Chapter 1
Introduction

1.1 Partial Differential Equations

Continuous phenomena, such as wave propagation or fluid flow, are generally modeled with *partial differential equations* (PDE), which express relationships between rates of change with respect to multiple independent variables. In contrast, phenomena that can be described with a single independent variable, such as the motion of a rigid body in classical physics, are modeled by *ordinary differential equations* (ODE).

A general PDE for a function u has the form

$$F\left(\boldsymbol{x}, u(\boldsymbol{x}), \frac{\partial u}{\partial x_j}(\boldsymbol{x}), \dots, \frac{\partial^m u}{\partial x_{j_1} \dots \partial x_{j_m}}(\boldsymbol{x})\right) = 0. \tag{1.1}$$

The *order* of this equation is m, the order of the highest derivative appearing (which is assumed to be finite). A *classical solution* u admits continuous partial derivatives up to order m and satisfies (1.1) at all points $\boldsymbol{x}$ in its domain. In certain situations the differentiability requirements can be relaxed, allowing us to define *weak solutions* that do not solve the equation literally.

A somewhat subtle aspect of the definition (1.1) is the fact that the equation is required to be *local*. This means that functions and derivatives appearing in the equation are all evaluated at the same point.

Although classical physics provided the original impetus for the development of PDE theory, PDE models have since played a crucial role in many other fields, including engineering, chemistry, biology, ecology, medicine, and finance. Many industrial applications of mathematics are based on the numerical analysis of PDE.

Most PDE are not solvable in the explicit sense that a simple calculus problem can be solved. That is, we typically cannot obtain a exact formula for $u(\boldsymbol{x})$. Therefore much of the analysis of PDE is focused on drawing meaningful conclusions from an equation without actually writing down a solution.

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1.2 Example: d'Alembert's Wave Equation

One of the earliest and most influential PDE models was the *wave equation*, developed by Jean d'Alembert in 1746 to describe the motion of a vibrating string. With physical constants normalized to 1, the equation reads

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where $u(t, x)$ denotes the vertical displacement of the string at position x and time t. If the string has length ℓ and is attached at both ends, then we also require that $u(t, 0) = u(t, \ell) = 0$ for all t. We will discuss the formulation of this model in Sect. 4.1.

D'Alembert also found a general formula for the solution of (1.2), based on the observation that (1.2) is solved by any function of the form $f(x \pm t)$, assuming f is twice-differentiable. Given two such functions on $\mathbb{R}$, we can write a general solution

$$u(t, x) := f_1(x + t) + f_2(x - t). \tag{1.3}$$

A similar formula applies in the case of a string with fixed ends. If f is 2ℓ-periodic on $\mathbb{R}$, meaning $f(x + 2\ell) = f(x)$ for all x, then it is easy to check that

$$u(t, x) := \frac{1}{2} [f(x + t) - f(t - x)] \tag{1.4}$$

satisfies $u(t, 0) = u(t, \ell) = 0$ for any t.

One curious feature of this formula is that it appears to give a sensible solution even in cases where f is not differentiable. For example, to model a plucked string we might take the initial displacement to be a simple piecewise linear function in the form of a triangle from the fixed endpoints, as shown in Fig. 1.1.

If we extend this to an odd, 2ℓ-periodic function on $\mathbb{R}$, then the formula (1.4) yields the result illustrated in Fig. 1.2. The initial kink splits into two kinks which travel in opposite directions on the string and and appear to rebound from the fixed ends.

This is not a classical solution because u is not differentiable at the kinks. However, u does satisfy the requirements for a weak solution, as we will see in Chap. 10.

Although a physical string could not exhibit sharp corners without breaking, the piecewise linear solutions are nevertheless physically reasonable. Direct observations of plucked and bowed strings were first made in the late 19th century by Hermann von Helmholtz, who saw patterns of oscillation quite similar to what is shown in Fig. 1.2. The appearance of kinks propagating along the string is striking, although the corners are not exactly sharp.

