Chapter 12
Distributions

To define weak derivatives in Chap. 10, we measured the values of a function $f \in L^1_{\mathrm{loc}}(\Omega)$ by integrating against test functions. One way to interpret this process is that f defines a functional $C^\infty_{\mathrm{cpt}}(\Omega) \to \mathbb{C}$ given by

$$\psi \mapsto \int_\Omega f\psi \, d^n\boldsymbol{x}.$$

A *distribution* on $\Omega \subset \mathbb{R}^n$ is a more general functional $C^\infty_{\mathrm{cpt}}(\Omega) \to \mathbb{C}$, not necessarily expressible as an integral. To qualify as a distribution, a functional is required to satisfy conditions that insure that weak derivatives and other basic operations are well defined.

As with weak derivatives, the concept of a distribution was inspired by idealized situations in physics. Indeed, the term "distribution" was inspired by charge distributions in electrostatics, an example that we will discuss in Sect. 12.1. Distributions generalize the notion of weak solutions, in the sense that every function in $L^1_{\mathrm{loc}}(\Omega)$ also defines a distribution. The trade-off for the increased generality is that some basic operations for functions cannot be applied to distributions. The product of two distributions is not generally well defined, for example.

There are some technicalities in the mathematical theory of distributions that require more background on the topology of function spaces than we assume for this text. We will treat these technicalities rather lightly; our focus will be on exploring the PDE applications.

12.1 Model Problem: Coulomb's Law

Coulomb's law of electrostatics is an empirical observation developed by 18th century physicist Charles-Augustin de Coulomb. It says that a particle with electric charge q_0, located at the origin, generates an electric field given by

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$$\boldsymbol{E}(\boldsymbol{x}) = \frac{kq_0\boldsymbol{x}}{|\boldsymbol{x}|^3}, \tag{12.1}$$

where k (Coulomb's constant) depends on the properties of the medium surrounding the charges.

In Sect. 11.1 we discussed another important empirical law of electrostatics, Gauss's law. With the same convention for physical constants as in (12.1), the differential form of the law says that

$$\nabla \cdot \boldsymbol{E} = 4\pi k\rho, \tag{12.2}$$

where ρ is the charge per unit volume as a function of position.

These two empirical laws present something of a mathematical conundrum, in that the field specified by Coulomb is not differentiable at $\boldsymbol{x} = 0$, not even weakly. On the other hand, for $\boldsymbol{x} \neq 0$,

$$\begin{aligned} \nabla \cdot \frac{\boldsymbol{x}}{r^3} &= \frac{\nabla \cdot \boldsymbol{x}}{r^3} - \frac{3\boldsymbol{x}}{r^4} \cdot \nabla r \\ &= \frac{3}{r^3} - \frac{3\boldsymbol{x}}{r^4} \cdot \frac{\boldsymbol{x}}{r} \\ &= 0 \end{aligned} \tag{12.3}$$

This is consistent with (12.2), in that Coulomb assumes the charge density is zero for $\boldsymbol{x} \neq 0$. However, if a function in L^1_{loc} vanishes except at a single point, then that function is zero by the equivalence (7.6). Thus a point charge density has no meaningful interpretation as a locally integrable function.

To reconcile (12.1) with Gauss's law, let us consider the weak form of (12.2),

$$\int_{\mathbb{R}^3} \boldsymbol{E} \cdot \nabla\psi \, d^3\boldsymbol{x} = -4\pi k \int_{\mathbb{R}^3} \rho\psi \, d^3\boldsymbol{x} \tag{12.4}$$

for all $\psi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^3)$. The left side of (12.4) is well defined because the components of $\boldsymbol{E}$ are locally integrable.

Since the Coulomb field is smooth away from the origin, we can integrate by parts as long as we exclude the origin from the region of integration by writing the integral as a limit,

$$\int_{\mathbb{R}^3} \boldsymbol{E} \cdot \nabla\psi \, d^3\boldsymbol{x} = \lim_{\varepsilon \to 0} \int_{\{r \geq \varepsilon\}} \boldsymbol{E} \cdot \nabla\psi \, d^3\boldsymbol{x}. \tag{12.5}$$

The region $\{r \geq \varepsilon\}$ has boundary given by the sphere $\{r = \varepsilon\}$. In this case the "outward" unit normal is a radial unit vector pointing towards the origin,

$$\boldsymbol{\nu} = -\frac{\boldsymbol{x}}{r}\bigg|_{r=\varepsilon}. \tag{12.6}$$

By the divergence theorem (Theorem 2.6), and the fact that $\nabla \cdot \boldsymbol{E} = 0$ for $r > 0$ by (12.3),

$$\int_{\{r\geq\varepsilon\}} \boldsymbol{E}\cdot\nabla\psi\, d^3\boldsymbol{x} = \int_{\{r=\varepsilon\}} \boldsymbol{\nu}\cdot\boldsymbol{E}\,\psi\, dS.$$

Hence by (12.1) and (12.6),

$$\int_{\{r\geq\varepsilon\}} \boldsymbol{E}\cdot\nabla\psi\, d^3\boldsymbol{x} = -\int_{\{r=\varepsilon\}} \frac{kq_0}{\varepsilon^2}\psi\, dS$$

Taking $\varepsilon \to 0$ now gives

$$\int_{\mathbb{R}^3} \boldsymbol{E}\cdot\nabla\psi\, d^3\boldsymbol{x} = -\lim_{\varepsilon\to 0}\int_{\{r=\varepsilon\}} \frac{kq_0}{\varepsilon^2}\psi\, dS. \tag{12.7}$$

Because ψ is continuous, the average of ψ over the sphere $\{r=\varepsilon\}$ approaches $\psi(0)$ as $\varepsilon \to 0$, i.e.,

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon^2}\int_{\{r=\varepsilon\}} \psi\, dS = \psi(0).$$

Applying this to (12.7) gives

$$\int_{\mathbb{R}^3} \frac{kq\boldsymbol{x}}{r^3}\cdot\nabla\psi\, d^3\boldsymbol{x} = -4\pi kq_0\psi(0). \tag{12.8}$$

The weak condition (12.4) thus requires that

$$\int_{\mathbb{R}^3} \rho\psi\, d^3\boldsymbol{x} = q_0\psi(0),$$

for every $\psi \in C^\infty_{\text{cpt}}(\mathbb{R}^3)$. This is consistent with the physical interpretation of ρ as a charge located exactly at the origin.

