Chapter 10
Renewal Processes

Abstract This is the first chapter in the book to deal with random processes in continuous time, namely, with the so-called renewal processes. Section 10.1 establishes the basic terminology and proves the integral renewal theorem in the case of non-identically distributed random variables. The classical Key Renewal Theorem in the arithmetic case is proved in Sect. 10.2, including its extension to the case where random variables can assume negative values. The limiting behaviour of the excess and defect of a random walk at a growing level is established in Sect. 10.3. Then these results are extended to the non-arithmetic case in Sect. 10.4. Section 10.5 is devoted to the Law of Large Numbers and the Central Limit Theorem for renewal processes. It also contains the proofs of these laws for the maxima of sums of independent non-identically distributed random variables that can take values of both signs, and a local limit theorem for the first hitting time of a growing level. The chapter ends with Sect. 10.6 introducing generalised (compound) renewal processes and establishing for them the Central Limit Theorem, in both integral and integro-local forms.

10.1 Renewal Processes. Renewal Functions

10.1.1 Introduction

The sequence of sums of random variables $\{S_n\}$, considered in previous chapters, is often called a *random walk*. It can be considered as the simplest *random process in discrete time n*. The further study of such processes is contained in Chaps. 11, 12 and 20.

In this chapter we consider the simplest processes in *continuous time t* that are also entirely determined by a sequence of independent random variables and do not require, for their construction, any special structures (in the general case such constructions will be needed; see Chap. 18).

Let $\tau_1, \{\tau_j\}_{j=2}^{\infty}$ be a sequence of independent random variables given on a probability space $\langle \Omega, \mathfrak{F}, \mathbf{P}\rangle$ (here we change our conventional notations ξ_j to τ_j for reasons that will become clear in Sect. 10.6, where ξ_j appear again). For the random variables $\tau_2, \tau_3, \ldots$ we will usually assume some homogeneity property: proximity

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of the expectations or identical distributions. The random variable τ_1 can be arbitrary.

Definition 10.1.1 A *renewal process* is a collection of random variables $\eta(t)$ depending on a parameter t and defined on $\langle \Omega, \mathfrak{F}, \mathbf{P}\rangle$ by the equality

$$\eta(t) := \min\{k \ge 0 : T_k > t\}, \quad t \ge 0, \tag{10.1.1}$$

where

$$T_k := \sum_{j=1}^{k} \tau_j, \qquad T_0 := 0.$$

The variables $\eta(t)$ are not completely defined yet. We do not know what $\eta(t)$ is for ω such that the level t is never reached by the sequence of sums T_k. In that case it is natural to put

$$\eta(t) := \infty \quad \text{if all } T_k \le t. \tag{10.1.2}$$

Clearly, $\eta(t)$ is a stopping time (see Sect. 4.4).

Usually the random variables $\tau_2, \tau_3, \dots$ are assumed to be identically distributed with a finite expectation. The distribution of the random variable τ_1 can be arbitrary.

We assume first that all the random variables τ_j are *positive*. Then definition (10.1.1) allows us to consider $\eta(t)$ as a random function that can be described as follows. If we plot the points $T_0 = 0, T_1, T_2, \dots$ on the real line, then one has $\eta(t) = 0$ on the semi-axis $(-\infty, 0)$, $\eta(t) = 1$ on the semi-interval $[0, T_1)$, $\eta(t) = 2$ on the semi-interval $[T_1, T_2)$ and so on.

The sequence $\{T_k\}_{k=0}^{\infty}$ is also often called a renewal process. Sometimes we will call the sequence $\{T_k\}$ a *random walk*. The quantity $\eta(t)$ can also be called the first passage time of the level t by the random walk $\{T_k\}_{k=0}^{\infty}$.

If, based on the sequence $\{T_k\}$, we construct a random walk $T(x)$ in continuous time:

$$T(x) := T_k \quad \text{for } x \in [k, k+1),\ k \ge 0,$$

then *the renewal process $\eta(t)$ will be the generalised inverse of $T(x)$*:

$$\eta(t) = \inf\bigl\{x \ge 0 : T(x) > t\bigr\}.$$

The term "renewal process" is related to the fact that the function $\eta(t)$ and the sequence $\{T_k\}$ are often used to describe the operation of various physical devices comprising replaceable components. If, say, τ_j is the failure-free operating time of such a component, after which the latter requires either replacement or repair ("renewal", which is supposed to happen immediately), then T_k will denote the time of the k-th "renewal" of the component, while $\eta(t)$ will be equal to the number of "renewals" which have occurred by the time t.

Remark 10.1.1 If the j-th renewal of the component does not happen immediately but requires a time $\tau'_j \ge 0$, then, introducing the random variables

$$\tau^*_j := \tau_j + \tau'_j, \qquad T^*_k := \sum_{j=1}^{k} \tau^*_j, \qquad \eta^*(t) := \min\{k : T^*_k > t\},$$

we get an object of the same nature as before, with nearly the same physical meaning. For such an object, a number of additional results can be obtained, see e.g., Remark 10.3.1.

Renewal processes are also quite often used in probabilistic research per se, and also when studying other processes for which there exist so-called "regeneration times" after which the evolution of the process starts anew. Below we will encounter examples of such use of renewal processes.

Now we return to the general case where τ_j may assume both positive and negative values.

Definition 10.1.2 The function

$$H(t) := \mathbf{E}\eta(t), \quad t \ge 0,$$

is called the *renewal function for the sequence* $\{T_k\}_{k=0}^{\infty}$.

In the existing literature, another definition is used more frequently.

Definition 10.1.2A The *renewal function for the sequence* $\{T_k\}_{k=0}^{\infty}$ is defined by

$$U(t) := \sum_{j=0}^{\infty} \mathbf{P}(T_j \le t).$$

The values of $H(u)$ and $T(u)$ can be infinite.

If $\tau_j \ge 0$ then the above definitions are equivalent. Indeed, for $t \ge 0$, consider the random variable

$$\nu(t) := \max\{k : T_k \le t\} = \eta(t) - 1.$$

Then clearly

$$\sum_{j=0}^{\infty} \mathrm{I}(T_j \le t) = 1 + \nu(t),$$

where $\mathrm{I}(A)$ is the indicator of the event A, and

$$U(t) = 1 + \mathbf{E}\nu(t) = \mathbf{E}\eta(t) = H(t).$$

The value $U(t) = \mathbf{E}\,\nu(t) + 1$ is the mean time spent by the trajectory $\{T_j\}_{j=0}^{\infty}$ in the interval $[0, t]$.

If τ_j can take values of different signs then clearly $\nu(t) \ge \eta(t)$ and, with a positive probability, $\nu(t) > \eta(t)$ (the trajectory $\{T_j\}$, after crossing the level t, can return to the region $(-\infty, t]$). Therefore in that case $U(t) > H(t)$. Thus *for τ_j taking* *values of different signs we have two versions of the renewal function* given in Definitions 10.1.2 and 10.1.2A. We will call them the first and the second versions, respectively. *In the present chapter we will consider the first version only* (Definition 10.1.2). The second version is discussed in Appendix 9.

Note that, for τ_j assuming values of both signs and $t < 0$, we have $H(t) = 0$, $U(t) > 0$, so the function $H(t)$ has a jump of magnitude 1 at the point $t = 0$.

Note also that the functions $H(t)$ and $U(t)$ we defined above are *right-continuous*. In the existing literature, one often considers *left-continuous* versions of renewal functions defined respectively as

$$H(t-0) = \mathbf{E}\min\{k : S_k \geq t) \quad \text{and} \quad U(t-0) = \sum_{j=0}^{\infty} \mathbf{P}(S_j < t).$$

If all τ_j are identically distributed and $F^{*k}(t)$ is the k-fold convolution of the distribution function $F(t) = \mathbf{P}(\xi_j < t)$, then the second left-continuous version of the renewal function can also be represented in the form

$$\sum_{k=0}^{\infty} F^{*k}(t),$$

where F^{*0} corresponds to the distribution degenerate at zero.

From the point of view of the exposition below, it makes no difference which version of continuity is chosen. For several reasons, in the present chapter it will be more convenient for us to deal with *right-continuous* renewal functions. Everything below will equally apply to left-continuous renewal functions as well.

10.1.2 The Integral Renewal Theorem for Non-identically Distributed Summands

In the case where τ_j, $j \geq 2$, are not necessarily identically distributed and do not possess other homogeneity properties, singling out the random variable τ_1 makes little sense.

Theorem 10.1.1 *Let τ_j, $j \geq 1$, be uniformly integrable from the right*, $\mathbf{E}|T_N| < \infty$ *for any fixed N and $a_k = \mathbf{E}\tau_k \to a > 0$ as $k \to \infty$. Then the following limit exists*

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{a}. \qquad (10.1.3)$$

Proof We will need the following definition.

Definition 10.1.3 The random variable

$$\chi(t) = T_{\eta(t)} - t > 0$$

is said to be the *excess* of the level t (or *overshoot* over the level t) for the random walk $\{T_j\}$.

Lemma 10.1.1 *If* $a_k \in [a_*, a^*]$, $a_* > 0$, *then*

$$\mathbf{E}\eta(t) > \frac{t}{a^*}, \qquad \limsup_{t\to\infty} \frac{\mathbf{E}\eta(t)}{t} \le \frac{1}{a_*}. \tag{10.1.4}$$

Proof By Theorem 4.4.2 (see also Example 4.4.3)

$$\mathbf{E}T_{\eta(t)} = t + \mathbf{E}\chi(t) \le a^*\mathbf{E}\eta(t).$$

This implies the first inequality in (10.1.4). Now introduce truncated random variables $\tau_j^{(s)} := \min(\tau_j, s)$. By virtue of the uniform integrability, one can choose an s such that, for a given $\varepsilon \in (0, a_*)$, we would have

$$a_{j,s} := \mathbf{E}\tau_j^{(s)} \ge a_* - \varepsilon.$$

Then, by Theorem 4.4.2,

$$t + s \ge \mathbf{E}T^{(s)}_{\eta^{(s)}(t)} \ge (a_* - \varepsilon)\mathbf{E}\eta^{(s)},$$

where

$$T_n^{(s)} := \sum_{j=1}^{n} \tau_j^{(s)}, \qquad \eta^{(s)}(t) := \min\{k : T_k^{(s)} > t\}.$$

Since $\eta(t) \le \eta^{(s)}(t)$, one has

$$H(t) = \mathbf{E}\eta(t) \le \mathbf{E}\eta^{(s)}(t) \le \frac{t+s}{a_* - \varepsilon}. \tag{10.1.5}$$

As $\varepsilon > 0$ can be chosen arbitrarily, we obtain that

$$\limsup_{t\to\infty} \frac{H(t)}{t} \le \frac{1}{a_*}.$$

The lemma is proved. □

We return to the proof of Theorem 10.1.1. For a given $\varepsilon > 0$, find an N such that $a_k \in [a - \varepsilon, a + \varepsilon]$ for all $k > N$ and denote by $H_N(t)$ the renewal function corresponding to the sequence $\{\tau_{N+k}\}_{k=1}^{\infty}$. Then

$$\begin{aligned} H(t) &= \mathbf{E}\big(\eta(t); T_N > t\big) + \int_{-\infty}^{t} \mathbf{P}(T_N \in du)\big[N + H_N(t-u)\big] \\ &= \mathbf{E}\big[H_N(t - T_N);\ T_N \le t\big] + r_N, \end{aligned} \tag{10.1.6}$$

where

$$r_N := \mathbf{E}\big(\eta(t);\ T_N > t\big) + N\mathbf{P}(T_N \le t) \le N\mathbf{P}(T_N > t) + N\mathbf{P}(T_N \le t) = N.$$

Relation (10.1.5) implies that there exist constants c_1, c_2, such that, for all t,

$$H_N(t) \le c_1 + c_2 t.$$

Therefore, for fixed N and M,

$$R_{N,M} := \mathbf{E}\big[H_N(t-T_N);\ |T_N| \ge M, T_N \le t\big]$$
$$\le (c_1 + c_2 t)\mathbf{P}\big(|T_N| \ge M, T_N \le t\big) + c_2\mathbf{E}|T_N|.$$

Choose an M such that $c_2\mathbf{P}(|T_N| \ge M) < \varepsilon$. Then

$$\limsup_{t\to\infty} \frac{r_N + R_{N,M}}{t} \le \varepsilon. \tag{10.1.7}$$

To bound $H(t)$ in (10.1.6) it remains to consider, for the chosen N and M, the function

$$H_{N,M}(t) := \mathbf{E}\big[H_N(t-T_N);\ |T_N| < M\big].$$

By Lemma 10.1.1,

$$\limsup_{t\to\infty} \frac{H_{N,M}(t)}{t} \le \frac{1}{a-\varepsilon},$$
$$\liminf_{t\to\infty} \frac{H_{N,M}(t)}{t} \ge \frac{\mathbf{P}(|T_N| < M)}{a+\varepsilon} \ge \frac{1+\varepsilon/c_1}{a+\varepsilon}.$$

This together with (10.1.6) and (10.1.7) yields

$$\limsup_{t\to\infty} \frac{H(t)}{t} \le \varepsilon + \frac{1}{a-\varepsilon}, \qquad \liminf_{t\to\infty} \frac{H(t)}{t} \ge \frac{(1-\varepsilon/c_2)}{a+\varepsilon}.$$

Since ε is arbitrary, the foregoing implies (10.1.3).