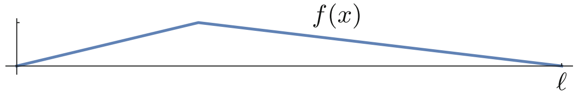

Fig. 1.1 Initial state of a plucked string

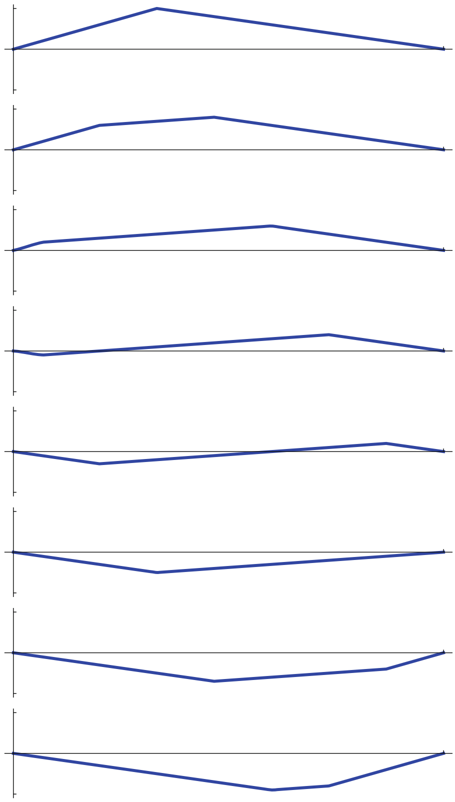

Fig. 1.2 Evolution of the plucked string, starting from $t = 0$ at the *top*

1.3 Types of Equations

There is no general theory of PDE that allows us to analyze all equations of the form (1.1). To make progress it is necessary to restrict our attention to certain classes of equations and develop methods appropriate to those.

The most fundamental distinction between PDE is the property of *linearity*. A PDE is called linear if it can be written in the form

$$Lu = f, \tag{1.5}$$

where f is some function independent of u, and L is a differential operator. Many of the important classical PDE that we will discuss in this book are linear and of first or second order. For such cases L has the general form

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c, \tag{1.6}$$

where the coefficients a_{ij}, b_j, and c are functions of $\boldsymbol{x}$. The second-order coefficients are assumed to be symmetric, $a_{ij} = a_{ji}$, because the mixed partials derivatives of a twice continuously differentiable function commute.

Linearity implies that a linear combination of solutions is still a solution, a fact that is referred to as the *superposition principle*. Superposition often lets us decompose problems into simpler components, which is the main reason that linear problems are much easier to handle than nonlinear. It also makes it possible to work with complex-valued solutions, which is sometimes more convenient, because the real and imaginary parts of a complex solution will solve the equation independently.

Most linear PDE are derived as approximations to more realistic, nonlinear models. We will focus primarily on the linear case in this book. The main reason for this is that nonlinear PDE are inherently more complicated, and for an introduction it makes sense to start with the more basic theory. Furthermore, the analysis of nonlinear problems frequently involves the study of associated linear approximations, so that one must understand at least some of the linear theory first.

Linear equations are further classified by the properties of the terms with the highest orders of derivatives, since this determines many qualitative properties of solutions. *Elliptic* equations of second order are associated to an operator L of the form (1.6), such that the eigenvalues of the symmetric matrix $[a_{ij}]$ are strictly positive at each point in the domain. The prototype of an elliptic operator is $L = -\Delta$ where Δ denotes the *Laplacian*,

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \tag{1.7}$$

named after the mathematician and physicist Pierre-Simon Laplace.

Equations that include time as an independent variable are called *evolution* equations. The time variable usually plays a very different role from the spatial variables, so in such cases we adapt the form (1.6) by separating out the time derivatives explicitly.

The two classic types of second-order evolution equations are *hyperbolic* and *parabolic*. Hyperbolic equations are exemplified by d'Alembert's wave equation (1.2). The general form is (1.5) with

$$L = \frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) \frac{\partial^2 u}{\partial x_i x_j^2} + \text{(lower order terms)}, \tag{1.8}$$

where once again $[a_{ij}]$ is assumed to be a strictly positive matrix. Hyperbolic equations are used to model oscillatory phenomena.

Parabolic evolution equations have the form (1.5) with

$$L = \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) \frac{\partial^2 u}{\partial x_i x_j^2} + \text{(lower order terms)}, \tag{1.9}$$

where $[a_{ij}]$ is a strictly positive matrix. The heat equation, whose derivation we will discuss in detail in Sect. 6.1, is the prototype for this type of equation. Parabolic equations are generally used to model phenomena of conduction and diffusion.

Note that hyperbolic and parabolic equations revert to elliptic equations in the spatial variables if the solution is independent of time. Elliptic equations thus serve to model the equilibrium states of evolution equations.

Because of their association with phenomenological properties of a system, the terms "elliptic", "hyperbolic", and "parabolic" are frequently applied more broadly than this simple classification would suggest. A nonlinear equation is typically described by the category of its linear approximations, which can change depending on the conditions.

