

Chapter 3

Conservation Equations and Characteristics

A conservation law for a physical system states that a certain quantity (e.g., mass, energy, or momentum) is independent of time. For continuous systems such as fluids or gases, these global quantities can be defined as integrals of density functions. The conservation law then translates into a local form, as a PDE for the density function.

In this section we will study some first-order PDE that arise from conservation laws. We introduce a classic technique, called the method of characteristics, for analyzing these equations.

3.1 Model Problem: Oxygen in the Bloodstream

To derive the conservation equation, we consider a simple model for the concentration of oxygen carried by the bloodstream. For this discussion we ignore any external effects that might break the conservation of mass, such as absorption of oxygen into the walls of a blood vessel. (Some examples of external effects will be considered in the exercises.)

Let us model an artery as a straight tube, as pictured in Fig. 3.1. We assume that the concentration is constant on cross-sections of the tube, so that the problem reduces to one spatial dimension. For the moment, suppose that the artery extends along the real line and is parametrized by $x \in \mathbb{R}$.

Let $u(t, x)$ denote the oxygen concentration, expressed in units of mass per unit length. Within a fixed interval $[a, b]$, as highlighted in Fig. 3.1, the total mass at time t is given by an integral,

$$m(t) := \int_a^b u(t, x) dx. \tag{3.1}$$

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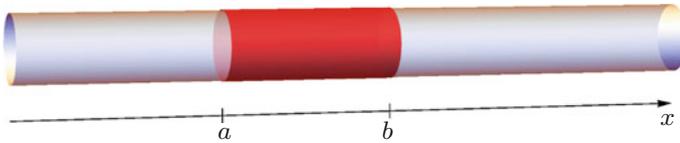


Fig. 3.1 One-dimensional model of an artery

The instantaneous flow rate at a given point x is called the *flux* $q(t, x)$, expressed as mass per unit time. The general relationship between flux and concentration is

$$\text{flux} = (\text{concentration}) \times (\text{velocity}).$$

For the bloodstream model we can reasonably assume that velocity is independent of the oxygen concentration (because oxygen accounts for a relatively small portion of the total density). This assumption implies that q has a linear dependence on u . In other models the velocity might depend on the concentration, making q a nonlinear function of u .

Conservation of mass implies that the total amount of oxygen within the segment changes only as oxygen flows across the boundary points at $x = a$ and $x = b$. Since the flow across these points is given by the flux, the corresponding equation is

$$\frac{dm}{dt}(t) = q(t, a) - q(t, b). \quad (3.2)$$

If q is continuously differentiable with respect to position, then the fundamental theorem of calculus allows us to write the right-hand side of (3.2) as an integral,

$$q(t, a) - q(t, b) = - \int_a^b \frac{\partial q}{\partial x} dx.$$

We can also differentiate the integral in (3.1) to obtain

$$\frac{dm}{dt} = \int_a^b \frac{\partial u}{\partial t} dx,$$

provided that $u(t, x)$ is continuously differentiable with respect to time. These calculations transform (3.2) into the integral equation

$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} \right) dx = 0. \quad (3.3)$$

Since the segment was arbitrary, (3.3) should hold for all values of a, b . This is only possible if the integrand is identically zero, which gives the local form of the law conservation of mass:

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0. \quad (3.4)$$

This relationship between concentration and flux is called the *continuity equation* (or *transport equation*). The continuity equation applies generally to the physical process of *advection*, which refers to the motion of particles in a bulk fluid flow.

To adapt (3.4) to a particular model, we need to specify the relationship between q and u . As we remarked above, for the bloodstream model it is reasonable to assume a linear relationship,

$$q = vu, \quad (3.5)$$

where the velocity $v(t, x)$ is part of the input data for the equation. Under this assumption (3.4) reduces to

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x} = 0, \quad (3.6)$$

which is called the *linear conservation equation*.

3.2 Lagrangian Derivative and Characteristics

In this section we will discuss the strategy for solving a first-order PDE such as (3.6). The basic idea is to adopt the perspective of an observer traveling with velocity v . This is like taking measurements in a river from a raft drawn by the current. Once we fix a starting point for the observer, the observed concentration depends only on the time variable, thus reducing the equation to an ODE.

This principle applies to any first-order PDE of the form

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + w = 0, \quad (3.7)$$

where $v = v(t, x)$ is independent of u . The zeroth-order term w could be a general function $w(t, x, u)$. A trajectory $t \mapsto x(t)$ is called a *characteristic* for the equation (3.7) if

$$\frac{dx}{dt}(t) = v(t, x(t)). \quad (3.8)$$

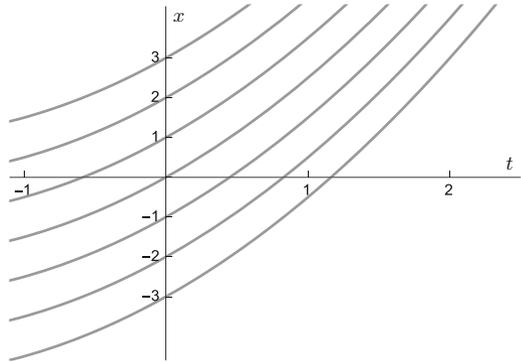
For v and $\partial v/\partial x$ continuous, Theorem 2.4 shows that a unique solution exists in the neighborhood of each starting point (t_0, x_0) .

