Chapter 8
Fourier Series

In his study of heat flow in 1807, Fourier made the radical claim that it should be possible to represent all solutions of the one-dimensional heat equation by trigonometric series. As we noted in the introduction to Chap. 7, trigonometric series had been studied earlier by other mathematicians. Fourier's innovation was to suggest that the general solution could be obtained this way.

This claim proved difficult to resolve, because the tools of functional analysis that we discussed in Chap. 7 were not yet available in Fourier's time. Indeed, the difficult problem of Fourier series convergence provided some of the strongest motivation for the development of these tools.

In this chapter we will analyze Fourier series in more detail, and show that the Fourier approach yields a general solution for the one-dimensional heat equation. The primary significance of this approach to PDE is the philosophy of *spectral analysis* that it inspired. The decomposition of functions with respect to the spectrum of a differential operator is a tool with enormous applications, both theoretical and practical.

8.1 Series Solution of the Heat Equation

Consider the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \tag{8.1}$$

on a domain $\Omega \subset \mathbb{R}^n$, with Dirichlet or Neumann boundary conditions. According to Lemma 5.1 the product solutions of (8.1) have the form

$$u(t, \boldsymbol{x}) = v(t)\phi(\boldsymbol{x}),$$

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where ϕ solves the Helmholtz equation

$$-\Delta\phi = \lambda\phi$$

on Ω. The temporal equation is a simple ODE,

$$\frac{\partial v}{\partial t} = -\lambda v,$$

with the family of solutions

$$v(t) = v(0)e^{-\lambda t}. \tag{8.2}$$

Let us assume that the equation for ϕ admits a sequence of solutions ϕ_k, with eigenvalues λ_k. We have seen specific examples of this in Chap. 5, including the trigonometric case in Theorem 5.2. By (8.2), the corresponding product solutions of the heat equation are

$$u_k(t, \boldsymbol{x}) := e^{-\lambda_k t}\phi_k(\boldsymbol{x}).$$

Fourier's strategy calls for us to express the general solution as a series,

$$u(t, x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(\boldsymbol{x}), \tag{8.3}$$

for some choice of coefficients a_n. To fix the coefficients a_n in (8.3) we assume an initial condition $u(0, \boldsymbol{x}) = h(\boldsymbol{x})$. Setting $t = 0$ gives

$$h(\boldsymbol{x}) = \sum_{n=1}^{\infty} a_n\phi_n(\boldsymbol{x}). \tag{8.4}$$

If we can show that $\{\phi_n\}$ forms an orthonormal basis of $L^2(\Omega)$, then this gives us a way to assign coefficients to h such that (8.4) holds, at least in the sense of L^2 convergence.

Even if the orthonormal basis property is established, some big issues still remain. The fact that each term u_k satisfies the heat equation does not guarantee that u does, because of the infinite series summation. Similarly, the limit of (8.3) as $t \to 0$ is not necessarily (8.4), because the limit cannot necessarily be taken inside the summation. In this chapter we will explain how to resolve these problems in the context of trigonometric series.

Example 8.1 Consider the case of a one-dimensional metal rod with insulated ends. For convenience take the length to be π, so that $x \in [0, \pi]$ and the Neumann boundary conditions are

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, \pi) = 0.$$

Let us consider the initial condition

$$h(x) = 3\pi x^2 - 2x^3, \tag{8.5}$$

as pictured on the left in Fig. 8.1.

The boundary conditions are satisfied by cosines,

$$\phi_n(x) = \cos(nx), \quad n \in \mathbb{N}_0.$$

Hence the strategy outlined above calls for us to represent the initial condition as a series

$$h(x) = \sum_{n=0}^{\infty} a_n \cos(nx). \tag{8.6}$$

To choose the coefficients, we recall the discussion of basis expansion from Sect. 7.5. The cosines satisfy an orthogonality condition with respect to the L^2 inner product on $[0, \pi]$,

$$\int_0^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & m \neq n, \\ \pi & m = n = 0, \\ \pi/2 & m = n \geq 1, \end{cases}. \tag{8.7}$$

This could be checked with trigonometric identities, but it is perhaps easier is to use the complex form $\cos(kx) = \frac{1}{2}(e^{ikx} + e^{-ikx})$.

Since the sequence $\{\phi_n\}$ is not normalized, the coefficient formula (7.18) must be interpreted as

$$a_n = \frac{1}{\|\phi_n\|^2} \langle h, \phi_n \rangle .$$

By (8.7) the Fourier coefficients are thus given by

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{\pi} h(x)\,dx, \\ a_n &= \frac{2}{\pi} \int_0^{\pi} h(x)\cos(nx)\,dx, \quad n \geq 1. \end{aligned} \tag{8.8}$$

After substituting (8.5) into (8.8), integration by parts yields

$$a_n = \begin{cases} \frac{\pi^3}{2}, & n = 0, \\ -\frac{48}{\pi n^4}, & n \geq 1, \text{ odd}, \\ 0, & n \geq 2, \text{ even}. \end{cases} \tag{8.9}$$

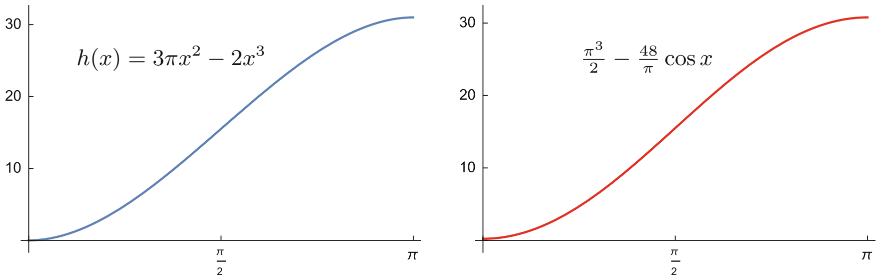

Fig. 8.1 Comparison of the initial condition $h(x)$ with the first two terms of its cosine series

Figure 8.1 shows a comparison between h and the partial sum $S_1[h]$. The close match between these functions is clearly evident. And since the higher coefficients decay by a factor of n^{-4}, convergence of this series seems quite plausible. The resulting solution would be given by

$$u(t, x) = \frac{\pi^3}{2} - \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{48}{\pi n^4} e^{-n^2 t} \cos(nx).$$

Note that the convergence rate improves dramatically as t increases. ◊

8.2 Periodic Fourier Series

We saw examples of Fourier series based on sines in Theorem 5.2 and cosines in Example 8.1. To account for both cases, it is convenient to consider periodic functions on $\mathbb{R}$. We define

$$\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}), \tag{8.10}$$

where the quotient notation means that points separated by an integer multiple of 2π are considered equivalent. The space $C^m(\mathbb{T})$ consists of the functions in $C^m(\mathbb{R})$ which are 2π-periodic.