The theorem is proved. □

Remark 10.1.2 One can obtain the following generalisation of Theorem 10.1.1, in which no restrictions on $\tau_1 \ge 0$ are imposed. *Let τ_1 be an arbitrary nonnegative random variable, and $\tau_j^* := \tau_{1+j}$ satisfy the conditions of Theorem* 10.1.1. *Then* (10.1.3) *still holds true*.

This assertion follows from the relations

$$H(t) = \mathbf{P}(\tau_1 > t) + \int_0^t \mathbf{P}(\tau_1 \in dv)H^*(t-v), \tag{10.1.8}$$

where $H^*(t)$ corresponds to the sequence $\{\tau_j^*\}$ and, for each fixed N and $v \le N$,

$$\frac{H^*(t-v)}{t} = \frac{H^*(t-v)}{t-v} \cdot \frac{t-v}{t} \to \frac{1}{a}$$

as $t \to \infty$. Therefore

$$\frac{1}{t}\int_0^N \mathbf{P}(\tau_1 \in dv)H^*(t-v) \to \frac{\mathbf{P}(\tau_1 \le N)}{a}.$$

For the remaining part of the integral in (10.1.8), we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_N^t \mathbf{P}(\tau_1 \in dv)H^*(t-v) \le \limsup_{t\to\infty} \frac{H^*(t)}{t}\mathbf{P}(\tau_1 > N) = \frac{\mathbf{P}(\tau_1 > N)}{a}.$$

Since the probability $\mathbf{P}(\tau_1 > N)$ can be made arbitrarily small by the choice of N, the assertion is proved. □

It is not difficult to verify that the condition $\tau_1 \ge 0$ can be relaxed to the condition $\mathbf{E}\min(0, \tau_1) > -\infty$. However, if $\mathbf{E}\min(0, \tau_1) = -\infty$, then $H(t) = \infty$ and relation (10.1.3) does not hold.

Obtaining an analogue of Theorem 10.1.1 for the second version $U(t)$ of the renewal function in the case of *uniformly integrable* τ_j *taking values of both signs* is accompanied by greater technical difficulties and additional conditions. For a fixed $\varepsilon > 0$, split the series $U(t) = \sum_{n=0}^{\infty} \mathbf{P}(T_n \le t)$ into the three parts

$$\sum\nolimits_1 = \sum_{n<\frac{t(1-\varepsilon)}{a}}, \qquad \sum\nolimits_2 = \sum_{|n-\frac{t}{a}|\le\frac{t\varepsilon}{a}}, \qquad \sum\nolimits_3 = \sum_{n>\frac{t(1+\varepsilon)}{a}}.$$

By the law of large numbers (see Corollary 8.3.2),

$$\frac{T_n}{n} \xrightarrow{p} a.$$

Therefore, for $n < \frac{t(1-\varepsilon)}{a}$,

$$\mathbf{P}(T_n \le t) \ge \mathbf{P}\biggl(T_n \le \frac{na}{1-\varepsilon}\biggr) \to 1$$

and hence

$$\frac{1}{t}\sum\nolimits_1 \to \frac{1-\varepsilon}{a}.$$

The second sum allows the trivial bound

$$\frac{1}{t}\sum\nolimits_2 < \frac{2\varepsilon}{a},$$

where the right-hand side can be made arbitrarily small by the choice of ε.

The main difficulties are related to estimating $\sum_3$. To illustrate the problems arising here we confine ourselves to the case of identically distributed $\tau_j \stackrel{d}{=} \tau$. In this case the required estimate for $\sum_3$ can only be obtained under the condition $\mathbf{E}(\tau^-)^2 < \infty$, $\tau^- := \max(0, -\tau)$. Assume without losing generality that $\mathbf{E}\tau^2 < \infty$. (If $\mathbf{E}(\tau^+)^2 = \infty$, $\tau^+ := \max(0, \tau)$, then introducing truncated random variables $\tau_j^{(s)} = \min(s, \tau_j)$, we obtain, using obvious conventions concerning notations, that $\mathbf{P}(T_n \le t) \le \mathbf{P}(T_n^{(s)} \le t)$, $U(t) \le U^{(s)}(t)$ and $\sum_3 \le \sum_3^{(s)}$, where $\mathbf{E}(\tau^{(s)})^2 < \infty$ and the value of $\mathbf{E}\tau^{(s)}$ can be made arbitrarily close to a by the choice of s.) In the case $\mathbf{E}\tau^2 < \infty$ we can use Theorem 9.5.1 by virtue of which, for a regularly varying left tail $W(t) = \mathbf{P}(\tau < -t) = t^{-\beta}L(t)$ ($L(t)$ is a slowly varying function) and $n > \frac{t}{a}(1+\varepsilon)$, we have

$$\mathbf{P}(T_n \le t) = \mathbf{P}\bigl(T_n - an \le -(an - t)\bigr) \sim nW(an - t).$$

By the properties of slowly varying functions (see Appendix 6), for the values $u = n/t$ comparable to 1, $n > \frac{t}{a}(1+\varepsilon)$ and $t \to \infty$, we have

$$\frac{W(an-t)}{W(\varepsilon t)} \sim \biggl(\frac{au-1}{\varepsilon}\biggr)^{-\beta}.$$

Thus for $\beta > 2$, as $t \to \infty$,

$$\sum\nolimits_3 = \sum_{n > \frac{(1+\varepsilon)t}{a}} \mathbf{P}(T_n \le t) \sim \int_{v > \frac{(1+\varepsilon)t}{a}} vW(av - t)\, dv$$

$$\sim t^2 W(\varepsilon t) \int_{\frac{1+\varepsilon}{a}}^{\infty} u \left[\frac{au-1}{\varepsilon}\right]^{-\beta} du \sim c(\varepsilon) t^2 W(t) = o(1).$$

Summarising, we have obtained that

$$\lim_{t\to\infty} \frac{U(t)}{t} = \frac{1}{a}.$$

Now if $\mathbf{E}(\tau^-)^2 = \infty$ then $U(t) = \infty$ for all t. In this case, instead of $U(t)$ one studies the "local" renewal function

$$U(t,h) = \sum_n \mathbf{P}\bigl(T_n \in (t, t+h]\bigr)$$

which is always finite provided that $a > 0$ and has all the properties of the increment $H(t+h) - H(t)$ to be studied below (see e.g. [12]).

In view of the foregoing and since the function $H(t)$ will be of principal interest to us, in what follows we will restrict ourselves to studying the first version of the renewal function, as was noted above. We will mainly pay attention to the asymptotic behaviour of the increments $H(t+h) - H(t)$ as $t \to \infty$. To this is closely related a more general problem that often appears in applications: the problem on the asymptotic behaviour as $t \to \infty$ of integrals (see e.g. Chap. 13)

$$\int_0^t g(t-y)\, dH(y) \tag{10.1.9}$$

for functions $g(v)$ such that

$$\int_0^\infty g(v)\, dv < \infty.$$

Theorems describing the asymptotic behaviour of (10.1.9) will be called the *key renewal theorems*. The next sections and Appendix 9 will be devoted to these theorems. Due to the technical complexity of the mentioned problems, we will confine ourselves to considering only the case where τ_j, $j \ge 2$, are *identically distributed*.

Note that in some special cases the above problems can be solved in a very simple way, since the renewal function $H(t)$ can be found there explicitly. To do this, as it follows from Wald's identity used above, it suffices to find $\mathbf{E}\chi(t)$ in explicit form. If, for instance, τ_j are integer-valued, $\mathbf{P}(\tau_j = 1) > 0$ and $\mathbf{P}(\tau_j \ge 2) = 0$, for all $j \ge 1$, then $\chi(t) \equiv 1$ and Wald's identity yields $H(t) = (t+1)/a$. Similar equalities will hold if $\mathbf{P}(\tau_j > t) = ce^{-\gamma t}$ for $t > 0$ and $\gamma > 0$ (if τ_j are integer-valued, then t takes only integer values in this formula). In that case the distribution of $\chi(t)$ will be exponential and will not depend on t (for more details, see the exposition below and also Chap. 15).

10.2 The Key Renewal Theorem in the Arithmetic Case

We will distinguish between two distribution types for τ_j: *arithmetic in an extended sense* (when the lattice span is not necessary 1; for the definition of arithmetic distributions see Sect. 7.1) and all other distributions that we will call *non-arithmetic*. It is clear that, say, a random variable taking values 1 and $\sqrt{2}$ with positive probabilities cannot be arithmetic.

In the present section, we will consider the arithmetic case. Without loss of generality, we will assume that the lattice span is 1. Then the functions $\mathbf{P}(\tau_j < t)$ and $H(t)$ will be completely determined by their values at integer points $t = k$, $k = 0, 1, 2 \dots$.

First we consider the case where the τ_j are *positive*, $\tau_j \stackrel{d}{=} \tau$ for $j \ge 2$. In that case, the difference

$$h(k) := H(k) - H(k-1) = \sum_{j=0}^{\infty} \mathbf{P}(T_j = k), \quad k \ge 1,$$

is equal to the expectation of the number of visits of the point k by the walk $\{T_j\}$. Put

$$q_k := \mathbf{P}(\tau_1 = k), \qquad p_k := \mathbf{P}(\tau = k).$$

Definition 10.2.1 A renewal process $\eta(t)$ will be called *homogeneous* and denoted by $\eta_0(t)$ if

$$q_k = \frac{1}{a} \sum_{k}^{\infty} p_j, \quad k = 1, 2, \dots, \quad a = \mathbf{E}\tau, \quad \left(\text{so that } \sum_{k=1}^{\infty} q_k = 1\right). \tag{10.2.1}$$

If we denote by $p(z)$ the generating function

$$p(z) = \mathbf{E} z^{\tau} = \sum_{k=1}^{\infty} p_k z^k,$$

then the generating function $q(z) = \mathbf{E} z^{\xi_1} = \sum_{k=1}^{\infty} q_k z^k$ will be equal to

$$q(z) = \frac{1}{a} \sum_{k=1}^{\infty} z^k \sum_{j=k}^{\infty} p_j = \frac{z}{a} \sum_{j=1}^{\infty} p_j \sum_{k=0}^{j-1} z^k = \frac{z}{a} \sum_{j=1}^{\infty} p_j \frac{1 - z^j}{1 - z} = \frac{z(1 - p(z))}{a(1 - z)}.$$

As we will see below, the term "homogeneous" for the process $\eta_0(t)$ is quite justified. One of the reasons for its use is the following exact (non-asymptotic) equality.

Theorem 10.2.1 *For a homogeneous renewal process* $\eta_0(t)$, *one has*

$$H_0(k) := \mathbf{E}\eta_0(k) = 1 + \frac{k}{a}.$$

Proof Consider the generating function $r(z)$ for the sequence $h_0(k) = H_0(k) - H_0(k-1)$:

$$r(z) = \sum_{1}^{\infty} z^k h_0(k) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} z^k \mathbf{P}(T_j = k)$$
$$= \sum_{j=1}^{\infty} \mathbf{E} z^{T_j} = q(z) \sum_{j=0}^{\infty} p^j(z) = \frac{q(z)}{1 - p(z)} = \frac{z}{a(1-z)}.$$

This implies that $h_0(k) = 1/a$. Since $H_0(0) = 1$, one has $H_0(k) = 1 + k/a$. The theorem is proved. □

Sometimes the process $\eta_0(t)$ is also called *stationary*. As we will see below, it would be more appropriate to call it a *process with stationary increments* (see Sect. 22.1).

The asymptotic regular behaviour of the function $h(k)$ as $k \to \infty$ persists in the case of arbitrary τ_1 as well.

Denote by d the greatest common divisor (g.c.d.) of the possible values of τ:

$$d := \text{g.c.d.}\{k : p_k > 0\},$$

and let $g(k)$, $k = 0, 1, \ldots$, be an arbitrary sequence such that

$$\sum_{k=0}^{\infty} \big|g(k)\big| < \infty.$$

Theorem 10.2.2 (The key renewal theorem) *If $d = 1$, τ_1 is an arbitrary integer-valued random variable and $\tau_j \stackrel{d}{=} \tau > 0$ for $j \ge 2$, then, as $k \to \infty$,*

$$h(k) := H(k) - H(k-1) \to \frac{1}{a}, \qquad \sum_{l=1}^{k} h(l) g(k-l) \to \frac{1}{a} \sum_{m=0}^{\infty} g(m).$$

These two relations are equivalent.

The first assertion of the theorem is also called the *local* renewal theorem.

To prove the theorem we will need two auxiliary assertions.

Lemma 10.2.1 *Let all τ_j be identically distributed and $\nu \ge 1$ be a Markov time with respect to the collection of σ-algebras $\{\mathfrak{F}_n\}$, where $\mathfrak{F}_n$ is independent of $\sigma(\tau_{n+1}, \tau_{n+2}, \ldots)$. Then the σ-algebra generated by the random variables ν, $\tau_1, \ldots, \tau_\nu$, and the σ-algebra $\sigma\{\tau_{\nu+1}, \tau_{\nu+2}, \ldots\}$ are independent. The sequence $\{\tau_{\nu+1}, \tau_{\nu+2}, \ldots\}$ has the same distribution as $\{\tau_1, \tau_2, \ldots\}$.*

Thus, in spite of their random numbers, the elements of the sequence $\tau_{\nu+j}$ are distributed as τ_j.