Example 3.1 Suppose $v(t, x) = at + b$, with a and b constant. Integration over t gives

$$x(t) = \frac{a}{2}t^2 + bt + x_0.$$

The characteristics are a family of curves indexed by the parameter x_0 , as illustrated in Fig. 3.2. ◇

Fig. 3.2 Sample characteristics for the velocity $v(t, x) = 1 + 2t$



From the point of view of an observer carried by the flow, the measured concentration is $u(t, x(t))$. The observed rate of change is the derivative of this quantity,

$$\frac{Du}{Dt}(t) := \frac{d}{dt}u(t, x(t)), \quad (3.9)$$

called the *Lagrangian derivative* (or *material derivative*). This concept was developed by the 18th century mathematician and physicist Joseph-Louis Lagrange. Note that Du/Dt depends also on the initial value (t_0, x_0) that determines the characteristic. For convenience we suppress the initial point from the notation.

Theorem 3.2 *On each characteristic, (3.7) reduces to the ODE*

$$\frac{Du}{Dt} + \tilde{w} = 0, \quad (3.10)$$

where \tilde{w} is the restriction of w to the characteristic,

$$\tilde{w}(t) := w(t, x(t), u(t, x(t))).$$

In particular, if $w = 0$ then u is constant on each characteristic.

Proof Applying the chain rule in (3.9) gives

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt},$$

with the understanding that the partial derivatives on the right are evaluated at the point $(t, x(t))$. Because $x(t)$ solves (3.8), this reduces to

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x}, \quad (3.11)$$

If we restrict the variables in (3.7) to $(t, x(t))$, then the first two terms match the right-hand side of (3.11), reducing the equation to (3.10).

If $w = 0$, then (3.10) becomes

$$\frac{Du}{Dt} = 0.$$

This is equivalent to the statement that $u(t, x(t))$ is independent of t . □

With Theorem 3.2 we can effectively reduce the PDE (3.7) to a pair of ODE, namely the characteristic equation (3.8) and the Lagrangian derivative equation (3.10). In many cases, solving these ODE will lead to an explicit formula for $u(t, x)$. This approach is referred to as the *method of characteristics*.

Example 3.3 For constants $a, b \in \mathbb{R}$, assume that $u(t, x)$ satisfies

$$\frac{\partial u}{\partial t} + (at + b) \frac{\partial u}{\partial x} = 0,$$

with the initial condition

$$u(0, x) = g(x),$$

for some function $g \in C^1(\mathbb{R})$. The characteristics for this velocity, $v(t, x) = at + b$, were computed in Example 3.1.

According to Theorem 3.2, u is constant along characteristics, implying that

$$u\left(t, \frac{a}{2}t^2 + bt + x_0\right) = u(0, x_0) = g(x_0), \quad (3.12)$$

for all $t \in \mathbb{R}$. This is not yet a formula for $u(t, x)$, but we can derive the solution formula by identifying

$$x = \frac{a}{2}t^2 + bt + x_0.$$

Solving for x_0 in terms of x and substituting this into (3.12) gives

$$u(t, x) = g\left(x - \frac{a}{2}t^2 - bt\right). \quad \diamond$$

Example 3.4 For steady flow through a pipe of changing diameter, the velocity would vary with position rather than time. Let $v(t, x) = a + bx$ for $x \geq 0$, with $a, b > 0$. The resulting characteristic equation (3.8) is

$$\frac{dx}{dt} = a + bx.$$

This can be solved by the standard ODE technique of separating the t and x variables to different sides of the equation:

$$\frac{dx}{a + bx} = dt.$$

Integration of both sides gives the general solution

$$\frac{1}{b} \ln(a + bx) = t + C,$$

with C a constant of integration. (Note that $a + bx > 0$ by our assumptions.) Solving for x gives

$$x(t) = \frac{1}{b} [e^{b(t+C)} - a].$$

Given the assumption $x \geq 0$, it is natural to index the characteristics by the start time t_0 such that $x(t_0) = 0$. With this convention, the family of solutions is

$$x(t) = \frac{a}{b} [e^{b(t-t_0)} - 1]. \quad (3.13)$$

These characteristic curves are illustrated in Fig. 3.3.