Integrals of functions on $\mathbb{T}$ are defined by restricting the range of integration to an arbitrary interval of length 2π in $\mathbb{R}$. We will write the inner product on $L^2(\mathbb{T})$ as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f \bar{g} \, dx,$$

but the range of integration could be shifted if needed.

The Helmholtz equation on $\mathbb{T}$ is

$$-\frac{\partial^2 \phi}{\partial x^2} = \lambda \phi,$$

for $\phi \in C^2(\mathbb{T})$, with no need for additional boundary conditions because of the periodicity. The eigenfunctions are the complex exponentials

$$\phi_k(x) := e^{ikx}, \tag{8.11}$$

for $k \in \mathbb{Z}$, with $\lambda_k = k^2$.

It is possible to recover cosine and sine Fourier series from the periodic case, by restricting our attention to even or odd functions on $\mathbb{T}$. We will demonstrate this specialization in the examples and exercises.

The complex exponentials satisfy a simple orthogonality relation,

$$\begin{aligned} \langle \phi_k, \phi_l \rangle &= \int_{-\pi}^{\pi} e^{i(k-l)x}\, dx \\ &= \begin{cases} 2\pi, & k = l, \\ 0, & k \neq l. \end{cases} \end{aligned}$$

We did not include a normalizing factor in (8.11), so $\|\phi_k\|^2 = 2\pi$ and the Fourier coefficients of an integrable function $f \in L^1(\mathbb{T})$ are defined by

$$\begin{aligned} c_k[f] &:= \frac{1}{2\pi} \langle f, \phi_k \rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx. \end{aligned} \tag{8.12}$$

Because the index set is $\mathbb{Z}$ rather than $\mathbb{N}$, we define the partial sums of the periodic Fourier series by truncating on both sides,

$$S_n[f](x) := \sum_{k=-n}^{n} c_k[f] e^{ikx}. \tag{8.13}$$

For the sequence $\{\phi_k\}$, Bessel's inequality (Theorem 7.9) takes the form

$$\sum_{k \in \mathbb{Z}} \left| c_k[f] \right|^2 \leq \frac{1}{2\pi} \|f\|_2^2, \tag{8.14}$$

with equality if and only if $S_n[f] \to f$ in $L^2(\mathbb{T})$.

In the specific example considered in Example 8.1, the Fourier series appeared to converge very quickly. To illustrate potential complications with the convergence, let us consider a function with a jump discontinuity.

Example 8.2 On the interval $[0, \pi]$, define the function

$$h(x) = \begin{cases} 0, & x \in [0, \frac{\pi}{2}], \\ 1, & x \in (\frac{\pi}{2}, \pi], \end{cases}$$

as pictured on the left in Fig. 8.2. As noted above, in order to represent h as a cosine series using the periodic eigenfunctions, we first extend h to $\mathbb{T}$ as an even function, i.e.,

$$h(x) := \begin{cases} 0, & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] + 2\pi\mathbb{Z}, \\ 1, & x \in (\frac{\pi}{2}, \frac{3\pi}{2}] + 2\pi\mathbb{Z}. \end{cases}$$

By (8.12), with a shift to the more convenient interval $[0, 2\pi]$, the Fourier coefficients of h are

$$\begin{aligned} c_k[h] &= \frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-ikx}\, dx \\ &= \begin{cases} \frac{1}{2}, & k = 0, \\ \frac{(-1)^k}{\pi k} \sin\left(\frac{\pi k}{2}\right), & k \neq 0. \end{cases} \end{aligned}$$

Since $c_{-k}[h] = c_k[h]$, we can combine terms in the partial sums (8.13) to give

$$S_n[h](x) = \frac{1}{2} + 2 \sum_{k=1}^{n} \frac{(-1)^k}{\pi k} \sin\left(\frac{\pi k}{2}\right) \cos(kx).$$

Figure 8.2 shows a sample of these partial sums. In contrast to the case of Example 8.1, where 2 terms of the Fourier series were enough to give a very convincing approximation, we can see significant issues with convergence in the vicinity of the jump, even with 40 terms. ♢

The Fourier series computed in Example 8.2 makes for a good illustration of some different notions of convergence. Consider the sequence of differences $h - S_n[h]$, as

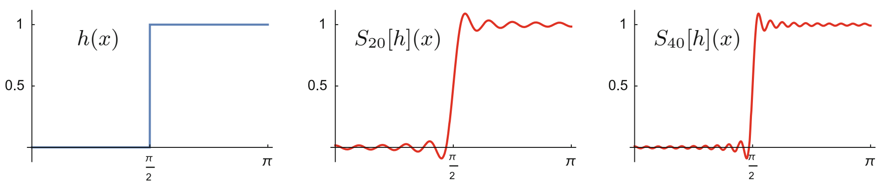

Fig. 8.2 Fourier series expansion for a step function

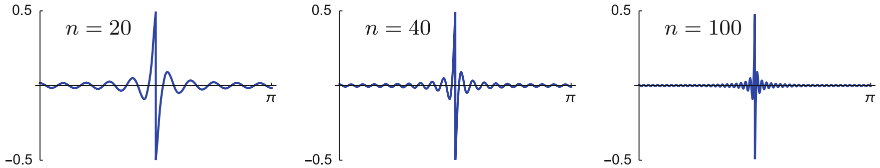

Fig. 8.3 Plots of the differences $h - S_n[h]$

illustrated in Fig. 8.3. If $\{\phi_k\}$ forms an $L^2(\mathbb{T})$ basis, then we would have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left|h - S_n[h]\right|^2 dx \overset{?}{=} 0.$$

It is not easy to judge such a limit visually, but this claim is true, as we will prove in Sect. 8.6.

