zeta_n \le 0)} \\ &= \frac{1 - \mathbf{E}(e^{i\lambda\chi_-} z^{\eta_-};\ \eta_- < \infty)}{\sum_{n=0}^{\infty} z^n \mathbf{E}(e^{i\lambda S_n};\ \gamma_n \ge 0)}. \end{aligned} \tag{12.1.10}$$

Combining these relations with (12.1.7) and (12.1.8), we will use below the same argument as above to prove the following assertion.

Theorem 12.1.3

$$\begin{aligned} 1 - \mathbf{E}\left(e^{i\lambda\chi_+^0} z^{\eta_+^0};\ \eta_+^0 < \infty\right) &= D(z)\left[1 - \mathbf{E}\left(e^{i\lambda\chi_+} z^{\eta_+};\ \eta_+ < \infty\right)\right], \\ 1 - \mathbf{E}\left(e^{i\lambda\chi_-^0} z^{\eta_-^0};\ \eta_-^0 < \infty\right) &= D(z)\left[1 - \mathbf{E}\left(e^{i\lambda\chi_-} z^{\eta_-};\ \eta_- < \infty\right)\right]. \end{aligned} \tag{12.1.11}$$

Here the function $D(z)$ defined in Theorem 12.1.2 also satisfies the relations

$$D^{-1}(z) = \sum_{n=0}^{\infty} z^n \mathbf{P}(S_n = 0,\ \zeta_n \le 0) = \sum_{n=0}^{\infty} z^n \mathbf{P}(S_n = 0, \gamma_n \ge 0). \tag{12.1.12}$$

Clearly, from Theorem 12.1.3 one can obtain some other versions of the factorisation identity. For instance, one has

$$\mathfrak{f}_z(\lambda) = \left[1 - \mathbf{E}\left(e^{i\lambda\chi_+} z^{\eta_+};\ \eta_+ < \infty\right)\right]\left[1 - \mathbf{E}\left(e^{i\lambda\chi_-^0} z^{\eta_-^0};\ \eta_-^0 < \infty\right)\right]. \tag{12.1.13}$$

Representations (12.1.12) for $D(z)$ imply, in particular, that

$$\mathbf{P}(S_n = 0, \zeta_n \le 0) = \mathbf{P}(S_n = 0, \gamma_n \ge 0)$$

and that $D(z) \equiv 1$ if $\mathbf{P}(S_n = 0) = 0$ for all $n \ge 1$.

Proof of Theorem 12.1.3 Let us derive the first relation in (12.1.11). Comparing (12.1.8) with (12.1.10) we find, as above, that

$$\left[1 - \mathbf{E}\left(e^{i\lambda\chi_+} z^{\eta_+};\ \eta_+ < \infty\right)\right]\mathfrak{b}_{z+}(\lambda) = \text{const} = 1, \tag{12.1.14}$$

since the product equals 1 for $\lambda = i\infty$. Therefore we obtain (12.1.13) by virtue of (12.1.8). It remains to compare (12.1.13) with the identity of Theorem 12.1.2.

Expressions (12.1.12) for $D(z)$ follow if we recall (see (12.1.8) and (12.1.10)) that the left-hand side of (12.1.14) equals

$$\left[\sum_{n=0}^{\infty} z^n \mathbf{E}\left(e^{i\lambda S_n};\ \zeta_n \le 0\right)\right]\left[1-\mathbf{E}\left(e^{i\lambda\chi_-^0} z^{\eta_-^0};\ \eta_-^0<\infty\right)\right].$$

Since this product also equals 1, letting $\lambda = -i\infty$ here and in the second identity of (12.1.11) we get the first equality in (12.1.12). The second equality is proved in a similar way. □

Remark 12.1.1 It is important to note that Theorems 12.1.2 and 12.1.3, as well as proving the existence of the identities, also provide a means of finding the characteristic function of the joint distribution of χ and η. That is, if we manage somehow to get a representation for $\mathfrak{f}_z(\lambda) = 1 - z\varphi(\lambda)$ of the form $\mathfrak{h}_{z+}(\lambda)\mathfrak{h}_{z-}(\lambda)$, where $\mathfrak{h}_{z\pm}(\lambda) \in \mathcal{K}_\pm$, then by uniqueness of the canonical factorisation we can, for instance, claim that, up to a constant factor, the function $1 - \mathbf{E}(e^{i\lambda\chi_+} z^{\eta_+};\ \eta_+)$ coincides with $\mathfrak{h}_{z+}(\lambda)$. For examples of how such arguments can be used, see Sects. 12.5 and 12.6.

12.2 Some Consequences of Theorems 12.1.1–12.1.3

12.2.1 Direct Consequences

Theorems 12.1.1–12.1.3 (and also their modifications of the form (12.1.13)) and the uniqueness of the canonical factorisation (see Lemma 12.1.1) directly imply the next result.

Corollary 12.2.1 *In the notation of Theorems* 12.1.1 *and* 12.1.2 *one has the following equalities.*

$$\begin{aligned} 1-\mathbf{E}\bigl(e^{i\lambda\chi_+} z^{\eta_+};\ \eta_+<\infty\bigr) &= \mathfrak{f}_{z+}(\lambda); \\ D(z) &= C(z); \\ 1-\mathbf{E}\bigl(e^{i\lambda\chi_-} z^{\eta_-};\ \eta_-<\infty\bigr) &= \mathfrak{f}_{z-}(\lambda). \end{aligned}$$

Now we will obtain, as corollaries of Theorems 12.1.1–12.1.3, some further identities in which the parameter z is fixed and equal to 1.

Corollary 12.2.2 *Letting* $z \to 1$ *in* (12.1.13) *we obtain*

$$\mathfrak{f}_1(\lambda) := 1-\varphi(\lambda) = \bigl[1-\mathbf{E}\bigl(e^{i\lambda\chi_+};\ \eta_+<\infty\bigr)\bigr]\bigl[1-\mathbf{E}\bigl(e^{i\lambda\chi_-^0};\ \eta_-^0<\infty\bigr)\bigr]. \quad (12.2.1)$$

It is obvious that one can similarly write other identities of such type corresponding to the identities that can be derived from Theorems 12.1.1–12.1.3.

Clearly, identity (12.2.1) delivers a factorisation of the function $\mathfrak{f}_1(\lambda) = 1-\varphi(\lambda)$, but this factorisation is not canonical since $\mathfrak{f}_1(0) = 0$ and $\mathfrak{f}_1(\lambda) \notin \mathcal{K}$.

Corollary 12.2.3 *If there exists* $\mathbf{E}\xi = a < 0$ *then* $\mathbf{P}(\eta_-^0 < \infty) = 1$, $\mathbf{E}\chi_-^0$ *exists*, *and* $\mathbf{P}(\zeta \le 0) = a/\mathbf{E}\chi_-^0 > 0$.

Proof The first relation follows from the law of large numbers, because

$$\mathbf{P}\big(\eta_-^0 > n\big) < \mathbf{P}(S_n > 0) \to 0$$

as $n \to \infty$. Therefore, in the case under consideration, one has

$$\mathbf{E}\big(e^{i\lambda\chi_-^0}; \eta_-^0 < \infty\big) = \mathbf{E}e^{i\lambda\chi_-^0}.$$

The existence of $\mathbf{E}\chi_-^0$ follows from Wald's identity $\mathbf{E}\chi_-^0 = a\mathbf{E}\eta_-^0$ and the theorems of Chap. 10, which imply that $\mathbf{E}\eta_-^0 \le \mathbf{E}\eta_- < \infty$, since $\mathbf{E}\eta_-$ is the value of the corresponding renewal function at 0.

Finally, dividing both sides of the identity in Corollary 12.2.2 by λ and taking the limit as $\lambda \to 0$, we obtain

$$a = \big(1 - \mathbf{P}(\eta_+ < \infty)\big)\mathbf{E}\chi_-^0 = \mathbf{P}(\zeta \le 0)\mathbf{E}\chi_-^0. \qquad \square$$

It is interesting to note that, as a consequence of this assertion, we can obtain the *strong law of large numbers*. Indeed, since $\{\zeta < \infty\}$ is a tail event and $\mathbf{P}(\zeta < \infty) \ge \mathbf{P}(\zeta \le 0)$, Corollary 12.2.3 implies that $\mathbf{P}(\zeta < \infty) = 1$ for $a < 0$. This means that the assertion of Theorem 10.5.3 holds, and it was this assertion that the strong law of large numbers was derived from.

Based on factorisation identities, we will obtain below a generalisation of this law.

In the remaining part of this chapter, to avoid trivial complications, we will be assuming that ξ takes, with positive probability, both positive and negative values.

Corollary 12.2.4 *If* $a = \mathbf{E}\xi = 0$ *then* $\mathbf{P}(\eta_+ < \infty) = \mathbf{P}(\eta_-^0 < \infty) = 1$, *so that*

$$1 - \varphi(\lambda) = \big(1 - \mathbf{E}e^{i\lambda\chi_+}\big)\big(1 - \mathbf{E}\,e^{i\lambda\chi_-^0}\big). \qquad (12.2.2)$$

If, moreover, $\mathbf{E}\xi^2 = \sigma^2 < \infty$ *then there exist* $\mathbf{E}\chi_+$ *and* $\mathbf{E}\chi_-^0$, *and*

$$\mathbf{E}\chi_+\mathbf{E}\chi_-^0 = -\frac{\sigma^2}{2}.$$

Proof Consider the sequence $\widetilde{\xi}_k = \xi_k - \varepsilon$, $\varepsilon > 0$. Denoting by $\widetilde{\zeta}$, $\widetilde{\chi}_-^0$ and $\widetilde{a}$ the corresponding characteristics for the newly introduced sequence, we obtain by Corollary 12.2.3 that

$$\mathbf{P}(\zeta \le 0) < \mathbf{P}(\widetilde{\zeta} \le 0) = \frac{\widetilde{a}}{\mathbf{E}\widetilde{\chi}_-^0} = -\frac{\varepsilon}{\widetilde{\chi}_-^0},$$

where

$$\mathbf{E}\widetilde{\chi}_-^0 \le \mathbf{E}(\widetilde{\xi}_1; \widetilde{\xi}_1 \le 0) = \mathbf{E}(\xi - \varepsilon; \xi \le \varepsilon) \le \mathbf{E}(\xi; \xi \le 0) < 0.$$

So we can make the probability $\mathbf{P}(\zeta \le 0)$ arbitrarily small by choosing an appropriate ε, and thus $\mathbf{P}(\zeta \le 0) = \mathbf{P}(\eta_+ = \infty) = 0$. Similarly, we find that $\mathbf{P}(\gamma \ge 0) = 0$ and hence

$$\mathbf{P}\big(\eta_-^0 = \infty\big) \le \mathbf{P}(\eta_- = \infty) = \mathbf{P}(\gamma \ge 0) = 0.$$

The obtained relations and Corollary 12.2.2 yield identity (12.2.2).

In order to prove the second assertion of the corollary, divide both sides of identity (12.2.2) by $\lambda^2 = -(i\lambda)^2$ and let $\lambda \in \Pi$ tend to zero. Then the limit of the left-hand side will be equal to $\sigma^2/2$ (see (7.1.1)), whereas that of the right-hand side will be equal to $-\mathbf{E}\chi_+\mathbf{E}\chi_-^0$, where $\mathbf{E}\chi_+ > 0$, $|\mathbf{E}\chi_-^0| > 0$. The corollary is proved. □

Corollary 12.2.5

1. *We always have* $\sum \frac{\mathbf{P}(S_k=0)}{k} < \infty$.
2. *The following three conditions are equivalent*:

   (a) $\mathbf{P}(\zeta < \infty) = 1$;
   (b) $\mathbf{P}(\zeta \le 0) = \mathbf{P}(\eta_+ = \infty) > 0$;
   (c) $\sum_{k=1}^\infty \frac{\mathbf{P}(S_k>0)}{k} < \infty$ *or* $\sum_{k=1}^\infty \frac{\mathbf{P}(S_k\ge 0)}{k} < \infty$.

Proof To obtain the first assertion, one should let $z \to 1$ in the second equality in Corollary 12.2.1 and recall that

$$D(1) = 1 - \mathbf{P}\big(\chi_+^0 = 0,\ \eta_+^0 < \infty\big) > \mathbf{P}(\xi > 0) > 0.$$

The equivalence of (b) and (c) follows from the equality

$$1 - \mathbf{P}(\eta_+ < \infty) = \mathbf{P}(\zeta \le 0) = \exp\left\{-\sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k > 0}{k}\right\},$$

which is derived by putting $\lambda = 0$ and letting $z \to 1$ in the first identity of Corollary 12.2.1.

Now we will establish the equivalence of (b) and (c). If $\mathbf{P}(\zeta \le 0) > 0$ then $\mathbf{P}(\zeta < \infty) > 0$ and hence $\mathbf{P}(\zeta < \infty) = 1$, since $\{\zeta < \infty\}$ is a tail event. Conversely, let ζ be a proper random variable. Choose an N such that $\mathbf{P}(\zeta \le N) > 0$, and $b > 0$ such that $k = N/b$ is an integer and $\mathbf{P}(\xi < -b) > 0$. Then

$$\{\zeta \le 0\} \supset \Big\{\xi_1 < -b, \dots, \xi_k < -b,\ \sup_{j\ge 1}(-bk + \xi_{k+1} + \cdots + \xi_{k+j}) \le 0\Big\}.$$

Since the sequence $\xi_{k+1}, \xi_{k+2}, \dots$ is distributed identically to $\xi_1, \xi_2, \dots$, one has

$$\mathbf{P}(\zeta \le 0) \ge \big[\mathbf{P}(\xi < -b)\big]^k \mathbf{P}(\zeta \le bk) > 0.$$ □

Corollary 12.2.6

1. $\mathbf{P}(\zeta < \infty, \gamma > -\infty) = 0$.

2. *If there exists* $\mathbf{E}\xi = a < 0$ *then*

$$\mathbf{P}(\eta_+ < \infty) < 1, \qquad \mathbf{P}(\zeta < \infty, \gamma = -\infty) = 1,$$
$$\left(\sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k > 0)}{k} < \infty, \sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k < 0)}{k} = \infty\right).$$

3. *If there exists* $\mathbf{E}\xi = a = 0$ *then*

$$\mathbf{P}(\zeta = \infty, \gamma = -\infty) = 1,$$
$$\left(\sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k > 0)}{k} = \infty, \sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k < 0)}{k} = \infty\right).$$

Here we do not consider the case $a > 0$ since it is "symmetric" to the case $a < 0$.

Proof The first assertion follows from the fact that at least one of the two series $\sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k<0)}{k}$ and $\sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k\ge 0)}{k}$ diverges. Therefore, by Corollary 12.2.5 either $\mathbf{P}(\gamma = -\infty) = 1$ or $\mathbf{P}(\zeta = \infty) = 1$.

The second and third assertions follow from Corollaries 12.2.3–12.2.5 in an obvious way. □

12.2.2 A Generalisation of the Strong Law of Large Numbers

The above mentioned generalisation of the strong law of large numbers consists of the following.

Theorem 12.2.1 (The one-sided law of large numbers) *Convergence of the series*

$$\sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k > \varepsilon k)}{k}$$

for every $\varepsilon > 0$ *is a necessary and sufficient condition for*

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{S_n}{n} \le 0\right) = 1. \tag{12.2.3}$$

Proof *Sufficiency*. If the series converges then by Corollary 12.2.5 we have

$$\mathbf{P}\Big(\sup_k \{S_k - \varepsilon k\} < \infty\Big) = 1.$$

Hence $\{\varepsilon n\}$ is an upper sequence for $\{S_n\}$ and

$$\mathbf{P}\left(\limsup_{k\to\infty} \frac{S_k}{k} \le \varepsilon\right) = 1.$$

But since ε is arbitrary, we see that

$$\mathbf{P}\left(\limsup_{k\to\infty}\frac{S_k}{k}\le 0\right)=1.$$

Necessity. Conversely, if equality (12.2.3) holds then, for any $\varepsilon>0$, with probability 1 we have $S_n/n<\varepsilon$ for all n large enough. This means that $\sup_k(S_k-\varepsilon k)<\infty$ with probability 1, and hence by Corollary 12.2.5 the series $\sum_{k=1}^{\infty}\frac{\mathbf{P}(S_k>\varepsilon k)}{k}$ converges. The theorem is proved. □

Corollary 12.2.7 *With probability* 1 *we have*

$$\limsup_{n\to\infty}\frac{S_n}{n}=\alpha,$$

where

$$\alpha=\inf\left\{b:\sum\frac{\mathbf{P}(S_k>bk)}{k}<\infty\right\}.$$

Proof For any $b>\alpha$, the series in the definition of the number α converges. Since $\{\limsup_{n\to\infty}S_n/n\le b\}$ is a tail event and $S_n'=S_n-bn$ again form a sequence of sums of independent identically distributed random variables, Theorem 12.2.1 immediately implies that

$$\mathbf{P}\left(\limsup\frac{S_n}{n}\le b\right)=1,$$
$$\mathbf{P}\left(\limsup\frac{S_n}{n}\le \alpha\right)=\mathbf{P}\left(\bigcap_{k=1}^{\infty}\left\{\limsup\frac{S_n}{n}\le\alpha+\frac{1}{k}\right\}\right)=1.$$

If we assume that $\mathbf{P}(\limsup S_n/n\le\alpha^*)=1$ for $\alpha^*<\alpha$ then, for $\xi_k^*=\xi_k-\alpha^*$ and $S_k^*=\sum_{j=1}^{k}\xi_j^*$, we will have $\limsup\frac{S_n^*}{n}\le 0$, and

$$\sum_{k=1}^{\infty}\frac{\mathbf{P}(S_k>(\alpha^*+\varepsilon)k)}{k}<\infty$$

for any $\varepsilon>0$, which contradicts the definition of α. The corollary is proved. □

In order to derive the conventional law of large numbers from Theorem 12.2.1 it suffices to use Corollary 12.2.7 and assertion 2 of Corollary 12.2.6. We obtain that in the case $\mathbf{E}\xi=0$ the value of α in Corollary 12.2.7 is 0 and hence $\limsup S_n/n=0$ with probability 1. One can establish in the same way that $\liminf S_n/n=0$. □

12.3 Pollaczek–Spitzer's Identity. An Identity for $S=\sup_{k\ge 0}S_k$

It is important to note that, besides Theorems 12.1.1 and 12.1.2, there exist a number of factorisation identities that give explicit representations (in terms of factorisation components) for ch.f.s of the so-called *boundary functionals* of the trajectory of the random walk $\{S_k\}$, i.e. functionals associated with the crossing by the trajectory of $\{S_k\}$ of certain levels (not just the zero level, as in Theorems 12.1.1–12.1.3). The functionals

$$\overline{S}_n = \max_{k\le n} S_k, \qquad \theta_n = \min\{k : S_k = \overline{S}_n\}$$

and some others are also among the boundary functionals. For instance, for the triple transform of the joint distribution of $(\overline{S}_n, \theta_n)$, the following representation is valid.

For $|z| < 1$, $|\rho| < 1/|z|$ and $\operatorname{Im}\lambda \ge 0$, one has

$$(1-z)\sum_{n=0}^{\infty} z^n \mathbf{E}\big(\rho^{\theta_n} e^{i\lambda \overline{S}_n}\big) = \frac{\mathfrak{f}_{z+}(0)}{\mathfrak{f}_{z\rho+}(\lambda)}.$$

(For more detail on factorisation identities, see [3].)