With $v = a + bx$ the linear conservation equation (3.6) becomes

$$\frac{\partial u}{\partial t} + (a + bx) \frac{\partial u}{\partial x} + bu = 0.$$

Let us find the solution under the boundary condition

$$u(t, 0) = f(t). \quad (3.14)$$

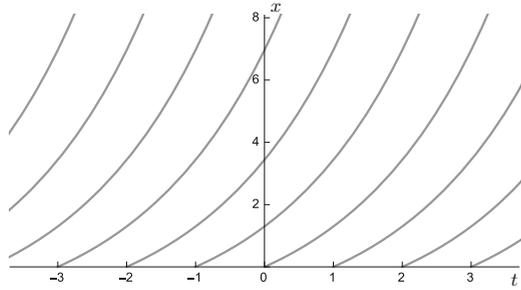
Since $\partial v / \partial x = b$, (3.10) gives

$$\frac{Du}{Dt} + bu = 0.$$

This is a decay equation, with the family of exponential solutions

$$u(t, x(t)) = Ae^{-bt}.$$

Fig. 3.3 Characteristic lines for the position-dependent velocity function of Example 3.4



To fix A , we substitute the starting point $(t_0, 0)$ into the equation and obtain

$$u(t, x(t)) = f(t_0)e^{-b(t-t_0)}. \tag{3.15}$$

Putting together (3.13) and (3.15) and applying the boundary condition (3.14) gives

$$u\left(t, \frac{a}{b} [e^{b(t-t_0)} - 1]\right) = f(t_0)e^{-b(t-t_0)}. \tag{3.16}$$

To express this as a function of (t, x) , we set

$$x = \frac{a}{b} [e^{b(t-t_0)} - 1],$$

and solve for t_0 to obtain

$$t_0 = t + \frac{1}{b} \ln\left(\frac{a}{a + bx}\right).$$

Substituting this expression into (3.16) gives the final form of the solution:

$$u(t, x) = \left(\frac{a}{a + bx}\right) f\left(t + \frac{1}{b} \ln\left(\frac{a}{a + bx}\right)\right).$$

A sample solution is illustrated in Fig. 3.4 for $a = 1, b = \frac{1}{2}$. For this example the boundary condition $f(t)$ was taken to have support between $t = -1$ and $t = 1$, with a maximum at $t = 0$. The plots of $u(t, x)$ on the right show concentrations at a succession of times. Mass conservation is reflected in the fact that the total area under each of these curves is independent of t . \diamond

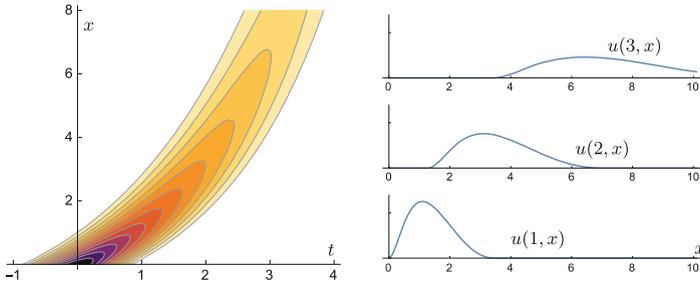


Fig. 3.4 Behavior of solutions for Example 3.4. In the contour plot on the *left*, darker regions correspond to higher concentration. The change in colors corresponds to exponential decay along the characteristics illustrated in Fig. 3.3

3.3 Higher-Dimensional Equations

For flow problems in more than one spatial dimension, we can develop a continuity equation analogous to (3.4) by the same reasoning as in Sect. 3.1. Suppose $u(t, \mathbf{x})$ represents a concentration defined for $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. Let $\mathcal{R} \subset \mathbb{R}^n$ be a bounded region with C^1 boundary. The total mass within this region is given by the volume integral

$$m(t) := \int_{\mathcal{R}} u(t, \mathbf{x}) d^n \mathbf{x}.$$

The flow of u is represented by a vector-valued flux density $\mathbf{q}(t, \mathbf{x})$. The interpretation of the flux density is that the rate at which mass passes through an $(n - 1)$ -dimensional surface is given by the surface integral of \mathbf{q} over this surface. In particular, the rate at which mass exits \mathcal{R} through the boundary is the quantity

$$\int_{\partial \mathcal{R}} \mathbf{v} \cdot \mathbf{q} dS,$$

where \mathbf{v} is the outward unit normal vector defined on $\partial \mathcal{R}$.

Conservation of mass dictates that the mass within \mathcal{R} can change only as mass enters or leaves through the boundary. In other words,

$$\frac{dm}{dt} = - \int_{\partial \mathcal{R}} \mathbf{v} \cdot \mathbf{q} dS. \tag{3.17}$$

Assuming that \mathbf{q} is C^1 with respect to \mathbf{x} , the Divergence Theorem (Theorem 2.6) allows us to rewrite the flux integral as

$$\int_{\partial \mathcal{R}} \mathbf{q} \cdot \mathbf{v} dS = \int_{\mathcal{R}} \nabla \cdot \mathbf{q} d^n \mathbf{x}. \tag{3.18}$$

Note that since \mathbf{q} depends on both t and \mathbf{x} , the notation $\nabla \cdot \mathbf{q}$ is slightly ambiguous. We follow the standard convention that vector calculus operators such as ∇ and Δ act only on spatial variables.