Proof For given Borel sets $B_1, B_2, \ldots, C_1, C_2, \ldots$ put

$$A := \{\nu \in N,\ \tau_1 \in B_1, \ldots, \tau_\nu \in B_\nu\}, \qquad D_\nu := \{\tau_{\nu+1} \in C_1, \ldots, \tau_{\nu+k} \in C_k\},$$

where **N** is a given set of integers and k is arbitrary. Since $\mathbf{P}(D_j) = \mathbf{P}(D_0)$ and the events D_j and $\{\nu = j\}$ are independent, the total probability formula yields

$$\mathbf{P}(D_\nu) = \sum_{j=1}^{\infty} \mathbf{P}(\nu = j, D_j) = \sum_{j=1}^{\infty} \mathbf{P}(\nu = j)\mathbf{P}(D_j) = \mathbf{P}(D_0).$$

Therefore, by Theorem 3.4.3, in order to prove the required independence of the σ-algebras, it suffices to show that $\mathbf{P}(D_\nu A) = \mathbf{P}(D_0)\mathbf{P}(A)$.

By the total probability formula,

$$\mathbf{P}(D_\nu A) = \sum_{j\in\mathbf{N}} \mathbf{P}\big(D_\nu A\{\nu = j\}\big) = \sum_{j\in\mathbf{N}} \mathbf{P}\big(D_j A\{\nu = j\}\big).$$

But the event $A\{\nu = j\}$ belongs to $\mathfrak{F}_j$, whereas $D_j \in \sigma(\tau_{j+1}, \dots, \tau_{j+k})$. Therefore D_j and $A\{\nu = j\}$ are independent events and

$$\mathbf{P}\big(D_j A\{\nu = j\}\big) = \mathbf{P}(D_j)\mathbf{P}\big(A\{\nu = j\}\big) = \mathbf{P}(D_0)\mathbf{P}\big(A\{\nu = j\}\big), \quad j \ge 1.$$

From here it clearly follows that $\mathbf{P}(D_\nu A) = \mathbf{P}(D_0)\mathbf{P}(A)$. The lemma is proved. □

Lemma 10.2.2 *Let $\zeta_1, \zeta_2, \dots$ be independent arithmetic identically and symmetrically distributed random variables with zero expectation* $\mathbf{E}\zeta_j = 0$. *Put* $Z_n := \sum_{j=1}^{n} \zeta_j$. *Then, for any integer k,*

$$\nu_k := \min\{n : Z_n = k\}$$

is a proper random variable: $\mathbf{P}(\nu_k < \infty) = 1$.

The proof of the lemma is given in Sect. 13.3 (see Corollary 13.3.1).

Proof of Theorem 10.2.2 Consider two independent sequences of random variables (we assume that they are given on a common probability space): a sequence $\tau_1, \tau_2, \dots$, where τ_1 has an arbitrary distribution, and a sequence $\tau'_1, \tau'_2, \dots$, where $\mathbf{P}(\tau'_1 = k) = q_k$ (see (10.2.1)), and $\mathbf{P}(\tau'_j = k) = \mathbf{P}(\tau_j = k) = p_k$ for $j \ge 2$ (so that $\tau'_j \stackrel{d}{=} \tau_j$ for $j \ge 2$; the process $\eta'(t)$ constructed from the sums $T'_k = \sum_{j=1}^{k} \tau'_j$ is homogeneous (see Definition 10.2.1)).

Set $\nu := \min\{n \ge 1 : T_n = T'_n\}$. It is clearly a Markov time with respect to the sequence $\{\tau_j, \tau'_j\}$. We show that $\mathbf{P}(\nu < \infty) = 1$. Put

$$Z_n := \sum_{j=2}^{n} \big(\tau_j - \tau'_j\big) \quad \text{for } n \ge 2,\ Z_1 := 0,\ Z_0 := \tau_1 - \tau'_1.$$

Then

$$\nu = \min\{n \ge 1 : Z_n = -Z_0\}.$$

By Lemma 10.2.2 ($\zeta_j = \tau_j - \tau'_j$ have a symmetric distribution for $j \ge 2$), for each integer k the variable $\nu_k = \min\{n \ge 1 : Z_n = k\}$ is proper. Since Z_n for $n \ge 1$ and Z'_1 are independent, we have

$$\mathbf{P}(\nu < \infty) = \sum_{k} \mathbf{P}(Z_0 = -k)\mathbf{P}(\nu_k < \infty) = 1.$$

Now we will "glue together" ("couple") the sequences $\{T_n\}$ and $\{T'_n\}$. Since $T_\nu = T'_\nu$ and ν is a Markov time, by Lemma 10.2.1 one can replace $\tau_{\nu+1}, \tau_{\nu+2}, \ldots$ with $\tau'_{\nu+1}, \tau'_{\nu+2}, \ldots$ (and thereby replace $T_{\nu+1}, T_{\nu+2}$ with $T'_{\nu+1}, T'_{\nu+2}, \ldots$) without changing the distribution of the sequence $\{T_n\}$.

Therefore, on the set $\{T_\nu < k\}$ one has $\eta(t) = \eta'(t)$ for $t \ge k-1$ and hence

$$\begin{aligned}
h(k) &= \mathbf{E}\big(\eta(k) - \eta(k-1)\big) \\
&= \mathbf{E}\big[\eta'(k) - \eta'(k-1);\ T_\nu < k\big] + \mathbf{E}\big[\eta(k) - \eta(k-1);\ T_\nu \ge k\big] \\
&= \frac{1}{a} - \mathbf{E}\big[\eta'(k) - \eta'(k-1);\ T_\nu \ge k\big] + \mathbf{E}\big[\eta(k) - \eta(k-1);\ T_\nu \ge k\big].
\end{aligned}$$

Since $|\eta(k) - \eta(k-1)| \le 1$, we have

$$\left| h(k) - \frac{1}{a} \right| \le \mathbf{P}(T_\nu \ge k) \to 0$$

as $k \to \infty$. The first assertion of Theorem 10.2.2 is proved.

Since $h(k) \le 1$, we can make the value of

$$\left| \sum_{l=1}^{k-N} h(l) g(k-l) \right| \le \sum_{l=N+1}^{k-1} \big|g(l)\big| \le \sum_{l=N+1}^{\infty} \big|g(l)\big|$$

arbitrarily small by choosing an appropriate N. Moreover, by virtue of the first assertion, for any fixed N,

$$\sum_{l=k-N}^{k} h(l) g(k-l) \to \frac{1}{a} \sum_{l=0}^{N} g(l) \quad \text{as } k \to \infty.$$

This implies the second assertion of the theorem. □

Remark 10.2.1 The coupling of $\{T_n\}$ and $\{T'_n\}$ in the proof of Theorem 10.2.2 could be done earlier, at the time $\gamma := \min\{n \ge 1 : T_n \in T'\}$, where T' is the set of points $T' = \{T'_1, T'_2, \ldots\}$.

Theorem 10.2.3 *The assertion of Theorem* 10.2.2 *remains true for arbitrary* (*assuming values of different signs*) τ_j.

Proof We will reduce the problem to the case $\tau_j > 0$. First let all τ_j be identically distributed. Consider the random variable $\chi_1 = \chi(0)$ that we will call the *first positive sum*. We will show in Chap. 12 (see Corollary 12.2.3) that $\mathbf{E}\chi_1 < \infty$ if $a = \mathbf{E}\tau_j < \infty$. According to Lemma 10.2.1, the sequence $\tau_{\eta(0)+1}, \tau_{\eta(0)+2}, \ldots$ will have the same distribution as $\tau_1, \tau_2, \ldots$. Therefore the "second positive sum" χ_2 or, which is the same, the first positive sum of the variables $\tau_{\eta(0)+1}, \tau_{\eta(0)+2}, \ldots$ will have the same distribution as χ_1 and will be independent of it. The same will be true for the subsequent "overshoots" over the already achieved levels $\chi_1, \chi_1 + \chi_2, \ldots$. Now consider the random walk

$$\left\{ H_k = \sum_{i=1}^{k} \chi_i \right\}_{k=1}^{\infty}$$

and put

$$\eta^*(t) := \min\{k : H_k > t\}, \qquad \chi^*(t) := H_{\eta^*(t)} - t, \qquad H^*(t) := \mathbf{E}\eta^*(t).$$

Since $\chi_k > 0$, Theorem 10.2.2 is applicable, and therefore by Wald's identity

$$H^*(k) - H^*(k-1) = \frac{1}{\mathbf{E}\chi_1}\big(1 + \mathbf{E}\chi^*(k) - \mathbf{E}\chi^*(k-1)\big) \to \frac{1}{\mathbf{E}\chi_1},$$
$$\mathbf{E}\chi^*(k) - \mathbf{E}\chi^*(k-1) \to 0.$$

Note now that the distributions of the random variables $\chi(t)$ (see Definition 10.1.3) and $\chi^*(t)$ coincide. Therefore

$$\begin{aligned} H(k) - H(k-1) &= \frac{1}{a}\big(1 + \mathbf{E}\chi(k) - \mathbf{E}\chi(k-1)\big) \\ &= \frac{1}{a}\big(1 + \mathbf{E}\chi^*(k) - \mathbf{E}\chi^*(k-1)\big) \to \frac{1}{a}. \end{aligned}$$

Now let the distributions of τ_1 and τ_j, $j \ge 2$, be different. Then the renewal function $H_1(t)$ for such a walk will be equal to

$$H_1(k) = 1 + \sum_{i=-\infty}^{k} \mathbf{P}(\tau_1 = i)\big[H(k-i) + 1\big] = 1 + \sum_{i=-\infty}^{k} \mathbf{P}(\tau_1 = i)H(k-i),$$

$$h_1(k) = H_1(k) - H_1(k-1) = \sum_{i=-\infty}^{k} \mathbf{P}(\tau_1 = i)h(k-i), \quad k \ge 0,$$

where $H_1(-1) = 0$, $h(0) = H(0)$ and the function $H(t)$ corresponds to identically distributed τ_j. If we had $h(k) < c < \infty$ for all k, that would imply convergence $h_1(k) \to 1/a$ and thus complete the proof of the theorem.

The required inequality $h(k) < c$ actually follows from the following general proposition which is true for arbitrary (not necessarily lattice) random variables τ_j. □

Lemma 10.2.3 *If all τ_j are identically distributed then, for all t and u,*

$$H(t+u) - H(t) \le H(u) \le c_1 + c_2 u.$$

Proof The difference $\eta(t+u) - \eta(t)$ is the number of jumps of the trajectory $\{\widetilde{T}_k\}$ that started at the point $t + \chi(t) > t$ until the first passage of the level $t+u$, where the sequence $\{\widetilde{T}_k\}$ has the same distribution as $\{T_k\}$ and is independent of it (see Lemma 10.2.1). In other words, $\eta(t+u) - \eta(u)$ has the same distribution as $\widetilde{\eta}(t - \chi(t)) \le \widetilde{\eta}(t)$, where $\widetilde{\eta}$ corresponds to $\{\widetilde{T}_k\}$ if $\chi(t) \le u$ and to $\eta(t+u) - \eta(t) = 0$ if $\chi(t) > u$. Therefore $H(t+u) - H(t) \le H(u)$. The inequality for $H(u)$ follows from Theorem 10.2.1. The lemma is proved. □

Theorem 10.2.3 is proved. □

10.3 The Excess and Defect of a Random Walk. Their Limiting Distribution in the Arithmetic Case

Along with the excess $\chi(t) = T_{\eta(t)} - t$ we introduce one more random variable closely related to $\chi(t)$.

Definition 10.3.1 The random variable

$$\gamma(t) := t - T_{\eta(t)-1} = t - T_{\nu(t)}$$

is called the *defect* (or *undershoot*) of the level t in the walk $\{T_n\}$.

The quantity $\chi(t)$ may be thought of as the time during which the component that was working at time t will continue working after that time, while $\gamma(t)$ is the time for which the component has already been working by that time.

One should not think that the sum $\chi(t) + \gamma(t)$ has the same distribution as τ_j—this sum is actually equal to the value of a τ with the *random* subscript $\eta(t)$. In particular, as we will see below, it may turn out that $\mathbf{E}\chi(t) > \mathbf{E}\tau_j$ for large t. The following apparent paradox is related to this fact. A passenger coming to a bus stop at which buses arrive with inter-arrival times $\tau_1 > 0, \tau_2 > 0, \ldots$ ($\mathbf{E}\tau_j = a$), will wait for the arrival of the next bus for a random time χ of which the mean $\mathbf{E}\chi$ could prove to be greater than a.

One of the principal facts of renewal theory is the assertion that, under broad assumptions, the joint distribution of $\chi(t)$ and $\gamma(t)$ has a limit as $t \to \infty$, so that for large t the distribution of $\chi(t)$ does not depend on t any more and becomes stationary. Denote this limiting distribution of $\chi(t)$ by $\mathbf{G}$ and its distribution function by G:

$$G(x) = \lim_{t\to\infty} \mathbf{P}\big(\chi(t) < x\big). \tag{10.3.1}$$

If we take the distribution of τ_1 to be $\mathbf{G}$ then, for such process, by its very construction the distribution of the variable $\chi(t)$ will be independent of t. Indeed, in that case we can think of the positive elements of $\{T_j\}$ as the renewal times for a process which is constructed from the sequence $\{\tau_j\}$ and of which the start is shifted to a point $-N$, where N is very large. Since by virtue of (10.3.1) we can assume that the distributions of $\chi(N)$ and $\chi(N+t)$ coincide with each other, the distribution of the variable $\chi(t)$ (which can be identified with $\chi(N+t)$) is independent of t and coincides with that of τ_1. A formal proof of this fact is omitted, since it will not be used in what follows. However, the reader could carry it out using the explicit form of $G(x)$ from (10.3.1) to be derived below.