The concept of a "point density" was widely used in physics applications in the 18th and 19th centuries. In a 1930 book on quantum mechanics, the physicist Paul Dirac described such densities in terms of a *delta function* $\delta(\boldsymbol{x})$, whose defining property is that

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\delta(\boldsymbol{x})\, d^n\boldsymbol{x} := f(0), \tag{12.9}$$

for a continuous function f. This terminology and notation are potentially misleading, because δ is not a function and (12.9) is not actually an integral. However, Dirac's formulation hints at the proper mathematical interpretation, which is that δ should be understood as a functional $f \mapsto f(0)$.

If we accept the intuitive definition of the delta function for the moment, then we can interpret the calculation (12.8) as showing that

$$\nabla\cdot\frac{\boldsymbol{x}}{r^3} = 4\pi\delta. \tag{12.10}$$

12.2 The Space of Distributions

A *distribution* on a domain $\Omega \subset \mathbb{R}^n$ is a continuous linear functional $C^\infty_{\text{cpt}}(\Omega) \to \mathbb{C}$. The map defined by a distribution u is usually written as a pairing of u with a test function, i.e.,

$$\psi \mapsto (u, \psi) \in \mathbb{C} \tag{12.11}$$

for $\psi \in C^\infty_{\text{cpt}}(\Omega)$. Linearity means that

$$(u, c_1\psi_1 + c_2\psi_2) = c_1(f, \psi_1) + c_2(f, \psi_2),$$

for all $c_1, c_2 \in \mathbb{C}$ and $\psi_1, \psi_2 \in C^\infty_{\text{cpt}}(\Omega)$.

The definition of distribution also includes the word "continuous". To define continuity for functionals we must first specify what convergence means in $C^\infty_{\text{cpt}}(\Omega)$. The standard definition is that for a sequence $\{\psi_k\}$ to converge to ψ in $C^\infty_{\text{cpt}}(\Omega)$ means that all ψ_k have support in some fixed compact set $K \subset \Omega$, and the sequence of functions and all sequences of partial derivatives converge uniformly on K. Continuity of the functional (12.11) is then defined by the condition that convergence of a sequence $\psi_k \to \psi$ in $C^\infty_{\text{cpt}}(\Omega)$ implies that

$$\lim_{k\to\infty} (u, \psi_k) = (u, \psi). \tag{12.12}$$

In finite dimensions continuity is implied by linearity. That is not the case here, but in practice it is quite difficult to come up with a functional that is linear but not continuous.

The set of distributions on Ω forms a vector space denoted by $\mathcal{D}'(\Omega)$. Linear combinations of distributions are defined in the obvious way by

$$(c_1u_1 + c_2u_2, \psi) := c_1(u_1, \psi) + c_2(u_2, \psi),$$

for $u_1, u_2 \in \mathcal{D}'(\Omega)$ and $c_1, c_2 \in \mathbb{C}$. The mathematical theory of distributions was developed independently in the mid-20th century by Sergei Sobolev and Laurent Schwartz. Schwartz used $\mathcal{D}$ as a notation for C^∞_{cpt}, and the prime accent on $\mathcal{D}'$ comes from the notation for the dual of a vector space in linear algebra.

A locally integrable function $f \in L^1_{\text{loc}}(\Omega)$ defines a distribution through the integral pairing

$$(f, \psi) := \int_\Omega f\psi \, d^n\boldsymbol{x}. \tag{12.13}$$

Under this convention there is an inclusion

$$L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega).$$

In particular, all L^p functions can be interpreted as distributions.

As we saw with the point charge density in Sect. 12.1, not all distributions are given by functions. We use the notation $\delta_{\boldsymbol{x}}$ for the delta function centered at $\boldsymbol{x} \in \Omega$, defined by

$$(\delta_{\boldsymbol{x}}, \psi) := \psi(\boldsymbol{x}). \tag{12.14}$$

By convention the subscript is dropped for $\boldsymbol{x} = 0$, i.e., $\delta := \delta_0$.

Multiplication by smooth functions preserves the space $C^\infty_{\text{cpt}}(\Omega)$. Therefore it makes sense to multiply a distribution $u \in \mathcal{D}'(\Omega)$ by a function $f \in C^\infty(\Omega)$. The product distribution is defined by

$$(fu, \psi) := (u, f\phi).$$

It does not make sense, however, to multiply two distributions together. This fact was intuitively clear in early applications: the product of two charge densities makes no physical sense.

Convergence of a sequence of distributions is defined in a very straightforward way. We say that $u_k \to u$ in $\mathcal{D}'(\Omega)$ if

$$\lim_{k\to\infty} (u_k, \psi) = (u, \psi)$$

for all $\psi \in C^\infty_{\text{cpt}}(\Omega)$. All distributions can in fact be approximated by smooth functions by such a limit, although we are not equipped to prove that here. We will present one useful special case, a construction of the delta function as a limit of integrable functions.

Lemma 12.1 *Given* $f \in L^1(\mathbb{R}^n)$ *satisfying*

$$\int_{\mathbb{R}^n} f\, d^n\boldsymbol{x} = 1, \tag{12.15}$$

define the rescaled function,

$$f_a(\boldsymbol{x}) := a^n f(a\boldsymbol{x})$$

for $a > 0$*. Then*

$$\lim_{a\to\infty} f_a = \delta, \tag{12.16}$$

as a distributional limit.

Proof For $\psi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$ we can evaluate the pairing with f_a using a change variables,

$$\begin{aligned}(f_a, \psi) &= \int_{\mathbb{R}^n} a^n f(a\boldsymbol{x})\psi(\boldsymbol{x})\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\psi(\boldsymbol{x}/a)\, d^n\boldsymbol{x}.\end{aligned}$$

By the assumption (12.15), we can also write

$$\psi(0) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\psi(0)\, d^n\boldsymbol{x},$$

which gives the estimate

$$\left|(f_a, \psi) - \psi(0)\right| \le \int_{\mathbb{R}^n} |f(\boldsymbol{x})| \left|\psi(\boldsymbol{x}/a) - \psi(0)\right| d^n\boldsymbol{x}. \tag{12.17}$$

Given $\varepsilon > 0$, the fact that f is integrable implies that exists R sufficiently large so that

$$\int_{|\boldsymbol{x}|\ge R} |f|\, d^n\boldsymbol{x} < \varepsilon. \tag{12.18}$$