If u is C^1 with respect to t , then we can also differentiate the integral for m to obtain

$$\frac{dm}{dt} = \int_{\mathcal{R}} \frac{\partial u}{\partial t} d^n \mathbf{x}.$$

Combining this with (3.17) and (3.18) gives

$$\int_{\mathcal{R}} \left(\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} \right) d^n \mathbf{x} = 0. \quad (3.19)$$

As in the one-dimensional case, we now observe that since (3.19) holds for an arbitrary region \mathcal{R} , the integrand must vanish. This is the higher-dimensional continuity equation:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0. \quad (3.20)$$

Suppose we make the linear assumption that $\mathbf{q} = \mathbf{v}u$ for a velocity field \mathbf{v} which is independent of u . The product rule for the divergence of a vector field is

$$\nabla \cdot (\mathbf{v}u) = (\nabla \cdot \mathbf{v})u + \mathbf{v} \cdot \nabla u.$$

Substituting this into (3.20) gives the higher-dimensional form of the linear conservation equation

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u + (\nabla \cdot \mathbf{v})u = 0. \quad (3.21)$$

In the special case where $\nabla \cdot \mathbf{v} = 0$ the velocity field is called *solenoidal* (or *divergence-free*). This situation arises frequently in applications, because incompressible fluids like blood or water have solenoidal velocity fields.

The method of characteristics from Sect. 3.2 can be adapted directly to (3.21). Consider a somewhat more general first-order PDE in the form

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u + w = 0, \quad (3.22)$$

with $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ and $w = w(t, \mathbf{x}, u)$. The characteristics associated to this equation are by definition the solutions of

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{v}(t, \mathbf{x}(t)). \quad (3.23)$$

Theorem 2.4 guarantees that characteristics exist in the neighborhood of each starting point (t_0, \mathbf{x}_0) provided $\mathbf{v}(t, \mathbf{x})$ and its partial derivatives with respect to \mathbf{x} are continuous.

The Lagrangian derivative of u along $\mathbf{x}(t)$ is defined as before by

$$\frac{Du}{Dt}(t) := \frac{d}{dt}u(t, \mathbf{x}(t)).$$

The higher-dimensional version of Theorem 3.2 is the following:

Theorem 3.5 *On each characteristic curve, the PDE (3.22) reduces to the ODE*

$$\frac{Du}{Dt} + \tilde{w} = 0, \tag{3.24}$$

where \tilde{w} denotes the restriction of w to the characteristic. In particular, if $w = 0$ then u is constant on each characteristic.

Proof By the chain rule,

$$\frac{Du}{Dt}(t) = \frac{\partial u}{\partial t}(t, \mathbf{x}(t)) + \nabla u(t, \mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt}(t).$$

Since $\mathbf{x}(t)$ satisfies (3.23), this gives

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u.$$

Substituting this into (3.22) reduces the equation to (3.24).

If $w = 0$ the equation becomes

$$\frac{Du}{Dt} = 0,$$

which means precisely that u is constant along the characteristic curves. □

Example 3.6 Consider a two-dimensional channel modeled as $\Omega = \mathbb{R} \times [-1, 1]$ with coordinates $\mathbf{x} = (x_1, x_2)$. The velocity field

$$\mathbf{v}(t, \mathbf{x}) := (1 - x_2^2, 0). \tag{3.25}$$

is solenoidal and vanishes on the boundary $\{x_2 = \pm 1\}$. The characteristic line originating from $(a, b) \in \Omega$ at $t = 0$ is

$$\mathbf{x}(t) = (a + (1 - b^2)t, b).$$

Let us consider the conservation equation (3.21) for $(t, \mathbf{x}) \in \mathbb{R} \times \Omega$, with \mathbf{v} given by (3.25), subject to the initial condition

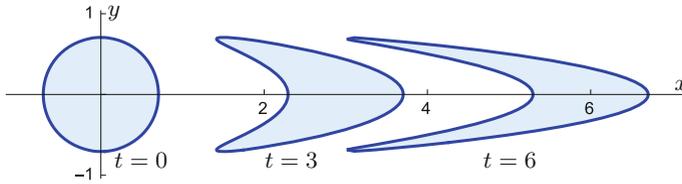


Fig. 3.5 Evolution of a circle according to the two-dimensional flow in Example 3.6

$$u(0, \mathbf{x}) = g(\mathbf{x}),$$

for $g \in C^1(\Omega)$. Since \mathbf{v} is solenoidal, Theorem 3.5 implies that u is constant on characteristics. This gives the relation

$$u(t, a + (1 - b^2)t, b) := g(a, b).$$

Rewriting this as a function of (t, x, y) gives

$$u(t, x, y) = g(x - (1 - b^2)t, y).$$

Figure 3.5 illustrates the evolution of a circular “ink spot” distribution under this flow. Conservation of mass is reflected in the fact that the area of the spot is independent of t . ◇

For applications of Theorem 3.5 on a bounded domain $\Omega \in \mathbb{R}^n$, the specification of boundary conditions can be quite a complicated problem, especially if the velocity is time-dependent. (We avoided this problem in Example 3.6 by taking \mathbf{v} tangent to $\partial\Omega$.) We will illustrate this issue in the exercises.