We could instead focus our attention the values of $S_n[h](x)$ for some fixed x. A sequence of functions f_n is said to converge *pointwise* to f (assuming these functions have a common domain) if for each fixed x in the domain,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

In Fig. 8.3, if we focus our attention on some point x away from the center, then the bumps at this point do seem to be decreasing in size as n gets larger. We will verify in Sect. 8.3 that this Fourier series converges pointwise except at $x = \frac{\pi}{2}$.

Another feature that is quite apparent in Fig. 8.3 is the spike near the center. It is possible to prove that such a spike persists, with height essentially constant, for all values of n. The historical term for this effect, which is caused by the jump discontinuity, is the *Gibbs phenomenon*. It was actually first observed in 1848 by Henry Wilbraham, but remained generally unknown until it was rediscovered independently by J. Willard Gibbs in 1899.

The Gibbs phenomenon relates to yet a third definition of convergence. A sequence of bounded functions f_n is said to converge *uniformly* to a function f on a set W if

$$\lim_{n\to\infty} \sup_{x\in W} \left| f_n(x) - f(x) \right| = 0. \tag{8.15}$$

In the cases plotted in Fig. 8.3 we can see that

$$\sup_{x\in\mathbb{T}} \left| h(x) - S_n[h](x) \right| \approx \frac{1}{2}.$$

Since this does not decrease, uniform convergence fails for this series. However, the sequence does converge uniformly on domains that exclude a neighborhood of the jump point, for example on the interval $[0, \frac{\pi}{2} - \varepsilon]$ for $\varepsilon > 0$.

8.3 Pointwise Convergence

The basic theory of pointwise convergence of Fourier series was worked out by Dirichlet in the mid-19th century. In this section we will establish a criterion for pointwise convergence of periodic Fourier series.

Theorem 8.3 *Suppose $f \in L^2(\mathbb{T})$, and that for $x \in \mathbb{T}$ the estimate,*

$$\operatorname*{ess\text{-}sup}_{y \in [-\varepsilon, \varepsilon]} \left| \frac{f(x) - f(x-y)}{y} \right| < \infty, \tag{8.16}$$

holds for some $\varepsilon > 0$. Then

$$\lim_{n \to \infty} S_n[f](x) = f(x).$$

The essential supremum was defined in (7.10). The inequality (8.16) means that, after possibly replacing f by an equivalent function in the sense of (7.6), we can assume that

$$\sup_{0 < |y| \le \varepsilon} \left| \frac{f(x) - f(x-y)}{y} \right| < \infty. \tag{8.17}$$

This bound holds automatically for $f \in C^1(\mathbb{T})$, by the estimate

$$\begin{aligned} \left| \frac{f(x) - f(x-y)}{y} \right| &= \left| \frac{1}{y} \int_{x-y}^{x} f'(t)\, dt \right| \\ &\le \sup_{t \in \mathbb{T}} \left| f'(t) \right|. \end{aligned}$$

Thus Theorem 8.3 shows that the Fourier series a C^1 function converges pointwise on all of $\mathbb{T}$. The same argument can be extended to functions on $\mathbb{T}$ which are merely piecewise C^1.

It is possible to prove pointwise convergence with a weaker hypothesis than that of Theorem 8.3. However, there are counterexamples, discovered by Fejér and Lebesgue, that show that pointwise convergence of the Fourier series may fail for $f \in C^0(\mathbb{T})$.

Before getting into the proof of Theorem 8.3, let us consider the structure of the partial sums in more detail. Plugging the coefficient formula (8.14) into (8.13) gives

$$\begin{aligned} S_n[f](x) &= \sum_{k=-n}^{n} e^{ikx} \frac{1}{2\pi} \int_0^{2\pi} f(y) e^{-iky}\, dy \\ &= \int_0^{2\pi} f(y) \left(\frac{1}{2\pi} \sum_{k=-n}^{n} e^{ik(x-y)} \right) dy. \end{aligned} \tag{8.18}$$

The function that appears in parentheses is called the *Dirichlet kernel*,

$$D_n(t) := \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikt}. \tag{8.19}$$

With this definition the formula for the partial sum becomes

$$S_n[f](x) = \int_0^{2\pi} f(y) D_n(x-y)\, dy. \tag{8.20}$$

This could be written as a convolution,

$$S_n[f] = f * D_n.$$

Because the sum (8.19) is finite, it is clear that the Dirichlet kernel is a smooth function on $\mathbb{T}$. It is also easy to compute that

$$\int_0^{2\pi} D_n(t)\, dt = 1 \tag{8.21}$$

for $n \in \mathbb{N}$, since only the $k = 0$ term in (8.19) contributes to the integral. Applying the polynomial identity

$$1 + z + z^2 + \cdots + z^m = \frac{z^{m+1} - 1}{z - 1}.$$

to (8.19) with $z = e^{it}$ gives the explicit formula

$$D_n(t) = \frac{1}{2\pi} \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1}. \tag{8.22}$$

Factoring $e^{it/2}$ out of the numerator and denominator reduces this to

$$D_n(t) = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)},$$

which makes it clear that D_n is real-valued.

A plot of $D_n(y)$ for various values of n, as shown in Fig. 8.4, gives some intuition as to why we might expect (8.23) to converge to $f(x)$ as $n \to \infty$. The function $D_n(y)$ concentrates at $y = 0$, and oscillates with increasing frequency away from this point. These oscillations will cause cancellation as $n \to \infty$, except at $y = 0$.

Proof of Theorem 8.3 Because both f and D_n are periodic, a change of variables $y \to x - y$ allows us to rewrite the convolution in the opposite order:

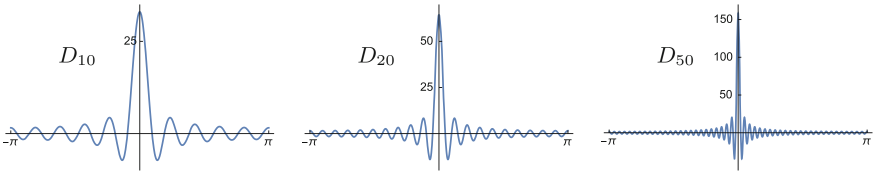

Fig. 8.4 The Dirichlet kernel for increasing values of n

$$S_n[f](x) = \int_{-\pi}^{\pi} D_n(y) f(x-y)\, dy. \tag{8.23}$$

Thus, by (8.21) and (8.23),

$$f(x) - S_n[f](x) = \int_{-\pi}^{\pi} [f(x) - f(x-y)] D_n(y)\, dy.$$

Substituting in with the explicit formula (8.22) for $D_n(t)$ gives

$$f(x) - S_n[f](x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x) - f(x-y)}{e^{iy} - 1} \left[e^{i(n+1)y} - e^{-iny} \right] dy. \tag{8.24}$$

The crucial observation here is that if we separate the terms inside the brackets, then this looks like a formula for Fourier coefficients.