By the continuity of ψ we can also choose $\delta > 0$ so that

$$|\psi(\boldsymbol{x}) - \psi(0)| < \varepsilon$$

for $|\boldsymbol{x}| < \delta$. For $a \ge R/\delta$ this implies that

$$\left|\psi(\boldsymbol{x}/a) - \psi(0)\right| < \varepsilon \tag{12.19}$$

for all $|\boldsymbol{x}| \le R$. Using (12.18) and (12.19) to estimate the difference (12.17) gives

$$\begin{aligned}\left|(f_a, \psi) - \psi(0)\right| &\le 2\|\psi\|_\infty \int_{|\boldsymbol{x}|\ge R} |f(\boldsymbol{x})|\, d^n\boldsymbol{x} + \varepsilon \int_{|\boldsymbol{x}|\le R} |f(\boldsymbol{x})|\, d^n\boldsymbol{x} \\ &\le \big(2\|\psi\|_\infty + \|f\|_1\big)\varepsilon,\end{aligned}$$

for $a \ge R/\delta$. Since ε was arbitrary, this shows that

$$\lim_{a\to\infty} (f_a, \psi) = \psi(0).$$

□

The rescaling used in Lemma 12.1 is illustrated in Fig. 12.1. Note that this looks very similar to Fig. 9.1, and in fact the proof of Lemma 12.1 uses essentially the same argument as that of Theorem 9.1. We saw another case of this construction in the proof of Theorem 6.2. Indeed, we can now interpret the result of Theorem 6.2 as a distributional limit of the heat kernel,

$$\lim_{t\to 0} H_t = \delta,$$

where H_t was defined in (6.16).

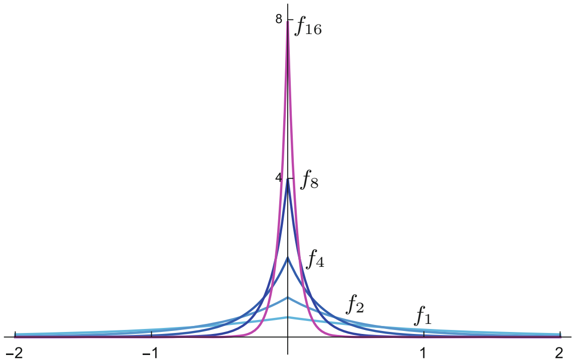

Fig. 12.1 Rescaled functions f_a for $f(x) = \frac{1}{2}e^{-|x|}$

12.3 Distributional Derivatives

The *distributional derivative* extends the concept of the weak derivative introduced in Sect. 10.1. By analogy with (10.7), for $u \in \mathcal{D}'(\Omega)$ and we define the distribution $D^\alpha u$ by

$$(D^\alpha u, \psi) := (-1)^{|\alpha|}(u, D^\alpha \psi), \tag{12.20}$$

with

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and $|\alpha| := \alpha_1 + \cdots + \alpha_n$, as before. The pairing (12.20) is well defined as a distribution because D^α is both linear and continuous as a map $C^\infty_{\mathrm{cpt}}(\Omega) \to C^\infty_{\mathrm{cpt}}(\Omega)$.

The terms "distributional" and "weak" are frequently used interchangeably to describe derivatives, since the definitions overlap to a considerable extent. The only difference is that a weak derivative is representable as a locally integrable function. Weak derivatives may not exist, whereas all distributions are infinitely differentiable.

Example 12.2 Let us reconsider Example 10.3, where we considered the derivative of $w \in L^1_{\mathrm{loc}}(\Omega)$ defined by

$$w(t) = \begin{cases} w_-(t), & t < 0, \\ w_+(t), & t \geq 0, \end{cases}$$

where $w_\pm \in C^1(\mathbb{R})$. As part of that calculation we showed that

$$-\int_{-\infty}^{\infty} w\psi'\, dt = \left[w_+(0) - w_-(0)\right]\psi(0) + \int_{-\infty}^{\infty} h\psi\, dt, \tag{12.21}$$

where h is the piecewise derivative

$$h(t) := \begin{cases} w'_-(t), & t < 0, \\ w'_+(t), & t > 0. \end{cases}$$

The left-hand side of (12.21) is the pairing (w', ψ) by the definition (12.20). From the right-hand side we can thus see that the distributional derivative is

$$w' = h + \big[w_+(0) - w_-(0)\big]\delta.$$

◊

Example 12.3 For $\delta_{\boldsymbol{x}} \in \mathcal{D}'(\mathbb{R}^n)$, the derivatives $D^\alpha \delta_{\boldsymbol{x}}$ are easily computed from the definition (12.20). For $\psi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$,

$$\begin{aligned}(D^\alpha \delta_{\boldsymbol{x}}, \psi) &= (-1)^{|\alpha|}(\delta_{\boldsymbol{x}}, D^\alpha \psi) \\ &= (-1)^{|\alpha|} D^\alpha \psi(\boldsymbol{x}).\end{aligned}$$

In other words, the distribution $D^\alpha \delta_{\boldsymbol{x}}$ evaluates the derivative of the test function at the point $\boldsymbol{x}$, up to a sign. ◊

Example 12.4 The function $\ln|x|$ is locally integrable on $\mathbb{R}$ and so defines a distribution in $\mathcal{D}'(\mathbb{R})$. Therefore $(\ln|x|)'$ exists in the distribution sense. This is puzzling because

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

for $x \neq 0$, and x^{-1} is not locally integrable.

To understand what is happening here, we must return to the distributional definition,

$$\begin{aligned}\big((\ln|x|)', \psi\big) &:= -(\ln|x|, \psi') \\ &= -\int_{-\infty}^{\infty} \psi'(x) \ln|x|\, dx\end{aligned}$$

for $\psi \in C^\infty_{\text{cpt}}(\mathbb{R})$. To compute this we avoid the singularity at 0 by writing

$$\big((\ln|x|)', \psi\big) = -\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \psi'(x) \ln|x|\, dx. \tag{12.22}$$

Integration by parts gives

$$\begin{aligned}-\int_{-\infty}^{-\varepsilon} \psi'(x) \ln|x|\, dx &= -\psi(x) \ln|x| \Big|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{\psi(x)}{x}\, dx \\ &= -\psi(-\varepsilon) \ln\varepsilon + \int_{-\infty}^{-\varepsilon} \frac{\psi(x)}{x}\, dx,\end{aligned}$$

and similarly

$$-\int_{\varepsilon}^{\infty} \psi'(x) \ln |x| \, dx = \psi(\varepsilon) \ln \varepsilon + \int_{\varepsilon}^{\infty} \frac{\psi(x)}{x} \, dx.$$