3.4 Quasilinear Equations

The method of characteristics remains an important tool for analysis of first-order PDE even in the nonlinear case. In this section we will illustrate the application of this method to the continuity equation (3.20) in the case of a flux term \mathbf{q} that depends on the concentration u .

To simplify the analysis, we assume that $\mathbf{q} = \mathbf{q}(u)$, with no explicit dependence on t and x . By the chain rule, (3.20) then reduces to the form

$$\frac{\partial u}{\partial t} + \mathbf{a}(u) \cdot \nabla u = 0, \tag{3.26}$$

where $\mathbf{a}(u) := d\mathbf{q}/du$. This type of PDE is called *quasilinear*, which means that the equation is linear in the highest-order derivatives (which are merely first order in this case).

A comparison of (3.26) to the linear conservation equation (3.21) shows that $a(u)$ is now playing the role of velocity. This suggests a definition for the characteristics, but we must keep in mind that $\mathbf{a}(u)$ depends on t and x implicitly through u .

Theorem 3.7 *Suppose that $u \in C^1([0, T] \times \Omega)$ is a solution of (3.26) for some region $\Omega \subset \mathbb{R}^n$, with $\mathbf{a} \in C^1(\mathbb{R}; \mathbb{R}^n)$. Then for each $\mathbf{x}_0 \in \Omega$, u is constant along the characteristic line defined by*

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{a}(u(0, \mathbf{x}_0))t.$$

Proof Suppose that a solution u exists. Let $\mathbf{x}(t)$ be the solution to the ODE

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{a}(u(t, \mathbf{x}(t))), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

for $t \in [0, T]$. Existence of such a characteristic is guaranteed by Theorem 2.4, at least for t near 0, because the composition $\mathbf{a} \circ u$ is C^1 as a function of (t, x) by the assumptions on \mathbf{a} and u .

To establish the claim that $u(t, \mathbf{x}(t))$ is independent of t , we use the chain rule to differentiate

$$\begin{aligned} \frac{d}{dt}u(t, \mathbf{x}(t)) &= \frac{\partial u}{\partial t}(t, \mathbf{x}(t)) + \nabla u(t, \mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt}(t) \\ &= \frac{\partial u}{\partial t}(t, \mathbf{x}(t)) + \mathbf{a}(u(t, \mathbf{x}(t))) \cdot \nabla u(t, \mathbf{x}(t)). \end{aligned}$$

The right-hand side vanishes by (3.26), so that

$$\frac{d}{dt}u(t, \mathbf{x}(t)) = 0.$$

This implies that

$$u(t, \mathbf{x}(t)) = u(0, \mathbf{x}_0),$$

which means that $\mathbf{a}(u(t, \mathbf{x}(t)))$ is also constant. The characteristic equation reduces to

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{a}(u(0, \mathbf{x}_0)),$$

and we can integrate over t to compute $\mathbf{x}(t)$. □

In contrast to the characteristic equation (3.8) in the linear case, the equation for $\mathbf{x}(t)$ here depends on the initial condition $u(0, \mathbf{x}_0)$. Furthermore, it is important to keep in mind that Theorem 3.7 does not imply that a solution to (3.26) exists; this

is assumed as a hypothesis. As we will see below, it is possible that the conclusion of the theorem will lead to a contradiction, in the form of multiple values for the solution at the same point. The implication in such a case is that a classical solution does not exist.

To illustrate the application of Theorem 3.7, let us consider a simple model for traffic on a single-lane road of infinite length, parametrized by $x \in \mathbb{R}$. Let $u(t, x)$ denote the linear density of cars at a given point and time. Cars are discrete objects, of course, but for modeling purposes we can assume that u is a C^1 function that describes the density in an aggregate sense.

In traffic flow, the density of cars affects the flow velocity, with traffic slowing down and possibly stopping as the density increases. A standard way to model this effect is to set a maximum value for the velocity v_m (presumably the speed limit). The velocity is assumed to take its maximum value at $u = 0$ and decrease linearly as u increases, up to some maximum value u_m for which $v = 0$. In other words, for this model $u \in [0, u_m]$ and

$$v(u) := v_m \left(1 - \frac{u}{u_m}\right).$$

Since $v \geq 0$, the model always assumes that traffic moves to the right.