Assuming that the hypothesis of the theorem is satisfied at $x \in \mathbb{T}$, consider the function

$$h(y) := \frac{f(x) - f(x-y)}{e^{iy} - 1}, \tag{8.25}$$

defined for $y \in \mathbb{T}$ with $y \neq 0$. We can split this into factors as

$$h(y) = \frac{f(x) - f(x-y)}{y} \frac{y}{e^{iy} - 1},$$

note that the first factor is essentially bounded near $y = 0$ by the assumption (8.16). Since $e^{iy} - 1 \sim iy$ as $y \to 0$ by Taylor's approximation, the second factor is also bounded near $y = 0$. The hypothesis (8.16) thus guarantees that $h(y)$ is equivalent to a function that is bounded on the interval $[-\varepsilon, \varepsilon]$. Since $f \in L^2(\mathbb{T})$ and $(e^{iy} - 1)^{-1}$ is bounded for $y \in \pm[\varepsilon, \pi]$, we conclude from this that $h \in L^2(\mathbb{T})$.

We can thus interpret (8.24) in terms of Fourier coefficients,

$$f(x) - S_n[f](x) = c_{-n-1}[h] - c_n[h]. \tag{8.26}$$

Bessel's inequality, which takes the form (8.14) here, implies that $c_k[h] \to 0$ as $k \to \pm\infty$. By (8.26) this establishes pointwise convergence at x. $\square$

8.4 Uniform Convergence

According to the definition (8.15) of uniform convergence, $f_n \to f$ uniformly on $\mathbb{T}$ if

$$\sup_{x\in\mathbb{T}} \left|f_n(x) - f(x)\right| \to 0$$

as $n \to \infty$. This is closely related to the convergence with respect the L^∞ norm, as introduced in Sect. 7.3. If a sequence converges in the L^∞ sense, then after possibly modifying the functions on a set of measure zero we can assume that the convergence is uniform.

Continuity is not necessarily preserved under pointwise limits. For example the sequence e^{-nx^2} converges pointwise on $\mathbb{R}$ but the limit function is discontinuous at $x = 0$. On the other hand, uniform convergence of continuous functions does guarantee continuity.

Lemma 8.4 *Suppose* $\{f_n\} \subset C^0(\Omega)$ *for a domain* $\Omega \subset \mathbb{R}^n$*. If* $\{f_n\}$ *converges uniformly to a function* $f : \Omega \to \mathbb{R}$*, then* f *is also continuous.*

Proof The goal is to show that $f(\boldsymbol{y})$ can be made close to $f(\boldsymbol{x})$ by taking $\boldsymbol{y}$ close to $\boldsymbol{x}$. To make use of the uniform convergence, we note that the triangle inequality implies

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le \left|f(\boldsymbol{x}) - f_n(\boldsymbol{x})\right| + \left|f_n(\boldsymbol{x}) - f_n(\boldsymbol{y})\right| + \left|f_n(\boldsymbol{y}) - f(\boldsymbol{y})\right|. \tag{8.27}$$

For n large we can control the first and third terms on the right by the assumption of uniform convergence. To control the middle term we can use the continuity of f_n.

Fix $\boldsymbol{x} \in \Omega$ and $\varepsilon > 0$. By uniform convergence there exists n so that

$$\sup_{\boldsymbol{y}\in\Omega} \left|f_n(\boldsymbol{y}) - f(\boldsymbol{y})\right| < \varepsilon. \tag{8.28}$$

The fact that f_n is continuous at $\boldsymbol{x}$ means that we can find $\delta > 0$ (depending on $\boldsymbol{x}$) such that for $\boldsymbol{y} \in \Omega$ satisfying $|\boldsymbol{x} - \boldsymbol{y}| < \delta$,

$$\left|f_n(\boldsymbol{x}) - f_n(\boldsymbol{y})\right| < \varepsilon. \tag{8.29}$$

Combining (8.28) and (8.29) with (8.27) shows that for $\boldsymbol{y} \in \Omega$ satisfying $|\boldsymbol{x} - \boldsymbol{y}| < \delta$,

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < 3\varepsilon.$$

Thus f is continuous at $\boldsymbol{x}$. □

Uniform convergence is particularly easy to check for periodic Fourier series, because the eigenfunctions ϕ_k satisfy

$$\left|e^{ikx}\right| = 1.$$

Theorem 8.5 *For $f \in C^1(\mathbb{T})$, the sequence of partial sums $S_n[f]$ convergences uniformly to f.*

Proof The assumption that $f \in C^1(\mathbb{T})$ implies $f' \in C^0(\mathbb{T})$. By integration by parts,

$$\begin{aligned} c_k[f'] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(y)e^{-iky}\,dy \\ &= \frac{1}{2\pi} f(y)e^{-iky}\Big|_{-\pi}^{\pi} + \frac{ik}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy. \end{aligned}$$