After combining these two halves, we obtain

$$\int_{|x| \geq \varepsilon} \psi'(x) \ln |x| \, dx = \big[\psi(\varepsilon) - \psi(-\varepsilon)\big] \ln \varepsilon + \int_{|x| \geq \varepsilon} \frac{\psi(x)}{x} \, dx.$$

By the definition of the derivative,

$$\lim_{\varepsilon \to 0} \frac{\psi(\varepsilon) - \psi(-\varepsilon)}{2\varepsilon} = \psi'(0).$$

Therefore

$$\begin{aligned} \lim_{\varepsilon \to 0} \big[\psi(\varepsilon) - \psi(-\varepsilon)\big] \ln \varepsilon &= 2\psi'(0) \lim_{\varepsilon \to 0} \varepsilon \ln \varepsilon \\ &= 0. \end{aligned}$$

Hence (12.22) reduces to

$$\big((\ln |x|)', \psi\big) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\psi(x)}{x} \, dx. \tag{12.23}$$

The limit on the right exists for $\psi \in C^{\infty}_{\text{cpt}}(\mathbb{R})$, even though x^{-1} is not integrable, because the limit is taken symmetrically. This limiting procedure defines a distribution called the *principal value* of x^{-1}, written as $\text{PV}[x^{-1}]$. We could rephrase (12.23) as

$$\frac{d}{dx} \ln |x| = \text{PV}\left[x^{-1}\right].$$

◊

Example 12.5 Let us reinterpret the discussion from Sect. 12.1 in terms of distributional derivatives. We already noted that the components of $\boldsymbol{x}/r^3$ are locally integrable, so we can consider the Coulomb formula (12.1) for $\boldsymbol{E}$ as the definition of a vector-valued distribution. The distributional divergence of $\boldsymbol{x}/r^3$ is defined by the condition that

$$\left(\nabla \cdot \frac{\boldsymbol{x}}{r^3}, \psi\right) := -\int_{\mathbb{R}^3} \frac{\boldsymbol{x}}{r^3} \cdot \nabla\psi \, d^3x,$$

for $\psi \in C^{\infty}_{\text{cpt}}(\mathbb{R}^3)$. The derivation of (12.8) thus shows that

$$\nabla \cdot \frac{\boldsymbol{x}}{r^3} = 4\pi\delta. \tag{12.24}$$

We can also consider the corresponding result for the Coulomb electric potential

$$\phi(\boldsymbol{x}) = \frac{1}{r}.$$

(ignoring the physical constants). The gradient of ϕ exists in the weak sense and is given by

$$\nabla\left(\frac{1}{r}\right) = -\frac{\boldsymbol{x}}{r^3}.$$

Since $\Delta = \nabla \cdot \nabla$, we deduce from (12.24) that

$$-\Delta\left(\frac{1}{4\pi r}\right) = \delta. \tag{12.25}$$

◊

12.4 Fundamental Solutions

Because the Poisson equation is linear, it makes sense to construct a solution with a continuous density by superimposing a field of point sources. With a change of variables, we can see from (12.25) that the potential function corresponding to a point source at $\boldsymbol{y} \in \mathbb{R}^3$ is

$$\phi_{\boldsymbol{y}}(\boldsymbol{x}) := \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}.$$

Weighting the point sources by the density ρ and summing them with an integral gives

$$u(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d^3\boldsymbol{y}. \tag{12.26}$$

This formula, which is often stated as the integral form of Coulomb's law, does indeed yield a solution of the Poisson equation on $\mathbb{R}^3$ under certain conditions. For example if $\rho \in C^1_{\text{cpt}}(\mathbb{R}^3)$ then one can confirm that $-\Delta u = \rho$ by direct computation. The C^1 condition is stronger than necessary here, but continuity alone would not be sufficient. (The precise notion of regularity needed for this problem is something called *Hölder continuity*.)

This idea of constructing of general solutions by superposition of point sources is the inspiration for the concept of a *fundamental solution*. For a constant-coefficient differential operator L acting on $\mathbb{R}^n$, of the form

$$L = \sum_{|\alpha|\le m} a_\alpha D^\alpha,$$

with $a^\alpha \in \mathbb{C}$, a fundamental solution is a distribution $\Phi \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$L\Phi = \delta. \tag{12.27}$$

For example, in the Coulomb case the calculation (12.25) gives the fundamental solution of $-\Delta$ on $\mathbb{R}^3$. Fundamental solutions are especially important for classical problems involving the Laplacian.

The solution formula (12.26) resembles the convolution used to solve the heat equation in Sect. 6.3. For $f, g \in L^1(\mathbb{R}^n)$ the convolution is defined as

$$f * g(\boldsymbol{x}) := \int_{\mathbb{R}^n} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y})\, d^n\boldsymbol{y}.$$

A simple change of variables shows that this product is symmetric,

$$f * g = g * f.$$

In order to produce solution formulas from fundamental solutions, we need to understand how to take convolutions with distributions.

For $f, g \in L^1(\mathbb{R}^n)$, the distributional pairing of $f * g$ with $\psi \in C^\infty_{\text{cpt}}(\Omega)$ gives

$$(f * g, \psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) \psi(\boldsymbol{x})\, d^n\boldsymbol{y}\, d^n\boldsymbol{x}. \tag{12.28}$$

The $\boldsymbol{x}$ integration looks almost like the convolution of ψ with g, except with the argument switched from $\boldsymbol{y} - \boldsymbol{x}$ to $\boldsymbol{x} - \boldsymbol{y}$. With the reflection defined by

$$g^-(\boldsymbol{x}) := g(-\boldsymbol{x}),$$

we have

$$g^- * \psi(\boldsymbol{y}) = \int_{\mathbb{R}^n} g(\boldsymbol{x} - \boldsymbol{y}) \psi(\boldsymbol{x})\, d^n\boldsymbol{x}.$$

Thus (12.28) reduces to

$$(f * g, \psi) := (f, g^- * \psi).$$

If $\phi, \psi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$, then it is easy to check that $\phi^- * \psi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$ also. Moreover, the map $\psi \mapsto \phi^- * \psi$ is linear and continuous. We can thus define $u * \phi$ for $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$ by

$$(u * \phi, \psi) := (u, \phi^- * \psi) \tag{12.29}$$

for $\psi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$.

The distribution δ plays a special role with regard to convolutions. By the definition (12.29),

$$\begin{aligned}(\delta * \phi, \psi) &:= (\delta, \phi^- * \psi)\\ &= \phi^- * \psi(0)\\ &= \int_\Omega \phi(\boldsymbol{x})\psi(\boldsymbol{x})\, d^n\boldsymbol{x}\\ &= (\phi, \psi).\end{aligned}$$

This shows that

$$\delta * \phi = \phi. \tag{12.30}$$

In other words, convolution by δ is the identity map.