Among many consequences of this identity we will highlight two results that can also be established using the already available Theorems 12.1.1–12.1.3.

12.3.1 Pollaczek–Spitzer's Identity

So far we have obtained several factorisation identities as relations for numerators in representations (12.1.7), (12.1.8) and (12.1.9). Now we turn to the denominators. We will obtain one more identity playing an important role in studying the distributions of

$$\overline{S}_n = \max(0, \zeta_n) = \max(0, S_1, \dots, S_n).$$

This is the so-called *Pollaczek–Spitzer identity* relating the ch.f.s of $\overline{S}_n$, $n = 1, 2, \dots,$ with those of $\max(0, S_n)$, $n = 1, 2, \dots$.

Theorem 12.3.1 *For* $|z| < 1$ *and* $\operatorname{Im}\lambda \ge 0$,

$$\sum_{n=0}^{\infty} z^n \mathbf{E} e^{i\lambda \overline{S}_n} = \exp\left\{\sum_{n=0}^{\infty} \frac{z^k}{k} \mathbf{E} e^{i\lambda \max(0, S_k)}\right\}.$$

Using the notation of Theorem 12.1.1, one could write the right-hand side of this identity as

$$\frac{\mathfrak{f}_{z+}(0)}{(1-z)\mathfrak{f}_{z+}(\lambda)}$$

(see the last relation in the proof of the theorem).

Proof Theorems 12.1.1–12.1.3 (as well as their modifications of the form (12.1.13)) and the uniqueness of the canonical factorisation imply that

$$\sum_{k=0}^{\infty} z^n \mathbf{E}\big(e^{i\lambda S_n};\ \zeta_n < 0\big) = \big[1 - \mathbf{E}\big(e^{i\lambda\chi_-} z^{\eta_-};\ \eta_- < \infty\big)\big]^{-1} = \mathfrak{f}_{z-}^{-1}(\lambda),$$

where we assume that $\mathbf{E}(e^{i\lambda S_0};\ \zeta_0<0)=1$, so all the functions in the above relation turn into 1 at $\lambda=-i\infty$. Set

$$S_k^* := S_{n-k}-S_n, \qquad \theta_n^* := \min\bigl\{k : S_k^* = \overline{S}_n^* := \max\bigl(0, S_1^*, \ldots, S_n^*\bigr)\bigr\}$$

(θ_n^* is time of the first maximum in the sequence $0, S_1^*, \ldots, S_n^*$). Then the event $\{S_n \in dx, \zeta_n<0\}$ can be rewritten as $\{S_n^* \in -dx, \theta_n^* = n\}$. This implies that

$$\begin{gathered}\mathbf{E}\bigl(e^{i\lambda S_n};\ \zeta_n<0\bigr)=\mathbf{E}\bigl(e^{-i\lambda S_n^*};\ \theta_n^*=n\bigr),\\ \sum_{n=0}^{\infty} z^n\mathbf{E}\bigl(e^{i\lambda S_n^*};\ \theta_n^*=n\bigr)=\mathfrak{f}_{z-}^{-1}(-\lambda).\end{gathered} \tag{12.3.1}$$

But the sequence $S_1^*, \ldots, S_n^*$ is distributed identically to the sequence of sums $\xi_1^*, \xi_1^*+\xi_2^*, \ldots, \xi_1^*+\cdots+\xi_n^*$, where $\xi_k^*=-\xi_k$. If we put $\theta_n := \min\{k : S_k = \overline{S}_n\}$ then identity (12.3.1) can be equivalently rewritten as

$$\sum_{n=0}^{\infty} z^n\mathbf{E}\bigl(e^{i\lambda S_n};\ \theta_n=n\bigr)=\bigl(\mathfrak{f}_{z-}^*(-\lambda)\bigr)^{-1},$$

where $\mathfrak{f}_{z-}^*(\lambda)$ is the negative factorisation component of the function $1-z\varphi^*(\lambda)=1-z\varphi(-\lambda)$ corresponding to the random variable $-\xi$. Since

$$1-z\varphi(-\lambda)=\mathfrak{f}_{z+}(-\lambda)C(z)\mathfrak{f}_{z-}(-\lambda)$$

and the function $\mathfrak{f}_{z+}(-\lambda)$ possesses all the properties of the negative component $\mathfrak{f}_{z-}^*(\lambda)$ of the factorisation of $1-z\varphi^*(\lambda)$, while the function $\mathfrak{f}_{z-}(-\lambda)$ has all the properties of a positive component, we see that $\mathfrak{f}_{z-}^*(\lambda)=\mathfrak{f}_{z+}(-\lambda)$ and

$$\sum_{n=0}^{\infty} z^n\mathbf{E}\bigl(e^{i\lambda S_n};\ \theta_n=n\bigr)=\frac{1}{\mathfrak{f}_{z+}(\lambda)}.$$

Now we note that

$$\begin{aligned}\mathbf{E}e^{i\lambda\overline{S}_n}&=\sum_{k=0}^{n}\mathbf{E}\bigl(e^{i\lambda\overline{S}_n};\ \theta_n=k\bigr)\\&=\sum_{k=0}^{n}\mathbf{E}\bigl(e^{i\lambda S_k};\ \theta_k=k,\ S_{k+1}-S_k\le 0,\ldots,S_n-S_k\le 0\bigr)\\&=\sum_{k=0}^{n}\mathbf{E}\bigl(e^{i\lambda S_k};\ \theta_k=k\bigr)\mathbf{P}(\overline{S}_{n-k}=0).\end{aligned}$$

Since the right-hand side is the convolution of two sequences, we obtain that

$$\sum_{n=0}^{\infty} z^n\mathbf{E}\,e^{i\lambda\overline{S}_n}=\sum_{n=0}^{\infty} z^n\mathbf{P}(\overline{S}_n=0)\frac{1}{\mathfrak{f}_{z+}(\lambda)}.$$

Putting $\lambda=0$ we get

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n\mathbf{P}(\overline{S}_n=0)\frac{1}{\mathfrak{f}_{z+}(0)}.$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty} z^n \mathbf{E}\, e^{i\lambda \overline{S}_n} &= \frac{\mathfrak{f}_{z+}(0)}{(1-z)\mathfrak{f}_{z+}(\lambda)} \\
&= \exp\left\{ -\ln(1-z) + \sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}\big(e^{i\lambda S_k};\; S_k > 0\big) - \sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{P}(S_k > 0) \right\} \\
&= \exp\left\{ \sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}\big(e^{i\lambda S_k};\; S_k > 0\big) + \sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{P}(S_k \le 0) \right\} \\
&= \exp\left\{ \sum_{k=1}^{\infty} \frac{z^k}{k} \mathbf{E}\big(e^{i\lambda \max(0, S_k)}\big) \right\}.
\end{aligned}$$

The theorem is proved. □

12.3.2 An Identity for $S = \sup_{k\ge 0} S_k$

The second useful identity to be discussed in this subsection is associated with the distribution of the random variable $S = \sup_{k\ge 0} S_k = \max(0, \zeta)$ (of course, we deal here with the cases when $\mathbf{P}(S < \infty) = 1$). This distribution is of interest in many applications. Two such illustrative applications will be discussed in the next subsection.

We will establish the relationship of the distribution of S with that of the vector (χ_+, η_+) and with the factorisation components of the function $1 - z\varphi(\lambda)$.

First of all, note that the random variable η_+ is a Markov time. For such variables, one can easily see (cf. Lemma 10.2.1) that the sequence $\xi_1^* = \xi_{\eta_+ + 1}, \xi_2^* = \xi_{\eta_+ + 2}, \dots$ on the set $\{\omega : \eta_+ < \infty\}$ (or given that $\eta_+ < \infty$) is distributed identically to $\xi_1, \xi_2, \dots$ and does not depend on $(\eta_+, \xi_1, \dots, \xi_{\eta_+})$. Indeed,

$$\begin{aligned}
&\mathbf{P}\big(\xi_1^* \in B_1, \dots, \xi_k^* \in B_k \,\big|\, \eta_+ = j, \xi_1 \in A_1, \dots, \xi_{\eta_+} \in A_{\eta_+}\big) \\
&\quad = \mathbf{P}(\xi_{j+1} \in B_1, \dots, \xi_{j+k} \in B_k \mid \xi_1 \in A_1, \dots, \xi_j \in A_j;\; \eta_+ = j) \\
&\quad = \mathbf{P}(\xi_1 \in B_1, \dots, \xi_k \in B_k).
\end{aligned}$$

Considering the new sequence $\{\xi_k^*\}_{k=1}^{\infty}$ we note that it will exceed level 0 (the level χ_+ for the original sequence) with probability $p = \mathbf{P}(\eta_+ < \infty)$, and that the distribution of $\zeta^* = \sup_{k\ge 1}(\xi_1^* + \cdots + \xi_k^*)$ coincides with the distribution of $\zeta = \sup_{k\ge 1} S_k$.

Thus, with $S^* := \max(0, \zeta^*)$, we have

$$S = S(\omega) = \begin{cases} 0 & \text{on } \{\omega : \eta_+ = \infty\}, \\ S_{\eta_+} + S^* = \chi_+ + S^* & \text{on } \{\omega : \eta_+ < \infty\}. \end{cases}$$

Since, as has already been noted, S^* does not depend on χ_+ and η_+, and the distribution of S^* coincides with that of S, we have

$$\begin{aligned}\mathbf{E}e^{i\lambda S} &= \mathbf{P}(\eta_+ = \infty) + \mathbf{E}\big(e^{i\lambda(\chi_+ + S^*)};\ \eta_+ < \infty\big) \\ &= (1-p) + \mathbf{E}e^{i\lambda S}\mathbf{E}\big(e^{i\lambda\chi_+};\ \eta_+ < \infty\big).\end{aligned}$$

This implies the following result.

Theorem 12.3.2 *If* $\sum \frac{\mathbf{P}(S_k>0)}{k} < \infty$ *or, which is the same,* $p = \mathbf{P}(\eta_+ < \infty) < 1$, *then*

$$\mathbf{E}e^{i\lambda S} = \frac{1-p}{1-\mathbf{E}(e^{i\lambda\chi_+}, \eta_+ < \infty)} = \frac{1-p}{\mathfrak{f}_{1+}(\lambda)}.$$

In exactly the same way we can obtain the relation

$$\mathbf{E}e^{i\lambda S} = \frac{1-p_0}{1-\mathbf{E}(e^{i\lambda\chi_+^0}, \eta_+^0 < \infty)}, \tag{12.3.2}$$

where $p_0 = \mathbf{P}(\eta_+^0 < \infty) < 1$.

In this case, one can write a factorisation identity in following form:

$$1-\varphi(\lambda) = \frac{(1-p_0)(1-\mathbf{E}e^{i\lambda\chi_-})}{\mathbf{E}e^{i\lambda S}} = \frac{(1-p)(1-\mathbf{E}e^{i\lambda\chi_-^0})}{\mathbf{E}e^{i\lambda S}}. \tag{12.3.3}$$

In Sects. 12.5–12.7 we will discuss the possibility of finding the explicit form and the asymptotic properties of the distribution of S.

12.4 The Distribution of S in Insurance Problems and Queueing Theory

In this section we show that the need to analyse the distribution of the variable S considered in Sect. 12.3 arises in insurance problems and also when studying queueing systems.

12.4.1 Random Walks in Risk Theory

Consider the following simplified model of an insurance business operation. Denote by x the initial surplus of the company and consider the daily dynamics of the surplus. During the k-th day the company receives insurance premiums at the rate $\xi_k^+ \ge 0$ and pays out claims made by insured persons at the rate $\xi_k^- \ge 0$ (in case of a fire, a traffic accident, and so on). The amounts $\xi_k = \xi_k^- - \xi_k^+$ are random since they depend on the number of newly insured persons, the size of premiums, claim amounts and so on. For a foreseeable "homogeneous" time period, the amount ξ_k can be assumed to be independent and identically distributed. If we put $S_n := \sum_{k=1}^n \xi_k$ then the company's surplus after n days will be $Z_n = x - S_n$, provided that we allow it to be negative. But if we assume that the company ruins at the time when Z_n first becomes negative, then the probability of no ruin during the first n days equals

$$\mathbf{P}\Big(\min_{k\le n} Z_k \ge 0\Big) = \mathbf{P}(\overline{S}_n \le x),$$

where, as above, $\overline{S}_n = \max_{k\le n} S_k$. Accordingly, the probability of ruin within n days is equal to $\mathbf{P}(\overline{S}_n > x)$, and the probability of ruin in the long run can be identified with $\mathbf{P}(S > x)$. It follows that, for the probability of ruin to be less than 1, it is necessary that $\mathbf{E}\xi_k < 0$ or, which is the same, that $\mathbf{E}\xi_k^- < \mathbf{E}\xi_k^+$. When this condition is satisfied, in order to make the probability of ruin small enough, one has to make the initial surplus x large enough. In this connection it is of interest to find the explicit form of the distribution of S, or at least the asymptotic behaviour of $\mathbf{P}(S > x)$ as $x \to \infty$. Sections 12.5–12.7 will be focused on this.

12.4.2 Queueing Systems

Imagine that "customers" who are to be served by a certain system arrive with time intervals $\tau_1, \tau_2, \ldots$ between successive arrivals. These could be phone calls, planes landing at an airport, clients in a shop, messages to be processed by a computer, etc. Assume that serving the k-th customer (the first customer arrived at time 0, the second at time τ_1, and so on) requires time s_k, $k = 1, 2, \ldots$ If, at the time of the k-th customer's arrival, the system was busy serving one of the preceding customers, the newly arrived customer joins the "queue" and waits for service which starts immediately after the system has finished serving all the preceding customers. The problem is to find the distribution of the waiting time w_n of the n-th customer—the time spent waiting for the service.

Let us find out how the quantities w_{n+1} and w_n are related to each other. The $(n+1)$-th customer arrived τ_n time units after the n-th customer, but will have to wait for an extra s_n time units during the service of the n-th customer. Therefore,

$$w_{n+1} = w_n - \tau_n + s_n,$$

only if $w_n - \tau_n + s_n \ge 0$. If $w_n - \tau_n + s_n < 0$ then clearly $w_{n+1} = 0$. Thus, if we put $\xi_{n+1} := s_n - \tau_n$, then

$$w_{n+1} = \max(0, w_n + \xi_{n+1}), \quad n \ge 1, \tag{12.4.1}$$

with the initial value of $w_1 \ge 0$. Let us find the solution to this recurrence equation. Let, as above, $S_n = \sum_{k=1}^n \xi_k$. Denote by $\theta(n)$ the time when the trajectory of $0, S_1, \ldots, S_n$ first attains its minimum:

$$\theta(n) := \min\{k : S_k = \underline{S}_n\}, \qquad \underline{S}_n := \min_{0\le j\le n} S_j.$$

Then clearly (for $w_0 := w_1$)

$$w_{n+1} = w_1 + S_n \quad \text{if } w_{\theta(n)} = w_1 + \underline{S}_n > 0 \tag{12.4.2}$$

(since in this case the right-hand side of (12.4.1) does not vanish and $w_{k+1} = w_k + \xi_k$ for all $k \le n$), and

$$w_{n+1} = S_n - S_{\theta(n)} \quad \text{if } w_1 + \underline{S}_n \le 0 \tag{12.4.3}$$

($w_{\theta(n)} = 0$ and $w_{k+1} = w_k + \xi_k$ for all $k \ge \theta(n)$). Put

$$S_{n,j} := \sum_{k=n-j+1}^{n} \xi_k, \qquad \overline{S}_{n,n} := \max_{0 \le j \le n} S_{n,j},$$

so that

$$S_{n,0} = 0, \qquad S_{n,n} = S_n.$$

Then

$$S_n - S_{\theta(n)} = S_n - \underline{S}_n = \max_{0 \le j \le n} (S_n - S_j) = \overline{S}_{n,n},$$

so that $w_1 + \underline{S}_n = w_1 + S_n - \overline{S}_{n,n}$ and the inequality $w_1 + \overline{S}_n \le 0$ in (12.4.3) is equivalent to the inequality $\overline{S}_{n,n} = S_n - S_{\theta(n)} \ge w_1 + S_n$. Therefore (12.4.2) and (12.4.3) can be rewritten as

$$w_{n+1} = \max(\overline{S}_{n,n}, w_1 + S_n). \tag{12.4.4}$$

This implies that, for each fixed $x > 0$,

$$\mathbf{P}(w_{n+1} > x) = \mathbf{P}(\overline{S}_{n,n} > x) + \mathbf{P}(\overline{S}_{n,n} \le x,\ w_1 + S_n > x).$$

Now assume that $\xi_k \stackrel{d}{=} \xi$ are independent and identically distributed with $\mathbf{E}\xi < 0$. Then $\overline{S}_{n,n} \stackrel{d}{=} \overline{S}_n$ and, as $n \to \infty$, we have $S_n \xrightarrow{a.s.} -\infty$, $\mathbf{P}(w_1 + S_n > x) \to 0$ and $\mathbf{P}(\overline{S}_n > x) \uparrow \mathbf{P}(S > x)$. We conclude that, for any initial value w_1, the following limit exists

$$\lim_{n\to\infty} \mathbf{P}(w_n > x) = \mathbf{P}(S > x).$$

This distribution is called the *stationary waiting time distribution*. We already know that it will be proper if $\mathbf{E}\xi = \mathbf{E}s_1 - \mathbf{E}\tau_1 < 0$. As in the previous section, here arises the problem of finding the distribution of S. If, on the other hand, $\mathbf{E}s_1 > \mathbf{E}\,\tau_1$ or $\mathbf{E}s_1 = \mathbf{E}\tau_1$ and $s_1 \not\equiv \tau_1$ then the "stationary" waiting time will be infinite.

12.4.3 Stochastic Models in Continuous Time

In the theory of queueing systems and risk theory one can equally well employ stochastic models in *continuous time*, when, instead of random walks $\{S_n\}$, one uses generalised renewal processes $Z(t)$ as described in Sect. 10.6. For a given sequence of independent identically distributed random vectors (τ_j, ζ_j), the process $Z(t)$ is defined by the equality

$$Z(t) := Z_{\nu(t)},$$

where

$$Z_n := \sum_{j=1}^{n} \zeta_j, \qquad \nu(t) := \max\{k : T_k \le t\}, \quad T_k := \sum_{j=1}^{k} \tau_j.$$

For instance, in risk theory, the capital inflow during time t that comes from regular premium payments can be described by the function qt, $q > 0$. The insurer covers claims of sizes $\zeta_1, \zeta_2, \dots$ with time intervals $\tau_1, \tau_2, \dots$ between them (the first claim is covered at time τ_1). Thus, if the initial surplus is x, then the surplus at time t will be

$$x + qt - Z_{\nu(t)} = x + qt - Z(t).$$

The insurer ruins if $\inf_t (x + qt - Z(t)) < 0$ or, which is the same,

$$\sup_t \bigl(Z(t) - qt\bigr) > x.$$

It is not hard to see that

$$\sup_t (Z_{\nu(t)} - qt) = \sup_{k \ge 0} S_k =: S,$$

where $S_k = \sum_{j=1}^{k} \xi_j$, $\xi_j = \zeta_j - q\tau_j$. Thus the continuous-time version of the ruin problem for an insurance company also reduces to finding the distribution of the maximums of the cumulative sums.