In the arithmetic case, the distribution $\mathbf{G}$ is just the law (10.2.1) used to construct the homogeneous renewal process $\eta_0(t)$. We will prove this in our next theorem.

It follows from the fact that, for the process $\eta_0(t)$, the distribution of $\chi(t)$ does not depend on t and coincides with that of τ_1, that the distribution of $\eta_0(t+u) - \eta_0(t)$ coincides with that of $\eta_0(u)$ and hence is also independent of t. It is this property that establishes the stationarity of the increments of the renewal process; we called this property homogeneity. It means that the distribution of the number of renewals over a time interval of length u does not depend on when we start counting, and therefore depends on u only.

Theorems on the limiting distribution of $\chi(t)$ and $\gamma(t)$ are of interest not only from the point of view of their applications. We will need them for a variety of other problems. Again we consider first the case when the variables $\tau_j > 0$ are arithmetic. In that case the "time" can also be assumed discrete and we will denote it, as before, by the letters n and k. Let, as before, $\tau_j \stackrel{d}{=} \tau$ for $j \geq 2$ and $p_k = \mathbf{P}(\tau = k)$.

Theorem 10.3.1 *Let the random variable $\tau > 0$ be arithmetic, $\mathbf{E}\tau = a$ exist, τ_1 be an arbitrary integer random variable, and the g.c.d. of the possible values of τ be equal to 1. Then the following limit exists*

$$\lim_{k\to\infty} \mathbf{P}\big(\gamma(k) = i,\ \chi(k) = j\big) = \frac{p_{i+j}}{a}, \quad i \geq 0,\ j > 0. \tag{10.3.2}$$

It follows from Theorem 10.3.1 that

$$\begin{aligned} &\lim_{k\to\infty} \mathbf{P}\big(\chi(k) = i\big) = \frac{1}{a}\sum_{j=i}^{\infty} p_j, \quad i > 0; \\ &\lim_{k\to\infty} \mathbf{P}\big(\gamma(k) = i\big) = \frac{1}{a}\sum_{j=i+1}^{\infty} p_j, \quad j \geq 0. \end{aligned} \tag{10.3.3}$$

Proof of Theorem 10.3.1 By the renewal theorem (see Theorem 10.2.2), for $k > i$,

$$\begin{aligned} \mathbf{P}\big(\gamma(k) = i,\ \chi(k) = j\big) &= \sum_{l=1}^{\infty} \mathbf{P}(T_l = k - i,\ \tau_{l+1} = i + j) \\ &= \sum_{l=1}^{\infty} \mathbf{P}(T_l = k - i)\mathbf{P}(\tau = i + j) = h(k-i)p_{i+j} \to \frac{p_{i+j}}{a} \end{aligned}$$

as $k \to \infty$. The theorem is proved. □

If $\mathbf{E}\tau^2 = m_2 < \infty$, then Theorem 10.3.1 allows a refinement of Theorem 10.2.2 (see Theorem 10.3.2 below).

Corollary 10.3.1 *If $m_2 < \infty$, then the random variables $\chi(k)$ are uniformly integrable and*

$$\mathbf{E}\chi(k) \to \frac{1}{a}\sum_{i=0}^{\infty} i \sum_{j=i}^{\infty} p_j = \frac{m_2 + a}{2a} \quad \text{as } k \to \infty. \tag{10.3.4}$$

Proof The uniform integrability follows from the inequalities $h(k) \leq 1$,

$$\mathbf{P}\big(\chi(k) = j\big) = \sum_{i=0}^{k} h(k-i)p_{i+j} \leq \sum_{i=j}^{\infty} p_i.$$

This implies (10.3.4) (see Sect. 6.1). □

Now we can state a refined version of the integral theorem that implies Theorem 10.2.2.

Theorem 10.3.2 *If all τ_j are identically distributed and $\mathbf{E}\tau^2 = m_2 < \infty$, then*

$$H(k) = \frac{k}{a} + \frac{m_2 + a}{2a^2} + o(1)$$

as $k \to \infty$.

The Proof immediately follows from the Wald identity

$$H(k) = \mathbf{E}\eta(k) = \frac{k + \mathbf{E}\chi(k)}{a}$$

and Corollary 10.3.1. □

Remark 10.3.1 For the process $\eta^*(t)$ corresponding to nonzero times τ'_j required for components' renewals (mentioned in Remark 10.1.1), the reader can easily find, similarly to Theorem 10.3.1, not only the asymptotic value p_{i+j}/a^* of the probability that at time $k \to \infty$ the current component has already worked for time i and will still work for time j, but also the asymptotics of the probability that the component has been "under repair" for time i and will stay in that state for time j, that is given by p'_{i+j}/a^*, where $p'_i = \mathbf{P}(\tau'_j = i)$, $a^* = \mathbf{E}(\tau_j + \tau'_j) = \mathbf{E}\tau^*_j$.

Now consider the question of under what circumstances the distribution of the random variable τ_1 for the homogeneous process (i.e. the distribution of what one could denote by $\chi(\infty)$) will coincide with that of τ_j for $j \ge 2$. Such a coincidence is equivalent to the equality

$$p_i = \frac{1}{a} \sum_{j=i}^{\infty} p_j$$

for $i = 1, 2, \ldots$, or, which is the same, to

$$a(p_i - p_{i-1}) = -p_{i-1}, \quad p_i = \frac{a-1}{a} p_{i-1}, \quad p_i = \frac{1}{a-1} \left(\frac{a-1}{a} \right)^i .$$

This means that the renewal process generated by the sequence of independent identically distributed random variables $\tau_1, \tau_2, \ldots$ is homogeneous if and only if τ_j (or, more precisely, τ_{j-1}) have the geometric distribution.

Denote by γ and χ the random variables having distribution (10.3.2). Using (10.3.1), it is not hard to show that γ and χ are independent also only in the case when τ_j, $j \ge 2$, have the geometric distribution. When all τ_j, $j \ge 1$, have such a distribution, $\gamma(n)$ and $\chi(n)$ are also independent, and $\chi(n) \stackrel{d}{=} \tau_1$. These facts can be proved in exactly the same way as for the exponential distribution (see Sect. 10.4).

We now return to the general case and recall that if $\mathbf{E}\tau^2 < \infty$ then (see Corollary 10.3.1)

$$\mathbf{E}\chi = \frac{\mathbf{E}\tau^2}{2a} + \frac{1}{2}.$$

This means, in particular, that if the distribution of τ is such that $\mathbf{E}\tau^2 > 2a^2 - a$, then, for large n, the excess mean value $\mathbf{E}\chi(n)$ will become greater than $\mathbf{E}\tau = a$.

10.4 The Renewal Theorem and the Limiting Behaviour of the Excess and Defect in the Non-arithmetic Case

Recall that in this chapter by the non-arithmetic case we mean that there exists no $h > 0$ such that $\mathbf{P}(\bigcup_k \{\tau = kh\}) = 1$, where k runs over all integers. To state the key renewal theorem in that case, we will need the notion of a *directly integrable function*.

Definition 10.4.1 A function $g(u)$ defined on $[0, \infty)$ is said to be *directly integrable* if:

(1) the function g is Riemann integrable[1] over any finite interval $[0, N]$; and
(2) $\sum_k g(k) < \infty$, where $g_k = \max_{k \le u \le k+1} |g(u)|$.

It is evident that any monotonically decreasing function $g(t) \downarrow 0$ having a finite Lebesgue integral

$$\int_0^\infty g(t)\, dt < \infty$$

is directly integrable. This also holds for differences of such functions.

The notion of directly integrable functions introduced in [12] differs somewhat from the one just defined, although it essentially coincides with it. It will be more convenient for us to use Definition 10.4.1, since it allows us to simplify to some extent the exposition and to avoid auxiliary arguments (see Appendix 9).

Theorem 10.4.1 (The key renewal theorem) *Let* $\tau_j \overset{d}{=} \tau \ge 0$ *for* $j \ge 2$ *and* g *be a directly integrable function. If the random variable* τ *is non-arithmetic, there exists* $\mathbf{E}\tau = a > 0$, *and the distribution of* τ_1 *is arbitrary, then, as* $t \to \infty$,

$$\int_0^t g(t-u)\, dH(u) \to \frac{1}{a} \int_0^\infty g(u)\, du. \tag{10.4.1}$$

There is a measure $\mathbf{H}$ on $[0, \infty)$ associated with H that is defined by $\mathbf{H}((x, y]) := H(y) - H(x)$. The integral

$$\int_0^t g(t-u)\, dH(u)$$

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[1] That is, the sums $n^{-1} \sum_k \underline{g}_k$ and $n^{-1} \sum_k \overline{g}_k$ have the same limits as $n \to \infty$, where $\underline{g}_k = \min_{u \in \Delta_k} g(u)$, $\overline{g}_k = \max_{u \in \Delta_k} g(u)$, $\Delta_k = [k\Delta, (k+1)\Delta)$, and $\Delta = N/n$. The usual definition of Riemann integrability over $[0, \infty)$ assumes that condition (1) of Definition 10.4.1 is satisfied and the limit of $\int_0^N g(u)\, du$ as $N \to \infty$ exists. This approach covers a wider class of functions than in Definition 10.4.1, allowing, for example, the existence of a sequence $t_k \to \infty$ such that $g(t_k) \to \infty$.

in (10.4.1) can also be written as

$$\int_0^t g(t-u)\,\mathbf{H}(du).$$

It follows from (10.4.1), in particular, that, for any fixed u,

$$H(t)-H(t-u)\to\frac{u}{a}. \tag{10.4.2}$$

It is not hard to see that this relation, which is called the *local renewal theorem*, is equivalent to (10.4.1).

The proof of Theorem 10.4.1 is technically rather difficult, so we have placed it in Appendix 9. One can also find there refinements of Theorem 10.4.1 and its analogue in the case where τ has a density.

The other assertions of Sects. 10.2 and 10.3 can also be extended to the non-arithmetic case without any difficulties. Let all τ_j be nonnegative.

Definition 10.4.2 In the non-arithmetic case, a renewal process $\eta(t)$ is called *homogeneous* (and is denoted by $\eta_0(t)$) if the distribution of the first jump has the form

$$\mathbf{P}(\tau_1>x)=\frac{1}{a}\int_x^\infty \mathbf{P}(\tau>t)\,dt.$$

The ch.f. of τ_1 equals

$$\varphi_{\tau_1}(\lambda):=\mathbf{E}e^{i\lambda\tau_1}=\frac{1}{a}\int_0^\infty e^{i\lambda x}\mathbf{P}(\tau>x)\,dx.$$

Since here we are integrating over $x>0$, the integral exists (as well as the function $\varphi(\lambda)=\varphi_\tau(\lambda):=\mathbf{E}e^{i\lambda\tau}$) for all λ with $\operatorname{Im}\lambda>0$ (for $\lambda=i\alpha+v$, $-\infty<v<\infty$, $\alpha\ge 0$, the factor $e^{i\lambda x}$ is equal to $e^{-\alpha x}e^{ivx}$; see property 6 of ch.f.s). Therefore, for $\operatorname{Im}\lambda\ge 0$,

$$\varphi_{\tau_1}(\lambda)=-\frac{1}{i\lambda a}\left[1+\int_0^\infty e^{i\lambda x}\,d\mathbf{P}(\tau>x)\right]=\frac{\varphi(\lambda)-1}{i\lambda a}.$$

Theorem 10.4.2 *For a homogeneous renewal process,*

$$H_0(t)\equiv\mathbf{E}\eta_0(t)=1+\frac{t}{a},\qquad t\ge 0.$$

Proof This theorem can be proved in the same way as Theorem 10.2.1. Consider the Fourier–Stieltjes transform of the function $H_0(t)$:

$$r(\lambda):=\int_0^\infty e^{i\lambda x}\,dH_0(x).$$

Note that this transform exists for $\operatorname{Im}\lambda>0$ and the uniqueness theorem established for ch.f.s remains true for it, since $\varphi^*(v):=r(i\alpha+v)/r(i\alpha)$, $-\infty<v<\infty$ (we put $\lambda=i\alpha+v$ for a fixed $\alpha>0$) can be considered as the ch.f. of a certain distribution being the "Cramér transform" (see Chap. 9) of $H_0(t)$.

Since $\tau_j \ge 0$, one has

$$H_0(x) = \sum_{j=0}^{\infty} \mathbf{P}(T_j \le x).$$

As $H_0(t)$ has a unit jump at $t = 0$, we obtain

$$r(\lambda) = \int_0^\infty e^{i\lambda x}\, dH_0(x) = 1 + \sum_{j=0}^{\infty} \varphi_{\tau_1}(\lambda)\varphi^j(\lambda) = 1 + \frac{\varphi(\lambda) - 1}{i\lambda a} \frac{1}{1 - \varphi(\lambda)}$$
$$= 1 - \frac{1}{i\lambda a}.$$

It is evident that this transform corresponds to the function $H_0(t) = 1 + t/a$. The theorem is proved. □

In the non-arithmetic case, one has the same connections between the homogeneous renewal process $\eta_0(t)$ and the limiting distribution of $\chi(t)$ and $\gamma(t)$ as we had in the arithmetic case. In the same way as in Sect. 10.3, we can derive from the renewal theorem the following.