Let Φ be the fundamental solution for the constant coefficient operator L. Our goal is to show that the equation $Lu = f$ is solved by the convolution $u = \Phi * f$, at least for $f \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$. To check this, we need to know how to evaluate derivatives of the convolution.

Lemma 12.6 *For $w \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$,*

$$D^\alpha(w * f) = (D^\alpha w) * \phi = w * (D^\alpha f).$$

Proof For $\phi, \psi \in C^\infty_{\text{cpt}}(\Omega)$, we compute directly that

$$\begin{aligned}D^\alpha(\phi * \psi)(\boldsymbol{x}) &= \int_\Omega \phi(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\, d^n\boldsymbol{y}\\ &= \int_\Omega D^\alpha\phi(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\, d^n\boldsymbol{y}\\ &= (D^\alpha\phi) * \psi(\boldsymbol{x}).\end{aligned}$$

Since the convolution is symmetric, the same formula holds with ψ and ϕ switched. Thus the formula

$$D^\alpha(\psi * \phi) = (D^\alpha\psi) * \phi = \psi * (D^\alpha\phi) \tag{12.31}$$

holds for test functions.

For $w \in \mathcal{D}'(\mathbb{R})$ and $\psi \in C^\infty_{\text{cpt}}(\mathbb{R}^n)$, it follows from the definitions that

$$(D^\alpha(f * \phi), \psi) = (-1)^{|\alpha|}(f, \phi^- * (D^\alpha\psi)).$$

By (12.31) this gives

$$\begin{aligned}(D^\alpha(f * \phi), \psi) &= (-1)^{|\alpha|}(f, D^\alpha(\phi^- * \psi))\\ &= (D^\alpha f, \phi^- * \psi)\\ &= ((D^\alpha f) * \phi, \psi),\end{aligned}$$

and also

$$(D^\alpha(f * \phi), \psi) = (-1)^{|\alpha|}(f, (D^\alpha\phi)^- * \psi))$$
$$= (f * (D^\alpha\phi), \psi).$$

□

Theorem 12.7 *If L is a constant coefficient operator on $\mathbb{R}^n$ with fundamental solution Φ, then for $f \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ the equation*

$$Lu = f$$

is solved by

$$u = \Phi * f.$$

Proof By Lemma 12.6,

$$\begin{aligned} L(\Phi * f) &= \sum_{|\alpha|\le m} a_\alpha D^\alpha(\Phi * f) \\ &= \sum_{|\alpha|\le m} a_\alpha (D^\alpha\Phi) * f \\ &= (L\Phi) * f. \end{aligned}$$

Note that the second step only works because the coefficients a_α are assumed to be constant. Since $L\Phi = \delta$, we see from (12.30) that

$$L(\Phi * f) = f.$$

□

A result called the Malgrange-Ehrenpreis theorem, proven in the 1950s, says that every constant coefficient differential operator on $\mathbb{R}^n$ admits a fundamental solution. The fundamental solution of the Laplacian, which we will now work out for any dimension, is the most important case.

Theorem 12.8 *On $\mathbb{R}^n$ the operator $-\Delta$ has the fundamental solution*

$$\Phi(\boldsymbol{x}) = \begin{cases} -\frac{1}{2\pi}\ln r, & n = 2, \\ \frac{1}{(n-2)A_n r^{n-2}}, & n \ge 3, \end{cases} \tag{12.32}$$

where A_n denotes the volume of the unit sphere in dimension n.

Proof We start from the distributional derivative,

$$(-\Delta\Phi, \psi) = -(\Phi, \Delta\psi)$$

for $\psi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$. To evaluate this, it is useful to first compute the gradient,

$$\nabla \Phi(\boldsymbol{x}) = -\frac{\boldsymbol{x}}{A_n r^n}.$$

The function $\boldsymbol{x}/r^n$ is locally integrable in $\mathbb{R}^n$ and ψ has compactly support. Therefore we can deduce from Green's first identity (Theorem 2.10) that

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi \Delta\psi \, d^n\boldsymbol{x} &= -\int_{\mathbb{R}^n} \nabla\Phi \cdot \nabla\psi \, d^n\boldsymbol{x} \\ &= \frac{1}{A_n}\int_{\mathbb{R}^n} \frac{\boldsymbol{x}}{r^n} \cdot \nabla\psi \, d^n\boldsymbol{x} \\ &= \frac{1}{A_n}\int_{\mathbb{R}^n} \frac{1}{r^{n-1}} \frac{\partial\psi}{\partial r} \, d^n\boldsymbol{x}.\end{aligned}$$

The integral can be evaluated using radial coordinates as in (2.10):

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi \Delta\psi \, d^n\boldsymbol{x} &= \frac{1}{A_n}\int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{\partial\psi}{\partial r} \, dr \, dS \\ &= -\frac{1}{A_n}\int_{\mathbb{S}^{n-1}} \psi(0) \, dS \\ &= -\psi(0).\end{aligned}$$

This shows that

$$(-\Delta\Phi, \psi) = \psi(0),$$

hence $-\Delta\Phi = \delta$. □

12.5 Green's Functions

Although fundamental solutions are defined only for the domain $\mathbb{R}^n$, one of their principle applications is to boundary value problems on a bounded domain $\Omega \subset \mathbb{R}^n$. The connection comes from a integral formula introduced in 1828 by George Green.

For this section, let Φ denote the fundamental solution of the Laplacian on $\mathbb{R}^n$, as given by (12.32). For $\boldsymbol{y} \in \mathbb{R}^n$ we set

$$\Phi_{\boldsymbol{y}}(\boldsymbol{x}) := \Phi(\boldsymbol{x} - \boldsymbol{y}). \tag{12.33}$$

Theorem 12.9 (Green's representation formula) *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary. For $u \in C^2(\overline{\Omega})$,*

$$u(\boldsymbol{y}) = -\int_{\Omega} \Phi_y \Delta u \, d^n\boldsymbol{x} + \int_{\partial\Omega} \left[\Phi_y \frac{\partial u}{\partial \nu} - u \frac{\partial \Phi_y}{\partial \nu} \right] dS$$

for $\boldsymbol{y} \in \Omega$.