12.5 Cases Where Factorisation Components Can Be Found in an Explicit Form. The Non-lattice Case

As was already noted, the boundary functionals of random walks that were considered in Sects. 12.1–12.3 appear in many applied problems (see e.g., Sect. 12.4). This raises the question: in what cases can one find, in an explicit form, the factorisation components and hence the explicit form of the boundary functionals distributions we need? Here we will deal with factorisation of the function $1 - \varphi(\lambda)$ and will be interested in the boundary functionals $\chi_\pm$ and S.

12.5.1 Preliminary Notes on the Uniqueness of Factorisation

As was already mentioned, the factorisation of the function $1 - \varphi(\lambda)$ obtained in Corollaries 12.2.2 and 12.2.4 is not canonical since that function vanishes at $\lambda = 0$. In this connection arises the question of whether a factorisation is unique. In other words, if, say, in the case $\mathbf{E}\xi < 0$, we obtained a factorisation

$$1 - \varphi(\lambda) = \mathfrak{f}_+(\lambda)\mathfrak{f}_-(\lambda),$$

where $\mathfrak{f}_\pm$ are analytic on $\Pi_\pm$ and continuous on $\Pi_\pm \cup \Pi$, then under what conditions can we state that

$$\mathbf{E}e^{i\lambda S} = \frac{\mathfrak{f}_+(0)}{\mathfrak{f}_+(\lambda)}$$

(cf. Theorem 12.3.2)? In order to answer this question, in contrast to the above, we will have to introduce here restrictions on the distribution of ξ.

1. We will assume that $\mathbf{E}\xi$ exists, and in the case $\mathbf{E}\xi = 0$ that $\mathbf{E}\xi^2$ also exists.
2. Regarding the structure of the distribution of ξ we will assume that either
   (a) the distribution $\mathbf{F}$ is non-lattice and the Cramér condition on ch.f. holds:
   $$\limsup_{\substack{|\lambda|\to\infty \\ \operatorname{Im}\lambda=0}} \bigl|\varphi(\lambda)\bigr| < 1, \tag{12.5.1}$$
   or
   (b) the distribution $\mathbf{F}$ is arithmetic.

Condition (12.5.1) always holds once the distribution $\mathbf{F}$ has a nonzero absolutely continuous component. Indeed, if $\mathbf{F} = \mathbf{F}_a + \mathbf{F}_s + \mathbf{F}_d$ is the decomposition of $\mathbf{F}$ into the absolutely continuous, singular and discrete components then, by the Lebesgue theorem, $\int e^{i\lambda x}\mathbf{F}_a(dx) \to 0$ as $|\lambda| \to \infty$ on $\operatorname{Im}\lambda = 0$, and so

$$\limsup_{|\lambda|\to\infty} \bigl|\varphi(\lambda)\bigr| \le \mathbf{F}_s\bigl((-\infty,\infty)\bigr) + \mathbf{F}_d\bigl((-\infty,\infty)\bigr) < 1.$$

For lattice distributions concentrated at the points $a + hk$, k being an integer, condition (12.5.1) is evidently not satisfied since, for $\lambda = 2\pi j/h$, we have $|\varphi(\lambda)| = |e^{i2\pi a/h}| = 1$ for all integers j. The condition is also not met for any discrete distribution, since any "part" of such a distribution, concentrated on a finite number of points, can be approximated arbitrarily well by a lattice distribution. For singular distributions, condition (12.5.1) can yet be satisfied.

Since, for non-lattice distributions, $|\varphi(\lambda)| < 1$ for $\lambda \ne 0$, under condition (12.5.1) one has

$$\sup_{|\lambda|>\varepsilon} \bigl|\varphi(\lambda)\bigr| < 1 \tag{12.5.2}$$

for any $\varepsilon > 0$. This means that the function $\mathfrak{f}(\lambda) = 1 - \varphi(\lambda)$ has no zeros on the real line Π (completed by the points $\pm\infty$) except at the point $\lambda = 0$.

In case (b), when the distribution of $\mathbf{F}$ is arithmetic, one can consider the ch.f. $\varphi(\lambda)$ on the segment $[0, 2\pi]$ only or, which is the same, consider the generating function $p(z) = \mathbf{E}z^\xi$, in which case we will be interested in the factorisation of the function $1 - p(z)$ on the unit circle $|z| = 1$.

Under the aforementioned conditions, we can "tweak" the function $1 - \varphi(\lambda)$ so that it allows canonical factorisation.

In this section we will confine ourselves to the non-lattice case. The arithmetic case will be considered in Sect. 12.6.

Lemma 12.5.1 *Let the distribution* $\mathbf{F}$ *be non-lattice and condition* (12.5.1) *hold. Then*:

1. *If* $\mathbf{E}\xi < 0$ *then the function*

$$\mathfrak{v}(\lambda) := \frac{1-\varphi(\lambda)}{i\lambda}(i\lambda+1) \tag{12.5.3}$$

belongs to $\mathcal{K}$ *and allows a unique canonical factorisation*

$$\mathfrak{v}(\lambda) = \mathfrak{v}_+(\lambda)\mathfrak{v}_-(\lambda),$$

where

$$\mathfrak{v}_+(\lambda) := 1 - \mathbf{E}\big(e^{i\lambda\chi_+};\ \eta_+ < \infty\big) = \frac{1-p}{\mathbf{E}e^{i\lambda S}}, \tag{12.5.4}$$

$$\mathfrak{v}_-(\lambda) := \frac{1-\mathbf{E}e^{i\lambda\chi_-^0}}{i\lambda}(i\lambda+1). \tag{12.5.5}$$

2. *If* $\mathbf{E}\xi = 0$ *and* $\mathbf{E}\xi^2 < \infty$ *then the function*

$$\mathfrak{v}^0(\lambda) := \frac{1-\varphi(\lambda)}{\lambda^2}\big(\lambda^2+1\big) \tag{12.5.6}$$

belongs to $\mathcal{K}$ *and allows a unique canonical factorisation*

$$\mathfrak{v}^0(\lambda) = \mathfrak{v}_+^0(\lambda)\mathfrak{v}_-^0(\lambda),$$

where

$$\begin{aligned} \mathfrak{v}_+^0(\lambda) &:= \frac{1-\mathbf{E}e^{i\lambda\chi_+}}{i\lambda}(i\lambda-1), \\ \mathfrak{v}_-^0(\lambda) &:= \frac{1-\mathbf{E}e^{i\lambda\chi_-^0}}{i\lambda}(i\lambda+1) \end{aligned} \tag{12.5.7}$$

(*cf. Corollaries* 12.2.2 *and* 12.2.4).

Here we do not consider the case $\mathbf{E}\xi > 0$ since it is "symmetric" to the case $\mathbf{E}\xi < 0$ and the corresponding assertion can be derived from the assertion 1 of the lemma by applying it to the random variables $-\xi_k$ (or by changing λ to $-\lambda$ in the identities), so that in the case $\mathbf{E}\xi > 0$, the function $\frac{1-\varphi(\lambda)}{i\lambda}(i\lambda-1)$ will allow a unique canonical factorisation.

The uniqueness of the canonical factorisation immediately implies the following result.

Corollary 12.5.1 *If, for* $\mathbf{E}\xi < 0$, *we have a canonical factorisation*

$$\mathfrak{v}(\lambda) = \mathfrak{w}_+(\lambda)\mathfrak{w}_-(\lambda),$$

then

$$\mathbf{E}e^{i\lambda S} = \frac{\mathfrak{w}_+(0)}{\mathfrak{w}_+(\lambda)}. \tag{12.5.8}$$

Proof of Lemma 12.5.1 Let $\mathbf{E}\xi < 0$. Since

$$\frac{1-\varphi(\lambda)}{i\lambda} \to -\mathbf{E}\xi > 0$$

as $\lambda \to 0$ and (12.5.1) is satisfied, we see that $\mathfrak{v}(\lambda)$ is bounded and continuous on Π and is bounded away from zero. This means that $\mathfrak{v}(\lambda) \in \mathcal{K}$.

Further, by Corollary 12.2.2 (see (12.2.1))

$$\mathfrak{v}(\lambda) = \frac{(1 - \mathbf{E}e^{i\lambda\chi_-^0})(i\lambda + 1)}{i\lambda} \bigl[1 - \mathbf{E}\bigl(e^{i\lambda\chi_+};\, \eta_+ < \infty\bigr)\bigr],$$

where $\mathbf{E}\chi_-^0 \in (-\infty, 0)$. Therefore, similarly to the above, we find that

$$\mathfrak{v}_-(\lambda) := \frac{(1 - \mathbf{E}e^{i\lambda\chi_-^0})(i\lambda + 1)}{i\lambda} \in \mathcal{K}.$$

Furthermore, $\mathfrak{v}_-(\lambda) \in \mathcal{K}_-$ (the factor $i\lambda + 1$ has a zero at the point $\lambda = i \in \Pi_+$). Evidently, we also have

$$\mathfrak{v}_+(\lambda) = 1 - \mathbf{E}\bigl(e^{i\lambda\chi_+};\, \eta_+ < \infty\bigr) \in \mathcal{K} \cap \mathcal{K}_+.$$

This proves the first assertion of the lemma. The last equality in (12.5.4) follows from Theorem 12.3.2. The uniqueness follows from Lemma 12.1.1.

The second assertion is proved in a similar way using Corollary 12.2.5, which implies that $\mathbf{E}\chi_+ \in (0, \infty)$, $\mathbf{E}\chi_-^0 \in (-\infty, 0)$, and

$$\mathfrak{v}^0(\lambda) = \biggl[\frac{(1 - \mathbf{E}e^{i\lambda\chi_+})(i\lambda - 1)}{i\lambda}\biggr]\biggl[\frac{(1 - \mathbf{E}e^{i\lambda\chi_-^0})(i\lambda + 1)}{i\lambda}\biggr], \qquad (12.5.9)$$

where, as before, we can show that $\mathfrak{v}^0(\lambda) \in \mathcal{K}$ and the factors on the right-hand side of (12.5.9) belong to $\mathcal{K} \cap \mathcal{K}_\pm$, correspondingly. The lemma is proved. □

12.5.2 Classes of Distributions on the Positive Half-Line with Rational Ch.F.s

As we saw in Example 7.1.5, the ch.f. of the exponential distribution with density $\beta e^{-\beta x}$ on $(0, \infty)$ is $\beta/(\beta - i\lambda)$. The j-th power of this ch.f. ocrresponds to the gamma-distribution $\mathbf{\Gamma}_{\beta,j}$ (the j-th convolution of the exponential distribution) with density (see Sect. 7.7)

$$\frac{\beta^k x^{j-1} e^{-\beta x}}{(j-1)!}, \quad x \ge 0.$$

This means that a density of the form

$$\sum_{k=1}^{K} \sum_{j=1}^{l_k} a_{kj} x^{j-1} e^{-\beta_k x} \qquad (12.5.10)$$

on $(0, \infty)$ (where all $\beta_k > 0$ are different) can then be considered as a mixture of gamma-distributions and its ch.f. will be a rational function $P_m(\lambda)/Q_n(\lambda)$, where

P_m and Q_n are polynomials of degrees m and n, respectively (for definiteness, we can put)

$$Q_n(\lambda) = \prod_{k=1}^{K} (\beta_k - i\lambda)^{l_k}, \tag{12.5.11}$$

and necessarily $m < n$ (see Property 7.1.8) with $n = \sum_{k=1}^{K} l_k$. Here all the zeros of the polynomial Q_n are real. But not only densities of the form (12.5.10) can have rational ch.f.s. Clearly, the Fourier transform of the function $e^{-\beta x} \cos \gamma x$, which can be rewritten as

$$\frac{1}{2} e^{-\beta x} \big(e^{i\gamma x} + e^{-i\gamma x}\big), \tag{12.5.12}$$

will also be a rational function. Complex-valued functions of this kind will have poles that are *symmetric* with respect to the imaginary line (in our case, at the points $\lambda = -i\beta \pm \gamma$). Convolutions of functions of the form (12.5.12) will have a more complex form but will not go beyond representation (12.5.10), where β_k are "symmetric" complex numbers. Clearly, densities of the form (12.5.10), where β_k are either real and positive or complex and symmetric, $\operatorname{Re} \beta_k > 0$, exhaust all the distributions with rational ch.f.s (the coefficients of the "conjugate" complex-valued exponentials must coincide to avoid the presence of irremovable complex terms).

It is obvious that the converse is also true: rational ch.f.s $P_m(\lambda)/Q_n(\lambda)$ correspond to densities of the form (12.5.10). In order to show this it suffices to decompose $P_m(\lambda)/Q_n(\lambda)$ into partial fractions, for which the inverse Fourier transforms are known.

We will call densities of the form (12.5.10) on $(0, \infty)$ *exponential polynomials* with exponents β_k. We will call the number l_k the *multiplicity* of the exponent β_k — it corresponds to the multiplicity of the pole of the Fourier transform at the point $\lambda = -i\beta_k$ (recall that $Q_n(\lambda) = \prod_{k=1}^{K} (\beta_k - i\lambda)^{l_k}$). One can approximate an arbitrary distribution on $(0, \infty)$ by exponential polynomials (for more details, see [3]).

12.5.3 Explicit Canonical Factorisation of the Function $\mathfrak{v}(\lambda)$ in the Case when the Right Tail of the Distribution $\mathbf{F}$ *Is an Exponential Polynomial*

Consider a distribution $\mathbf{F}$ on the whole real line $(-\infty, \infty)$ with $\mathbf{E}\xi < 0$ and such that, for $x > 0$, the distribution has a density that is an exponential polynomial (12.5.10). Denote by $\mathcal{EP}$ the class of all such distributions. The ch.f. of a distribution $\mathbf{F} \in \mathcal{EP}$ can be represented as

$$\varphi(\lambda) = \varphi^+(\lambda) + \varphi^-(\lambda),$$

where the function

$$\varphi^-(\lambda) = \mathbf{E}\big(e^{i\lambda\xi};\, \xi \le 0\big), \quad \xi \Subset \mathbf{F},$$

is analytic on Π_- and continuous on $\Pi_- \cup \Pi$, and $\varphi^+(\lambda)$ is a rational function

$$\varphi^+(\lambda) = \frac{P_m(\lambda)}{Q_n(\lambda)}, \quad m < n, \tag{12.5.13}$$

analytic on Π_+. Here $\varphi^+(\lambda)$ is a ch.f. up to the factor $\mathbf{P}(\xi > 0) > 0$.

It is important to note that, for real μ, the equality

$$\psi^+(\mu) := \varphi^+(-i\mu) = \mathbf{E}\big(e^{\mu\xi};\, \xi > 0\big) = \frac{P_m(-i\mu)}{Q_n(-i\mu)} \tag{12.5.14}$$

only makes sense for $\mu < \beta_1$, where β_1 is the minimal zero of the polynomial $Q_n(-i\mu)$ (i.e. the pole of $\psi^+(\lambda)$). It is necessarily a simple and real root since the function $\psi^+(\mu)$ is real and monotonically increasing. Further, $\psi^+(\mu) = \infty$ for $\mu \geq \beta_1$. Therefore the function $\mathbf{E}(e^{i\lambda\xi};\, \xi > 0)$ is undefined for $\operatorname{Re} i\lambda \geq \beta_1$ ($\operatorname{Im}\lambda \leq -\beta_1$). However, the right-hand side of (12.5.14) (and hence $\varphi^+(\lambda)$) can be analytically continued onto the lower half-plane $\operatorname{Im}\lambda \leq -\beta_1$ to a function defined on the whole complex plane. In what follows, when we will be discussing zeros of the function $1 - \varphi(\lambda)$ on Π_-, we will mean *zeros of this analytical continuation*, i.e. of the function $\varphi^-(\lambda) + P_m(\lambda)/Q_n(\lambda)$.

Further, note that, for distributions from the class $\mathcal{EP}$, the Cramér condition (12.5.1) on ch.f.s always holds, since $\varphi^+(\lambda) \to 0$ as $|\lambda| \to \infty$, and

$$\limsup_{|\lambda|\to\infty,\, \lambda\in\Pi} \big|\varphi(\lambda)\big| = \limsup_{|\lambda|\to\infty,\, \lambda\in\Pi} \big|\varphi_-(\lambda)\big| \leq \mathbf{P}(\xi \leq 0) < 1.$$

For a distribution $\mathbf{F} \in \mathcal{EP}$, the canonical factorisation of the functions $\mathfrak{v}(\lambda)$ and $\mathfrak{v}^0(\lambda)$ (see (12.5.3) and (12.5.6)) can be obtained in explicit form expressed in terms of the zeros of the function $1 - \varphi(\lambda)$.

Theorem 12.5.1 *Let there exist* $\mathbf{E}\xi < 0$. *In order for the positive component* $\mathfrak{w}_+(\lambda)$ *of a canonical factorisation*

$$\mathfrak{v}(\lambda) = \mathfrak{w}_+(\lambda)\mathfrak{w}_-(\lambda), \qquad \mathfrak{w}_\pm \in \mathcal{K}_\pm \cap \mathcal{K},$$

to be a rational function, it is necessary and sufficient that the function

$$\varphi^+(\lambda) = \mathbf{E}\big(e^{i\lambda\xi};\, \xi > 0\big)$$

is rational.

If $\varphi^+ = P_m/Q_n$ *is an uncancellable ratio of polynomials* P_m *and* Q_n *of degrees* m *and* n, *respectively,* $m < n$, *then the function* $1 - \varphi(\lambda)$ *has precisely* n *zeros on* Π_- (*we denote them by* $-i\mu_1, \ldots, -i\mu_n$), *and*

$$\mathfrak{w}_+(\lambda) = \frac{\prod_{k=1}^n (\mu_k - i\lambda)}{Q_n(\lambda)}, \tag{12.5.15}$$

where $Q_n(-i\mu_k) \neq 0$ (*i.e. ratio* (12.5.15) *is uncancellable*).

If all zeros $-i\mu_k$ *are arranged in descending order of their imaginary parts*:

$$\operatorname{Re}\mu_1 \leq \operatorname{Re}\mu_2 \leq \cdots \leq \operatorname{Re}\mu_n,$$

then the zero $-i\mu_1$ *will be simple and purely imaginary,* $\mu_1 < \min(\operatorname{Re}\mu_2, \beta_1)$, *where* β_1 *is the minimal zero of* $Q_n(-i\mu)$.

The theorem implies that the component $\mathfrak{w}_-(\lambda)$ can also be found in an explicit form:

$$\mathfrak{w}_-(\lambda) = \frac{(1-\varphi(\lambda))(i\lambda+1)Q_n(\lambda)}{i\lambda\prod_{k=1}^n(\mu_k - i\lambda)}.$$

From Corollary 12.5.1 we obtain the following assertion.

Corollary 12.5.2 *If* $\mathbf{E}\xi < 0$ *and* $\varphi = P_m/Q_m$ *then*

$$\mathbf{E}e^{i\lambda S} = \frac{\mathfrak{w}_+(0)}{\mathfrak{w}_+(\lambda)} = \frac{Q_n(\lambda)}{\prod_{k=1}^n(\mu_k - i\lambda)}\,\frac{\prod_{k=1}^n \mu_k}{Q_n(0)}.$$

By Theorem 12.5.1 and (12.3.3) we also have

$$\begin{aligned}\mathbf{E}e^{i\lambda\chi_-^0} &= 1 - \frac{(1-\varphi(\lambda))}{1-p}\,\frac{Q_n(\lambda)}{\prod_{k=1}^\infty(\mu_k - i\lambda)}\,\frac{\prod_{k=1}^n \mu_k}{Q_n(0)},\\ \mathbf{E}\big(e^{i\lambda\chi_+};\ \eta_+ < \infty\big) &= 1 - \frac{(1-p)\prod_{k=1}^n(\mu_k - i\lambda)Q_n(0)}{Q_n(\lambda)\prod_{k=1}^n \mu_k}.\end{aligned} \tag{12.5.16}$$

Proof of Theorem 12.5.1 The proof of *sufficiency* will be divided into several stages.