Theorem 10.4.3 *If $\tau \ge 0$ is non-arithmetic*, $\mathbf{E}\tau = a$, *and the distribution of $\tau_1 \ge 0$ is arbitrary*, *then the following limit exists*

$$\lim_{t\to\infty} \mathbf{P}\big(\gamma(t) > u,\ \chi(t) > v\big) = \frac{1}{a} \int_{u+v}^{\infty} \mathbf{P}(\tau > x)\, dx. \tag{10.4.3}$$

Proof For $t > u$, by the total probability formula,

$$\begin{aligned}
&\mathbf{P}\big(\gamma(t) > u, \chi(t) > v\big) \\
&\quad = \mathbf{P}(\tau_1 > t + v) + \sum_{j=1}^{\infty} \int_0^{t-u} \mathbf{P}\big(\eta(t) = j + 1, T_j \in dx, \gamma(t) > u, \chi(t) > v\big) \\
&\quad = \mathbf{P}(\tau_1 > t + v) + \sum_{j=1}^{\infty} \int_0^{t-u} \mathbf{P}(T_j \in dx, \tau_{j+1} > t - x + v) \\
&\quad = \mathbf{P}(\tau_1 > t + v) - \mathbf{P}(\tau > t + v) + \int_0^{t-u} dH(x)\mathbf{P}(\tau > t - x + v).
\end{aligned} \tag{10.4.4}$$

Here the first two summands on the right-hand side converge to 0 as $t \to \infty$. By the renewal theorem for $g(x) = \mathbf{P}(\tau > x + u + v)$ (see (10.4.1)), the last integral converges to

$$\frac{1}{a} \int_0^\infty \mathbf{P}(\tau > x + u + v)\, dx.$$

The theorem is proved. □

As was the case in the previous section (see Theorem 10.3.2), in the case $\mathbf{E}\tau^2 = m_2 < \infty$ Theorem 10.4.3 allows us to refine the key renewal theorem.

Theorem 10.4.4 *If all* $\tau_j \overset{d}{=} \tau \geq 0$ *are identically distributed and* $\mathbf{E}\tau^2 = m_2 < \infty$, *then, as* $t \to \infty$,

$$H(t) = \frac{t}{a} + \frac{m_2}{2a^2} + o(1).$$

Proof From (10.4.4) for $u = 0$ and Lemma 10.2.3 it follows that $\chi(t)$ are uniformly integrable, for

$$\mathbf{P}\big(\chi(t) > v\big) = \int_0^t dH(x)\,\mathbf{P}(\tau > t - x + v) < (c_1 + c_2)\sum_{k\geq 0}\mathbf{P}(\tau > k + v), \tag{10.4.5}$$

and therefore by (4.4.3)

$$\mathbf{E}\chi(t) \to \frac{1}{a}\int_0^\infty \int_v^\infty \mathbf{P}(\tau > u)\,du\,dv = \frac{m_2}{2a}. \tag{10.4.6}$$

It remains to make use of Wald's identity. The theorem is proved. □

One can add to relation (10.4.6) that, under the conditions of Theorem 10.4.4, one has

$$\mathbf{E}\chi^2(t) = o(t) \tag{10.4.7}$$

as $t \to \infty$. Indeed, (10.4.5) and Lemma 10.2.3 imply

$$\mathbf{P}\big(\chi(t) > v\big) < (c_1 + c_2)\sum_{k\leq t}\mathbf{P}(\tau > k + v) < c\int_0^t \mathbf{P}(\tau > z + v)\,dz.$$

Further, integrating by parts, we obtain

$$\begin{aligned}\mathbf{E}\chi^2(t) &= -\int_0^\infty v^2\,d\mathbf{P}\big(\chi(t) > v\big)\\ &= 2\int_0^\infty v\mathbf{P}\big(\chi(t) > v\big)\,dv < 2c\int_0^t \int_0^\infty v\mathbf{P}(\tau > z + v)\,dv\,dz,\end{aligned} \tag{10.4.8}$$

where the inner integral converges to zero as $z \to \infty$:

$$\int_0^\infty v\mathbf{P}(\tau > z + v)\,dv = \frac{1}{2}\int_0^\infty v^2\,d\mathbf{P}(\tau < z + v) < \frac{1}{2}\mathbf{E}\big(\tau^2; \tau > z\big) \to 0.$$

This and (10.4.8) imply (10.4.7).

Note also that if only $\mathbf{E}\tau$ exists, then, by Theorem 10.1.1, we have $\mathbf{E}\chi(t) = o(t)$ and, by Theorem 10.4.1 (or 10.4.3),

$$\mathbf{P}\big(\chi(t) > v\big) \to \frac{1}{a}\int_0^\infty \mathbf{P}(\tau > u + v)\,du.$$

Now let, as before, γ and χ denote random variables distributed according to the limiting distribution (10.4.3). Similarly to the above, it is not hard to establish that if $\mathbf{E}\tau^k < \infty$, $k \ge 1$, then, as $t \to \infty$,

$$\mathbf{E}\chi^{k-1}(t) \to \mathbf{E}\chi^{k-1} < \infty, \qquad \mathbf{E}\chi^k(t) = o(t).$$

Further, it is seen from Theorem 10.4.3 that each of the random variables γ and χ has density equal to $a^{-1}\mathbf{P}(\tau > x)$. The joint distribution of γ and χ may have no density. If τ has density $f(x)$ then there exists a joint density of γ and χ equal to $a^{-1}f(x+y)$. It also follows from Theorem 10.4.3 that γ and χ are independent if and only if

$$\int_x^\infty \mathbf{P}(\tau > u)\, du = \frac{1}{\alpha} e^{-\alpha x}$$

for some $\alpha > 0$, i.e. independence takes place only for the exponential distribution $\tau \Subset \mathbf{\Gamma}_\alpha$.

Moreover, for *homogeneous* renewal processes the coincidence of $\mathbf{P}(\tau_1 > x)$ and $\mathbf{P}(\tau > x)$ takes place only when $\tau \Subset \mathbf{\Gamma}_\alpha$. In other words, the renewal process generated by a sequence of identically distributed random variables $\tau_1, \tau, \ldots$ will be homogeneous if and only if $\tau_j \Subset \mathbf{\Gamma}_\alpha$. In that case $\eta_0(t)$ is called (see also Sect. 19.4) a *Poisson process*. This is because for such a process, for each t, the variable $\eta(t) = \eta_0(t)$ has the Poisson distribution with parameter t/α.

The Poisson process has some other remarkable properties as well (see also Sect. 19.4). Clearly, one has $\chi(t) \Subset \mathbf{\Gamma}_\alpha$ for such a process, and moreover, the variables $\gamma(t)$ and $\chi(t)$ are independent. Indeed, by (10.4.4), taking into account that $H(x)$ has a jump of magnitude 1 at the point $x = 0$, we obtain for $u < t$ that

$$\begin{aligned}
\mathbf{P}\big(\gamma(t) > u, \chi(t) > v\big) &= e^{-\alpha(t+v)} + \alpha \int_0^{t-u} e^{-\alpha(t-x+v)}\, dx \\
&= e^{-\alpha(u+v)} = \mathbf{P}\big(\gamma(t) > u\big)\mathbf{P}\big(\chi(t) > v\big); \\
\mathbf{P}\big(\gamma(t) = t, \chi(t) > v\big) &= \mathbf{P}(\tau_1 > t + v) = e^{-\alpha(t+v)} = \mathbf{P}\big(\gamma(t) = t\big)\mathbf{P}\big(\chi(t) > v\big); \\
\mathbf{P}\big(\gamma(t) > t\big) &= 0.
\end{aligned}$$

These relations also imply that the random variable $\tau_{\eta(t)} = \gamma(t) + \chi(t)$ has the same distribution as $\min(t, \tau_1) + \tau_2$, where $\tau_j \Subset \mathbf{\Gamma}_\alpha$, $j = 1, 2$, are independent so that $\tau_{\eta(t)} \Leftrightarrow \mathbf{\Gamma}_{\alpha,2}$ as $t \to \infty$.

The fact that $\gamma(t)$ and $\chi(t)$ are independent of each other deserves attention from the point of view of its interpretation. It means the following. The residual lifetime of the component operating at a given time t has the same distribution as the lifetime of a *new* component (recall that $\tau_j \Subset \mathbf{\Gamma}_\alpha$) and is independent of how long this component has already been working (which at first glance is a paradox). Since the lifetime distributions of devices consisting of large numbers of reliable elements are close to the exponential law (see Theorem 20.3.2), the above-mentioned fact is of significant practical interest.

If τ_i can assume *negative* values as well, the problems related to the distributions of $\gamma(t)$ and $\chi(t)$ become much more complicated. To some extent such problems can be reduced to the case of nonnegative variables, since the distribution of $\chi(t)$ coincides with that of the variable $\chi^*(t)$ constructed from a sequence $\{\tau_j^* \ge 0\}$, where τ_j^* have the same distribution as $\chi(0)$. The distribution of $\chi(0)$ can be found using the methods of Chap. 12.

In particular, for random variables $\tau_1, \tau_2, \dots$ *taking values of both signs*, Theorems 10.4.1 and 10.4.3 imply the following assertion.

Corollary 10.4.1 *Let $\tau_1, \tau_2, \dots$ be non-arithmetic independent and identically distributed and $\mathbf{E}\tau_1 = a$. Then the following limit exists*

$$\lim_{t\to\infty} \mathbf{P}\big(\chi(t) > v\big) = \frac{1}{\mathbf{E}\chi(0)} \int_v^\infty \mathbf{P}\big(\chi(0) > t\big)\, dt, \qquad v > 0.$$

For arithmetic τ_j,

$$\lim_{k\to\infty} \mathbf{P}\big(\chi(k) = i\big) = \frac{1}{\mathbf{E}\chi(0)} \mathbf{P}\big(\chi(0) > i\big), \qquad i > 0.$$

10.5 The Law of Large Numbers and the Central Limit Theorem for Renewal Processes

In this section we return to the general case where τ_j are not necessarily identically distributed (cf. Sect. 10.1).

10.5.1 The Law of Large Numbers

First assume that $\tau_j \ge 0$ and put

$$a_k := \mathbf{E}\tau_k, \qquad A_n := \sum_{k=1}^n a_k.$$

Theorem 10.5.1 *Let $\tau_k \ge 0$ be independent, $\tau_k - a_k$ uniformly integrable, and $n^{-1}A_n \to a > 0$ as $n \to \infty$. Then, as $t \to \infty$,*

$$\frac{\eta(t)}{t} \xrightarrow{p} \frac{1}{a}.$$

Proof The basic relation we shall use is the equality

$$\big\{\eta(t) > n\big\} = \{T_n \le t\}, \tag{10.5.1}$$

which implies

$$\mathbf{P}\bigg(\frac{\eta(t)}{t} - \frac{1}{a} > \varepsilon\bigg) = \mathbf{P}\bigg(\eta(t) > \frac{t}{a}(+\varepsilon)\bigg) = \mathbf{P}(T_n \le t),$$

where for simplicity we assume that $n = \frac{t}{a}(1+\varepsilon)$ is an integer. Further,

$$\begin{aligned}\mathbf{P}(T_n \le t) &= \mathbf{P}\left(\frac{T_n}{n} \le \frac{a}{1+\varepsilon}\right)\\ &= \mathbf{P}\left(\frac{T_n - A_n}{n} \le \frac{a}{1+\varepsilon} - \frac{A_n}{n}\right) \le \mathbf{P}\left(\frac{T_n - A_n}{n} \le -\frac{a\varepsilon}{2}\right)\end{aligned}$$

for n large enough and ε small enough. Applying the law of large numbers to the right-hand side of this relation (Theorem 8.3.3), we obtain that, for any $\varepsilon > 0$, as $t \to \infty$,

$$\mathbf{P}\left(\frac{\eta(t)}{t} - \frac{1}{a} > \frac{\varepsilon}{a}\right) \to 0.$$

The probability $\mathbf{P}(\frac{\eta(t)}{t} - \frac{1}{a} < -\frac{\varepsilon}{a})$ can be bounded in the same way. The theorem is proved. □

10.5.2 The Central Limit Theorem

Put

$$\sigma_k^2 := \mathbf{E}(\tau_k - a_k)^2 = \operatorname{Var}\tau_k, \qquad B_n^2 := \sum_{k=1}^{n} \sigma_k^2.$$

Theorem 10.5.2 *Let $\tau_k \ge 0$ and the random variables $\tau_k - a_k$ satisfy the Lindeberg condition: for any $\delta > 0$ and $n \to \infty$,*

$$\sum_{k=1}^{n} \mathbf{E}\big(|\tau_k - a_k|^2;\ |\tau_k - a_k| > \delta B_n\big) = o\big(B_n^2\big).$$

Let, moreover, there exist $a > 0$ and $\sigma > 0$ such that, as $n \to \infty$,

$$A_n := \sum_{k=1}^{n} a_k = an + o(\sqrt{n}), \qquad B_n^2 = \sigma^2 n + o(n). \tag{10.5.2}$$

Then

$$\frac{\eta(t) - t/a}{\sigma\sqrt{t/a^3}} \Leftrightarrow \mathbf{\Phi}_{0,1}. \tag{10.5.3}$$

Proof From (10.5.1) we have

$$\mathbf{P}\big(\eta(t) > n\big) = \mathbf{P}(T_n \le t) = \mathbf{P}\left(\frac{T_n - A_n}{B_n} \le \frac{t - A_n}{B_n}\right). \tag{10.5.4}$$