Proof Because the point $\boldsymbol{y} \in \Omega$ is fixed, for notational convenience we can change variables to assume $\boldsymbol{y} = 0$. For $\varepsilon > 0$ set

$$B_\varepsilon := B(0; \varepsilon),$$

and assume that ε is small enough that $\overline{B_\varepsilon} \subset \Omega$.

On $\overline{\Omega} - B_\varepsilon$, Φ is smooth and satisfies $\Delta\Phi = 0$. Therefore, applying Green's second identity (Theorem 2.11) on this domain with $v = \Phi$ gives

$$\int_{\Omega - \overline{B}_\varepsilon} \Phi \Delta u \, d^n\boldsymbol{x} = \int_{\partial\Omega} \left(\Phi \frac{\partial u}{\partial \nu} - u \frac{\partial \Phi}{\partial \nu} \right) dS + \int_{\partial B_\varepsilon} \left(\Phi \frac{\partial u}{\partial \nu} - u \frac{\partial \Phi}{\partial \nu} \right) dS. \tag{12.34}$$

Because Δu is continuous and Φ is locally integrable,

$$\lim_{\varepsilon \to 0} \int_{\Omega - \overline{B}_\varepsilon} \Phi \Delta u \, d^n\boldsymbol{x} = \int_{\Omega} \Phi \Delta u \, d^n\boldsymbol{x},$$

To prove the representation formula we must therefore show that

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \left(\Phi \frac{\partial u}{\partial r} - u \frac{\partial \Phi}{\partial r} \right) dS = u(0). \tag{12.35}$$

To handle the first term in (12.35), note that

$$\left| \frac{\partial u}{\partial r} \right| \le |\nabla u|,$$

for $r > 0$. Therefore, since ∇u is continuous by assumption, we have a bound

$$\max_{\partial B_\varepsilon} \left| \frac{\partial u}{\partial r} \right| \le M$$

for $\varepsilon > 0$, with M independent of ε. Using the fact that $\mathrm{vol}(\partial B_\varepsilon) = A_n \varepsilon^{n-1}$ and the formula (12.32) for Φ, we can estimate

$$\left| \int_{\partial B_\varepsilon} \Phi \frac{\partial u}{\partial r} \, dS \right| \le M \begin{cases} \varepsilon \ln \varepsilon, & n = 2, \\ \frac{\varepsilon}{n-2}, & n \ge 3. \end{cases}$$

This shows that

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \Phi \frac{\partial u}{\partial r} \, dS = 0.$$

For the second term in (12.35), we use the fact that

$$\frac{\partial \Phi}{\partial r} = -\frac{1}{A_n r^{n-1}},$$

for $r > 0$, to compute

$$-\int_{\partial B_\varepsilon} u \frac{\partial \Phi}{\partial r}\, dS = \frac{1}{A_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon} u\, dS.$$

The right-hand side is the average value of u over the sphere ∂B_ε. By continuity,

$$\lim_{\varepsilon \to 0} \frac{1}{\operatorname{vol}(\partial B_\varepsilon)} \int_{\partial B_\varepsilon} u\, dS = u(0).$$

This proves (12.35), and thus establishes the representation formula. □

The representation formula of Theorem 12.9 has many applications. The original goal that Green had in mind was a solution formula for the Poisson problem with inhomogeneous Dirichlet boundary conditions, which we will now describe.

Suppose there exists a family of functions $H_{\boldsymbol{y}} \in C^2(\overline{\Omega})$, for $\boldsymbol{y} \in \Omega$, satisfying

$$\Delta H_{\boldsymbol{y}} = 0, \qquad H_{\boldsymbol{y}}\big|_{\partial\Omega} = \Phi_{\boldsymbol{y}}\big|_{\partial\Omega}. \tag{12.36}$$

Then the *Green's function* of Ω is

$$G_{\boldsymbol{y}} := \Phi_{\boldsymbol{y}} - H_{\boldsymbol{y}}. \tag{12.37}$$

It is possible to show that $H_{\boldsymbol{y}}$ exists under general regularity conditions on $\partial\Omega$, but this is too technical for us to get into here. We will focus on cases where $H_{\boldsymbol{y}}$ can be computed explicitly, which requires the geometry of Ω to be very simple.

Theorem 12.10 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary that admits a Green's function $G_{\boldsymbol{y}}$, Then the Poisson problem on Ω,*

$$-\Delta u = f, \qquad u\big|_{\partial\Omega} = g,$$

for $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega)$, is solved by the function

$$u(\boldsymbol{y}) = -\int_\Omega f G_{\boldsymbol{y}}\, d^n\boldsymbol{x} - \int_{\partial\Omega} g \frac{\partial G_{\boldsymbol{y}}}{\partial \nu}\, dS.$$

Proof Setting $v = H_{\boldsymbol{y}}$ in Green's second identity (Theorem 2.11) gives

$$\int_\Omega \Big(H_{\boldsymbol{y}} \Delta u - u \Delta H_{\boldsymbol{y}} \Big) d^n\boldsymbol{y} = \int_{\partial\Omega} H_{\boldsymbol{y}} \frac{\partial u}{\partial \nu}\, dS. \tag{12.38}$$

The result then follows by subtracting (12.38) from the representation formula of Theorem 12.9. □

Example 12.11 The Green's function for the unit disk $\mathbb{D} \subset \mathbb{R}^2$ can be derived using a trick from electrostatics called the *method of images*. This involves placing charges outside the domain in order to solve the boundary value problem. For the unit disk, in order to find $H_{\boldsymbol{y}}$ we consider a charge placed at the point $\tilde{\boldsymbol{y}}$ given by "reflecting" $\boldsymbol{y} \in \mathbb{C}\backslash\{0\}$ across the unit circle, i.e.,

$$\tilde{\boldsymbol{y}} := \frac{\boldsymbol{y}}{|\boldsymbol{y}|^2}. \tag{12.39}$$

Note that $\Phi_{\tilde{\boldsymbol{y}}}$ is harmonic on $\mathbb{D}$ because $\tilde{\boldsymbol{y}} \notin \mathbb{D}$.