1. In the vicinity of the point $\lambda = 0$ on the line $\varPi$, the value of

$$\mathfrak{v}(\lambda) = \frac{(1-\varphi(\lambda))(i\lambda+1)}{i\lambda}$$

lies in the vicinity of the point $-\mathbf{E}\xi > 0$. By virtue of (12.5.2), outside a neighbourhood of zero one has

$$\arg\big(1-\varphi(\lambda)\big) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \arg\frac{i\lambda+1}{i\lambda} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{12.5.17}$$

where, for a complex number $z = |z|e^{i\gamma}$, $\arg z$ denotes the exponent γ. In (12.5.17) $\arg z$ means the principal value of the argument from $(-\pi, \pi]$. Clearly, $\arg z_1 z_2 = \arg z_1 + \arg z_2$. This implies that, when λ changes from $-T$ to T for large T, the values of $\arg \mathfrak{v}(\lambda)$ do not leave the interval $(-\pi, \pi)$ and do not come close to its boundaries. Moreover, the initial and final values of $\mathfrak{v}(\lambda)$ lie in the sector $\arg z \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This means that, for any T, the following relation is valid for the *index* of the function $\mathfrak{v}$ on $[-T, T]$:

$$\operatorname{ind}_T \mathfrak{v} := \frac{1}{2\pi}\int_{-T}^{T} d\big(\arg \mathfrak{v}(\lambda)\big) \in \left(-\frac{b}{2}, \frac{b}{2}\right), \quad b < 1. \tag{12.5.18}$$

(If the distribution $\mathbf{F}$ has a density on $(-\infty, 0]$ as well then $\varphi(\pm T) \to 0$ and $\operatorname{ind}_T \mathfrak{v} \to 0$ as $T \to \infty$.)

2. Represent the function $\mathfrak{v}$ as the product $\mathfrak{v}(\lambda) = \mathfrak{v}_1(\lambda)\mathfrak{v}_2(\lambda)$, where

$$\begin{aligned}\mathfrak{v}_1(\lambda) &= \frac{(i\lambda+1)^n}{Q_n(\lambda)},\\ \mathfrak{v}_2(\lambda) &= \frac{Q_n(\lambda)(1-\varphi(\lambda))}{i\lambda(i\lambda+1)^{n-1}} = \frac{Q_n(\lambda) - P_m(\lambda) - Q_n(\lambda)\varphi^-(\lambda)}{i\lambda(i\lambda+1)^{n-1}}.\end{aligned} \tag{12.5.19}$$

We show that

$$|n+\mathrm{ind}_T\,\mathfrak{v}_2|<\frac{1}{2}. \tag{12.5.20}$$

In order to do this, we first note that the function $\mathfrak{v}_1$ is analytic on Π_+ and has there a zero of multiplicity n at the point $\lambda=i$. Consider a closed contour $\mathcal{T}_T^+$ consisting of the segment $[-T,T]$ and the semicircle $|\lambda|=T$ lying in Π_+. According to the argument principle in complex function theory, the number of zeros of the function $\mathfrak{v}_1$ inside $\mathcal{T}_T$ equals the increment of the argument of $\mathfrak{v}_1(\lambda)$ divided by 2π when moving along the contour $\mathcal{T}_T^+$ in the positive direction, i.e.

$$\frac{1}{2\pi}\int_{\mathcal{T}_T^+} d\arg\mathfrak{v}_1(\lambda)=n.$$

As, moreover, $\mathfrak{v}_1(\lambda)\to(-1)^n=\mathrm{const}$ as $|\lambda|\to\infty$ (see (12.5.11) and (12.5.19)), we see that the increment of $\arg\mathfrak{v}_1$ on the semicircle tends to 0 as $T\to\infty$, and hence

$$\mathrm{ind}_T\,\mathfrak{v}_1\to\mathrm{ind}\,\mathfrak{v}_1:=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\arg\mathfrak{v}_1(\lambda)=n.$$

It remains to note that $\mathrm{ind}_T\,\mathfrak{v}=\mathrm{ind}_T\,\mathfrak{v}_1+\mathrm{ind}_T\,\mathfrak{v}_2$ and make use of (12.5.18).

3. We show that $1-\varphi(\lambda)$ has precisely n zeros in Π_-. To this end, we first show that the function $\mathfrak{v}_2(\lambda)$, which is analytic in Π_- and continuous on $\Pi_-\cup\Pi$, has n zeros in Π_-. Consider the positively oriented closed contour $\mathcal{T}_T^-$ consisting of the segment $[-T,T]$ (traversed in the negative direction) and the lower half of the circle $|\lambda|=T$, and compute

$$\frac{1}{2}\int_{\mathcal{T}_T^-} d\arg\mathfrak{v}_2(\lambda). \tag{12.5.21}$$

Since $\mathfrak{v}_2(\lambda)\sim(-1)^n(1-\varphi^-(\lambda))$ (see (12.5.11) and (12.5.19)), $|\varphi^-(\lambda)|<1$ as $|\lambda|\to\infty$, $\mathrm{Im}\,\lambda\le 0$, for large T the part of integral (12.5.21) over the semicircle will be less than $1/2$ in absolute value. Comparing this with (12.5.20) we obtain that integral (12.5.21), being an integer, is necessarily equal to n. This means that $\mathfrak{v}_2(\lambda)$ has exactly n zeros in Π_-, which we will denote by $-i\mu_1,\dots,-i\mu_n$. Since $Q_n(-i\mu_k)\neq 0$ (otherwise we would have, by (12.5.19), $P_m(-i\mu_k)=0$, which would mean cancellability of the fraction P_m/Q_n), the function $1-\varphi(\lambda)$ has in Π_- the same zeros as $\mathfrak{v}_2(\lambda)$ (see (12.5.19)).

4. It remains to put

$$\mathfrak{w}_+(\lambda)=\frac{\prod_{k=1}^n(\mu_k-i\lambda)}{Q_n(\lambda)},$$

$$\mathfrak{w}_-(\lambda)=\frac{(Q_n(\lambda)-P_m(\lambda)-Q_n(\lambda)\varphi^-(\lambda))(i\lambda+1)}{i\lambda\prod_{k=1}^n(\mu_k-i\lambda)}$$

and note that $\mathfrak{w}_\pm\in\mathcal{K}_\pm\cap\mathcal{K}$.

The last assertion of the theorem follows from the fact that the real function $\psi(\mu)=\varphi(-i\mu)$ for $\mathrm{Im}\,\mu=0$ is convex on $[0,\beta_1)$, $\psi'(0)=\mathbf{E}\xi<0$ and

$\psi(\mu) \to \infty$ as $\mu \to \beta_1$. Therefore on $[0, \beta_1)$ there exists a unique real solution to the equation $\psi(\mu) = 1$. There are no complex zeros in the half-plane $\operatorname{Re} \mu \le \mu_1$ since in this region, for $\operatorname{Im} \mu \ne 0$, one has

$$\left|\psi(\mu)\right| < \psi(\operatorname{Re} \mu) \le \psi(\mu_1) = 1$$

because of the presence of an absolutely continuous component.

Necessity. Now let $\mathfrak{w}_+(\lambda)$ be rational. This means that

$$\mathfrak{w}_+(\lambda) = c_1 + \int_0^\infty e^{i\lambda x} g(x)\, dx,$$

where $c_1 = \mathfrak{w}_+(i\infty)$ and $g(x)$ is an exponential polynomial. It follows from the equality (see (12.5.5))

$$\begin{aligned} 1 - \varphi(\lambda) = \mathfrak{w}_+(\lambda) \frac{\mathfrak{w}_-(\lambda) i\lambda}{i\lambda + 1} &= c_2 \mathfrak{w}_+(\lambda)\big(1 - \mathbf{E} e^{i\lambda \chi_-^0}\big) \\ &= c_2 \biggl(c_1 + \int_0^\infty e^{i\lambda x} g(x)\, dx\biggr) \int_{-\infty}^0 e^{i\lambda x}\, dW(x), \end{aligned}$$

where $W(x) = -\mathbf{P}(\chi_-^0 < x)$ for $x < 0$, $c_2 = \text{const}$, that ξ has a density for $x > 0$ that is equal to

$$\int_{-\infty}^0 dW(t)\, g(x - t).$$

Since the integral

$$\int_{-\infty}^0 dW(t)\, (x - t)^k e^{-\beta(x-t)} = e^{-\beta x} \sum_{j=0}^k (-1)^j \binom{k}{j} x^j c_{kj},$$

$$c_{kj} = \int_{-\infty}^0 dW(t)\, t^{k-j} e^{\beta t},$$

is an exponential polynomial, the integral $\int_{-\infty}^0 dW(t)\, g(x - t)$ is also an exponential polynomial, which implies the rationality of $\mathbf{E}(e^{i\lambda\xi};\, \xi > 0)$. The theorem is proved. □

Example 12.5.1 Let the distribution $\mathbf{F}$ be exponential on the positive half-line:

$$\mathbf{P}(\xi > x) = q e^{-\beta x}, \quad \beta > 0,\ q < 1.$$

Then $\varphi^+(\lambda) = q\beta/(\beta - i\lambda)$ and we can put $m = 0$, $n = 1$, $P_0(\lambda) = q\beta$, $Q_1(\lambda) = \beta - i\lambda$. The equation $\psi(\mu) := \mathbf{E} e^{\mu\xi_1} = 1$ has, in the half-plane $\operatorname{Re} \mu > 0$, the unique solution μ_1,

$$\mathfrak{w}_+(\lambda) = \frac{\mu_1 - i\lambda}{Q_1(\lambda)}$$

(see (12.5.15)). By Corollary 12.5.2,

$$\mathbf{E} e^{i\lambda S} = \frac{\mu_1}{Q_1(0)} \frac{Q_1(\lambda)}{(\mu_1 - i\lambda)} = \frac{\mu_1(\beta - i\lambda)}{\beta(\mu_1 - i\lambda)} = \frac{\mu_1}{\beta} + \frac{\beta - \mu_1}{\beta} \frac{\mu_1}{\mu_1 - i\lambda}.$$

This yields $\mathbf{P}(S=0)=\mu_1/\beta$,

$$\frac{\mathbf{P}(S\in dx)}{dx}=\left(1-\frac{\mu_1}{\beta}\right)\mu_1 e^{-\mu_1 x}\quad \text{for } x>0,$$

i.e. the distribution of S is exponential on $(0,\infty)$ with parameter μ_1 and has a positive atom at zero.

Example 12.5.2 (A generalisation of Example 12.5.1) Let $\mathbf{F}$ have, on the positive half-line, the density $\sum_{k=1}^n a_k e^{-\beta_k x}$ (a sum of exponentials), where $0<\beta_1<\beta_2<\cdots<\beta_n$, $a_k>0$. Then

$$Q_n(\lambda)=\prod_{k=1}^n(\beta_k-i\lambda).$$

As was already noted in Theorem 12.5.1, the equation $\psi(\mu):=\varphi(-i\lambda)=1$ has, on the interval $(0,\beta_1)$, a unique zero μ_1. The function $\psi^-(\mu):=\varphi^-(-i\mu)$ is continuous, positive, and bounded for $\mu>0$. On each interval (β_k,β_{k+1}), $k=1,\dots,n-1$, the function

$$\psi^+(\mu):=\varphi^+(-i\mu)=\sum_{k=1}^n\frac{a_k\beta_k}{(\beta_k-\mu)}$$

is continuous and changes from $-\infty$ to ∞. Therefore, on each of these intervals, there exists at least one root μ_{k+1} of the equation $\psi(\mu)=1$. Since by Theorem 12.5.1 there are only n roots of this equation in $\operatorname{Re}\mu>0$, we obtain that μ_{k+1} is the unique root in (β_k,β_{k+1}) and

$$\mathfrak{w}_+(\lambda)=\frac{\prod_{k=1}^n(\mu_k-i\lambda)}{Q_n(\lambda)},\qquad \mathbf{E}\,e^{i\lambda S}=\prod_{k=1}^n\frac{(\beta_k-i\lambda)\mu_k}{(\mu_k-i\lambda)\beta_k}.\tag{12.5.22}$$

This means that $1-p:=\mathbf{P}(S=0)=\prod_{k=1}^n\frac{\mu_k}{\beta_k}$, and

$$\frac{\mathbf{P}(S\in dx)}{dx}=\sum_{k=1}^n b_k e^{-\mu_k x}\quad \text{for } x>0,$$

where $\mu_k\in(\beta_{k-1},\beta_k)$, $k=1,\dots,n$, $\beta_0=0$, and the coefficients b_k are defined by the decomposition of (12.5.22) into partial fractions.

By (12.5.16),

$$\mathbf{E}\big(e^{i\lambda\chi_+};\ \eta_+<\infty\big)=1-\frac{1-p}{\mathbf{E}e^{i\lambda S}}=1-\prod_{k=1}^n\frac{(\mu_k-i\lambda)}{(\beta_k-i\lambda)},\tag{12.5.23}$$

so the conditional distribution of χ_+ given $\chi_+<\infty$ has a density which is equal to

$$\sum_{k=1}^n c_k e^{-\beta_k x},\tag{12.5.24}$$

where the coefficients c_k, similarly to the above, are defined by the expansion of the right-hand side of (12.5.23) into partial fractions. Relation (12.5.24) means that the density of χ_+ has the same "structure" as the density of ξ does for $x > 0$, but differs in coefficients of the exponentials only. By (12.5.16) this property of the density of χ_+ holds in the general case as well.

12.5.4 *Explicit Factorisation of the Function* $\mathfrak{v}(\lambda)$ *when the Left Tail of the Distribution* $\mathbf{F}$ *Is an Exponential Polynomial*

Now consider the case where the *left* tail of the distribution $\mathbf{F}$ has a density which is an exponential polynomial (belongs to the class $\mathcal{EP}$). In this case,

$$\varphi^{-}(\lambda) = \mathbf{E}\big(e^{i\lambda\xi};\ \xi < 0\big) = \frac{P_m(\lambda)}{Q_n(\lambda)},$$

where

$$Q_n(\lambda) = \prod_{k=1}^{K} (\beta_k - i\lambda)^{l_k}, \quad n = \sum_{k=1}^{K} l_k, \quad \operatorname{Re}\beta_k < 0,\ m < n.$$

Theorem 12.5.2 *Let there exist* $\mathbf{E}\xi < 0$. *For the positive component of the canonical factorisation* $\mathfrak{v}(\lambda) = \mathfrak{w}_+(\lambda)\mathfrak{w}_-(\lambda)$ *of the function*

$$\mathfrak{v}(\lambda) = \frac{(1 - \varphi(\lambda))(i\lambda + 1)}{i\lambda}$$

to be representable as

$$\mathfrak{w}_+(\lambda) = \big(1 - \varphi(\lambda)\big)R(\lambda),$$

where $R(\lambda)$ *is a rational function, it is necessary and sufficient that the function* $\varphi^{-}(\lambda)$ *is rational. If* $\varphi^{-}(\lambda) = P_m(\lambda)/Q_n(\lambda)$ *then the function* $1 - \varphi(\lambda)$ *has precisely* $n - 1$ *zeros in the half-plane on* $\operatorname{Im}\lambda > 0$ *which we denote by* $-i\mu_1, \dots, -i\mu_{n-1}$, *and*

$$R(\lambda) = \frac{Q_n(\lambda)}{i\lambda \prod_{k=1}^{n-1}(\mu_k - i\lambda)}.$$

Theorem 12.5.2, Corollary 12.5.1 and (12.3.3) imply the following assertion.

Corollary 12.5.3 *If* $\mathbf{E}\xi < 0$ *and* $\varphi^{-}(\lambda) = P_m(\lambda)/Q_n(\lambda)$ *then*

$$\mathbf{E}e^{i\lambda S} = \frac{\mathfrak{w}_+(0)}{\mathfrak{w}_+(\lambda)} = -\frac{\mathbf{E}\xi\, Q_n(0) i\lambda \prod_{k=1}^{n-1}(\mu_k - i\lambda)}{(1 - \varphi(\lambda))Q_n(\lambda)\prod_{k=1}^{n-1}\mu_k},$$

$$\mathbf{E}e^{i\lambda\chi_-} = 1 + \frac{(1 - p_0)\mathbf{E}\xi\, Q_n(0) i\lambda \prod_{k=1}^{n-1}(\mu_k - i\lambda)}{\prod_{k=1}^{n-1}\mu_k Q_n(\lambda)}.$$

Here the density of χ_- has the same "structure" as the density of ξ does for $x < \infty$.

Proof of Theorem 12.5.2 The proof is close to that of Theorem 12.5.1, but unfortunately is not its direct consequence. We present here a brief proof of Theorem 12.5.2 under the simplifying assumption that the distribution $\mathbf{F}$ is absolutely continuous. Using the scheme of the proof of Theorem 12.5.1, the reader can easily reconstruct the argument in the general case.

Sufficiency. As in Theorem 12.5.1, we verify that the trajectory of $\mathfrak{v}(\lambda)$, $-\infty < \lambda < \infty$, does not intersect the ray $\arg \mathfrak{v} = -\pi$, so in our case there exists

$$\operatorname{ind} \mathfrak{v} := \lim_{T\to\infty} \operatorname{ind}_T \mathfrak{v} = 0.$$

Put $\mathfrak{v} := \mathfrak{v}_1\mathfrak{v}_2$, where

$$\mathfrak{v}_1 := \frac{Q_n - P_m - Q_n\varphi^+}{i\lambda(i\lambda - 1)^{n-1}}, \qquad \mathfrak{v}_2 := \frac{(i\lambda+1)(i\lambda-1)^{n-1}}{Q_n}.$$

Clearly, $\mathfrak{v}_2 \in \mathcal{K}_- \cap \mathcal{K}$ and has exactly $n-1$ zeros in Π_-. Hence, by the argument principle, $\operatorname{ind} \mathfrak{v}_2 = -(n-1)$, and

$$\operatorname{ind} \mathfrak{v}_1 = -\operatorname{ind} \mathfrak{v}_2 = n - 1.$$

Since $\mathfrak{v}_1 \in \mathcal{K}_+ \cap \mathcal{K}$, again using the argument principle we obtain that $\mathfrak{v}_1$, as well as $1-\varphi$, has exactly $n-1$ zeros $-i\mu_1, \dots, -i\mu_{n-1}$ in Π_+. Putting

$$\mathfrak{w}_+ := \frac{(1-\varphi)Q_n}{i\lambda\prod_{k=1}^{n-1}(\mu_k - i\lambda)}, \qquad \mathfrak{w}_- := \frac{(i\lambda+1)\prod_{k=1}^{n-1}(\mu_k - i\lambda)}{Q_n},$$

we obtain a canonical factorisation.

Necessity. Similarly to the preceding arguments, the necessity follows from the factorisation identity

$$\begin{aligned} 1-\varphi(\lambda) &= c_1\big(1 - \mathbf{E}\big(e^{i\lambda\chi_+};\ \eta_+ < \infty\big)\big)\mathfrak{w}_-(\lambda) \\ &= c_1\int_0^\infty e^{i\lambda x}\,dV(x)\left(c_2 + \int_{-\infty}^0 e^{i\lambda x} g(x)\,dx\right), \end{aligned}$$

where $V(x) = \mathbf{P}(\chi_+ > x;\ \eta_+ < \infty)$ for $x > 0$, $c_i = \text{const}$ and $g(x)$ is an exponential polynomial. The theorem is proved. □

As in Sect. 12.5.1, we do not consider the case $\mathbf{E}\xi > 0$ since it reduces to applying the aforementioned argument to the random variable $-\xi$.