Let n vary as $t \to \infty$ so that

$$\frac{t - A_n}{B_n} \to v$$

for a fixed v. To find such an n, solve for n the equation

$$\frac{t - an}{\sigma\sqrt{n}} = v.$$

This is a quadratic equation in n, and its solution has the form

$$n = \frac{t}{a} \pm \frac{v\sigma}{a^2}\sqrt{at}\left(1 + O\left(\frac{1}{\sqrt{t}}\right)\right). \tag{10.5.5}$$

For such n, by (10.5.2),

$$\frac{t - A_n}{B_n} = \left[\mp\frac{v\sigma}{a}\sqrt{at} + o(\sqrt{t})\right]\frac{(1 + o(1))}{\sigma\sqrt{t/a}} = \mp v + o(1).$$

This equality means that we have to choose the minus sign in (10.5.5). Therefore, by (10.5.4) and the central limit theorem,

$$\mathbf{P}\big(\eta(t) > n\big) = \mathbf{P}\left(\frac{\eta(t) - t/a}{\sigma\sqrt{t/a^3}} > -v + o(1)\right) \to \Phi(v) = 1 - \Phi(-v).$$

Changing $-v$ to u, by the continuity theorems (see Lemma 6.2.2) we get

$$\mathbf{P}\left(\frac{\eta(t) - t/a}{\sigma\sqrt{ta^{-3}}} < u\right) \to \Phi(u).$$

The theorem is proved. □

Remark 10.5.1 In Theorems 10.5.1 and 10.5.2 we considered the case where A_n grows asymptotically linearly as $n \to \infty$. Then the centring parameter t/a for $\eta(t)$ changes asymptotically linearly as well. However, nothing prevents us from considering a more general case where, say, $A_n \sim cn^\alpha$, $\alpha > 0$. Then the centring parameter for $\eta(t)$ will be the solution to the equation $cn^\alpha = t$, i.e. the function $(t/c)^{1/\alpha}$ (under the conditions of Theorem 10.5.2, in this case we have to assume that $B_n = o(A_n)$). The asymptotics of the renewal function will have the same form.

In order to extend the assertions of Theorems 10.5.1 and 10.5.2 to τ_j assuming values of both signs, we need some auxiliary assertions that are also of independent interest.

10.5.3 A Theorem on the Finiteness of the Infimum of the Cumulative Sums

In this subsection we will consider *identically distributed* independent random variables $\tau_1, \tau_2, \ldots$. We first state the following simple assertion in the form of a lemma.

Lemma 10.5.1 *One has* $\mathbf{E}|\tau| < \infty$ *if and only if*

$$\sum_{j=1}^{\infty}\mathbf{P}\big(|\tau| > j\big) < \infty.$$

The Proof follows in an obvious way from the equality

$$\mathbf{E}|\tau| = \int_0^\infty \mathbf{P}\big(|\tau| > x\big)\,dx$$

and the inequalities

$$\sum_{j=1}^\infty \mathbf{P}\big(|\tau| > j\big) \le \int_0^\infty \mathbf{P}\big(|\tau| > x\big)\,dx \le 1 + \sum_{j=1}^\infty \mathbf{P}\big(|\tau| > j\big). \qquad \square$$

Let, as before,

$$T_n = \sum_{j=1}^n \tau_j.$$

Theorem 10.5.3 *If* $\tau_j \stackrel{d}{=} \tau$ *are identically distributed and independent and* $\mathbf{E}\tau > 0$, *then the random variable* $Z := \inf_{k\ge 0} T_k$ *is proper* (*finite with probability* 1).

Proof Let $\eta_1 = \eta(1)$ be the number of the first sum T_k to exceed level 1. Consider the sequence $\{\tau_k^* = \tau_{\eta_1+k}\}$ that, by Lemma 10.2.1, has the same distribution as $\{\tau_k\}$ and is independent of $\eta_1, \tau_1, \dots, \tau_{\eta_1}$. For this sequence, denote by η_2 the subscript k for which the sum $T_k^* = \sum_{j=1}^k \tau_j^*$ first exceeds level 1. It is clear that the random variables η_1 and η_2 are identically distributed and independent. Next, construct for the sequence $\{\tau_k^{**} = \tau_{\eta_1+\eta_2+k}\}$ the random variable η_3 following the same rule, and so on. As a result we will obtain a sequence of Markov times $\eta_1, \eta_2, \dots$ that determine the times of "renewals" of the original sequence $\{T_k\}$, associated with attaining level 1.

Now set

$$Z_1 := \min_{k<\eta_1} T_k, \qquad Z_2 := \min_{k<\eta_2} T_k^*, \dots$$

Clearly, the Z_j are identically distributed and

$$Z = \inf\{Z_1, T_{\eta_1} + Z_2, T_{\eta_1+\eta_2} + Z_3, \dots\},$$

where by definition $T_{\eta_1} > 1$, $T_{\eta_1+\eta_2} > 2$ and so on. Hence

$$\{Z < -N\} = \bigcup_{k=0}^\infty \{Z_{k+1} + T_{\eta_1+\dots+\eta_k} < -N\} \subset \bigcup_{k=0}^\infty \{Z_k + k < -N\},$$

$$\mathbf{P}(Z < -N) \le \sum_{k=1}^\infty \mathbf{P}(Z_k + k < -N) = \sum_{j=N+1}^\infty \mathbf{P}(Z_1 < -j).$$

This expression tends to 0 as $N \to \infty$ provided that $\mathbf{E}|Z_1| < \infty$ (see Lemma 10.5.1). It remains to verify the finiteness of $\mathbf{E}Z_1$, which follows from the finiteness of $\mathbf{E}\eta_1 = \mathbf{E}\eta(1) = H(1) < c$ (see Example 4.4.5) and the relations

$$\mathbf{E}|Z_1| \le \mathbf{E}\sum_{j=1}^{\eta_1} |\tau_j| = \mathbf{E}\eta_1 \mathbf{E}|\tau_1| < \infty$$

(see Theorem 4.4.2). $\square$

10.5.4 Stochastic Inequalities. The Law of Large Numbers and the Central Limit Theorem for the Maximum of Sums of Non-identically Distributed Random Variables Taking Values of Both Signs

In this subsection we extend the assertions of some theorems of Chap. 8 to maxima of sums of random variables with a positive “mean drift”. To do this we will have to introduce some additions restrictions that are always satisfied when the summands are identically distributed. Here we will need the notion of stochastic inequalities (or inequalities in distribution). Let ξ and ζ be given random variables.

Definition 10.5.1 We will say that ζ majorises (minorises) ξ in distributionand denote this by $\xi \overset{d}{\le} \zeta$ ($\xi \overset{d}{\ge} \zeta$) if, for all t,

$$\mathbf{P}(\xi \ge t) \le \mathbf{P}(\zeta \ge t) \qquad \big(\mathbf{P}(\xi \ge t) \ge \mathbf{P}(\zeta \ge t)\big).$$

Clearly, if $\xi \overset{d}{\le} \zeta$ then $-\xi \overset{d}{\ge} -\zeta$. We show that stochastic inequalities possess some other properties of ordinary inequalities.

Lemma 10.5.2 *If $\{\xi_k\}_{k=1}^{\infty}$ and $\{\zeta\}_{k=1}^{\infty}$ are sequences of independent (in each sequence) random variables and $\xi_k \overset{d}{\le} \zeta_k$, then, for all n,*

$$S_n \overset{d}{\le} Z_n, \qquad \overline{S}_n \overset{d}{\le} \overline{Z}_n,$$

where

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad Z_n = \sum_{k=1}^{n} \zeta_k, \qquad \overline{S}_n = \max_{k \le n} S_k, \qquad \overline{Z}_n = \max_{k \le n} Z_k.$$

Similarly, if $\xi_k \overset{d}{\ge} \zeta_k$, then $\min_{k\le n} S_k \overset{d}{\ge} \min_{k\le n} Z_k$.

Proof Let $F_k(t) := \mathbf{P}(\xi_k < t)$ and $G_k(t) := \mathbf{P}(\zeta_k < t)$. Using quantile transformations $F_k^{(-1)}$ and $G_k^{(-1)}$ (see Definition 3.2.4) and a sequence of independent random variables $\{\omega_k\}_{k=1}^{\infty}$, $\omega_k \in \mathbf{U}_{0,1}$, we can construct on a common probability space the sequences $\xi_k^* = F_k^{(-1)}(\omega_k)$ and $\zeta_k^* = G_k^{(-1)}(\omega_k)$ such that $\xi_k^* \overset{d}{=} \xi_k$ and $\zeta_k^* \overset{d}{=} \zeta_k$ (the distributions of ξ_k^* and ξ_k and of ζ_k^* and ζ_k^* coincide). Moreover, $\xi_k^* \le \zeta_k^*$, which is a direct consequence of the inequality $F_k(t) \ge G_k(t)$ for all t. Endowing with the superscript * all the notations for sums and maximum of sums of random variables with asterisks, we obviously obtain that

$$S_n \overset{d}{=} S_n^* \le Z_n^* \overset{d}{=} Z_n, \qquad \overline{S}_n \overset{d}{=} \overline{S}_n^* \le \overline{Z}_n^* \overset{d}{=} \overline{Z}_n.$$

The last assertion of the lemma follows from the previous ones. The lemma is proved. □

Below we will need the following corollary of Theorem 10.5.3.

Lemma 10.5.3 *Let ξ_k be independent, $\xi_k \overset{d}{\geq} \zeta$ for all k and* $\mathbf{E}\zeta > 0$. *Then, for all n, the random variable*

$$D_n := \overline{S}_n - S_n \geq 0$$

is majorised in distribution by the random variable $-Z$: $D_n \overset{d}{\leq} -Z$, where $Z :=$ $\inf Z_k$, $Z_k := \sum_{j=1}^{k} \zeta_j$ *and ζ_j are independent copies of ζ.*

Proof We have

$$\begin{aligned}\overline{S}_n = \max(0, S_1, \ldots, S_n) &= S_n + \max(0, -\xi_n, -\xi_n - \xi_{n-1}, \ldots, -S_n) \\ &= S_n - \min(0, \xi_n, \xi_n + \xi_{n-1}, \ldots, S_n),\end{aligned}$$

where, by the last assertion of Lemma 10.5.2,

$$-D_n \equiv \min(0, \xi_n, \xi_n + \xi_{n-1}, \ldots, S_n) \overset{d}{\geq} \min_{k \leq n} Z_k \geq Z, \qquad D_n \overset{d}{\leq} -Z.$$

The fact that Z is a proper random variable follows from Theorem 10.5.3 on the finiteness of the infimum of partial sums. The lemma is proved. □

If $\xi_k \overset{d}{=} \xi$ are identically distributed and $a = \mathbf{E}\xi > 0$, then we can put $\xi = \zeta$. The above reasoning shows that in this case the limit distribution of $\overline{S}_n - S_n$ as $n \to \infty$ exists and coincides with the distribution of the random variable Z (the random variables $\overline{S}_n - S_n$ themselves do not have a limit, and, by the way, neither do the variables $\frac{S_n - an}{\sqrt{n}}$ in the central limit theorem).

Lemma 10.5.3 shows that, for $\xi_k \overset{d}{\geq} \zeta$ and $\mathbf{E}\zeta > 0$, the random variables $\overline{S}_n$ and S_n differ from each other by a proper random variable only. This makes the limit theorems for $\overline{S}_n$ and S_n essentially the same.

We proceed to the law of large numbers and the central limit theorem for $\overline{S}_n$.

Theorem 10.5.4 *Let $a_k = \mathbf{E}\xi_k > 0$, $A_n = \sum_{k=1}^{n} a_k$ and $A_n \sim an$ as $n \to \infty$, $a > 0$. Let, moreover, $\xi_k - a_k$ be uniformly integrable for all k and $\xi_k \overset{d}{\geq} \zeta$ with $\mathbf{E}\zeta > 0$. Then, as $n \to \infty$,*

$$\frac{\overline{S}_n}{n} \xrightarrow{p} a.$$

Note that the *left* uniform integrability of $\xi_k - a_k$ follows from the inequalities $\xi_k \overset{d}{\geq} \zeta$.