For $\boldsymbol{x} \in \partial\mathbb{D}$, let φ denote the angle from $\boldsymbol{y}$ to $\boldsymbol{x}$, as shown in Fig. 12.2. By the law of cosines on the triangle made by 0, $\boldsymbol{y}$ and $\boldsymbol{x}$,

$$|\boldsymbol{x} - \boldsymbol{y}|^2 = 1 + |\boldsymbol{y}|^2 - 2|\boldsymbol{y}|\cos\varphi.$$

If $\boldsymbol{y}$ is replaced by $\tilde{\boldsymbol{y}}$, the corresponding formula is

$$\begin{aligned} |\boldsymbol{x} - \tilde{\boldsymbol{y}}|^2 &= 1 + |\tilde{\boldsymbol{y}}|^2 - 2|\tilde{\boldsymbol{y}}|\cos\varphi \\ &= 1 + |\boldsymbol{y}|^{-2} - 2|\boldsymbol{y}|^{-1}\cos\varphi. \end{aligned}$$

Solving for $\cos\varphi$ in these expressions gives the relation

$$\frac{|\boldsymbol{x} - \boldsymbol{y}|^2 - 1 - |\boldsymbol{y}|^2}{2|\boldsymbol{y}|} = \frac{|\boldsymbol{x} - \tilde{\boldsymbol{y}}|^2 - 1 - |\boldsymbol{y}|^{-2}}{2|\boldsymbol{y}|^{-1}},$$

which simplifies to

$$|\boldsymbol{x} - \tilde{\boldsymbol{y}}| = \frac{|\boldsymbol{x} - \boldsymbol{y}|}{|\boldsymbol{y}|} \tag{12.40}$$

for $\boldsymbol{x} \in \partial\mathbb{D}$.

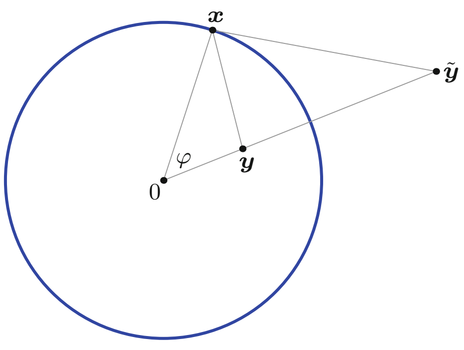

Fig. 12.2 Geometry for the method of images on the unit disk

Since $\Phi(\boldsymbol{x}) = -\frac{1}{2\pi}\ln|\boldsymbol{x}|$ in $\mathbb{R}^2$, taking the logarithm of (12.40) gives

$$\Phi_{\tilde{\boldsymbol{y}}}(\boldsymbol{x}) = \Phi_{\boldsymbol{y}}(\boldsymbol{x}) + \frac{1}{2\pi}\ln|\boldsymbol{y}|$$

for $\boldsymbol{y} \neq 0$ or $\boldsymbol{x}$. Thus we can solve (12.36) for $\boldsymbol{y} \neq 0$ by setting

$$H_{\boldsymbol{y}} := \Phi_{\tilde{\boldsymbol{y}}} - \frac{1}{2\pi}\ln|\boldsymbol{y}|.$$

For $\boldsymbol{y} = 0$ the obvious solution is $H_0 := 0$ because $\Phi|_{\partial\mathbb{D}} = 0$.

The Green's function is thus

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = \begin{cases} -\frac{1}{2\pi}\ln\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{|\boldsymbol{x}-\tilde{\boldsymbol{y}}|\,|\boldsymbol{y}|}\right), & \boldsymbol{y} \neq 0 \\ -\frac{1}{2\pi}\ln|\boldsymbol{x}|, & \boldsymbol{y} = 0. \end{cases}$$

To apply this in the solution formula, we need the radial derivative of G. For $\boldsymbol{y}$ fixed and $r := |\boldsymbol{x}|$ we compute

$$\begin{aligned} \frac{\partial}{\partial r}\ln|\boldsymbol{x}-\boldsymbol{y}| &= \boldsymbol{x}\cdot\nabla\ln|\boldsymbol{x}-\boldsymbol{y}| \\ &= \boldsymbol{x}\cdot\frac{(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} \\ &= \frac{1-\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2}. \end{aligned}$$

Applying the corresponding result for $|\boldsymbol{x}-\tilde{\boldsymbol{y}}|$ and subtracting gives

$$\frac{\partial G_{\boldsymbol{y}}}{\partial r}(\boldsymbol{x}) = -\frac{1}{2\pi}\left(\frac{1-\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2} - \frac{1-\boldsymbol{x}\cdot\tilde{\boldsymbol{y}}}{|\boldsymbol{x}-\tilde{\boldsymbol{y}}|^2}\right),$$

which by (12.40) simplifies to

$$\frac{\partial G_{\boldsymbol{y}}}{\partial r}(\boldsymbol{x}) = -\frac{1-|\boldsymbol{y}|^2}{2\pi|\boldsymbol{x}-\boldsymbol{y}|^2}. \tag{12.41}$$

Conveniently, the calculation for $\boldsymbol{y} = 0$ leads to the same expression.

In the case of the Laplace equation on $\mathbb{D}$, Theorem 12.10 gives the formula for a harmonic function u with boundary value g as

$$u(\boldsymbol{y}) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} \frac{1-|\boldsymbol{y}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^2}\, g(\boldsymbol{x})\, dS(\boldsymbol{x}).$$

This will look more familiar in polar coordinates. With $\boldsymbol{y} = (r\cos\theta, r\sin\theta)$ and $\boldsymbol{x} = (\cos\eta, \sin\eta)$, the formula becomes

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\eta - \theta)}\, g(\cos\eta, \sin\eta)\, d\eta.$$

This is the classical Poisson formula (9.4) that we derived from Fourier series. ♢

12.6 Time-Dependent Fundamental Solutions

To adapt the concept of a fundamental solution to evolution equations, we need to consider time-dependent distributions on $\mathbb{R}^n$. We will use a subscript to denote the time dependence, to avoid confusion with the spatial variables. Thus a map $\mathbb{R} \to \mathcal{D}'(\mathbb{R}^n)$ will be written

$$t \mapsto w_t.$$

For $\psi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ the pairing (w_t, ψ) is a complex-valued function of t.

The function $t \mapsto w_t$ is differentiable with respect to time if there exists a family of distributions $\frac{\partial w_t}{\partial t} \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\frac{d}{dt}(w_t, \psi) = \left(\frac{\partial w_t}{\partial t}, \psi\right), \tag{12.42}$$

for all $\psi \in C^\infty_{\mathrm{cpt}}(\Omega)$. Higher derivatives are defined in the same way.