12.5.5 Explicit Canonical Factorisation for the Function $\mathfrak{v}^0(\lambda)$

The goal of this subsection, as it was in Sects. 12.5.3 and 12.5.4, is to find an explicit form of the components $\mathfrak{w}^0_\pm(\lambda)$ in the canonical factorisation of the function

$\mathfrak{v}^0(\lambda) = \frac{1-\varphi(\lambda)}{\lambda^2}(\lambda^2+1)$ in (12.5.6) in terms of the zeros of the function $1-\varphi(\lambda)$ in the case where $\mathbf{E}\xi = 0$ and either $\varphi^+(\lambda)$ or $\varphi^-(\lambda)$ is a rational function. When $\mathbf{E}\xi = 0$, it is sufficient to consider the case where $\varphi^+(\lambda)$ is rational, i.e. the distribution $\mathbf{F}$ has on the positive half-line a density which is an exponential polynomial, so that

$$\varphi^+(\lambda) = \frac{P_m(\lambda)}{Q_n(\lambda)}, \quad Q_n(\lambda) = \prod_{k=1}^{K}(\beta_k - i\lambda)^{l_k}, \quad n = \sum_{k=1}^{K} l_k.$$

The case where it is the function $\varphi^-(\lambda)$ that is rational is treated by switching to random variable $-\xi$.

Theorem 12.5.3 *Let* $\mathbf{E}\xi = 0$ *and* $\mathbf{E}\xi^2 = \sigma^2 < \infty$. *For the positive component* $\mathfrak{w}^0_+(\lambda)$ *of the canonical factorisation*

$$\mathfrak{v}^0(\lambda) = \mathfrak{w}^0_+(\lambda)\mathfrak{w}^0_-(\lambda), \qquad \mathfrak{w}_\pm \in \mathcal{K}_\pm \cap \mathcal{K},$$

to be a rational function it is necessary and sufficient that the function $\varphi^+(\lambda) = \mathbf{E}(e^{i\lambda\xi};\ \xi > 0)$ *is rational. If* $\varphi^+(\lambda) = P_m(\lambda)/Q_n(\lambda)$ *is an uncancellable ratio of polynomials of degrees* m *and* n, *respectively,* $m < n$, *then the function* $1-\varphi(\lambda)$ *has exactly* $n-1$ *zeros in* Π_- *which we denote by* $-i\mu_1, \dots, -i\mu_{n-1}$, *and*

$$\mathfrak{w}^0_+(\lambda) = \frac{\prod_{k=1}^{n-1}(\mu_k - i\lambda)(i\lambda - 1)}{Q_n(\lambda)}, \qquad \mathfrak{w}^0_-(\lambda) = \frac{(1-\varphi(\lambda))(i\lambda+1)Q_n(\lambda)}{\lambda^2\prod_{k=1}^{n-1}(\mu_k - i\lambda)}. \tag{12.5.25}$$

Relation (12.5.3) and the uniqueness of canonical factorisation imply the following representation.

Corollary 12.5.4 *Under the conditions of Theorem* 12.5.3,

$$\mathbf{E}e^{i\lambda\chi_+} = 1 - \frac{i\lambda\mathbf{E}\chi_+ Q_n(0)\prod_{k=1}^{n-1}(1-\frac{i\lambda}{\mu_k})}{Q_n(\lambda)}.$$

Proof The corollary follows from (12.5.7), (12.5.25), the uniqueness of canonical factorisation and the equalities

$$\mathfrak{v}^0_+(0) = \mathbf{E}\chi_+, \qquad \mathfrak{v}^0_+(\lambda) = \frac{\mathfrak{w}^0_+(\lambda)\mathbf{E}\chi_+}{\mathfrak{w}^0_+(0)},$$

$$1 - \mathbf{E}e^{i\lambda\chi_+} = \frac{\mathfrak{v}^0_+(\lambda)i\lambda}{i\lambda - 1} = \frac{i\lambda\mathbf{E}\chi_+ Q_n(0)\prod_{k=1}^{n-1}(\mu_k - i\lambda)}{Q_n(\lambda)\prod_{k=1}^{n-1}\mu_k}.$$

Thus, here the "structure" of the density of χ_+ again repeats the structure of the density of ξ for $x > 0$. □

Proof of Theorem 12.5.3 The proof is similar to that of Theorem 12.5.1.

Sufficiency.

1. In the vicinity of the point $\lambda = 0$, $\lambda \in \Pi$, the value of $\mathfrak{v}^0(\lambda)$ lies in the vicinity of the point $\sigma^2/2 > 0$ by Property 7.1.5 of ch.f.s. Outside of a neighbourhood of

zero, similarly to (12.5.17), we have

$$\arg\bigl(1-\varphi(\lambda)\bigr) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \arg\frac{\lambda^2+1}{\lambda^2} = 0.$$

This, analogously to (12.5.18), implies

$$\operatorname{ind}_T\, \mathfrak{v}^0 := \frac{1}{2\pi}\int_{-T}^{T} d\bigl(\arg \mathfrak{v}^0(\lambda)\bigr) \in (-b/2, b/2), \quad b<1.$$

2. Represent $\mathfrak{v}^0$ as $\mathfrak{v}^0 = \mathfrak{v}_1\mathfrak{v}_2$, where

$$\begin{aligned} \mathfrak{v}_1 &:= \frac{(i\lambda+1)^n}{Q_n}, \\ \mathfrak{v}_2 &:= \frac{(1-\varphi)(1-i\lambda)\,Q_n}{\lambda^2(i\lambda+1)^{n-1}} = \frac{(Q_n-P_m-Q_n\varphi^-)(1-i\lambda)}{\lambda^2(i\lambda+1)^{n-1}}. \end{aligned} \tag{12.5.26}$$

Then, similarly to (12.5.20), we find that

$$\operatorname{ind}_T\, \mathfrak{v}_1 \to n \quad \text{as } T\to\infty, \qquad |n+\operatorname{ind}_T\, \mathfrak{v}_2| < \frac{1}{2}.$$

3. We show that $1-\varphi(\lambda)$ has exactly $n-1$ zeros in Π_-. To this end, note that the function $\mathfrak{v}_2$, which is analytic in Π_- and continuous on $\Pi_-\cup\Pi$ has exactly n zeros in Π_-. As in the proof of Theorem 12.5.1, consider the contour $\mathfrak{T}_T^-$. In the same way as in the argument in this proof, we obtain that

$$\frac{1}{2\pi}\int_{\mathfrak{T}_T^-} d\bigl(\arg \mathfrak{v}_2(\lambda)\bigr) = n,$$

so that $\mathfrak{v}_2$ has exactly n zeros in Π_-. Further, by (12.5.26) we have $\mathfrak{v}_2 = \mathfrak{v}_3\mathfrak{v}_4$, where the function $\mathfrak{v}_3 = (1-i\lambda)/(i\lambda+1)$ has one zero in Π_- at the point $\lambda=-i$. Therefore the function

$$\mathfrak{v}_4 = \frac{(Q_n-P_m-Q_n\varphi^-)}{\lambda^2(i\lambda+1)^{n-2}},$$

which is analytic in Π_-, has $n-1$ zeros there. Since the zeros of $1-\varphi(\lambda)$ and those of $\mathfrak{v}_4(\lambda)$ in Π_- coincide, the assertion concerning the zeros of $1-\varphi(\lambda)$ is proved.

4. It remains to put

$$\mathfrak{w}_+^0(\lambda) := \frac{\prod_{k=1}^{n-1}(\mu_k-i\lambda)(1-i\lambda)}{Q_n(\lambda)}, \qquad \mathfrak{w}_-^0(\lambda) := \frac{(1-\varphi(\lambda))(i\lambda+1)Q_n(\lambda)}{\lambda^2\prod_{k=1}^{n-1}(\mu_k-i\lambda)}$$

and note that $\mathfrak{w}_\pm^0 \in \mathcal{K}_\pm\cap\mathcal{K}$.

Necessity is proved in exactly the same way as in Theorems 12.5.1 and 12.5.2.

The theorem is proved. □

12.6 Explicit Form of Factorisation in the Arithmetic Case

The content of this section is similar to that of Sect. 12.5 and has the same structure, but there are also some significant differences.

12.6.1 Preliminary Remarks on the Uniqueness of Factorisation

As was already noted in Sect. 12.5, for arithmetic distributions defined by collections of probabilities $p_k = \mathbf{P}(\xi = k)$, we should use, instead of the ch.f.s $\varphi(\lambda)$, the generating functions

$$p(z) = \mathbf{E}z^{\xi} = \sum_{k=-\infty}^{\infty} z^k p_k$$

defined on the unit circle $|z| = 1$, which will be denoted by Π, as the axis $\operatorname{Im}\lambda = 0$ was in Sect. 12.5. The symbols Π_+ (Π_-) will denote the interior (exterior) of Π. For arithmetic distributions we will discuss the factorisation

$$1 - p(z) = \mathfrak{f}_+(z)\mathfrak{f}_-(z)$$

on the unit circle, where $\mathfrak{f}_\pm$ are analytic on $\Pi_\pm$ and continuous including the boundary Π. Similarly to the non-lattice case, the classes of such functions, that, moreover, are bounded and bounded away from zero on $\Pi_\pm$, we will denote by $\mathcal{K}_\pm$. Continuous bounded functions on Π, which are also bounded away from zero, form the class $\mathcal{K}$. The notion of *canonical factorisation* on Π is introduced in exactly the same way as above. Factorisation components must belong to the classes $\mathcal{K}_\pm$. The uniqueness of factorisation components (up to a constant factor) is proved in the same way as in Lemma 12.1.1.

We now show that if, similarly to the above, we "tweak" the function $1 - p(z)$ then it will admit a canonical factorisation. We will denote the tweaked function and its factorisation components by the same symbols as in Sect. 12.5. This will not lead to any confusion.

Lemma 12.6.1 1. *If* $\mathbf{E}\xi < 0$ *then the function*

$$\mathfrak{v}(z) := \frac{(1 - p(z))z}{1 - z}$$

belongs to $\mathcal{K}$ *and admits a unique canonical factorisation*

$$\mathfrak{v}(z) = \mathfrak{v}_+(z)\mathfrak{v}_-(z),$$

where

$$\mathfrak{v}_+(z) := 1 - \mathbf{E}\big(z^{\chi_+};\ \eta_+ < \infty\big) = \frac{1 - p}{\mathbf{E}z^S}, \qquad p := \mathbf{P}(\eta_+ < \infty),$$

$$\mathfrak{v}_-(z) := \frac{(1 - \mathbf{E}z^{\chi_-^0})z}{1 - z}.$$

2. *If* $\mathbf{E}\xi = 0$ *and* $\mathbf{E}\xi^2 < \infty$ *then the function*

$$\mathfrak{v}^0(z) := \frac{(1 - p(z))z}{(1 - z)^2}$$

belongs to $\mathcal{K}$ *and admits a unique canonical factorisation*

$$\mathfrak{v}^0(z) = \mathfrak{v}_+^0(z)\mathfrak{v}_-^0(z),$$

where

$$\mathfrak{v}_+^0 := \frac{1 - \mathbf{E} z^{\chi_+}}{1 - z}, \qquad \mathfrak{v}_-^0 := \frac{(1 - \mathbf{E} z^{\chi_-^0}) z}{1 - z}.$$

Here we do not discuss the case $\mathbf{E}\xi > 0$ since it reduces to the case $\mathbf{E}\xi < 0$. We will also not present an analogue of Corollary 12.5.1 in view of its obviousness.

Proof of Lemma 12.6.1 Let $\mathbf{E}\xi < 0$. Since

$$\frac{(1 - p(z))z}{1 - z} \to -\mathbf{E}\xi > 0$$

as $z \to 1$, $p(z)$ is continuous on the compact Π and, furthermore, $|p(z)| < 1$ for $z \neq 1$, we see that $\mathfrak{v}(z)$ is bounded away from zero on Π and bounded, and hence belongs to $\mathcal{K}$. Further, by Corollary 12.2.2 (see (12.2.1) for $i\lambda = z$),

$$\mathfrak{v}(z) = \frac{(1 - \mathbf{E} z^{\chi_-^0}) z}{1 - z} \big[1 - \mathbf{E}\big(z^{\chi_+};\ \eta_+ < \infty\big)\big],$$

where $\mathbf{E}\chi_-^0 \in (-\infty, 0)$. Therefore, similarly to the above, we get

$$\mathfrak{v}_-(z) = \frac{(1 - \mathbf{E} z^{\chi_-^0}) z}{1 - z} \in \mathcal{K}.$$

Moreover, it is obvious that $\mathfrak{v}_-(z) \in \mathcal{K}_-$. In the same way as above, we obtain that

$$\mathfrak{v}_+(z) = 1 - \mathbf{E}\big(z^{\chi_+};\ \eta_+ < \infty\big) \in \mathcal{K}_+ \cap \mathcal{K}.$$

This proves the first assertion of the lemma.

The second assertion is proved similarly by using Corollary 12.2.4, by which

$$\mathfrak{v}^0(z) = \frac{1 - \mathbf{E} z^{\chi_+}}{1 - z} \cdot \frac{(1 - \mathbf{E} z^{\chi_-^0}) z}{1 - z}.$$

Next, as before, we establish that $\mathfrak{v}^0 \in \mathcal{K}$ and that the factors on the right-hand side, denoted by $\mathfrak{v}_\pm^0(z)$, belong to $\mathcal{K}_\pm \cap \mathcal{K}$. The lemma is proved. □

12.6.2 The Classes of Distributions on the Positive Half-Line with Rational Generating Functions

The content of Sect. 12.5.2 is mostly preserved here. Now by exponential polynomials we mean the sequences

$$p_x = \sum_{k=1}^{K} \sum_{j=1}^{l_k} a_{kj} x^{j-1} q_k^x, \quad x = 1, 2, \ldots, \tag{12.6.1}$$

where $q_k < 1$ are different (cf. (12.5.10)). To probabilities p_x of such type will correspond rational functions

$$p^+(z) = \mathbf{E}\big(z^\xi;\ \xi > 0\big) = \sum_{x=1}^{\infty} z^x p_x = \frac{P_m(z)}{Q_n(z)},$$

where $1 \le m < n$, $n = \sum_{k=1}^{K} l_k$, and, for definiteness, we put

$$Q_n(z) = \prod_{k=1}^{K} (1 - q_k z)^{l_k}. \qquad (12.6.2)$$

Here a significant difference from the non-lattice case is that, for $p^+(z)$ to be rational, we do not need (12.6.1) to be valid *for all* $x > 0$. It is sufficient that (12.6.1) holds for all x, *starting from some* $r + 1 \ge 1$. The first r probabilities $p_1, \ldots, p_r$ can be *arbitrary*. In this case $p^+(z)$ will have the form

$$p^+(z) = \frac{P_m(z)}{Q_n(z)} + T_r(z) = \frac{P_M(z)}{Q_n(z)}, \qquad (12.6.3)$$

where T_r is a polynomial of degree r (for $r = 0$ we put $T_0 = 0$), so that p^+ is again a rational function, but now the degree of the polynomial P_M

$$M = \begin{cases} m, & \text{if } r = 0, \\ n + r, & \text{if } r \ge 1 \end{cases} \qquad (12.6.4)$$

in the numerator can be greater than the degree n of the polynomial in the denominator. In what follows, we only assume that $n + r > 0$, so that *the value* $n = 0$ *is allowed* (in this case there will be no exponential part in (12.6.1)). In that case we will assume that $Q_0 = 1$ and $P_m = 0$. The distributions corresponding to (12.6.3) will also be called exponential polynomials.

12.6.3 Explicit Canonical Factorisation of the Function $\mathfrak{v}(z)$ *in the Case when the Right Tail of the Distribution* $\mathbf{F}$ *Is an Exponential Polynomial*

Consider an arithmetic distribution $\mathbf{F}$ on the whole real line $(-\infty, \infty)$, $\mathbf{E}\xi < 0$, which is an exponential polynomial on the half-line $x > 0$. As before, denote the class of all such distributions by $\mathcal{EP}$. The generating function $p(z)$ of the distribution $\mathbf{F} \in \mathcal{EP}$ can be represented as

$$p(z) = p^+(z) + p^-(z),$$

where the function

$$p^-(z) = \mathbf{E}\big(z^\xi;\ \xi \le 0\big)$$

is analytic in Π_- and continuous including the boundary Π, and $p^+(z)$ is a rational function

$$p^+(z) = \mathbf{E}\big(z^\xi;\ \xi > 0\big) = \frac{P_M(z)}{Q_n(z)}$$

analytic in Π_+.

As above, in this case the canonical factorisation of the function

$$\mathfrak{v}(z) = \frac{(1-p(z))z}{1-z}$$

can be found in explicit form in terms of the zeros of the function $1 - p(z)$.

Theorem 12.6.1 *Let there exist* $\mathbf{E}\xi < 0$. *For the positive component* $\mathfrak{w}_+(z)$ *of the canonical factorisation*

$$\mathfrak{v}(z) = \mathfrak{w}_+(z)\mathfrak{w}_-(z), \qquad w_\pm \in \mathcal{K}_\pm \cap \mathcal{K},$$

to be a rational function it is necessary and sufficient that $p^+(z) = \mathbf{E}(z^\xi;\ \xi > 0)$ *is a rational function.*

If $p^+ = P_M/Q_n$, *where* M *is defined in* (12.6.4), *is an uncancellable ratio of polynomials then the function* $1 - p(z)$ *has in* Π_- *exactly* $n + r$ *zeros, which will be denoted by* $z_1, \ldots, z_{n+r}$, *and*

$$\mathfrak{w}_+(z) = \frac{\prod_{k=1}^{n+r}(z_k - z)}{Q_n(z)},$$

where $Q_n(z_k) \neq 0$.

If we arrange the zeros $\{z_k\}$ *according to the values of* $|z_k|$ *in ascending order, then the point* $z_1 > 1$ *is a simple real zero.*

The theorem implies that

$$\mathfrak{w}_-(z) = \frac{(1-p(z))z\, Q_n(z)}{(1-z)\prod_{k=1}^{n+r}(z_k - z)}.$$

By Lemma 12.6.1, from Theorem 12.6.1 we obtain the following representation.

Corollary 12.6.1 *If* $\mathbf{E}\xi < 0$ *and* $p^+ = P_M/Q_n$ *then*

$$\mathbf{E}z^S = \frac{\mathfrak{w}_+(1)}{\mathfrak{w}_+(z)} = \frac{Q_n(z)\prod_{k=1}^{n+r}(z_k - 1)}{Q_n(1)\prod_{k=1}^{n+r}(z_k - z)}.$$

Similarly to (12.5.16), we can also write down the explicit form of $\mathbf{E}z^{\chi_-^0}$ and $\mathbf{E}(z^{\chi_+};\ \eta_+ < \infty)$ as well.

Proof of Theorem 12.6.1 The proof is similar to that of Theorem 12.5.1.

Sufficiency.