The boundary term cancels by periodicity, leaving

$$c_k[f'] = ikc_k[f]. \tag{8.30}$$

Since $f' \in L^2(\mathbb{T})$ also, applying Bessel's inequality in the form (8.14) to the coefficients (8.30) implies that

$$\sum_{k\in\mathbb{Z}} \left|kc_k[f]\right|^2 < \infty. \tag{8.31}$$

Let $\ell^2(\mathbb{Z}\backslash\{0\})$ denote the discrete L^2 space on the set consisting of functions $\mathbb{Z}\backslash\{0\} \to \mathbb{C}$. (The ℓ^p spaces were introduced in Sect. 7.4.) The sequence

$$a_k := |kc_k[f]|$$

defines an element of $\ell^2(\mathbb{Z}\backslash\{0\})$ by (8.31). If we define $b \in \ell^2(\mathbb{Z}\backslash\{0\})$ by $b_k := k^{-1}$, then the sum of the coefficients $c_k[f]$ with $k \neq 0$ can be expressed as an ℓ^2 pairing,

$$\sum_{k\neq 0} |c_k[f]| = \langle a, b\rangle_{\ell^2}.$$

By the Cauchy-Schwarz inequality on $\ell^2(\mathbb{Z}\backslash\{0\})$,

$$\langle a, b\rangle_{\ell^2} \le \|a\|_{\ell^2}\,\|b\|_{\ell^2} < \infty.$$

Since the norms of a and b are finite, we conclude that

$$\sum_{k\in\mathbb{Z}} |c_k[f]| < \infty. \tag{8.32}$$

Note that we already know that $S_n[f] \to f$ pointwise by Theorem 8.10. This implies that

$$S_n[f](x) - f(x) = \sum_{|k|>n} c_k[f]\phi_k(x)$$

for each $x \in \mathbb{T}$. Because $|\phi_k(x)| = 1$,

$$\left|S_n[f](x) - f(x)\right| \le \sum_{|k|>n} \left|c_k[f]\right|. \tag{8.33}$$

The right-hand side of (8.33) is independent of x and tends to zero as $n \to \infty$ by (8.32), proving that $S_n[f] \to f$ uniformly. □

8.5 Convergence in L^2

The uniform convergence provided by Theorem 8.5 proves to be very helpful in resolving the L^2 basis question. This is because uniform convergence on $\mathbb{T}$ implies L^2 convergence also, by the integral estimate

$$\begin{aligned} \|f_n - f\|_2^2 &= \int_{-\pi}^{\pi} |f_n - f|^2 \, dx \\ &\le 2\pi \sup_{x\in\mathbb{T}} \left|f_n(x) - f(x)\right|^2. \end{aligned}$$

Hence Theorem 8.5 gives convergence of Fourier series in L^2 for C^1 functions. In this section we will extend this result to all of L^2.

Theorem 8.6 *The normalized periodic Fourier eigenfunctions*

$$\frac{1}{\sqrt{2\pi}} e^{ikx}, \quad k \in \mathbb{Z},$$

form an orthonormal basis for $L^2(\mathbb{T})$.

Proof Suppose that $u \in L^2(\mathbb{T})$ satisfies

$$\langle u, \phi_k \rangle = 0 \tag{8.34}$$

for each $k \in \mathbb{Z}$. By Theorem 7.10 the conclusion will follow if we can deduce that this implies $u = 0$.

As noted above, for $\psi \in C^1(\mathbb{T})$ Theorem 8.5 implies that $S_n[\psi] \to \psi$ in $L^2(\mathbb{T})$. In particular, this gives

$$\lim_{n\to\infty} \langle u, S_n[\psi] \rangle = \langle u, \psi \rangle. \tag{8.35}$$

However, since $S_n[\psi]$ is a finite linear combination of the ϕ_k, the assumption (8.34) implies that

$$\langle u, S_n[\psi] \rangle = 0.$$

Hence from (8.35) we deduce that

$$\langle u, \psi \rangle = 0.$$

Now recall Theorem 7.5, which says that $C^\infty_{\text{cpt}}(0, 2\pi)$ forms a dense subset of $L^2(0, 2\pi)$. This implies also that $C^1(\mathbb{T})$ is dense in $L^2(\mathbb{T})$. Therefore we can choose a sequence $\{\psi_l\}$ in $C^1(\mathbb{T})$ such that $\psi_l \to u$ in $L^2(\mathbb{T})$. Thus

$$\|u\|_2^2 = \lim_{l \to \infty} \langle u, \psi_l \rangle ,$$

and we just showed that all terms on the right are zero under the assumption (8.34). Therefore $u = 0$. □

The combination of Theorems 8.6 and 7.9 immediately yields the following:

Corollary 8.7 (Parseval's identity) *For $f \in L^2(\mathbb{T})$, the periodic Fourier coefficients satisfy*

$$\sum_{k \in \mathbb{Z}} |c_k[f]|^2 = \frac{1}{2\pi} \|f\|_{L^2} .$$

Applying Parseval's identity to $f + g$, where $f, g \in L^2(\mathbb{T})$, and separating out the cross-term yields the corresponding result for the inner product,

$$\langle f, g \rangle = 2\pi \sum_{k \in \mathbb{Z}} c_k[f] \overline{c_k[g]}. \tag{8.36}$$

Example 8.8 In Example 8.2, we found for the step function h that $c_k[h] = \pm \frac{1}{\pi k}$ for k odd, $c_0[h] = \frac{1}{2}$, and otherwise $c_k[h] = 0$. So for this case,

$$\sum_{k \in \mathbb{Z}} |c_k[h]|^2 = \frac{1}{4} + 2 \sum_{k \in \mathbb{N}_{\text{odd}}} \frac{1}{\pi^2 k^2}.$$

On the other hand, $\|h\|_2^2 = \pi$, so Parseval's identity implies

$$\frac{1}{4} + \frac{2}{\pi^2} \sum_{k \in \mathbb{N}_{\text{odd}}} \frac{1}{k^2} = \frac{1}{2}.$$

Thus we obtain the summation formula

$$\sum_{k \in \mathbb{N}_{\text{odd}}} \frac{1}{k^2} = \frac{\pi^2}{8}. \tag{8.37}$$

◊

The space $L^2(0, 2\pi)$ can be identified with $L^2(\mathbb{T})$ by extending functions periodically. Hence Theorem 8.6 also implies that $\left\{\frac{1}{2\pi}e^{ikx}\right\}$ is an orthonormal basis for $L^2(0, 2\pi)$. We can also specialize the periodic results to show that cosine or sine series give orthonormal bases for $L^2(0, \ell)$ with basis functions that satisfy Dirichlet or Neumann boundary conditions, respectively. We will discuss these cases in the exercises.

8.6 Regularity and Fourier Coefficients

In the preceding sections we have made some progress in understanding the representation of a function by Fourier series. However, we still have not addressed one of the primary questions raised in Sect. 8.1: when does a Fourier series yield a classical solution to the original PDE? In this section we will resolve this issue by studying the relationship between the regularity of a function and the decay of its Fourier coefficients.