For problems on a bounded domain, the application usually dictates some restriction on the solutions at the boundary. Two very common types are *Dirichlet boundary conditions*, specifying the values of u at the boundary, and *Neumann conditions*, specifying the normal derivatives of u at the boundary. These conditions are named for Gustave Lejeune Dirichlet and Carl Neumann, respectively. By default we will use these terms in the homogeneous sense, meaning that the boundary values of the function or derivative are set equal to zero. For evolution equations, we also impose *initial conditions*, specifying the values of u and possibly its time derivatives at some initial time.

1.4 Well Posed Problems

The set of functions used to formulate a PDE, which might include coefficients or terms in the equation itself as well as boundary and initial conditions, is collectively referred to as the input *data*. The most basic question for any PDE is whether a solution exists for a given set of data. However, for most purposes we want to require something more. A PDE problem is said to be *well posed* if, for a given set of data:

1. A solution exists.
2. The solution is uniquely determined by the data.
3. The solution depends continuously on the data.

These criteria were formulated by Jacques Hadamard in 1902. The first two properties hold for ODE under rather general assumptions, but not necessarily for PDE. It is easy to find nonlinear equations that admit no solutions, and even in the linear case there is no guarantee.

The third condition, continuous dependence on the input data, is sometimes called *stability*. One practical justification for this requirement is it is not possible to specify input data with absolute accuracy. Stability implies that the effects of small variations in the data can be controlled.

For certain PDE, especially the classical linear cases, we have a good understanding of the requirements for well-posedness. For other important problems, for

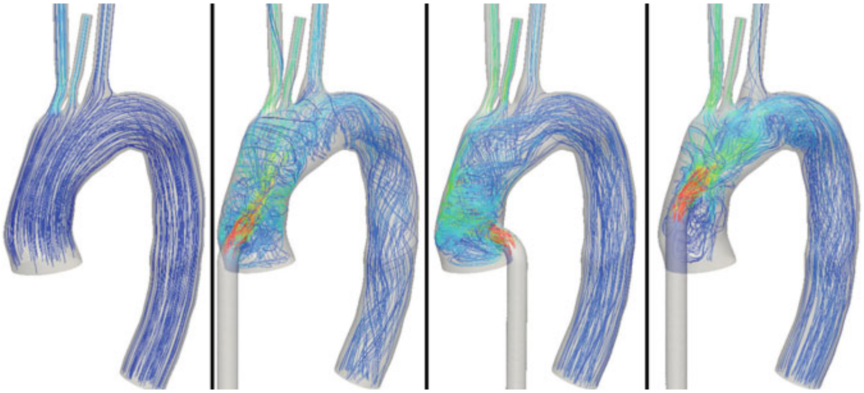

Fig. 1.3 Numerical simulations of blood flow in the aorta. Courtesy of D. Gupta, Emory University Hospital, and T. Passerini, M. Piccinelli and A. Veneziani, Emory Mathematics and Computer Science

example in fluid mechanics, well-posedness remains a difficult unsolved conjecture. Furthermore, many interesting problems are known not to be well posed. For example, problems in image processing are frequently ill posed, because information is lost due to noise or technological limitations.

1.5 Approaches

We can organize the methods for handling PDE problems according to three basic goals:

1. *Solving*: finding explicit formulas for solutions.
2. *Analysis*: understanding general properties of solutions.
3. *Approximation*: calculating solutions numerically.

Solving PDE is certainly worth understanding in those special cases where it is possible. The solution formulas available for certain classical PDE provide insight that is important to the development of the theory.

The goals of theoretical analysis of PDE are extremely broad. We wish to learn as much as we can about the qualitative and quantitative properties of solutions and their relationship to the input data.

Finally, numerical computation is the primary means by which applications of PDE are carried out. Computational methods rely on a foundation of theoretical analysis, but also bring up new considerations such as efficiency of calculation.

Example 1.1 Figure 1.3 shows a set of numerical simulations modeling the insertion in the aorta of a pipe-like device designed to improve blood flow. The leftmost frame shows the aorta before surgery, and the three panes on the right model the insertion at different locations. The PDE model is a complex set of fluid equations called the *Navier-Stokes* equations. These fluid equations are famously difficult to analyze and an exact solution is almost never possible. However, the cylinder is one case that can be handled explicitly. For the numerical simulations, exact solutions for a cylindrical pipe were used to provide boundary data at the point where the pipe meets the aorta.

Theoretical analysis also plays an important role here, in that the regularity theory for the fluid equations is used to predict the accuracy of the simulation. (The complete well-posedness analysis of the Navier-Stokes equations remains a famously unsolved problem, however.)

The simulated flows displayed in Fig. 1.3 were computed numerically by a technique called the *finite element method*. This involves discretizing the problem to reduce the PDE to a system of linear algebraic equations. Modeling a single heartbeat in this simulation require solving a linear system of about 500 million equations.