Proof By Lemma 10.5.3,

$$\overline{S}_n = S_n + D_n, \quad \text{where } D_n \geq 0,\ D_n \overset{d}{\leq} -Z. \tag{10.5.6}$$

Therefore,

$$\frac{\overline{S}_n}{n} = \frac{S_n - A_n}{n} + \frac{A_n}{n} + \frac{D_n}{n},$$

where by Theorem 8.3.3, as $n \to \infty$,

$$\frac{S_n - A_n}{n} \xrightarrow{p} 0.$$

It is also clear that

$$\frac{A_n}{n} \to a, \qquad \frac{D_n}{n} \xrightarrow{p} 0.$$

The theorem is proved. □

In addition to the notation from Theorem 10.5.3, put

$$\sigma_k^2 := \mathbf{E}(\xi_k - a_k)^2, \qquad B_n^2 := \sum_{k=1}^{n} \sigma_k^2.$$

Theorem 10.5.5 *Let, for some* $a > 0$ *and* $\sigma > 0$,

$$A_n = an + o(\sqrt{n}), \qquad B_n^2 = \sigma^2 n + o(n),$$

and let the random variables $\xi_k - a_k$ *satisfy the Lindeberg condition,* $\xi_k \overset{d}{\geq} \zeta$ *with* $\mathbf{E}\zeta > 0$. *Then*

$$\frac{\overline{S}_n - an}{\sigma\sqrt{n}} \Subset\Rightarrow \mathbf{\Phi}_{0,1}. \tag{10.5.7}$$

Proof By virtue of (10.5.6),

$$\frac{\overline{S}_n - an}{\sigma\sqrt{n}} = \frac{S_n - A_n}{B_n} \cdot \frac{B_n}{\sigma\sqrt{n}} + \frac{A_n - an}{\sigma\sqrt{n}} + \frac{D_n}{\sigma\sqrt{n}}, \tag{10.5.8}$$

where, by the central limit theorem,

$$\frac{S_n - A_n}{B_n} \Subset\Rightarrow \mathbf{\Phi}_{0,1}.$$

Moreover,

$$\frac{B_n}{\sigma\sqrt{n}} \xrightarrow{p} 1, \qquad \frac{A_n - an}{\sigma\sqrt{n}} \to 0, \qquad \frac{D_n}{\sigma\sqrt{n}} \xrightarrow{p} 0.$$

This and (10.5.8) imply (10.5.7). The theorem is proved. □

10.5.5 Extension of Theorems 10.5.1 and 10.5.2 to Random Variables Assuming Values of Both Signs

We return to renewal processes and limit theorems for them. In Theorems 10.5.1 and 10.5.2 we obtained the law of large numbers and the central limit theorem for the renewal process $\eta(t)$ defined in (10.1.1) with jumps $\tau_k \ge 0$. Now we drop the last assumption and assume that τ_j can take values of both signs.

Theorem 10.5.6 *Let the conditions of Theorem 10.5.1 be met, the condition $\tau_k \ge 0$ being replaced with the condition $\tau_k \overset{d}{\ge} \zeta$ with $\mathbf{E}\zeta > 0$. Then*

$$\frac{\eta(t)}{t} \xrightarrow{p} \frac{1}{a}. \tag{10.5.9}$$

If $\tau_k \overset{d}{=} \tau$ are identically distributed and $\mathbf{E}\tau > 0$, then we can put $\zeta = \tau$. Therefore Theorem 10.5.6 implies the following result.

Corollary 10.5.1 *If τ_k are independent and identically distributed and $\mathbf{E}\tau = a > 0$, then (10.5.9) holds true.*

Proof of Theorem 10.5.6 Here instead of (10.5.1) we should use the relation

$$\bigl\{\eta(t) > n\bigr\} = \{\overline{T}_n \le t\}, \qquad \overline{T}_n = \max_{k \le n} T_k, \qquad T_k = \sum_{j=1}^{k} \tau_j. \tag{10.5.10}$$

Then we repeat the argument from the proof of Theorem 10.5.1, changing in it T_n to $\overline{T}_n$ and using Theorem 10.5.4, which implies that $\overline{T}_n$ and T_n satisfy the law of large numbers. The theorem is proved. □

Theorem 10.5.7 *Let the conditions of Theorem 10.5.2 be met, the condition $\tau_k \ge 0$ being replaced with the condition $\tau_k \overset{d}{\ge} \zeta$ with $\mathbf{E}\zeta > 0$. Then (10.5.3) holds true.*

Proof Here we again have to use (10.5.10), instead of (10.5.1), and then repeat the argument proving Theorem 10.5.2 using Theorem 10.5.5, which implies that the distribution of $\frac{\overline{T}_n - an}{\sigma\sqrt{n}}$, as well as the distribution of $\frac{T_n - an}{\sigma\sqrt{n}}$, converges to the standard normal law $\mathbf{\Phi}_{0,1}$. The theorem is proved. □

Remark 10.5.2 (An analogue of Remarks 8.3.3, 8.4.1 and 10.1.1) The assertions of Theorems 10.5.6 and 10.5.7 can be generalised as follows. *Let τ_1 be an arbitrary random variable and random variables $\tau_k^* := \tau_{1+k}$, $k \ge 1$, satisfy the conditions of Theorem* 10.5.6 (Theorem 10.5.7). Then convergence (10.5.9) (10.5.3) still takes place.

Consider, for example, Theorem 10.5.7. Denote by A_x the event

$$A_x := \left\{ \frac{\eta(t) - a/t}{\sigma\sqrt{t/a^3}} < x \right\}.$$

Then the foregoing assertion follows from the relations

$$\mathbf{P}(A_x) = \mathbf{E}\bigl[\mathbf{P}(A_x | \tau_1);\ |\tau_1| \le N\bigr] + r_N,$$

where $r_N \le \mathbf{P}(|\tau_1| > N)$ can be made arbitrarily small by the choice of N, and by Theorem 10.5.7

$$\mathbf{P}(A_x|\tau_1) = \mathbf{P}\left(\frac{\eta^*(t-\tau_1) - \frac{t-\tau_1}{a}}{\sigma\sqrt{(t-\tau_1)/a^3}} + O\left(\frac{1}{\sqrt{t}}\right) < x \right) \to \mathbf{\Phi}(x)$$

as $t \to \infty$ for each fixed τ_1, $|\tau_1| \le N$. Here $\eta^*(t)$ is the renewal process that corresponds to the sequence $\{\tau_k^*\}$. □

10.5.6 The Local Limit Theorem

If we again narrow our assumptions and return to identically distributed $\tau_k \stackrel{d}{=} \tau \ge 0$ then we can derive local theorems more precise than Theorem 10.5.2. In this subsection we will find an asymptotic representation for $\mathbf{P}(\eta(t) = n)$ as $t \to \infty$. We know from Theorem 10.5.2 what range of values of n the bulk of the distribution of $\eta(t)$ is concentrated in. Therefore we will from the start consider not arbitrary n, but the values of n that can be represented as

$$n = \left[\frac{t}{a} + v\sigma\sqrt{\frac{t}{a^3}} \right], \qquad \sigma^2 = \operatorname{Var}(\tau), \tag{10.5.11}$$

for "proper" values of v ($[s]$ in (10.5.11) is the integer part of s), so that

$$\frac{(t-an)}{\sigma\sqrt{n}} = v + O\left(\frac{1}{\sqrt{t}}\right) \tag{10.5.12}$$

(see (10.5.5)). For the proof, it will be more convenient to consider the probabilities $\mathbf{P}(\eta(t) = n+1)$. Changing $n+1$ to n amends nothing in the argument below.

Theorem 10.5.8 *If $\tau \ge 0$ is either non-lattice or arithmetic and* $\operatorname{Var}(\tau) = \sigma^2 < \infty$, *then, for the values of n defined in* (10.5.11), *as $t \to \infty$,*

$$\mathbf{P}\big(\eta(t) = n+1\big) \sim \frac{a^{3/2}}{\sigma\sqrt{2\pi t}} e^{-v^2/2}, \tag{10.5.13}$$

where in the arithmetic case t is assumed to be integer.

Proof First let, for simplicity, τ have a density and satisfy the conditions of the local limit Theorem 8.7.2. Then

$$\mathbf{P}\big(\eta(t) = n+1\big) = \int_0^t \mathbf{P}(T_n \in du)\mathbf{P}(\tau > t-u), \tag{10.5.14}$$

where by Theorem 8.7.2, as $n \to \infty$,

$$\mathbf{P}\big(T_n - na \in d(u-na)\big) = \frac{du}{\sigma\sqrt{2\pi n}} \left[\exp\left\{ -\frac{(u-na)^2}{2n\sigma^2} \right\} + o(1) \right]$$

uniformly in u. Change the variable $u = t - z$. Since for the values of n we are dealing with one has (10.5.12), the exponential

$$\exp\left\{-\frac{(u-na)^2}{2n\sigma^2}\right\} = \exp\left\{-\frac{1}{2}\left(v - \frac{z}{\sigma\sqrt{n}}\right)^2\right\}$$

remains "almost constant" and asymptotically equivalent to $e^{-v^2/2}$ for $|z| < N$, $N \to \infty$, $N = o(\sqrt{n})$. Hence the integral in (10.5.14) is asymptotically equivalent to

$$\frac{1}{\sigma\sqrt{2\pi n}} e^{-v^2/2} \int_0^N \mathbf{P}(\tau > z)\, dz \sim \frac{a}{\sigma\sqrt{2\pi n}} e^{v^2/2}.$$

Since $n \sim t/a$ as $t \to \infty$, we obtain (10.5.13).

If τ has no density, but is non-lattice, then we should use the integro-local Theorem 8.7.1 for small Δ and, in a quite similar fashion, bound the integral in (10.5.14) (with t, which is a multiple of Δ) from above and from below by the sums

$$\sum_{k=0}^{t/\Delta-1} \mathbf{P}\big(T_n \in \Delta[k\Delta)\big)\mathbf{P}\big(\tau > t - (k+1)\Delta\big)$$

and

$$\sum_{k=0}^{t/\Delta-1} \mathbf{P}\big(T_n \in \Delta[k\Delta)\big)\mathbf{P}(\tau > t - k\Delta),$$

respectively. For small Δ both bounds will be close to the right-hand side of (10.5.13).

If τ has an arithmetic distribution then we have to replace integral (10.5.14) with the corresponding sum and, for integer u and t, make use of Theorem 8.7.3.

The theorem is proved. □

If examine the arguments in the proof concerning the behaviour of the correction term, then, in addition to (10.5.13), we can also obtain the representation

$$\mathbf{P}\big(\eta(t) = n\big) = \frac{a^{3/2}}{\sigma\sqrt{2\pi t}} e^{-v^2/2} + o\left(\frac{1}{\sqrt{t}}\right) \tag{10.5.15}$$

uniformly in v (or in n).

10.6 Generalised Renewal Processes

10.6.1 Definition and Some Properties

Let, instead of the sequence $\{\tau_j\}_{j=1}^{\infty}$, there be given a sequence of two-dimensional independent vectors (τ_j, ξ_j), $\tau_j \geq 0$, having the same distribution as (τ, ξ). Let, as before,

$$S_k = \sum_{j=1}^{k} \xi_j, \qquad T_k = \sum_{j=1}^{k} \tau_j, \qquad S_0 = T_0 = 0,$$
$$\eta(t) = \min\{k : T_k > t\}, \qquad \nu(t) = \max\{k : T_k \le t\} = \eta(t) - 1.$$

Definition 10.6.1 The process

$$S_{(\nu)}(t) = qt + S_{\nu(t)}$$

is called *a generalised renewal process with linear drift q*.

The process $S_{(\nu)}(t)$, as well as $\nu(t)$, is *right-continuous*. Clearly, $S_{(\nu)}(t) = qt$ for $t < \tau_1$. At time $t = \tau_1$ the first jump in the process $S_{(\nu)}(t)$ occurs, which is of size ξ_1:

$$S_{(\nu)}(\tau_1 - 0) = q\tau_1, \qquad S_{(\nu)}(\tau_1) = q\tau_1 + \xi_1.$$

After that, on the interval $[T_1, T_2)$ the value of $S_{(\nu)}(t)$ varies linearly with slope q. At the point T_2, the second jump occurs, which is of size ξ_2, and so on.

Generalised renewal processes are evidently a generalisation of random walks S_k (for $\tau_j \equiv 1$, $q = 0$) and renewal processes $\eta(t) = \nu(t) + 1$ (for $\xi_j \equiv 1$, $q = 0$). They are widespread in applications, as mathematical models of various physical systems.

Along with the process $S_{(\nu)}(t)$, we will consider generalised renewal processes of the form

$$S(t) = qt + S_{\eta(t)} = S_{(\nu)}(t) + \xi_{\eta(t)},$$

that are in a certain sense more convenient to analyse since $\eta(t)$ is a Markov time with respect to $\mathfrak{F}_n = \sigma(\tau_1, \ldots, \tau_n;\ \xi_1, \ldots, \xi_n)$ and has already been well studied.

The fact that the asymptotic properties of the processes $S(t)$ and $S_{(\nu)}(t)$, as $t \to \infty$, (the law of large numbers, the central limit theorem) are identical follows from the next assertion, which shows that the difference $S(t) - S_{(\nu)}(t)$ has a proper limiting distribution.

Lemma 10.6.1 *If* $\mathbf{E}\tau < \infty$, *then the following limiting distribution exists*

$$\lim_{t\to\infty} \mathbf{P}(\xi_{\eta(t)} < v) = \frac{\mathbf{E}(\tau;\ \xi < v)}{\mathbf{E}\tau}.$$

The lemma implies that $\xi_{\eta(t)}/b(t) \xrightarrow{p} 0$ for any function $b(t) \to \infty$ as $t \to \infty$.

Proof By virtue of the key renewal theorem,

$$\begin{aligned}\mathbf{P}(\xi_{\eta(t)} < v) &= \sum_{k=0}^{\infty} \int_0^t \mathbf{P}(T_k \in du)\mathbf{P}(\tau > t - u, \xi < v) \\ &= \int_0^t dH(t)\,\mathbf{P}(\tau > t - u, \xi < v) \to \frac{1}{\mathbf{E}\tau} \int_0^\infty \mathbf{P}(\tau > u, \xi < v)\,du \\ &= \frac{\mathbf{E}(\tau;\ \xi < v)}{\mathbf{E}\tau}.\end{aligned}$$

The lemma is proved. □

As was already noted, $\eta(t)$ is a stopping time with respect to

$$\mathfrak{F}_n = \sigma(\tau_1, \ldots, \tau_n;\ \xi_1, \ldots, \xi_n).$$

Therefore, if (τ_j, ξ_j) are identically distributed, then by the Wald identity (see Theorem 4.4.2 and Example 4.4.5)

$$\mathbf{E}S(t) = qt + a_\xi \mathbf{E}\eta(t) \sim qt + \frac{a_\xi t}{a} \tag{10.6.1}$$

as $t \to \infty$, where $a_\xi = \mathbf{E}\xi$ and $a = \mathbf{E}\tau$. The second moments of $S(t)$ will be found in Sect. 15.2. The laws of large numbers for $S(t)$ will be established in Sect. 11.5.