Example 12.12 In $\mathbb{R}$, consider the derivatives of δ_t, the delta function supported at t. By definition,

$$(\delta_t, \psi) := \psi(t),$$

so that

$$\left(\frac{\partial^n}{\partial t^n}\delta_t, \psi\right) = \psi^{(n)}(t).$$

Compare this to the spatial derivatives, defined according to (12.20),

$$\begin{aligned}\left(\frac{\partial^n}{\partial x^n}\delta_t, \psi\right) &:= (-1)^n \left(\delta_t, \psi^{(n)}\right) \\ &= (-1)^n \psi^{(n)}(t).\end{aligned}$$

We conclude that

$$\frac{\partial^n}{\partial t^n}\delta_t = (-1)^n \frac{\partial^n}{\partial x^n}\delta_t. \tag{12.43}$$

♢

Let us try to deduce the fundamental solution for the one-dimensional wave equation from d'Alembert's formula (4.8),

$$u(t, x) = \frac{1}{2}\big[g(x+t) + g(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} h(\tau)\, d\tau, \tag{12.44}$$

The second term in (12.44) could be interpreted as a convolution

$$\begin{aligned}\int_{x-t}^{x+t} h(\tau)\, d\tau &= \int_{-\infty}^{\infty} \chi_{[-t,t]}(x-\tau)h(\tau)\, d\tau \\ &= \chi_{[-t,t]} * h(x),\end{aligned}$$

where χ_I denotes a characteristic function as in (7.5). Therefore it makes sense to define this component of the fundamental solution as

$$W_t := \frac{1}{2}\chi_{(-t,t)}. \tag{12.45}$$

The time derivatives of W_t are computed from the pairing

$$(W_t, \psi) = \frac{1}{2}\int_{-t}^{t} \psi\, dx$$

for $\psi \in C^\infty_{\text{cpt}}(\mathbb{R})$. By the fundamental theorem of calculus,

$$\frac{d}{dt}(W_t, \psi) = \frac{1}{2}\big[\psi(t) + \psi(-t)\big], \tag{12.46}$$

which shows that

$$\frac{\partial W_t}{\partial t} = \frac{1}{2}\big(\delta_t + \delta_{-t}\big). \tag{12.47}$$

Differentiating again using (12.43) gives

$$\frac{\partial^2 W_t}{\partial t^2} = \frac{1}{2}\left(-\delta_t' + \delta_{-t}'\right). \tag{12.48}$$

On the other hand, x-derivatives of W_t are defined by (12.20). In particular,

$$\left(\frac{\partial^2 W_t}{\partial x^2}, \psi\right) := \big(W_t, \psi''\big)$$

This can be evaluated by direct integration,

$$\begin{aligned}\left(\frac{\partial^2 W_t}{\partial x^2}, \psi\right) &:= \frac{1}{2}\int_{-t}^{t} \psi''\, dx \\ &= \frac{1}{2}\big[\psi'(t) - \psi'(-t)\big].\end{aligned}$$

This shows that

$$\frac{\partial^2 W_t}{\partial x^2} = \frac{1}{2}\left(-\delta_t' + \delta_{-t}'\right). \tag{12.49}$$

By (12.48) and (12.49), W_t is a distributional solution of the wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) W_t = 0.$$

In contrast to the definition (12.27) of a fundamental solution in the spatial case, W_t satisfies a homogeneous equation. The delta function appears only in the boundary conditions,

$$W_0 = 0, \quad \left.\frac{\partial W_t}{\partial t}\right|_{t=0} = \delta.$$

The distribution W_t, which is analogous to a fundamental solution, is called the *wave kernel*. By (12.47), the g component of the d'Alembert solution formula (12.44) could be written in terms of W_t as

$$\frac{1}{2}\big[g(x+t) + g(x-t)\big] = \frac{\partial W_t}{\partial t} * g(x).$$

Thus the full convolution formula for the solution reads

$$u(t,\cdot) = \frac{\partial W_t}{\partial t} * g + W_t * h.$$

12.7 Exercises

12.1 Define the distribution $u \in \mathcal{D}'(\mathbb{R})$ by

$$(u,\psi) := \int_{-1}^{1} \frac{\psi(x) - \psi(0)}{x}\,dx + \int_{|x|\geq 1} \frac{\psi(x)}{x}\,dx.$$

Show that $u = \mathrm{PV}[x^{-1}]$.

12.2 Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be the function

$$f(x) = \begin{cases} \log x, & x > 0, \\ -\log(-x), & x < 0. \end{cases}$$

For $x \neq 0$, $f'(x) = |x|^{-1}$, but this is not locally integrable. Show that the distributional derivative is

$$(f', \psi) = \int_{-1}^{1} \frac{\psi(x) - \psi(0)}{|x|} + \int_{|x|\geq 1} \frac{\psi(x)}{|x|}\,dx.$$

12.3 Let $\mathbb{H}$ denote the upper half-plane $\{x_2 > 0\} \subset \mathbb{R}^2$. The goal of this problem is to show that the Laplace equation on $\mathbb{H}$,

$$\Delta u = 0, \qquad u(\cdot, 0) = g,$$

has the solution

$$u(\boldsymbol{y}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y_2}{(x - y_1)^2 + y_2^2}\, g(x)\,dx$$

for $g \in C^\infty_{\mathrm{cpt}}(\mathbb{R})$.

(a) Derive this formula from Theorem 12.10 using the method of images as in Example 12.11. In this case the reflection of $\boldsymbol{y} \in \mathbb{H}$ is given by $\overline{(y_1, y_2)} = (y_1, -y_2)$ (the complex conjugate).
(b) Show that the fact that $u(\cdot, 0) = g$ could also be derived by using Lemma 12.1 to deduce that

$$\lim_{x\to 0} \frac{y}{\pi(x^2 + y^2)} = \delta(y).$$

12.4 In $\mathbb{R}^3$ show that

$$(-\Delta - k^2)\frac{e^{ikr}}{4\pi r} = \delta$$

for all $k \in \mathbb{R}$.

12.5 For $n \geq 3$ let $\mathbb{B}^n$ denote the unit ball $\{r < 1\} \subset \mathbb{R}^n$.

(a) Apply the method of images as in Example 12.11 to derive the solution $H_{\boldsymbol{y}}$ of (12.36) for $\boldsymbol{y} \in \mathbb{B}^n$. and compute the Green's function. (Note that the formulas (12.39) and (12.40) remain valid in any dimension.)
(b) Show that the radial derivative of the Green's function satisfies

$$\frac{\partial G_{\boldsymbol{y}}}{\partial r}(\boldsymbol{x}) = -\frac{1 - |\boldsymbol{y}^2|}{A_n|\boldsymbol{x} - \boldsymbol{y}|^{n-1}}.$$

(c) Find the resulting solution formula from Theorem 12.10, and show that this generalizes the mean value formula for harmonic functions obtained in Theorem 9.3.