The starting point for this discussion is the computation used in the proof of Theorem 8.5,

$$c_k[f'] = ikc_k[f]$$

for $f \in C^1(\mathbb{T})$. Repeating this computation inductively gives the following:

Lemma 8.9 *Suppose that $f \in C^m(\mathbb{T})$. Then*

$$c_k[f^{(m)}] = (ik)^m c_k[f]. \tag{8.38}$$

To describe the decay rates of coefficients, we introduce some convenient *order notation*. For $\alpha \in \mathbb{R}$,

$$a_k = o(k^\alpha) \quad \text{means} \quad \lim_{|k|\to\infty} \frac{|a_k|}{|k|^\alpha} = 0.$$

This is commonly referred to as the “little-o” notion of order. There is a corresponding “big-O” definition,

$$a_k = O(k^\alpha) \quad \text{means} \quad |a_k| \le C\,|k|^\alpha\,,$$

for all sufficiently large $|k|$, with C independent of k. Note that the little-o condition is stronger. The content of the statement $a_k = o(k^\alpha)$ is that the ratio a_k/k^α tends to zero, while $a_k = O(k^\alpha)$ says only that the ratio is bounded.

Theorem 8.10 *For $f \in C^m(\mathbb{T})$ with $m \in \mathbb{N}_0$,*

$$\sum_{k\in\mathbb{Z}} k^{2m} \left|c_k[f]\right|^2 < \infty, \tag{8.39}$$

and

$$c_k[f] = o(k^{-m}).$$

Proof The inequality (8.39) follows immediately from a combination of Lemma 8.9 and and Bessel's inequality in the form (8.14). Since the terms in a convergent series must approach zero,

$$\lim_{|k|\to\infty} k^m c_k[f] = 0,$$

which gives the claimed decay estimate. □

Example 8.11 Consider the cosine series (8.6) computed in Example 8.1 for the function $h(x) = 3\pi x^2 - 2x^3$ on $(0, \pi)$. Although $h \in C^\infty(0, \pi)$, the extension of h to $\mathbb{T}$ as an even function is merely C^2. Theorem 8.10 thus implies that $\sum k^4 |c_k[h]|^2 < \infty$.

The periodic Fourier coefficients corresponding to (8.9) are

$$c_k[h] = \begin{cases} \frac{\pi^3}{2}, & k = 0, \\ -\frac{24}{\pi k^4}, & k \text{ odd}, \\ 0, & k \neq 0, \text{ even}. \end{cases}$$

This shows much faster decay than predicted, but not the rapid decay we would have seen if the even periodic extension had been smooth. ♢

Our next goal is to develop a converse to Theorem 8.10 that says that a certain level of decay rate of Fourier coefficients guarantees a corresponding level of differentiability for the function. In fact, the first stage of this result has already been worked out. Suppose $f \in L^2(\mathbb{T})$ and its coefficients satisfy

$$\sum_{k\in\mathbb{Z}} |c_k[f]| < \infty. \tag{8.40}$$

We know that $S_n[f] \to f$ in the L^2 sense by Theorem 8.6. By Theorem 8.5 we also know that $\{S_n[f]\}$ converges uniformly, and so the limit is continuous by Lemma 8.4. Hence we can conclude that (8.40) implies $f \in C^0(\mathbb{T})$. Recall from Sect. 7.3 that when we say an L^2 function is C^m we mean this only up to equivalence, i.e., the original function might require modification on a set of measure zero to make it C^m.

Theorem 8.12 *Suppose $f \in L^2(\mathbb{T})$ has Fourier coefficients satisfying*

$$\sum_{k\in\mathbb{Z}} \left|k^m c_k[f]\right| < \infty, \tag{8.41}$$

for $m \in \mathbb{N}_0$. Then $f \in C^m(\mathbb{T})$.

Proof As remarked above, the $m = 0$ case is already taken care of by Theorems 8.5 and 8.6.

Assume that (8.41) is satisfied for $m = 1$. For convenience, let $c_k := c_k[f]$ and $f_n = S_n[f]$. Since f_n is a (finite) linear combination of smooth functions, the derivatives are given by

$$f_n'(x) = \sum_{k=-n}^{n} ikc_k e^{ikx}.$$

By the $m = 0$ result, the sequence $\{f_n'\}$ converges uniformly to some $g \in C^0(\mathbb{T})$.

Our goal is to show that $g = f'$, which means

$$g(x) = \lim_{y \to 0} \frac{f(x+y) - f(x)}{t},$$

for every $x \in \mathbb{T}$. To argue this we will decompose the difference quotient as

$$\begin{aligned} \frac{f(x+t) - f(x)}{t} - g(x) = {} & \left[\frac{f_n(x+y) - f_n(x)}{y} - f_n'(x) \right] \\ & + \left(f_n'(x) - g(x) \right) + R_n(x, y). \end{aligned} \tag{8.42}$$

The first term on the right approaches zero by the definition of f_n', and the second term approaches zero as $n \to \infty$ by the construction of g. The remainder term is

$$R_n(x, y) := \sum_{|k|>n} c_k \frac{(e^{ik(x+y)} - e^{ikx})}{y},$$

which converges absolutely for each $y \neq 0$ by (8.41).

We can estimate the remainder by

$$|R_n(x, y)| \leq \sum_{|k|>n} |c_k| \left| \frac{e^{iky} - 1}{y} \right|.$$

By noting that $\left| e^{iky} - 1 \right| = 2\sin(ky/2)$, a simple calculus estimate gives

$$\left| \frac{e^{iky} - 1}{y} \right| \leq |k|$$

for all $y \neq 0$. This implies a uniform estimate on the remainder term,

$$|R_n(x, y)| \leq \sum_{|k|>n} |kc_k|. \tag{8.43}$$

In particular, by the assumption (8.41) the remainder term is arbitrarily small for n large.

Fix $x \in \mathbb{T}$ and $\varepsilon > 0$. By (8.43) and the fact that $f_n'(x) \to g(x)$, we can pick n so that

$$\left|f_n'(x) - g(x)\right| < \varepsilon \quad \text{and} \quad |R_n(x, y)| < \varepsilon.$$

for all $y \neq 0$. For this n and x, the definition of $f_n'(x)$ says that we can choose δ such that $0 < |y| < \delta$ implies

$$\left|\frac{f_n(x+y) - f_n(x)}{y} - f_n'(x)\right| < \varepsilon.$$

Applying these estimates to (8.42) shows that for $0 < |y| < \delta$,

$$\left|\frac{f(x+y) - f(x)}{y} - g(x)\right| \leq 3\varepsilon.$$

Since ε was arbitrarily small, this shows that $f'(x) = g(x)$. And since g is continuous, we conclude that $f \in C^1(\mathbb{T})$.