To eliminate the constants and focus on the equation itself, let us set $v_m = 1$ and $u_m = 1$, reducing the velocity equation to

$$v(u) = 1 - u$$

for $u \in [0, 1]$. The corresponding flux is

$$q(u) = u - u^2.$$

Substituting these assumptions into (3.26), we obtain a quasilinear equation called the *traffic equation*:

$$\frac{\partial u}{\partial t} + (1 - 2u) \frac{\partial u}{\partial x} = 0. \quad (3.27)$$

Suppose we impose a general initial condition of the form

$$u(0, x) = h(x),$$

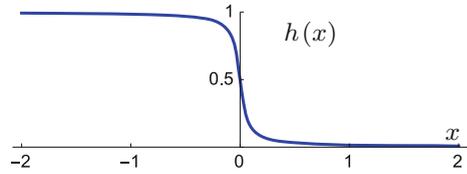
for some $h : \mathbb{R} \rightarrow [0, 1]$. Assuming a solution exists, Theorem 3.7 gives the family of characteristics

$$x(t) = x_0 + (1 - 2h(x_0))t. \quad (3.28)$$

Therefore, the solution u must satisfy

$$u(t, x_0 + (1 - 2h(x_0))t) = h(x_0). \quad (3.29)$$

Fig. 3.6 Initial traffic density modeling a line of cars stopped at a traffic light



As we will demonstrate in the examples below, (3.29) leads to a solution formula for some choices of h , while for others it leads to a contradiction.

Example 3.8 Figure 3.6 shows a plot of the initial condition

$$h(x) = \frac{1}{2} - \frac{1}{\pi} \arctan(20x),$$

which could represent a line of cars stopped at a traffic light at the point $x = 0$. The corresponding characteristic lines as given by (3.28) are plotted in Fig. 3.7.

To derive a formula for $u(t, x)$ from (3.29), we need to invert the equation

$$x = x_0 + (1 - 2h(x_0))t,$$

to express x_0 as a function of t and x . For the function h given above it is not possible to do this explicitly. However, there is a unique solution for each (t, x) , which can easily be calculated numerically. The resulting solutions are shown in Fig. 3.8. \square

Example 3.9 In order to solve the traffic equation explicitly, let us simplify the initial condition to the piecewise linear function

$$h(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

This is not C^1 , but the resulting solution could be interpreted as a weak solution in the sense described in Sect. 1.2. We will discuss the precise definition in Chap. 10.

By the formula from Theorem 3.7, the characteristic lines are

$$x(t) = \begin{cases} x_0 - t, & x_0 \leq 0, \\ x_0 + (2x_0 - 1)t, & 0 < x_0 < 1, \\ x_0 + t, & x_0 \geq 1. \end{cases} \quad (3.30)$$

Solving these equations for x_0 gives

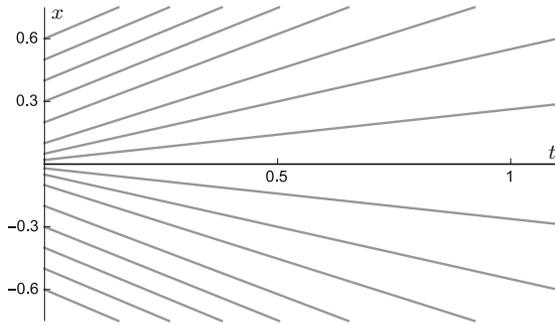


Fig. 3.7 Characteristic lines for the initial density shown in Fig. 3.6

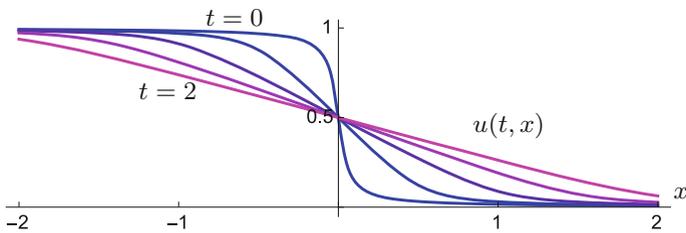


Fig. 3.8 Solutions for the traffic light problem

$$x_0 = \begin{cases} x + t, & x \leq -t, \\ \frac{x+t}{1+2t}, & -t < x < 1 + t, \\ x - t. & x \geq 1 + t. \end{cases}$$

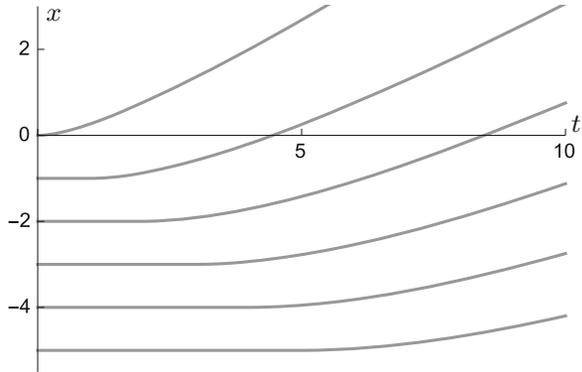
Therefore, by the solution formula (3.29), the solution is

$$u(t, x) = \begin{cases} 1, & x \leq -t, \\ 1 - \frac{x+t}{1+2t}, & -t < x < 1 + t, \\ 0. & x \geq 1 + t. \end{cases} \tag{3.31}$$

This is a continuous function, but differentiability fails on the lines $x = -t$ and $x = 1 + t$. Away from these lines it is easy to check that u solves (3.27).