1. In the vicinity of the point $z = 1$ in Π the value of $-\mathfrak{v}(z)$ lies in the vicinity of the point $-\mathbf{E}\xi > 0$. Outside a neighbourhood of the point $z = 1$ we have for $z \in \Pi$,

$$\arg\bigl(1-p(z)\bigr)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\quad \arg\left(\frac{z}{z-1}\right)=-\arg\left(1-\frac{1}{z}\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

This implies that, for $z\in\varPi$,

$$\arg\bigl(-\mathfrak{v}(z)\bigr)\in(-\pi,\pi),$$

and hence the trajectory of $-\mathfrak{v}(z)$, $z\in\varPi$, never intersects the ray $\arg\mathfrak{v}=-\pi$,

$$\operatorname{ind}\mathfrak{v}:=\frac{1}{2\pi}\int_0^{2\pi}d\bigl(\arg\mathfrak{v}\bigl(e^{i\lambda}\bigr)\bigr)=0.$$

2. Represent the function $\mathfrak{v}$ as $\mathfrak{v}=\mathfrak{v}_1\mathfrak{v}_2$, where

$$\mathfrak{v}_1(z):=\frac{z^{n+r}}{Q_n(z)},\qquad \mathfrak{v}_2(z):=\frac{Q_n(z)-P_M(z)-p^-(z)Q_n(z)}{(1-z)z^{n+r-1}}.$$

We show that

$$\operatorname{ind}\mathfrak{v}_2=-n-r. \tag{12.6.5}$$

In order to do this, we first note that the function $\mathfrak{v}_1$ is analytic in $\varPi_+$ and has there a zero of multiplicity $n+r$. Hence by the argument principle $\operatorname{ind}\mathfrak{v}_1=n+r$. Since $0=\operatorname{ind}\mathfrak{v}=\operatorname{ind}\mathfrak{v}_1+\operatorname{ind}\mathfrak{v}_2$, we obtain the desired relation.

3. We show that $1-p(z)$ has exactly $n+r$ zeros in $\varPi_-$. The function $\mathfrak{v}_2(z)$ is analytic on $\varPi_-$ and continuous including the boundary $\varPi$. The positively oriented contour $\varPi$, which contains $\varPi_+$, corresponds to the negatively oriented contour with respect to $\varPi_-$. By (12.6.5) this means that $\mathfrak{v}_2(z)$ has precisely $n+r$ zeros on $\varPi_-$ while the point $z=\infty$ is not a zero since the numerator and the denominator of $\mathfrak{v}(z)$ grow as $|z|^{n+r}$ as $|z|\to\infty$.

4. Denote the zeros of $\mathfrak{v}_2$ by $z_1,\dots,z_{n+r}$ and put

$$\mathfrak{w}_+(z):=\frac{\prod_{k=1}^{n+r}(z_k-z)}{Q_n(z)},\qquad \mathfrak{w}_-(z):=\frac{Q_n(z)(1-p(z))z}{(1-z)\prod_{k=1}^{n+r}(z_k-z)}.$$

It is easy to see that $\mathfrak{w}_\pm\in\mathcal{K}_\pm\cap\mathcal{K}$. The fact that $Q_n(z_k)\neq 0$ and z_1 is a simple real zero of $1-p(z)$ is proved in the same way as in Theorem 12.5.1.

Necessity is also established in the same fashion as in Theorem 12.5.1. The theorem is proved. □

Clearly, in the arithmetic case we have complete analogues of Examples 12.5.1 and 12.5.2. In particular, if

$$\mathbf{P}(\xi=k)=cq^{k-1},\quad c<(1-q),\ k=1,2,\dots,$$

then

$$\mathfrak{w}_+(z)=\frac{z_1-z}{1-qz},\qquad \mathbf{E}z^S=\frac{(1-qz)(z_1-1)}{(z_1-z)(1-q)},$$

$$\mathbf{P}(S=0)=\frac{1-z_1^{-1}}{1-q},\qquad \mathbf{P}(S=k)=\frac{(z_1^{-1}-q)(z_1-1)z_1^k}{1-q},\quad k\geq 1.$$

In contrast to Sect. 12.5, here one can give another example where the distribution of S is geometric.

Example 12.6.1 Let $\mathbf{P}(\xi = 1) = p_1 > 0$ and $\mathbf{P}(\xi \ge 2) = 0$. In this case $\chi_+ \equiv 1$ on the set $\{\eta_+ < \infty\}$, and to find the distribution of S there is no need to use Theorem 12.6.1. Indeed, $\mathbf{P}(S = 0) = 1 - p = \mathbf{P}(\eta_+ = \infty)$. If $\eta_+ < \infty$ then the trajectory $\xi_{\eta_+ +1}, \xi_{\eta_+ =2}, \ldots$ is distributed identically to $\xi_1, \xi_2, \ldots$ and hence

$$S = \begin{cases} 0 & \text{with probability } 1 - p, \\ \chi_+ + S_{(1)} & \text{with probability } p, \end{cases}$$

where the variable $S_{(1)}$ is distributed identically to S, $\chi_+ \equiv 1$. This yields

$$\mathbf{E}z^S = (1 - p) + pz\mathbf{E}z^S, \qquad \mathbf{E}z^S = \frac{1 - p}{1 - pz},$$

$$\mathbf{P}(S = k) = (1 - p)p^k, \quad k = 0, 1, \ldots$$

By virtue of identity (12.3.3) (for $e^{i\lambda} = z$) the point $z_1 = p^{-1}$ is necessarily a zero of the function $1 - p(z)$.

12.6.4 *Explicit Canonical Factorisation of the Function $\mathfrak{v}(z)$ when the Left Tail of the Distribution* F *Is an Exponential Polynomial*

We now consider the case where the distribution **F** on the *negative* half-line can be represented as an exponential polynomial, up to the values of $\mathbf{P}(\xi = -k)$ at finitely many points $0, -1, -2, \ldots, -r$. In this case, the value of $p^-(z)$ is derived similarly to that of $p^+(z)$ in (12.6.3) by replacing z with z^{-1}:

$$p^-(z) = \mathbf{E}\big(z^\xi;\ \xi < 0\big) = \frac{z^{n-M} P_M(z)}{Q_n(z)},$$

where Q_n and P_M are polynomials (which differ from (12.6.3)),

$$M = \begin{cases} m, & \text{if } r = 0, \\ n + r, & \text{if } r \ge 1, \end{cases} \qquad Q_n(z) = \prod_{k=1}^{K} (z - q_k)^{l_k},$$

and all $q_k < 1$ are distinct.

Theorem 12.6.2 *Let there exist* $\mathbf{E}\xi < 0$. *For the positive component of the canonical factorisation*

$$\mathfrak{v}(z) = \mathfrak{w}_+(z)\mathfrak{w}_-(z)$$

to be representable as

$$\mathfrak{w}_+(z) = \big(1 - p(z)\big)R(z),$$

where $R(z)$ is a rational function, it is necessary and sufficient that $p^-(z)$ is a rational function. If

$$p^-(z) = \frac{z^{n-M} P_M(z)}{Q_n(z)},$$

where P_M and Q_n are defined in (12.6.2) *and* (12.6.3), *then the function* $1 - p(z)$ *has in* Π_+ *exactly* $n + r - 1$ *zeros that we denote by* $z_1, \ldots, z_{n+r-1}$, *and*

$$R(z) := \frac{Q_n(z)}{(1-z)\prod_{k=1}^{n+r-1}(z - z_k)}.$$

Proof The proof is very close to that of Theorems 12.5.2 and 12.6.1. Therefore we will only present a brief proof of *sufficiency*.

1. As in Theorem 12.6.1, one can verify that

$$\operatorname{ind} \mathfrak{v} = 0.$$

2. Represent $\mathfrak{v}(z)$ as $\mathfrak{v} = \mathfrak{v}_1\mathfrak{v}_2$, where

$$\mathfrak{v}_1 := \frac{(Q_n(z) - z^{n-M}P_M(z) - p^+(z)Q_n(z))z^r}{(1-z)}, \qquad \mathfrak{v}_2(z) := \frac{z^{1-r}}{Q_n(z)}.$$

The function $\mathfrak{v}_2$ is analytic in Π_-, continuous including the boundary Π, and has a zero at $z = \infty$ of multiplicity $n + r - 1$, so that

$$\operatorname{ind} \mathfrak{v}_2 = n + r - 1.$$

The function $\mathfrak{v}_1$ is analytic in Π_+ and, by the argument principle, has there $n + r - 1$ zeros $z_1, \ldots, z_{n+r-1}$. The function $1 - p(z)$ has the same zeros.

3. By putting

$$\mathfrak{w}_+(z) := \frac{(1-p(z))Q_n(z)}{(1-z)\prod_{k=1}^{n+r-1}(z - z_k)}, \qquad \mathfrak{w}_-(z) := \frac{z\prod_{k=1}^{n+r-1}(z - z_k)}{Q_n(z)}$$

we obtain $\mathfrak{w}_\pm \in \mathcal{K}_\pm \cap \mathcal{K}$. The theorem is proved. □

12.6.5 Explicit Factorisation of the Function $\mathfrak{v}^0(z)$

By virtue of the remarks at the beginning of Sect. 12.5.5 it is sufficient to consider factorisation of the function

$$\mathfrak{v}^0(z) := \frac{(1-p(z))z}{(1-z)^2}$$

for $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 < \infty$ just in the case when the function

$$p^+(z) = \mathbf{E}\big(z^\xi;\ \xi > 0\big) = \frac{P_M(z)}{Q_n(z)}$$

is rational, where $Q_n(z) = \prod_{k=1}^{K}(1 - q_k z)^{l_k}$, $n = \sum_{k=1}^{K} l_k$ (see (12.6.2), (12.6.3)).

Theorem 12.6.3 *Let* $\mathbf{E}\xi = 0$ *and* $\mathbf{E}\xi^2 = \sigma^2 < \infty$. *For the positive component* $\mathfrak{w}_+^0(z)$ *of the canonical factorisation*

$$\mathfrak{v}^0(z) = \mathfrak{w}_+^0(z)\mathfrak{w}_-^0(z), \qquad w_\pm \in \mathcal{K}_\pm \cap \mathcal{K},$$

to be rational, it is necessary and sufficient that the function $p^+(z)$ *is rational. If* $p^+(z) = P_M(z)/Q_n(z)$, *where* M *is defined in* (12.6.4), *is an uncancellable ratio of polynomials then the function* $1 - p(z)$ *has in* Π_- *exactly* $n + r - 1$ *zeros that we denote by* $z_1, \dots, z_{n+r-1}$, *and*

$$\mathfrak{w}^0_+(z) = \frac{\prod_{k=1}^{n+r-1}(z_k - z)}{Q_n(z)}, \qquad \mathfrak{w}^0_-(z) = \frac{(1 - p(z))zQ_n(z)}{(1 - z)^2 \prod_{k=1}^{n+r-1}(z_k - z)}.$$

Corollaries similar to Corollary 12.5.4 hold true here as well.

Proof of Theorem 12.6.3 The proof is similar to those of Theorems 12.5.3, 12.6.1 and 12.6.2. Therefore, as in the previous theorem, we restrict ourselves to the key elements of the proof of sufficiency.

1. In the vicinity of the point $z = 1$ in Π, the value of $-\mathfrak{v}^0(z)$ lies in the vicinity of the point $\sigma^2/2 > 0$. Outside of a neighbourhood of the point $z = 1$, for $z \in \Pi$ we have

$$\arg\bigl(1 - p(z)\bigr) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$\arg \frac{-z}{(1 - z)^2} = -\arg\left((1 - z)\left(1 - \frac{1}{z}\right)\right) = -\arg\left(2 - z - \frac{1}{z}\right) = 0.$$

Hence

$$\operatorname{ind} \mathfrak{v}^0 := \frac{1}{2\pi} \int_0^{2\pi} d\bigl(\arg \mathfrak{v}^0\bigl(e^{i\lambda}\bigr)\bigr) = 0.$$

2. Represent the function $\mathfrak{v}^0(z)$ as

$$\mathfrak{v}^0(z) = \mathfrak{v}_1(z)\mathfrak{v}_2(z),$$

where

$$\mathfrak{v}_1(z) := \frac{z^{n+r-1}}{Q_n(z)}, \qquad \mathfrak{v}_2(z) := \frac{Q_n - P_M - p^-(z)Q_n}{(1 - z)^2 z^{n+r-2}}.$$

As before, we show that $\operatorname{ind} \mathfrak{v}_1 = n + r - 1$ and that $1 - p(z)$ has, on Π_-, exactly $n + r - 1$ zeros, which are denoted by $z_1, \dots, z_{n+r-1}$. It remains to put

$$\mathfrak{w}^0_+(z) = \frac{\prod_{k=1}^{n+r-1}(z_k - z)}{Q_n(z)}, \qquad \mathfrak{w}^0_-(z) = \frac{Q_n(z)(1 - p(z))z}{(1 - z)^2 \prod_{k=1}^{n+r-1}(z_k - z)}.$$

The theorem is proved. □

12.7 Asymptotic Properties of the Distributions of $\chi_\pm$ and S

We saw in the previous sections that one can find the distributions of the variables S and $\chi_\pm$ in explicit form only in some special cases. Meanwhile, in applied problems of, say, risk theory (see Sect. 12.4) one is interested in the values of $\mathbf{P}(S > x)$ for large x (corresponding to small ruin probabilities). In this connection there arises the problem on the asymptotic behaviour of $\mathbf{P}(S > x)$ as $x \to \infty$, as well as related problems on the asymptotics of $\mathbf{P}(|\chi_\pm| > x)$. It turns out that these problems can be solved under rather broad conditions.

12.7.1 *The Asymptotics of* $\mathbf{P}(\chi_+ > x \,|\, \eta_+ < \infty)$ *and* $\mathbf{P}(\chi_-^0 < -x)$ *in the Case* $\mathbf{E}\xi \le 0$

We introduce some classes of functions that will be used below.

Definition 12.7.1 A function $G(t)$ is called (asymptotically) *locally constant* (l.c.) if, for any fixed v,

$$\frac{G(t+v)}{G(t)} \to 1 \quad \text{as } t \to \infty. \tag{12.7.1}$$

It is not hard to see that, say, the functions $G(t) = t^\alpha [\ln(1+t)]^\gamma$, $t > 0$, are l.c.

We denote the class of all l.c. functions by $\mathcal{L}$. The properties of functions from $\mathcal{L}$ are studied in Appendix 6. In particular, it is established that (12.7.1) holds uniformly in v on any fixed segment, and that $G(t) = e^{o(t)}$ and $G(t) = o(G^I(t))$ as $t \to \infty$, where

$$G^I(t) := \int_t^\infty G(u)\,du. \tag{12.7.2}$$

Denote by $\mathcal{E}$ the class of distributions satisfying the right-hand side Cramér condition (the exponential class). The class $\mathcal{E}^* \subset \mathcal{E}$ of distributions $\mathbf{G}$ whose "tails" $G(t) = \mathbf{G}((t,\infty))$ satisfy, for any fixed $v > 0$, the relation

$$\frac{G(t+v)}{G(t)} \to 0 \quad \text{as } t \to \infty, \tag{12.7.3}$$

could be called the "superexponential" class. For example, the normal distribution belongs to $\mathcal{E}^*$. In the arithmetic case, one has to put $v = 1$ in (12.7.3) and consider integer-valued t.

In the case $\mathbf{E}\xi \le 0$ it is convenient to introduce a random variable χ with the distribution

$$\mathbf{P}(\chi \in dv) = \mathbf{P}(\chi_+ \in dv \,|\, \eta_+ < \infty) = \frac{\mathbf{P}(\chi_+ \in dv;\ \eta_+ < \infty)}{p}, \qquad p = \mathbf{P}(\eta_+ < \infty).$$

If $\mathbf{E}\xi = 0$ then the distributions of χ and χ_+ coincide. In the sequel we will confine ourselves to non-lattice ξ (then $\chi_\pm$ will also be non-lattice). In the arithmetic case everything will look quite similar.

Denote by $F_+(t)$ the right "tail" of the distribution $\mathbf{F}$: $F_+(t) := \mathbf{F}((t,\infty))$ and put

$$F_+^I(t) := \int_t^\infty F_+(u)\,du.$$

Theorem 12.7.1 *Let there exist* $\mathbf{E}\xi \le 0$ *and, in the case* $\mathbf{E}\xi = 0$, *assume* $\mathbf{E}\xi^2 < \infty$ *holds.*

1. *If* $F_+(t) = o(F_+^I(t))$ *as* $t \to \infty$ *then, as* $x \to \infty$,

$$\mathbf{P}(\chi > x) \sim -\frac{F_+^I(x)}{p\mathbf{E}\chi_-^0}. \tag{12.7.4}$$

2. *If* $F_+(t) = V(t)e^{-\beta t}$, $\beta > 0$, $V \in \mathcal{L}$ *then*

$$\mathbf{P}(\chi > x) \sim \frac{F_+(x)}{p(1 - \mathbf{E}e^{\beta\chi_-^0})}. \tag{12.7.5}$$

3. *If* $F_+ \in \mathcal{E}^*$ *then*

$$\mathbf{P}(\chi > x) \sim \frac{F_+(x)}{p\mathbf{P}(\chi_-^0 < 0)}. \tag{12.7.6}$$

Proof The proof is based on identity (12.2.1) of Corollary 12.2.2, which can be rewritten as

$$1 - p\mathbf{E}e^{i\lambda\chi} = \frac{1-\varphi(\lambda)}{1-\varphi_-^0(\lambda)}, \qquad \varphi_-^0(\lambda) := \mathbf{E}e^{i\lambda\chi_-^0}. \tag{12.7.7}$$

Introduce the renewal function $H_-(t)$ corresponding to the random variable $\chi_-^0 \le 0$:

$$H_-(t) = \sum_{k=0}^{\infty} \mathbf{P}(H_k \ge t), \qquad H_k = \chi_-^{(1)} + \cdots + \chi_-^{(k)},$$

where $\chi_-^{(k)}$ are independent copies of χ_-^0, $a_- := \mathbf{E}\chi_-^0 > -\infty$. As was noted in Sect. 10.1, the function $1/(1-\varphi_-^0(\lambda))$ can be represented as

$$\frac{1}{1-\varphi_-^0(\lambda)} = -\int_{-\infty}^{0} e^{i\lambda t}\, dH_-(t)$$

(the function $H_-(t)$ decreases). Therefore, for $x > 0$ and any $N > 0$, we obtain from (12.7.7) that

$$p\mathbf{P}(\chi > x) = -\int_{-\infty}^{0} dH_-(t)\, F_+(x-t) = -\int_{-N}^{0} - \int_{-\infty}^{-N}. \tag{12.7.8}$$

Here, by the condition of assertion 1,

$$-\int_{-N}^{0} \le F_+(x)\big[H_-(-N) - H_-(0)\big] = o\big(F_+^I(x)\big) \quad \text{as } x \to \infty.$$

Evidently, this relation will still be true when $N \to \infty$ slowly enough as $x \to \infty$. Furthermore, by the local renewal theorem, as $N \to \infty$,

$$-\int_{-\infty}^{-N} dH_-(t)\, F_+(x-t) \sim \int_{-\infty}^{-N} F_+(x-t)\frac{dt}{|a_-|} = \frac{F_+^I(x+N)}{|a_-|}. \tag{12.7.9}$$

For a formal justification of this relation, the interval $(-\infty, -N]$ should be divided into small intervals $(-N_{k+1}, -N_k]$, $k = 0, 1, \ldots$, $N_0 = N$, $N_{k+1} > N_k$, on each of which we use the local renewal theorem, so that

$$\frac{F_+(x-N_k)(N_{k+1}-N_k)}{|a_-|}\bigl(1+o(1)\bigr) \le -\int_{-N_{k+1}}^{-N_k} dH_-(t)\,F_+(x-t) \le \frac{F_+(x-N_{k+1})(N_{k+1}-N_k)}{|a_-|}\bigl(1+o(1)\bigr).$$

From here it is not difficult to obtain the required bounds for the left-hand side of (12.7.9) that are asymptotically equivalent to the right-hand side. Since, for N growing slowly enough,

$$F_+^I(x)-F_+^I(x+N)=\int_x^{x+N} F_+(u)\,du < F_+(x)N = o\bigl(F_+^I(x)\bigr)$$

one has $F_+^I(x+N)\sim F_+^I(x)$, and we finally obtain the relation

$$p\mathbf{P}(\chi > x)\sim \frac{F_+^I(x+N)}{|a_-|}.$$

This proves (12.7.4).