The same argument can now be repeated for higher derivatives, assuming (8.41) holds for larger m. □

The hypothesis (8.41) can be reformulated in terms of a decay condition on the coefficients, although this gives a slightly weaker result. If $c_k[f] = O(k^{-\alpha})$, then

$$\sum_{k \in \mathbb{Z}} \left|k^m c_k[f]\right| \leq C \sum_{k \in \mathbb{Z}} |k|^{-\alpha+m},$$

and this series converges provided $\alpha > m + 1$. Hence Theorem 8.12 implies that $f \in C^m(\mathbb{T})$ under the condition that

$$c_k[f] = O(k^{-m-1-\epsilon})$$

for some $\varepsilon > 0$.

Let us finally return to the one-dimensional heat equation that motivated this discussion, first considering the periodic case.

Theorem 8.13 *For $h \in C^0(\mathbb{T})$, the heat equation on $[0, \infty) \times \mathbb{T}$,*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

admits a solution $u \in C^\infty((0, \infty) \times \mathbb{T})$, defined for $t > 0$ by

$$u(t, x) := \sum_{k \in \mathbb{Z}} c_k[h] e^{-k^2 t} e^{ikx}, \tag{8.44}$$

and satisfying

$$\lim_{t\to 0} u(t,x) = h(x) \tag{8.45}$$

for each $x \in \mathbb{T}$.

Proof For $t > 0$, the Fourier coefficients of $u(t,\cdot)$ decay exponentially, and Theorem 8.12 shows that $u(t,\cdot) \in C^\infty(\mathbb{T})$ for each t. The same arguments used in the proof of that theorem apply to the t derivatives. To see this, let u_n denote the partial sum of (8.44),

$$u_n(t,x) := \sum_{k=-n}^{n} c_k e^{-k^2 t} e^{ikx}.$$

where $c_k := c_k[h]$. As a finite sum, this can be differentiated directly,

$$\frac{\partial u_n}{\partial t}(t,x) = \sum_{k=-n}^{n} (-k^2) c_k e^{-k^2 t} e^{ikx}. \tag{8.46}$$

By Theorem 8.10 the Fourier coefficients of h satisfy $c_k = o(k^{-1})$, so that

$$\left| -k^2 c_k e^{-k^2 t} e^{ikx} \right| \le Cke^{-k^2 t}. \tag{8.47}$$

As $n \to \infty$ the series (8.46) thus converges absolutely for $t > 0$, allowing us to define a function

$$g := \lim_{n\to\infty} \frac{\partial u_n}{\partial t}.$$

For $\varepsilon > 0$, the estimate (8.47) shows that the convergence is uniform for $t \ge \varepsilon$. Lemma 8.4 shows that the limit is continuous for $t \ge \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies $g \in C^0((0,\infty) \times \mathbb{T})$.

We can argue that $g = \partial u/\partial t$ by considering

$$\begin{aligned} \frac{u(t+s,x) - u(t,x)}{s} - h(t,x) = {} & \left[\frac{u_n(t+s,x) - u_n(t,x)}{s} - \frac{\partial u_n}{\partial t}(t,x) \right] \\ & + \left(\frac{\partial u_n}{\partial t}(t,x) - h(t,x) \right) + R_n(t,s,x), \end{aligned}$$

where

$$R_n(t,s,x) := \sum_{|k|>n} c_k e^{-k^2 t} e^{ikx} \left(\frac{e^{-k^2 s} - 1}{s} \right).$$

At this point the argument becomes essentially parallel to the analysis of (8.42), so we will omit the details. The conclusion is that $\partial u/\partial t$ is continuous on $(0,\infty) \times \mathbb{T}$.

The argument can be repeated for higher t derivatives, allowing us to conclude that $u \in C^\infty((0,\infty) \times \mathbb{T})$. Moreover, the partial derivatives of u_n converge to the corresponding derivatives of u, pointwise on $(0, \infty) \times \mathbb{T}$, and uniformly if we restrict to $t \geq \varepsilon$ for some $\varepsilon > 0$. Since u_n satisfies the wave equation for each n by construction, this shows that u satisfies the wave equation also.

At the moment we only have the tools to prove (8.45) under the stronger assumption that $h \in C^1(\mathbb{T})$. In this case we can argue exactly in the proof of Theorem 8.5 that (8.44) converges uniformly for $(t, x) \in [0, \infty) \times \mathbb{T}$. By Lemma 8.4 this shows $u \in C^0([0, \infty) \times \mathbb{T})$ and we can just set $t = 0$ to obtain (8.45).

If h is merely continuous, then this approach breaks down, because the series (8.44) may actually diverge for $t = 0$. We will cover the C^0 case in Chap. 13, after developing an alternate formula for (8.44). □

The one-dimensional heat equation derived in Sect. 6.1 involved Dirichlet or Neumann boundary conditions on an interval $[0, \ell]$. By rescaling the interval to $[0, \pi]$ and then extending functions to $\mathbb{T}$ with either even or odd symmetry, we apply the results for periodic Fourier series results to these cases. In particular, from Theorem 8.13 we deduce the following:

Corollary 8.14 *Suppose $h \in C[0, \ell]$ and satisfies Dirichlet or Neumann boundary conditions. The heat equation on $[0, \infty) \times [0, \ell]$ admits a solution $u \in C^\infty((0, \infty) \times [0, \ell])$, under the same boundary condition, such that*

$$\lim_{t \to 0} u(t, x) = h(x)$$

for each $x \in [0, \ell]$.

The solutions obtained in Theorem 8.13 and Corollary 8.14 are uniquely determined by the initial condition h. (See Exercise 6.4.) The fact that solutions are smooth for $t > 0$, even when h is merely continuous, is a characteristic property of diffusion equations. In fact the smoothing phenomenon carries over to cases where h is L^2 but not even continuous. This is illustrated in Fig. 8.5 for the case considered in Example 8.2.