Despite the lack of smoothness, this solution is quite reasonable. To illustrate this, let us trace the motion of a particular car starting from the position $x_0 \leq 0$. The velocity of the car is given by the flow rate $v(u) = 1 - u$. The initial density at x_0 is $u = 1$, so the car is stationary for a time. According to (3.31), at $t = -x_0$ the value of (t, x) enters the region where $-t < x < 1 + t$ and so at this time the density starts to decrease and the car starts to move. For (t, x) in this range, (3.31) gives

Fig. 3.9 Trajectories of individual cars according to the model of Example 3.9



$$v(t, x) = 1 - u(t, x) = \frac{x + t}{1 + 2t}. \tag{3.32}$$

Let $s(t)$ denote the position of the car at time t . For $t \geq -x_0$ the velocity formula (3.32) gives the equation

$$\frac{ds}{dt} = \frac{s + t}{1 + 2t}, \tag{3.33}$$

The initial condition at $t = -x_0$ is the original starting point $s(-x_0) = x_0$. The standard ODE method of integrating factors can be used to solve (3.33), yielding

$$s(t) = \begin{cases} x_0, & 0 \leq t \leq -x_0, \\ 1 + t - \sqrt{(1 - 2x_0)(1 + 2t)}, & t \geq -x_0. \end{cases}$$

These trajectories are illustrated in Fig. 3.9. As we might expect, the cars further back in the line wait longer before moving, but each car eventually moves forward and gradually accelerates. ◇

Example 3.10 Consider the initial condition

$$h(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(20x),$$

as shown in Fig. 3.10. This is the reverse of the initial condition of Example 3.8. The characteristics specified in Theorem 3.7 now cross each other, as illustrated in Fig. 3.11. The existence of crossings implies that a classical solution with this initial condition cannot exist beyond the time of the first crossing.

If we were to trace the trajectories of individual cars, as we did in Example 3.9, we would see that these also intersect each other at the points where characteristics cross. In effect, the model predicts the formation of a traffic jam. ◇

Fig. 3.10 Initial traffic density with a near-maximum density of cars to the right

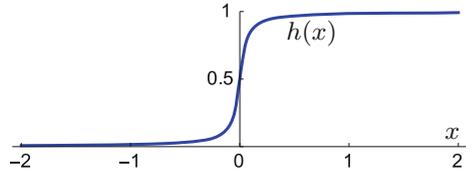
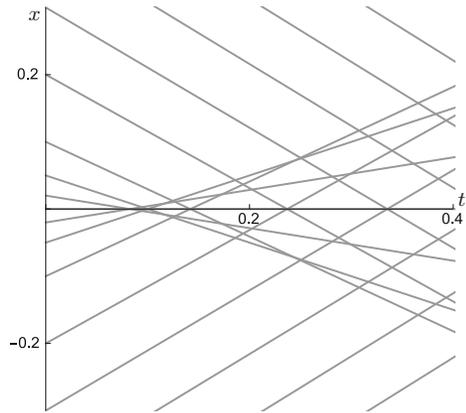


Fig. 3.11 Conflicting characteristic lines for the initial density shown in Fig. 3.10



A crossing of characteristics as observed in Example 3.10 is called a *shock*. After the shock, the solution is forced to have discontinuities. The proper interpretation of this situation requires weak solutions, for which discontinuities are allowed. We will return to this issue in Chap. 10.

3.5 Exercises

3.1 Consider the conservation equation with a constant velocity $c > 0$,

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0,$$

on the quadrant $t \geq 0, x \geq 0$. Suppose the boundary and initial conditions are

$$\begin{cases} u(0, x) = g(x), & x \geq 0, \\ u(t, 0) = h(t), & t \geq 0, \end{cases}$$

for $g, h \in C^1[0, \infty)$.

- (a) Find a formula for the solution $u(t, x)$ in terms of g and h .
- (b) Find a matching condition for g and h that will ensure that $u(t, x)$ is a C^1 function.

3.2 In the continuity equation (3.4), external factors that break the conservation of mass are accounted for by adding terms to the right-hand side.

- (a) A *forcing term* $f(t, x)$ is independent of the existing concentration. (In the bloodstream model of Sect. 3.1, this could represent intravenous injection, for example.) Assume that c is constant, $f \in C^1(\mathbb{R}^2)$, and $g \in C^1(\mathbb{R})$. Solve the equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = f, \quad u(0, x) = g(x),$$

to find an explicit formula for $u(t, x)$ in terms of f and g .