If $F_+(t)=V(t)e^{-\beta t}$, $V\in\mathcal{L}$, then we find from (12.7.8) that

$$p\mathbf{P}(\chi > x)\sim -V(x)e^{-\beta x}\int_{-\infty}^0 dH_-(t)\,e^{t\beta} = \frac{F_+(x)}{1-\mathbf{E}e^{\beta\chi_-^0}}.$$

This proves (12.7.5).

Now let $F_+\in\mathcal{E}^*$. If we denote by $h_0>0$ the jump of the function $H_-(t)$ at the point 0 then, clearly,

$$-\int_{-\infty}^0 dH_-(t)\,\frac{F_+(x-t)}{F_+(x)}\to h_0 \quad \text{as } x\to\infty,$$

and hence

$$p\mathbf{P}(\chi > x)\sim F_+(x)h_0.$$

If we put $q:=\mathbf{P}(\chi_-^0=0)$ then h_0, being the average time spent by the random walk $\{H_k\}$ at the point 0, equals

$$h_0=\sum_{k=0}^\infty q^k=\frac{1}{1-q}.$$

The theorem is proved. □

Now consider the asymptotics of $\mathbf{P}(\chi_-^0<-x)$ as $x\to\infty$.

Put $F_-(t):=\mathbf{F}((-\infty,-t))=\mathbf{P}(\xi<-t)$.

Theorem 12.7.2 *Let* $\mathbf{E}\xi<0$.

1. *If* $F_-\in\mathcal{L}$ *then, as* $x\to\infty$,

$$\mathbf{P}\bigl(\chi_-^0<-x\bigr)\sim\frac{F_-(x)}{1-p}.$$

2. *If* $F_-(t) = e^{-\gamma t} V(t)$, $V(t) \in \mathcal{L}$, *then*

$$\mathbf{P}\big(\chi_-^0 < -x\big) \sim \frac{\mathbf{E}e^{-\gamma S} F_-(x)}{1-p}.$$

3. *If* $F_- \in \mathcal{E}^*$ *then*

$$\mathbf{P}\big(\chi_-^0 < -x\big) \sim \frac{F_-(x)\mathbf{P}(S=0)}{1-p}.$$

Proof Making use of identity (12.3.3):

$$1 - \varphi_-^0(\lambda) = \frac{(1-\varphi(\lambda))\mathbf{E}\, e^{i\lambda S}}{1-p}, \qquad \varphi_-^0(\lambda) = \mathbf{E}e^{i\lambda\chi_-^0}.$$

This implies that $\mathbf{P}(\chi_-^0 < -x)$ is the weighted mean of the value $F_-(x+t)$ with the weight function $\mathbf{P}(S \in dt)/(1-p)$:

$$\mathbf{P}\big(\chi_-^0 < -x\big) = \frac{1}{1-p}\int_0^\infty \mathbf{P}(S \in dt)\, F_-(x+t).$$

From here the assertions of the theorem follow in an obvious way. □

If $\mathbf{E}\xi = 0$ then the asymptotics of $\mathbf{P}(\chi_-^0 < -x)$ will be different.

12.7.2 The Asymptotics of $\mathbf{P}(S > x)$

We will study the asymptotics of $\mathbf{P}(S > x)$ in the two non-overlapping and mutually complementary cases where $F_+ \in \mathcal{E}$ (the Cramér condition holds) and where F_+^I belongs to the class $\mathcal{S}$ of subexponential functions.

Definition 12.7.2 A distribution $\mathbf{G}$ on $[0,\infty)$ with the tail $G(t) := \mathbf{G}([t,\infty))$ belongs to the class $\mathcal{S}_+$ of *subexponential distributions on the positive half-line* if

$$G^{2*}(t) \sim 2G(t) \quad \text{as } t \to \infty. \tag{12.7.10}$$

A distribution $\mathbf{G}$ on the whole real line belongs to the class $\mathcal{S}$ of *subexponential distributions* if the distribution $\mathbf{G}^+$ of the positive part $\zeta^+ = \max\{0,\zeta\}$ of a random variable $\zeta \Subset \mathbf{G}$ belongs to $\mathcal{S}_+$. A random variable is called *subexponential* if its distribution is subexponential.

As we will see later (see Theorem A6.4.3 in Appendix 6), the subexponentiality distribution $\mathbf{G}$ is in essence a property of the asymptotics of the *tail* of $G(t)$ as $t \to \infty$. Therefore we can also talk about *subexponential functions*. A nonincreasing function $G_1(t)$ on $(0,\infty)$ is called *subexponential* if the distribution $\mathbf{G}$ with a tail $G(t)$ such that $G(t) \sim cG_1(t)$ as $t \to \infty$ for some $c > 0$ is subexponential. (For example, distributions with tails $G_1(t)/G_1(0)$ or $\min(1, G_1(t))$ if $G_1(0) > 1$.)

The properties of subexponential distributions are studied in Appendix 6. In particular, it is established that $\mathcal{S} \subset \mathcal{L}$, $\mathcal{R} \subset \mathcal{S}$ ($\mathcal{R}$ is the class of regularly varying functions) and that $G(t) = o(G^I(t))$ if $G^I \in \mathcal{S}$.

Theorem 12.7.3 *If $F_+^I(t) \in \mathcal{S}$ and $a = \mathbf{E}\xi < 0$, then, as $x \to \infty$,*

$$\mathbf{P}(S > x) \sim \frac{1}{|a|} F_+^I(x). \tag{12.7.11}$$

Proof Making use of the identity from Theorem 12.3.2:

$$\mathbf{E}e^{i\lambda S} = \frac{1-p}{1-p\varphi_\chi(\lambda)}, \qquad \varphi_\chi(\lambda) := \mathbf{E}e^{i\lambda\chi}, \tag{12.7.12}$$

it follows that

$$\mathbf{E}e^{i\lambda S} = (1-p)\sum_{k=0}^{\infty} p^k \varphi_\chi^k(\lambda),$$

and hence, for $x > 0$,

$$\mathbf{P}(S > x) = (1-p)\sum_{k=1}^{\infty} p^k \mathbf{P}(H_k > x), \qquad H_k := \sum_{j=1}^{k} \chi_j, \tag{12.7.13}$$

where χ_j are independent copies of χ. By assertion 1 of Theorem 12.7.1 the distribution of χ is subexponential, while by Theorem A6.4.3 of Appendix 6, as $x \to \infty$, for each fixed k one has

$$\mathbf{P}(H_k > x) \sim k\mathbf{P}(\chi > x). \tag{12.7.14}$$

Moreover, again by Theorem A6.4.3 of Appendix 6, for any $\varepsilon > 0$, there exists a $b = b(\varepsilon)$ such that, for all x and $k \ge 2$,

$$\frac{\mathbf{P}(H_k > x)}{\mathbf{P}(\chi > x)} < b(1+\varepsilon)^k.$$

Therefore, for $(1+\varepsilon)p < 1$, the series

$$\sum_{k=1}^{\infty} p^k \frac{\mathbf{P}(H_k > x)}{\mathbf{P}(\chi > x)}$$

converges uniformly in x. Passing to the limit as $x \to \infty$, by virtue of (12.7.14) we obtain that

$$\lim_{x\to\infty} \frac{\mathbf{P}(S > x)}{\mathbf{P}(\chi > x)} = (1-p)\sum_{k=1}^{\infty} kp^k = \frac{p}{1-p}$$

or, which is the same, that

$$\mathbf{P}(S > x) \sim \frac{p\mathbf{P}(\chi > x)}{1-p} \quad \text{as } x \to \infty,$$

where, by Theorem 12.7.1,

$$\mathbf{P}(\chi > x) \sim -\frac{F_+^I(x)}{p\mathbf{E}\chi_-^0}.$$

Since, by Corollary 12.2.3,

$$(1-p)\mathbf{E}\chi_-^0 = \mathbf{E}\xi,$$

we obtain (12.7.11). The theorem is proved. □

Now consider the case when $\mathbf{F}$ satisfies the right-hand side Cramér condition ($\mathbf{F}_+ \in \mathcal{E}$). For definiteness, we will again assume that the distribution $\mathbf{F}$ is non-lattice. Furthermore, we will assume that there exists an $\mu_1 > 0$ such that

$$\psi(\mu_1) := \mathbf{E}e^{\mu_1 \xi} = 1, \qquad b := \mathbf{E}\xi e^{\mu_1 \xi} = \psi'(\mu_1) < \infty. \tag{12.7.15}$$

In this case the Cramér transform of the distribution of $\mathbf{F}$ at the point μ_1 will be of the form

$$\mathbf{F}_{(\mu_1)}(dt) = \frac{e^{\mu_1 t}\mathbf{F}(dt)}{\psi(\mu_1)} = e^{\mu_1 t}\mathbf{F}(dt). \tag{12.7.16}$$

A random variable $\xi_{(\mu_1)}$ with the distribution $\mathbf{F}_{(\mu_1)}$ has, by (12.7.15), a finite expectation equal to b. Denote the size of the first overshoot of the level x by a random walk with jumps $\xi_{(\mu_1)}$ by $\chi_{(\mu_1)}(x)$. By Corollary 10.4.1, the distribution of $\chi_{(\mu_1)}(x)$ converges, as $x \to \infty$, to the limiting distribution: $\chi_{(\mu_1)}(x) \Rightarrow \chi_{(\mu_1)}$, so that

$$\mathbf{E}e^{-\mu_1 \chi_{(\mu_1)}(x)} \to \mathbf{E}e^{-\mu_1 \chi_{(\mu_1)}}. \tag{12.7.17}$$

Theorem 12.7.4 *Let $\mathbf{F}_+ \in \mathcal{E}$ and (12.7.15) be satisfied. Then, as $x \to \infty$,*

$$\mathbf{P}(S > x) \sim c e^{-\mu_1 x}, \tag{12.7.18}$$

where $c = \mathbf{E}e^{-\mu_1 \chi_{(\mu_1)}} < 1$.

There is a somewhat different interpretation of the constant c in Remark 15.2.3. Exact upper and lower bounds for $e^{\mu_1 x}\mathbf{P}(S > x)$ are contained in Theorem 15.3.5.

Note that the finiteness of $\mathbf{E}\xi < 0$ is not assumed in Theorem 12.7.4. In the arithmetic case, we have to consider only integer x.

Proof Put $\eta(x) := \min\{n \ge 1 : S_n > x\}$, $X_n := x_1 + \cdots + x_n$ and $\overline{X}_n := \max_{k \le n} X_k$. Then

$$\mathbf{P}(S > x) = \mathbf{P}\big(\eta(x) < \infty\big) = \sum_{n=1}^{\infty} \mathbf{P}\big(\eta(x) = n\big), \tag{12.7.19}$$

where

$$\begin{aligned}
\mathbf{P}\big(\eta(x) = n\big) &= \underbrace{\int \ldots \int}_{n} \mathbf{F}(dx_1)\ldots\mathbf{F}(dx_n)\,\mathrm{I}(\overline{X}_{n-1} \le x, X_n > x) \\
&= \underbrace{\int \ldots \int}_{n} \mathbf{F}_{(\mu_1)}(dx_1)\ldots\mathbf{F}_{(\mu_1)}(dx_n) e^{-\mu_1 X_n}\,\mathrm{I}(\overline{X}_{n-1} \le x, X_n > x) \\
&= \mathbf{E}_{(\mu_1)} e^{-\mu_1 S_n}\mathrm{I}\big(\eta(x) = n\big).
\end{aligned} \tag{12.7.20}$$

Here $\mathbf{E}_{(\mu_1)}$ denotes the expectation when taken assuming that the distribution of the summands ξ_i is $\mathbf{F}_{(\mu_1)}$. By the convexity of the function $\psi(\mu) = \mathbf{E}e^{\mu\xi}$,

$$\mathbf{E}_{(\mu_1)}\xi = \int x e^{\mu_1 x}\mathbf{F}(dx) = \psi'(\mu_1) = b > 0,$$

and hence

$$\mathbf{P}_{(\mu_1)}\big(\eta(x) < \infty\big) = 1.$$

Therefore, returning to (12.7.19), we obtain

$$\mathbf{P}(S > x) = \mathbf{E}_{(\mu_1)}\sum_{k=1}^{\infty} e^{-\mu_1 S_n}\,\mathrm{I}\big(\eta(x) = n\big) = \mathbf{E}_{(\mu_1)}e^{-\mu_1 S_{\eta(x)}}, \qquad (12.7.21)$$

where $S_{\eta(x)} = x + \chi_{(\mu_1)}(x)$ and, by (12.7.17),

$$e^{\mu_1 x}\mathbf{P}(S > x) \to c = \mathbf{E}e^{-\mu_1\chi_{(\mu_1)}} < 1.$$

This proves (12.7.18). For arithmetic ξ the proof is the same. We only have to replace $\mathbf{F}(dt)$ in (12.7.15) and (12.7.16) by $p_k = \mathbf{P}(\xi = k)$, as well as integration by summation. The theorem is proved. □

Corollary 12.7.1 *If, in the arithmetic case,* $\mathbf{E}\xi < 0$, $p_1 = \mathbf{P}(\xi = 1) > 0$, $\mathbf{P}(\xi \ge 2) = 0$ *then the conditions of Theorem* 12.7.4 *are satisfied and one has*

$$\mathbf{P}(S > x) = e^{-\mu_1(k+1)}, \quad k \ge 0.$$

Proof The proof follows immediately from (12.7.21) if we note that, in the case under consideration, $\chi_{(\mu_1)}(x) \equiv 1$ and $S_{\eta(x)} = x + 1$. This assertion repeats the result of Example 12.6.1. □

Remark 12.7.1 The asymptotics (12.7.18), obtained by a probabilistic argument, admits a simple analytic interpretation. From (12.7.18) it follows that, as $\mu \uparrow \mu_1$, we have

$$\mathbf{E}e^{\mu S} \sim \frac{c\mu_1}{\mu_1 - \mu}.$$

But that $\mathbf{E}e^{\mu S}$ has precisely this form follows from identity (12.3.3):

$$\mathbf{E}e^{\mu S} = \frac{(1-p)(1-\mathbf{E}e^{\mu\chi_-^0})}{1-\psi(\mu)}.$$

Indeed, since, by assumption, $\psi(\mu) = \mathbf{E}e^{\mu\xi}$ is left-differentiable at the point μ_1 and

$$\psi(\mu) = 1 - b(\mu_1 - \mu) + o\big((\mu_1 - \mu)\big), \qquad (12.7.22)$$

one has

$$\mathbf{E}e^{\mu S} \sim \frac{(1-p)(1-\mathbf{E}e^{\mu_1\chi_-^0})}{b(\mu_1 - \mu)} \qquad (12.7.23)$$

as $\mu \uparrow \mu_1$. This implies, in particular, yet another representation for the constant c in (12.7.18):

$$c = \frac{(1-p)(1-\mathbf{E}e^{\mu_1 \chi_-^0})}{b}.$$

Since

$$\mathbf{E}e^{\mu S} = \frac{\mathfrak{w}_+(0)}{\mathfrak{w}_+(\lambda)}$$

and $\mathfrak{w}_+(\lambda)$ has a zero at the point μ_1, we can obtain representations similar to (12.7.22) and (12.7.23) in terms of the values of $\mathfrak{w}_+(0)$ and $\mathfrak{w}'(\mu_1)$.

We should also note that the proof of asymptotics (12.7.18) with the help of relations of the form (12.7.23) is based on certain facts from mathematical analysis and is relatively simple only under the additional condition (12.5.1).

There are other ways to prove (12.7.18), but they also involve additional restrictions. For instance, (12.3.3) implies

$$\mathbf{E}e^{i\lambda S} = (1-p)\sum_{k=0}^{\infty}\bigl[\varphi^k(\lambda) - \varphi^k(\lambda)\mathbf{E}\,e^{i\lambda\chi_-^0}\bigr],$$

$$\begin{aligned}\mathbf{P}(S > x) &= (1-p)\sum_{k=0}^{\infty}\bigl[\mathbf{P}(S_k > x) - \mathbf{P}\bigl(S_k + \chi_-^0 > x\bigr)\bigr]\\ &= (1-p)\int_0^{\infty}\mathbf{P}\bigl(\chi_-^0 \in dt\bigr)\sum_{k=0}^{\infty}\mathbf{P}\bigl(S_k \in (x, x+t]\bigr),\end{aligned}$$

and the problem now reduces to integro-local theorems for large deviations of S_k (see Chap. 9) or to local theorems for the renewal function in the region where the function converges to zero.

12.7.3 The Distribution of the Maximal Values of Generalised Renewal Processes

Let $\{(\tau_i, \zeta_i)\}_{j=1}^{\infty}$ be a sequence of independent identically distributed random vectors,

$$Z(t) = Z_{\nu(t)},$$

where

$$Z_n := \sum_{j=1}^{n}\zeta_j, \qquad \nu(t) := \max\{k : T_k \le t\}, \qquad T_k := \sum_{j=1}^{k}\tau_j.$$

In Sect. 12.4.3 we reduced the problem of finding the distribution of $\sup_t (Z(t) - qt)$ to that of the distribution of $S := \sup_{k\ge 0} S_k$, $S_k := \sum_{j=1}^{k}\xi_j$, $\xi_j := \zeta_j - q\tau_j$ in the case $q > 0$, $\zeta_k \ge 0$. We show that such a reduction takes place in the general case as well. If $q \ge 0$ and the ζ_k can take values of both signs, then the reduction is the same as in Sect. 12.4.3. Now if $q < 0$ then

$$\sup_t (Z_{\nu(t)} - qt) = \sup(-qT_1, Z_1 - qT_2, Z_2 - qT_3, \ldots)$$
$$= -q\tau_1 + \sup_{k \ge 1}\big[Z_{k-1} - q(T_k - \tau_1)\big] \overset{d}{=} S - q\tau,$$

where the random variables τ_1 and S are independent.

12.8 On the Distribution of the First Passage Time

12.8.1 The Properties of the Distributions of the Times $\eta_\pm$

In this section we will establish a number of relations between the random variables $\eta_\pm$ and the time θ when the global maximum $S = \sup S_k$ is attained for the first time:

$$\theta := \min\{k : S_k = S\} \quad (\text{if } S < \infty \text{ a.s.}).$$

Put

$$P(z) := \sum_{k=0}^{\infty} z^k \mathbf{P}\big(\eta_-^0 > k\big), \qquad q(z) := \mathbf{E}\big(z^{\eta_+} \big| \eta_+ < \infty\big),$$
$$D_+ := \sum_{k=1}^{\infty} \frac{\mathbf{P}(S_k > 0)}{k}.$$

Further, let η be a random variable with the distribution

$$\mathbf{P}(\eta = k) = \mathbf{P}(\eta_+ = k \mid \eta_+ < \infty)$$

(and the generating function $q(z)$), $\eta_1, \eta_2, \ldots$ be independent copies of η,

$$H_k := \eta_1 + \cdots + \eta_k, \qquad H_0 = 0,$$

and ν be a random variable independent of $\{\eta_k\}$ with the geometric distribution $\mathbf{P}(\nu = k) = (1-p)p^k$, $k \ge 0$.