As these applications show, the Fourier series approach (and spectral analysis in general) is well suited to analyzing the regularity of solutions. However, other qualitative features are perhaps obscured from this viewpoint. For example, we would expect solutions to reflect the physical principle that heat flows from hot to cold. We can see this behavior quite clearly in the plots of Fig. 8.5, but it is not at all apparent in the series formula (8.44).

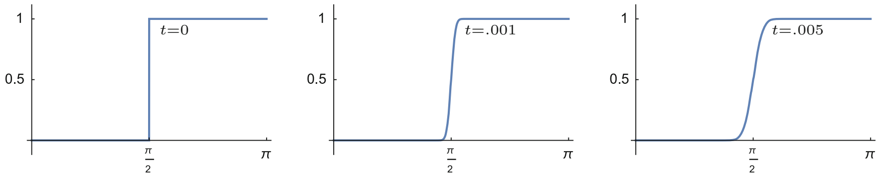

Fig. 8.5 Solutions of the heat equation become smooth for $t > 0$

8.7 Exercises

8.1 For $x \in (0, \pi)$, let
$$f(x) = x.$$

(a) Extend f to an odd function on $\mathbb{T}$ and compute the periodic Fourier coefficients $c_k[f]$ according to (8.12). (Note that the case $k = 0$ needs to be treated separately.) Show that the periodic series reduces to a sine series in this case.
(b) Show that the convergence of the Fourier series at $x = \frac{\pi}{2}$, which is guaranteed by Theorem 8.3, yields the summation formula
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$
(c) Show the Parseval identity (Corollary 8.7) leads to the formula
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

8.2 For $x \in (0, \pi)$, let
$$g(x) = x.$$

(a) Extend g to an even function on $\mathbb{T}$ and compute the periodic Fourier coefficients $c_k[g]$ according to (8.12). (Note that the case $k = 0$ needs to be treated separately.) Show that the periodic series reduces to a cosine series in this case.
(b) Show that the convergence of the Fourier series at $x = 0$, which is guaranteed by Theorem 8.3, reproduces the formula (8.37).
(c) Show the Parseval identity (Corollary 8.7) implies the formula
$$\sum_{k \in \mathbb{N}_{\text{odd}}} \frac{1}{k^4} = \frac{\pi^4}{96}.$$

8.3 Consider the periodic wave equation
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$
for $t \in \mathbb{R}$ and $x \in \mathbb{T}$. Suppose the initial conditions are
$$u(0, x) = g(x), \qquad \frac{\partial u}{\partial t}(0, x) = h(x),$$
for $g \in C^{m+1}(\mathbb{T})$ and $h \in C^m(\mathbb{T})$, for $m \in \mathbb{N}$.

(a) Assuming that $u(t, x)$ can be represented as a Fourier series

$$u(t, x) = \sum_{k \in \mathbb{Z}} a_k(t) e^{ikx}, \tag{8.48}$$

find an expression for $a_k(t)$ in terms of the Fourier coefficients of g and h.

(b) Using the assumptions on g and h, together with Theorem 8.10, show that the coefficients $a_k(t)$ satisfy an estimate

$$\sum_{k \in \mathbb{Z}} k^{2m} |a_k(t)|^2 \leq M < \infty,$$

uniformly for $t \in \mathbb{R}$.

(c) By the arguments used in Theorem 8.13, (b) implies that the series (8.48) converges to a solution u satisfying the initial conditions. What could you conclude about the differentiability of u?

8.4 In $L^2(0, \pi)$ consider the sequence

$$\psi_k(x) := \sqrt{\frac{2}{\pi}} \sin kx,$$

for $k \in \mathbb{N}$.

(a) Show that $\{\psi_k\}$ is an orthonormal sequence.

(b) Suppose that $f \in L^2(0, \pi)$ and $\langle f, \psi_k \rangle = 0$ for all $k \in \mathbb{N}$. Show that $f \equiv 0$. (Hint: extend f to an odd 2π-periodic function on $\mathbb{R}$, which can be regarded as an element of $L^2(\mathbb{T})$. Then apply Theorem 8.6.)

(c) Conclude that $\{\psi_k\}$ is an orthonormal basis for $L^2(0, \pi)$.

8.5 Suppose that $f \in L^2(-\pi, \pi)$ satisfies

$$\int_{-\pi}^{\pi} x^l f(x)\, dx = 0$$

for all $l \in \mathbb{N}_0$.

(a) Show that $\langle q_{m,k}, f \rangle = 0$ for all $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, where

$$q_{m,k}(x) := \sum_{l=0}^{m} \frac{(-ikx)^l}{l!}.$$

(b) Note that

$$\lim_{m \to \infty} q_{m,k}(x) = e^{-ikx}$$

by the definition of the complex exponential. Show that this convergence is uniform for $x \in [-\pi, \pi]$ (with k fixed).

(c) Use (a) and (b) to show that for $f \in L^2(-\pi, \pi)$,

$$\int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx = 0$$

for all $k \in \mathbb{Z}$.

(d) Conclude from Theorem 8.6 that $f \equiv 0$. (In other words, the monomials $1, x, x^2, \ldots$ form a basis for $L^2(-\pi, \pi)$, although not an orthonormal one.)

8.6 The *Legendre polynomials* are functions of $z \in [-1, 1]$ defined by

$$P_k(z) := \frac{1}{2^k k!} \frac{d^k}{dz^k} (z^2 - 1)^k,$$

for $k \in \mathbb{N}_0$. (This corresponds to the case $m = 0$ in (5.31).) These are solutions of the eigenvalue equation

$$L P_k = k(k+1) P_k,$$

where

$$L := \frac{d}{dz}\left[(z^2 - 1)\frac{d}{dz}\right].$$

(a) For $u, v \in C^2[-1, 1]$, check that L satisfies a formal self-adjointness condition,

$$\langle u, Lv \rangle_{L^2} = \langle Lu, v \rangle_{L^2}.$$

Conclude that the P_k's with distinct values of k are orthogonal in $L^2(-1, 1)$.

(b) Use the result of Exercise 8.5 to show that $\{P_k\}$ forms an orthogonal basis for $L^2(-1, 1)$. (The P_k are normalized by the condition $P_k(1) = 0$, rather than by unit L^2 norm.)