10.6.2 The Central Limit Theorem

In order to simplify the exposition, we first assume that the components τ_j and ξ_j of the vectors $(\tau_j, \xi_j) \stackrel{d}{=} (\tau, \xi)$ are independent. Moreover, without losing generality, we assume that $q = 0$.

Theorem 10.6.1 *Let there exist* $\sigma^2 = \operatorname{Var}\tau < \infty$, $\sigma_\xi^2 = \operatorname{Var}(\xi) < \infty$ *with* $\sigma + \sigma_\xi > 0$. *If the coordinates* τ *and* ξ *are independent then, as* $t \to \infty$,

$$\frac{S(t) - rt}{\sigma_S \sqrt{t}} \Subset\!\!\!\Rightarrow \mathbf{\Phi}_{0,1},$$

where $r = a_\xi / a$ *and* $\sigma_S^2 = a^{-1}(\sigma_\xi^2 + r^2\sigma^2) = a^{-1}\operatorname{Var}(\xi - r\tau)$. *The same assertion holds true for* $S_{(\nu)}(t)$ *as well.*

Proof If one of the values of σ and σ_ξ is zero, then the assertion of the theorem follows from Theorems 8.2.1 and 10.5.2. Therefore we can assume that $\sigma > 0$ and $\sigma_\xi > 0$. Denote by $\mathfrak{G} = \sigma(\tau_1, \tau_2, \ldots)$ the σ-algebra generated by the sequence $\{\tau_j\}$ and by $A_t \subset \mathfrak{G}$ the set

$$A_t = \bigl\{ \bigl|\eta(t) - t/a\bigr| < t^{1/2+\varepsilon} \bigr\}, \quad \varepsilon \in (0, 1/2).$$

Since by the central limit theorem $\mathbf{P}(A_t) \to 1$ as $t \to \infty$, for any trajectory $\eta(\cdot)$ in A_t we have $\eta(t) \to \infty$ as $t \to \infty$, and the random variables

$$Z(t) = \frac{S(t) - a_\xi \eta(t)}{\sigma_\xi \sqrt{\eta(t)}}$$

are asymptotically normal with parameters $(0, 1)$ by the independence of $\{\xi_j\}$ and $\{\tau_j\}$. In other words, on the sets A_t,

$$\mathbf{E}\bigl(e^{i\lambda Z(t)} \bigm| \mathfrak{G}\bigr) \to e^{-\lambda^2/2} \quad \text{as } t \to \infty.$$

Since

$$\eta(t) = \frac{t}{a} + \frac{\sigma\sqrt{t}}{a^{3/2}}\zeta_t, \quad \zeta_t \Subset\!\!\!\Rightarrow \mathbf{\Phi}_{0,1}, \quad \text{and} \quad \eta(t) \sim \frac{t}{a}$$

on the sets $A_t \in \mathfrak{G}$, we also have on the sets A_t the relation

$$\mathbf{E}\left(\exp\left\{\frac{i\lambda(S(t) - rt - \frac{a_\xi \sigma \sqrt{t}}{a^{3/2}}\zeta_t)}{\sigma_\xi \sqrt{t/a}}\right\}\Big|\mathfrak{G}\right) \to e^{-\lambda^2/2}.$$

Since the random variables ζ_t and $\eta(t)$ are measurable with respect to $\mathfrak{G}$, the corresponding factor can be taken outside of the conditional expectation, so that

$$\mathbf{E}\left(\exp\left\{\frac{i\lambda(S(t) - rt)}{\sigma_\xi \sqrt{t/a}}\right\}\Big|\mathfrak{G}\right) \sim \exp\left\{-\frac{\lambda^2}{2} + \frac{i\lambda r\sigma}{\sigma_\xi}\zeta_t\right\}.$$

Hence

$$\begin{aligned}\mathbf{E}\exp\left\{\frac{i\lambda(S(t) - rt)}{\sigma_\xi \sqrt{t/a}}\right\} &= o(1) + \mathbf{E}\left(\exp\left\{-\frac{\lambda^2}{2} + \frac{i\lambda\sigma}{\sigma_\xi}\zeta_t\right\};\ A_t\right) \\ &= o(1) + \exp\left\{-\frac{\lambda^2}{2}\left[1 + \left(\frac{r\sigma}{\sigma_\xi}\right)^2\right]\right\}.\end{aligned}$$

This means that

$$\frac{1}{\sqrt{t}}\left(S(t) - \frac{ta_\xi}{a}\right) \Leftrightarrow \mathbf{\Phi}_{0,\sigma_S^2},$$

where

$$\sigma_S^2 = \frac{\sigma_\xi^2}{a}\left[1 + \left(\frac{r\sigma}{\sigma_\xi}\right)^2\right] = a^{-1}[\sigma_\xi^2 + r^2\sigma^2].$$

The assertion corresponding to $S_{(\nu)}(t)$ follows from Lemma 10.6.1. The theorem is proved. □

Note that Theorems 8.2.1 and 10.5.2 are special cases of Theorem 10.6.1. If $a_\xi = 0$, then $S(t)$ is distributed identically to $S_{[t/a]}$ and is independent of σ.

Now consider the general case where τ and ξ are, generally speaking, dependent. Since $T_{\eta(t)} = t + \chi(t)$, we have the representation

$$S(t) - rt = Z_{\eta(t)} + r\chi(t), \tag{10.6.2}$$

where

$$Z_n = \sum_{j=1}^{n} \zeta_j, \quad \zeta_j = \xi_j - r\tau_j, \quad \mathbf{E}\zeta_j = 0, \quad \frac{\chi(t)}{\sqrt{t}} \xrightarrow{p} 0$$

as $t \to \infty$ ($\chi(t)$ has a proper limiting distribution as $t \to \infty$). Moreover, we will use yet another Wald identity

$$\mathbf{E}Z_{\eta(t)}^2 = d^2\mathbf{E}\eta(t), \quad d^2 = \mathbf{E}\zeta^2, \quad \zeta = \xi - r\tau, \tag{10.6.3}$$

that is derived below in Sect. 15.2.

Theorem 10.6.2 *Let* $(\tau_j, \xi_j) \stackrel{d}{=} (\tau, \xi)$ *be independent identically distributed and such that* $\sigma^2 = \mathrm{Var}(\tau) < \infty$ *and* $\sigma_\xi^2 = \mathrm{Var}(\xi) < \infty$ *exist. Then*

$$\frac{S(t) - rt}{\sigma_S\sqrt{t}} \Subset\Rightarrow \mathbf{\Phi}_{0,1},$$

where $r = a_\xi/a$ *and* $\sigma_S^2 = a^{-1}d^2$. *The random variables* $\frac{S_{(\nu)}(t)-rt}{\sigma_S\sqrt{t}}$ *and* $\frac{Z_{\eta(t)}}{\sigma_S\sqrt{t}}$ *have the same limiting distribution.*

Proof It is seen from (10.6.2) that it suffices to prove that

$$\frac{Z_{\eta(t)}}{\sigma_S\sqrt{t}} \Subset\Rightarrow \mathbf{\Phi}_{0,1}.$$

The main contribution to $Z_{\eta(t)}$ comes from Z_m with $m = [\frac{t}{a} - 2N\sqrt{t}]$, $N \to \infty$, $N = o(\sqrt{t}\,)$, where

$$\frac{\sqrt{a}\,Z_m}{d\sqrt{t}} = \frac{Z_m}{d\sqrt{m}}\sqrt{\frac{ma}{t}} \Subset\Rightarrow \mathbf{\Phi}_{0,1}.$$

The remainder $Z_{\eta(t)} - Z_m$, for each fixed

$$T_m \in I_N := [t - 3aN\sqrt{t},\ t - aN\sqrt{t}\,], \qquad \mathbf{P}(T_m \in I_N) \to 1,$$

has the same distribution as $Z_{\eta(t-T_m)}$, and its variance (see (10.6.3)) is equal to

$$d^2\mathbf{E}\eta(t - T_m) \sim d^2\frac{t - T_m}{a} < 3d^2N\sqrt{t} = o(t).$$

Since $\mathbf{E}Z_{\eta(t-T_m)} = 0$, we have

$$\frac{Z_{\eta(t-T_m)}}{\sqrt{t}} \xrightarrow{p} 0 \qquad (10.6.4)$$

as $t \to \infty$. The theorem is proved. □

Note that, for $N \to \infty$ slowly enough, relation (10.6.4) can be derived using not (10.6.3), but the law of large numbers for generalised renewal processes that was obtained in Sect. 11.5.

Theorem 10.6.1 could be proved in a somewhat different way—with the help of the local Theorem 10.5.3. We will illustrate this approach by the proof of the integro-local theorem for $S(t)$.

10.6.3 The Integro-Local Theorem

In this section we will obtain the integro-local theorem for $S(t)$ in the case of non-lattice ξ. In a quite similar way we can obtain local theorems for densities (if they exist) and for the probability $\mathbf{P}(S(t) = k)$ for $q = 0$ for arithmetic ξ_j.

Theorem 10.6.3 *Let the conditions of Theorem 10.6.1 hold and, moreover, ξ be non-lattice. Then, for any fixed $\Delta > 0$, as $t \to \infty$,*

$$\mathbf{P}\big(S(t) - rt \in \Delta[x)\big) = \frac{\Delta}{\sigma_S\sqrt{t}}\phi\left(\frac{x}{\sigma_S\sqrt{t}}\right) + o\left(\frac{1}{\sqrt{t}}\right), \qquad (10.6.5)$$

where the remainder term $o(1/\sqrt{t})$ is uniform in x.

Proof Since ξ is non-lattice, one has $\sigma_\xi > 0$. If $\sigma = 0$ then the assertion of the theorem follows from Theorem 8.7.1. Therefore we will assume that $\sigma > 0$. By the independence of $\{\xi_j\}$ and $\{\tau_j\}$,

$$\mathbf{P}\big(S(t) - rt \in \Delta[x)\big) = \sum_{n=1}^{\infty} \mathbf{P}\big(\eta(t) = n\big)\mathbf{P}\big(S_n - rt \in \Delta[x)\big) = \sum_{n\in M_t} + \sum_{n\notin M_t},$$

where $M_t = \{n : |n - t/a| < t^{1/2}N(t)\}$, $N(t) \to \infty$, $N(t) = o(\sqrt{t})$ as $t \to \infty$. We know the asymptotics of both factors of the terms in the sum from Theorems 8.7.1 and 10.5.8 (see also (10.5.15)). It remains to do the summation, which is unfortunately somewhat cumbersome. At the same time, it presents no substantial difficulties, so we will sketch this part of the proof. If we put $an - t =: u$,

$$P_1(t) := \frac{\Delta}{\sigma_\xi\sqrt{2\pi n}}\exp\left\{-\frac{(x - ru)^2}{2n\sigma_\xi^2}\right\}, \qquad P_2(t) := \frac{a^{3/2}}{\sigma\sqrt{2\pi t}}\exp\left\{-\frac{u^2}{2\sigma^2 n}\right\},$$

then

$$\mathbf{P}\big(S_n - rt \in \Delta[x)\big) = P_1(t) + o\left(\frac{1}{\sqrt{n}}\right).$$

Furthermore,

$$\mathbf{P}\big(\eta(t) = n\big) = P_2(t) + o\left(\frac{1}{\sqrt{t}}\right)$$

for $n \in M_t$ and $N(t) \to \infty$ slowly enough as $t \to \infty$. Clearly,

$$\sum_{n\notin M_t} = o\left(\frac{1}{\sqrt{t}}\right).$$

Since the sums of $P_1(t)$ and $P_2(t)$ are bounded in n by a constant, we have

$$\sum_{n\in M_t} = o\left(\frac{1}{\sqrt{t}}\right) + \sum_{n\in M_t} P_1(t)P_2(t).$$

The exponent in the product $P_1(t)P_2(t)$, taken with the negative sign, is equal to

$$\frac{1}{2n}\left[\frac{(x - ru)^2}{\sigma_\xi^2} + \frac{u^2}{\sigma^2}\right] \sim \frac{a}{2t}\left[\frac{(d^2u - rx\sigma^2)^2}{d^2\sigma^2\sigma_\xi^2} + \frac{x^2}{d^2}\right],$$

where $d^2 = r^2\sigma^2 + \sigma_\xi^2$. Since, for $x = o(\sqrt{t}\,N(t))$,

$$\sum_{n\in A_t} \frac{a^{3/2}d}{\sqrt{2\pi t}\,\sigma\sigma_\xi}\exp\left\{-\frac{a(d^2u - rx\sigma^2)^2}{2td^2\sigma^2\sigma_\xi^2}\right\} \to 1$$

as $t \to \infty$ and this sum does not exceed $1 + o(1)$ for all x (this is an integral sum that corresponds to the integral of the density of the normal law), it is easy to derive (10.6.5) from the foregoing. □

We will continue the study of generalised renewal processes in Sect. 11.5.