- (b) A *reaction term* depends on the concentration u . The simplest case is a linear term γu where the coefficient is some function $\gamma(t, x)$. (This could represent absorption of oxygen into the walls of the artery, for example.) Assume that c is constant, $\gamma \in C^1(\mathbb{R}^2)$, and $g \in C^1(\mathbb{R})$. Solve the equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \gamma u, \quad u(0, x) = g(x),$$

to find an explicit formula for $u(t, x)$ in terms of γ and g .

3.3 Assume that u satisfies the linear conservation equation

$$\frac{\partial u}{\partial t} + 2t \frac{\partial u}{\partial x} = 0,$$

for $t \in \mathbb{R}$ and $x \in [0, 1]$. Suppose the boundary conditions are given by

$$u(t, 0) = h_0(t), \quad u(t, 1) = h_1(t).$$

Find a relation between h_0 and h_1 . (This shows that we can only impose a boundary condition at one side of the interval $[0, 1]$.)

3.4 If the spatial domain in the linear conservation equation (3.21) is a bounded region $\Omega \subset \mathbb{R}^n$, then for a given velocity field \mathbf{v} , the *inflow boundary* $\partial\Omega_{\text{in}} \in \partial\Omega$ is defined as the set of boundary points where \mathbf{v} points into Ω . Fixing boundary conditions on the inflow boundary will generally determine the solution in the interior. Suppose $\Omega = (-1, 1) \times (-1, 1) \in \mathbb{R}^2$ with coordinates (x_1, x_2) . For the velocity fields below, determine the characteristics and specify the inflow boundary. Draw a sketch of Ω for each case, indicating these features.

(a) $\mathbf{v}(x_1, x_2) = (x_2, 1)$.

(b) $\mathbf{v}(x_1, x_2) = (1, -x_2)$.

3.5 Suppose that a section of of a river is modeled as a rectangle $\Omega = (0, \ell) \times (0, 1) \subset \mathbb{R}^2$, parametrized by (x_1, x_2) . Assume the flow is parallel to the x_1 -axis, with velocity

$$\mathbf{v}(x_1, x_2) = (f(x_2), 0),$$

for some positive function f on $(0, 1)$. Assume also that the concentration on the left boundary $\{x_1 = 0\}$ is given by

$$u(t, 0, x_2) = h(t, x_2).$$

Find a formula for $u(t, x_1, x_2)$ in terms of the functions h and f .

3.6 *Burgers' equation* is a simple quasilinear equation that appears in models of gas dynamics,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

(a) Use the method of characteristics as described in Sect. 3.4 to find a formula for the solution $u(t, x)$ given the initial condition

$$u(0, x) = \begin{cases} 0, & x \leq 0, \\ \frac{x}{a}, & 0 < x < a, \\ 1, & x \geq a. \end{cases}$$

(b) Suppose $a > b$ and

$$u(0, x) = \begin{cases} a, & x \leq 0, \\ a(1-x) + bx, & 0 < x < 1, \\ b, & x \geq 1. \end{cases}$$

Show that all of the characteristics originating from $x_0 \in [0, 1]$ meet at the same point (thus creating a shock).

3.7 In the mid-19th century, William Hamilton and Carl Jacobi developed a formulation of classical mechanics based on ideas from geometric optics. In this approach the dynamics of a free particle in \mathbb{R} are described by a generating function $u(t, x)$ satisfying the *Hamilton-Jacobi equation*:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 = 0. \quad (3.34)$$

Assume that $u \in C^1([0, \infty) \times \mathbb{R}^n)$ is a solution of (3.34). By analogy with Theorem 3.7, a characteristic of (3.34) is defined as a solution of

$$\frac{dx}{dt}(t) = \frac{\partial u}{\partial x}(t, x(t)), \quad x(0) = x_0. \quad (3.35)$$

- (a) Assuming that $x(t)$ solves (3.35), use the chain rule to compute d^2x/dt^2 .
 (b) Differentiate (3.34) with respect to x and then restrict the result to $(t, x(t))$, where $x(t)$ solves (3.35). Conclude from (a) that to

$$\frac{d^2x}{dt^2} = 0.$$

Hence, for some constant v_0 (which depends on the characteristic),

$$x(t) = x_0 + v_0 t.$$

- (c) Show that the Lagrangian derivative of u along $x(t)$ satisfies

$$\frac{Du}{Dt} = \frac{1}{2}v_0^2,$$

implying that

$$u(t, x_0 + v_0 t) = u(0, x_0) + \frac{1}{2}v_0^2 t.$$

- (d) Use this approach to find the solution $u(t, x)$ under the initial condition

$$u(0, x) = x^2.$$

(For the characteristic starting at $(0, x_0)$, note that you can compute v_0 by evaluating (3.35) at $t = 0$.)