Theorem 12.8.1 *If $p = \mathbf{P}(\eta_+ < \infty) < 1$ then*

1. $$1 - p = \frac{1}{\mathbf{E}\eta_-^0} = e^{-D_+}. \tag{12.8.1}$$
2. $$P(z) = \frac{1}{1 - pq(z)} = \frac{\mathbf{E}z^\theta}{1-p}. \tag{12.8.2}$$
3. $$\mathbf{P}\big(\eta_-^0 > n\big) = (1-p)\mathbf{P}(H_\nu = n) > \mathbf{P}(\eta_+ = n) \tag{12.8.3}$$

for all $n \ge 0$.

Recall that, for the condition $p < 1$ to hold, it is sufficient that $\mathbf{E}\xi < 0$ (see Corollary 12.2.6).

The second assertion of the theorem implies that the distributions of η_-^0, η_+ and θ uniquely determine each other, so that if at least one of them is known then, to find the other two, it is not necessary to know the original distribution $\mathbf{F}$. In particular, $\mathbf{P}(\theta = n) = (1-p)\mathbf{P}(\chi_-^0 > n)$.

Proof of Theorem 12.8.1 The arguments in this subsection are based on the following identities which follow from Theorems 12.1.1–12.1.3 if we put there $\lambda = 0$ and $|z| < 1$:

$$1 - z = \big[1 - \mathbf{E}z^{\eta_-^0}\big]\big[1 - \mathbf{E}\big(z^{\eta_+};\ \eta_+ < \infty\big)\big], \tag{12.8.4}$$

$$1 - \mathbf{E}z^{\eta_-^0} = \exp\Bigg\{-\sum_{k=1}^{\infty}\frac{z^k}{k}\mathbf{P}(S_k \le 0)\Bigg\}, \tag{12.8.5}$$

$$1 - \mathbf{E}\big(z^{\eta_+};\ \eta_+ < \infty\big) = \exp\Bigg\{-\sum_{k=1}^{\infty}\frac{z^k}{k}\mathbf{P}(S_k > 0)\Bigg\}. \tag{12.8.6}$$

Since

$$\frac{1 - \mathbf{E}z^{\eta_-^0}}{1 - z} = P(z), \qquad P(1) = \mathbf{E}\eta_-^0$$

we obtain from (12.8.4) the first equalities in (12.8.1) and (12.8.2). The second equality in (12.8.1) follows from (12.8.6).

To prove the second equality in (12.8.2), we make use of the relation

$$\theta = \begin{cases} 0 & \text{on } \{\omega : \eta_+ = \infty\}, \\ \eta_+ + \theta^* & \text{on } \{\omega : \eta_+ < \infty\}, \end{cases}$$

where θ^* is distributed on $\{\eta_+ < \infty\}$ identically to θ and does not depend on η_+. It follows that

$$\mathbf{E}z^{\theta} = (1-p) + \mathbf{E}z^{\theta}\mathbf{E}\big(z^{\eta_+};\ \eta_+ < \infty\big).$$

This implies the second equality in (12.8.2). The last assertion of the theorem follows from the first equality in (12.8.2), which implies

$$P(z) = \sum_{k=0}^{\infty} p^k q^k(z) = (1-p)\sum_{k=0}^{\infty}\mathbf{P}(\nu = k)\sum_{n=0}^{\infty}\mathbf{P}(H_k = n)z^n$$
$$= (1-p)\sum_{n=0}^{\infty} z^n \mathbf{P}(H_\nu = n).$$

The theorem is proved. □

The second equality in (12.8.2) and identity (12.7.12) mean that the representations

$$\theta = \eta_1 + \cdots + \eta_\nu \quad \text{and} \quad S = \chi_1 + \cdots + \chi_\nu,$$

respectively, hold true, where ν has the geometric distribution $\mathbf{P}(\nu = k) = (1-p)p^k$, $k \ge 0$, and does not depend on $\{\eta_j\}$, $\{\chi_j\}$.

Note that the probabilities $\mathbf{P}(S_k > 0) = \mathbf{P}(S_k - ak > -ak)$ on the right-hand sides of (12.8.5) and (12.8.6) are, for large k and $a = \mathbf{E}\xi < 0$, the probabilities of large deviations that were studied in Chap. 9. The results of that chapter on the asymptotics of these probabilities together with relations (12.8.5) and (12.8.6) give us an opportunity to find the asymptotics of $\mathbf{P}(\eta_+ = n)$ and $\mathbf{P}(\eta_-^0 = n)$ as $n \to \infty$ (see [8]).

Now consider the case where the both random variables η_-^0 and η_+ are proper. That is always the case if $\mathbf{E}\xi = 0$ (see Corollary 12.2.6). Here identities (12.8.4)–(12.8.6) hold true (with $\mathbf{P}(\eta_+ < \infty) = 1$). As before, (12.8.4) implies that the distributions of η_-^0 and η_+ uniquely determine each other.

Let $\eta_1, \eta_2, \ldots$ be independent copies of η_+, $H_k = \eta_1 + \cdots + \eta_k$ and $H_0 = 0$. For the sums H_k, define the local renewal function

$$h_n := \sum_{n=0}^{\infty} \mathbf{P}(H_k = n).$$

Theorem 12.8.2 *If* $\mathbf{P}(\eta_-^0 < \infty) = \mathbf{P}(\eta_+ < \infty) = 1$ *then*:

1. $\mathbf{E}\eta_-^0 = \mathbf{E}\eta_+ = \infty$.
2. $\mathbf{P}(\eta_-^0 > n) = h_n$.

Proof From (12.8.4) it follows that

$$P(z) = \frac{1 - \mathbf{E}z^{\eta_-^0}}{1 - z} = \frac{1}{1 - \mathbf{E}z^{\eta_+}} \to \infty \tag{12.8.7}$$

as $z \to 1$. Since $P(z) \to \mathbf{E}\eta_-^0$ as $z \to 1$, we have proved that $\mathbf{E}\eta_-^0$ is infinite. That $\mathbf{E}\eta_+$ is also infinite is shown in the same way. The second assertion also follows from (12.8.7) since the right-hand side of (12.8.7) is $\sum_{n=0}^{\infty} z^n h_n$. The theorem is proved. □

Now we turn to the important class of symmetric distributions. We will say that the *distribution of a random variable* ξ *is symmetric* if it coincides with the distribution of $-\xi$, and will call the *distribution of* ξ *continuous* if the distribution function of ξ is continuous. For such random variables, $\mathbf{E}\xi = 0$ (if $\mathbf{E}\xi$ exists), the distributions of S_n are also symmetric continuous for all n, and

$$\mathbf{P}(S_n > 0) = \mathbf{P}(S_n < 0) = \frac{1}{2}, \qquad \mathbf{P}(S_n = 0) = 0,$$

and hence $D(z) \equiv 1$, $\mathbf{P}(\chi_+^0 = 0) = 0$, and $\eta_+ = \eta_+^0$, $\chi_+ = \chi_+^0$ with probability 1.

Theorem 12.8.3 *If the distribution of* ξ *is symmetric and continuous then*

$$\begin{aligned} \mathbf{P}(\eta_+ = n) = \mathbf{P}\big(\eta_-^0 = n\big) &= \frac{(2n)!}{(2n-1)(n!)^2 2^{2n}} \sim \frac{1}{2\sqrt{\pi}\, n^{3/2}}, \\ \mathbf{P}(\gamma_n > 0) = \mathbf{P}(\zeta_n < 0) &\sim \frac{1}{\sqrt{\pi n}} \end{aligned} \tag{12.8.8}$$

as $n \to \infty$ (γ_n *and* ζ_n *are defined in Section* 12.1.3).

Proof Since $\mathbf{E}z^{\eta_-^0} = \mathbf{E}z^{\eta_+}$, by virtue of (12.8.4) one has

$$1 - \mathbf{E}z^{\eta_+} = \sqrt{1-z}.$$

Expanding $\sqrt{1-z}$ into a series, we obtain the second equality in (12.8.8). The asymptotic equivalence follows from Stirling's formula.

The second assertion of the theorem follows from the first one and the equality

$$\mathbf{P}(\zeta_n < 0) = \sum_{k=n+1}^{\infty} \mathbf{P}(\eta_+ = k).$$

The assertions concerning η_-^0 and γ_n follow by symmetry.

The theorem is proved. □

Note that, under the conditions of Theorem 12.8.3, the distributions of the variables η_+, η_-, γ_n, ζ_n do not depend on the distribution of ξ. Also note that the asymptotics

$$\mathbf{P}(\eta_+ = n) \sim \frac{1}{2\sqrt{\pi}\, n^{3/2}}$$

persists in the case of non-symmetric distributions as well provided that $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 < \infty$ (see [8]).

12.8.2 The Distribution of the First Passage Time of an Arbitrary Level x by Arithmetic Skip-Free Walks

The main object in this section is the time

$$\eta(x) = \min\{k : S_k \ge x\}$$

of the first passage of the level x by the random walk $\{S_k\}$. Below we will consider the class of arithmetic random walks for which $\chi_+ \equiv 1$.

By an *arithmetic skip-free walk* we will call a sequence $\{S_k\}_{k=0}^{\infty}$, where the distribution of ξ is arithmetic and $\max_\omega \xi(\omega) = 1$ (i.e. $p_1 > 0$ and $p_k = 0$ for $k \ge 2$, where $p_k = \mathbf{P}(\xi = k)$). The term "skip-free walk" appears due to the fact that the walk $\{S_k\}$, $k = 0, 1, \dots$, cannot skip any integer level $x > 0$: if $S_n > x$ then necessarily there is a $k < n$ such that $S_k = x$.

As we already know from Example 12.6.1, for skip-free walks with $\mathbf{E}\xi < 0$ the distribution of S is geometric:

$$\mathbf{P}(S = k) = (1-p)p^k, \quad k = 0, 1, \dots,$$

where $p = \mathbf{P}(\eta_+ < \infty)$ and $z_1 = p^{-1}$ is the zero of the function $1 - p(z)$ with $p(z) = \sum_k p_k z^k$.

It turns out that one can find many other explicit formulas for skip-free walks. In this section we will be interested in the distribution of the maximum

$\overline{S}_n = \max(0, S_1, \dots, S_n)$; as we already noted, knowing the distribution is important for many problems of mathematical statistics, queueing theory, etc. Note that finding the distribution of $\overline{S}_n$ is the same as finding the distribution of $\eta(x)$, since

$$\{\overline{S}_n < x\} = \big\{\eta(x) > n\big\}. \tag{12.8.9}$$

Here we put $\eta(x) := \infty$ if $S < x$.

The Pollaczek–Spitzer identity (see Theorem 12.3.1) provides the double transform of the distribution of $\overline{S}_n$. Analysing this identity shows that the distribution of $\overline{S}_n$ (or $\eta(x)$) itself typically cannot be expressed in terms of the distribution of ξ_k in explicit form. However, for discrete skip-free walks one has remarkable "duality" relations which we will now prove with the help of Pollaczek–Spitzer's identity.

Theorem 12.8.4 *If ξ is integer-valued then $\mathbf{P}(\xi_k \ge 2) = 0$ is a necessary and sufficient condition for*

$$n\mathbf{P}\big(\eta(x) = n\big) = x\mathbf{P}(S_n = x), \quad x \ge 1. \tag{12.8.10}$$

Using the Wald identity, it is also not hard to verify that if the expectation $\mathbf{E}\xi_1 = a > 0$ exists then the *walk $\{S_n\}$ will be skip-free if and only if* $\mathbf{E}\eta(x) = x/a$. (Note that the definition of $\eta(x)$ in this section somewhat differs from that in Chap. 10. One obtains it by changing x to $x+1$ on the right-hand side of the definition of $\eta(x)$ from Chap. 10.)

The asymptotics of the local probabilities $\mathbf{P}(S_n = x)$ was studied in Chap. 9 (see e.g., Theorem 9.3.4). This together with (12.8.10) enables us to find the asymptotics of $\mathbf{P}(\eta(x) = n)$.

Proof of Theorem 12.8.4 Set

$$r_x := \mathbf{P}\big(\eta(x) = \infty\big) = \mathbf{P}(S < x), \qquad q_{x,n} := \mathbf{P}\big(\eta(x) = n\big),$$
$$Q_{x,n} := \mathbf{P}\big(\eta(x) > n\big) = \sum_{k=n+1}^{\infty} q_{x,k} + r_x.$$

Since for each y, $0 \le y \le x$,

$$\big\{\eta(x) = n\big\} \subset \bigcup_{k=0}^{n} \big\{\eta(y) = k\big\},$$

using the fact that the walk is skip-free, by the total probability formula one has

$$q_{x,n} = \sum_{k=0}^{n} q_{y,k} q_{x-y,n-k},$$

where $q_{0,0} = 1$, and $q_{y,0} = 0$ for $y > 0$. Hence for $|z| \le 1$ using convolution we have

$$q_x(z) := \sum_{k=0}^{\infty} q_{x,n} z^n = \mathbf{E}\big(z^{\eta(x)};\ \eta(x) < \infty\big) = q_y(z) q_{x-y}(z).$$

Putting $y = 1$ and $q_0(z) = 1$, we obtain

$$q_x(z) = q(z)q_{x-1}(z) = q^x(z), \quad x \ge 0.$$

From here one can find the generating function $Q_x(z)$ of the sequence $Q_{x,n}$:

$$Q_x(z) := \sum_{n=0}^{\infty} z^n \left(r_x + \sum_{k=n+1}^{\infty} q_{x,k} \right) = \frac{r_x}{1-z} + \sum_{n=1}^{\infty} q_{x,n} \sum_{k=0}^{n-1} z^k$$
$$= \frac{r_x}{1-z} + \sum_{n=1}^{\infty} q_{x,n} \frac{1-z^n}{1-z} = \frac{r_x}{1-z} + \frac{q_x(1) - q_x(z)}{1-z} = \frac{1-q_x(z)}{1-z}.$$

Note that here the quantity $q_x(1) = \mathbf{P}(\eta(x) < \infty) = \mathbf{P}(S \ge x)$ can be less than 1. Using (12.8.9) we obtain that

$$\mathbf{P}(\overline{S}_n = x) = \mathbf{P}\big(\eta(x+1) > n\big) - \mathbf{P}\big(\eta(x) > n\big),$$
$$\sum_{n=0}^{\infty} z^n \mathbf{P}(\overline{S}_n = x) = \frac{(1-q^{x+1}(z)) - (1-q^x(z))}{1-z} = \frac{q^x(z)(1-q(z))}{1-z}.$$

Finally, making use of the absolute summability of the series below, we find that, for $|v| < 1$ and $|z| < 1$,

$$\sum_{n=0}^{\infty} z^n \mathbf{E} v^{\overline{S}_n} = \sum_{x=0}^{\infty} v^x \sum_{n=0}^{\infty} z^n \mathbf{P}(\overline{S}_n = x) = \frac{1-q(z)}{(1-z)(1-vq(z))}.$$

Turning now to the Pollaczek–Spitzer formula, we can write that

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{E} v^{\max(0,S_n)} = \ln \frac{1-q(z)}{1-z} - \ln\big(1 - vq(z)\big) = \ln \frac{1-q(z)}{1-z} + \sum_{x=1}^{\infty} \frac{(vq(z))^x}{x}.$$

Comparing the coefficients of v^x, $x \ge 1$, we obtain

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{P}(S_n = x) = \frac{q^x(z)}{x}, \quad x \ge 1. \tag{12.8.11}$$

Taking into account that $q^x(z) = q_x(z)$ and comparing the coefficients of z^n, $n \ge 1$, in (12.8.11) we get

$$\frac{1}{n} \mathbf{P}(S_n = x) = \frac{1}{x} \mathbf{P}(\eta_x = n), \quad x \ge 1, \ n \ge 1.$$

Sufficiency is proved.

The necessity of the condition $\mathbf{P}(\xi \ge 2) = 0$ follows from equality (12.8.10) for $x = n = 1$:

$$p_1 = q_{1,1} = \sum_{k=1}^{\infty} p_k, \qquad \sum_{k=2}^{\infty} p_k = \mathbf{P}(\xi \ge 2) = 0.$$

The theorem is proved. □

Using the obtained formulas one can, for instance, find in Example 4.2.3 the distribution of the time to ruin in a game with an infinitely rich adversary (the total capital being infinite). If the initial capital of the first player is x then, for the time $\eta(x)$ of his ruin, we obtain

$$\mathbf{P}\bigl(\eta(x)=n\bigr)=\frac{x}{n}\mathbf{P}(S_n=x),$$

where

$$S_n=\sum_{j=1}^{n}\xi_j;\qquad \mathbf{P}(\xi_j=1)=q,\qquad \mathbf{P}(\xi_j=-1)=p$$

(p is the probability for the first player to win in a single play). Therefore, if n and x are both either odd or even then

$$\mathbf{P}\bigl(\eta(x)=n\bigr)=\frac{x}{n}\binom{n}{(n-x)/2}q^{(n+x)/2}p^{(n-x)/2}, \tag{12.8.12}$$

and $\mathbf{P}(\eta(x)=n)=0$ otherwise.

It is interesting to ask how fast $\mathbf{P}(\eta(x)>n)$ decreases as n grows in the case when the player will be ruined with probability 1, i.e. when $\mathbf{P}(\eta(x)<\infty)=1$. As we already know, this happens if and only if $p\le q$. (The assertion also follows from the results of Sect. 13.3.)

Applying Stirling's formula, as was done when proving the local limit theorem for the Bernoulli scheme, it is not difficult to obtain from (12.8.12) that, for each fixed x, as $n\to\infty$ (n and x having the same parity), for $p\le q$,

$$\mathbf{P}\bigl(\eta(x)=n\bigr)\sim\frac{x}{n^{3/2}}\sqrt{\frac{2}{\pi}}(4pq)^{n/2}\left(\frac{q}{p}\right)^{x/2};$$

$$\mathbf{P}\bigl(\eta(x)\ge n\bigr)\sim\frac{x}{n^{3/2}(p-q)^2}\sqrt{\frac{2}{\pi}}(4pq)^{n/2}\left(\frac{q}{p}\right)^{x/2}\quad\text{for } p<q$$

and

$$\mathbf{P}\bigl(\eta(x)\ge n\bigr)\sim x\sqrt{\frac{2}{\pi n}}\quad\text{for } p=q.$$

The last relation allowed us, under the conditions of Sect. 8.8, to obtain the limiting distribution for the number of intersections of the trajectory $S_1,\dots,S_n$ with the strip $[u,v]$ (see (8.8.24)). Up to the normalising constants, this assertion also remains true for arbitrary random walks such that $\mathbf{E}\xi_k=0$ and $\mathbf{E}\xi_k^2<\infty$. However, even in the case of a skip-free walk, the proof of this assertion requires additional efforts, despite the fact that, for such walks, an upward intersection of the line $x=0$ by the trajectory $\{S_n\}$ divides the trajectory, as in Sect. 8.8, into independent identically distributed cycles.