Chapter 9
Maximum Principles

We saw in Sect. 4.7 that conservation of energy can be used to derive uniqueness for solutions of the wave equation. In this chapter we will consider another approach to issues of uniqueness and stability, based on maximum values. This method applies generally to elliptic equations, which describe equilibrium states, and to parabolic equations, which are generally used to model diffusion.

9.1 Model Problem: The Laplace Equation

As noted in Sect. 5.2, the classical evolution equations such as the heat or wave equation have the form

$$P_t u - \Delta u = 0,$$

where P_t denotes some combination of time derivatives. In an equilibrium state, for which the solution is independent of time, these equations all reduce to

$$\Delta u = 0,$$

which is called the *Laplace equation*. A solution of the Laplace equation is also called a *harmonic function*. The Laplace equation on a bounded domain Ω is generally formulated with an inhomogeneous Dirichlet boundary condition,

$$u|_{\partial\Omega} = f$$

for $f : \partial\Omega \to \mathbb{R}$.

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The Laplace equation frequently appears in applications involving vector fields. A conservative vector field $\boldsymbol{v} \in C^0(\Omega; \mathbb{R}^n)$ can be represented as the gradient of a potential function $\phi \in C^1(\Omega; \mathbb{R})$,

$$\boldsymbol{v} = \nabla\phi,$$

If the vector field $\boldsymbol{v}$ is also solenoidal ($\nabla \cdot \boldsymbol{v} = 0$), then the potential satisfies the Laplace equation

$$\Delta\phi = 0.$$

In fluid dynamics in $\mathbb{R}^3$, for example, the velocity field is solenoidal for an incompressible fluid, such as water, and conservative precisely when the flow is *irrotational* ($\nabla \times \boldsymbol{v} = 0$).

Electrostatics provides another important source of Laplace problems. In the absence of charges, the electric field $\boldsymbol{E}$ is conservative and is commonly written as

$$\boldsymbol{E} = -\nabla\phi$$

where ϕ is the electric potential. On the other hand, Gauss's law of electrostatics says that $\nabla \cdot \boldsymbol{E}$ is proportional to the electric charge density. Hence, the electric potential for a charge-free region satisfies the Laplace equation.

In the remainder of this section we will consider a particular classical case, the Laplace problem on the unit disk. Circular symmetry allows us to solve the equation explicitly using Fourier series, and the resulting formula gives some insight into the general behavior of harmonic functions.

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{R}^2$. Given $g \in C^0(\partial\mathbb{D})$, our goal is to solve

$$\Delta u = 0, \quad u|_{\partial\mathbb{D}} = g. \tag{9.1}$$

In Sect. 5.3 we used separation of variables in polar coordinates to find the family of harmonic functions,

$$\phi_k(r, \theta) := r^{|k|}e^{ik\theta},$$

for $k \in \mathbb{Z}$. The boundary $\partial\mathbb{D}$ is naturally identified with the space $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ introduced in Sect. 8.2, parametrized by θ.

Consider the periodic Fourier series expansion,

$$g(\theta) = \sum_{k\in\mathbb{Z}} c_k[g]e^{ik\theta}, \tag{9.2}$$

where

$$c_k[g] := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}g(\theta)\,d\theta.$$

Given that

$$e^{ik\theta} = \phi_k(1, \theta),$$

we might hope to construct a solution of (9.1) by setting

$$u(r, \theta) = \sum_{k\in\mathbb{Z}} c_k[g]\phi_k(r, \theta). \tag{9.3}$$

Theorem 8.10 shows that the sequence $\{c_k[g]\}$ is bounded for g continuous. Note also that

$$|\phi_k(r, \theta)| = r^{|k|}$$

and

$$\sum_{k\in\mathbb{Z}} r^{|k|} < \infty$$

for $r < 1$ by geometric series. This implies that (9.3) converges absolutely for $r < 1$. In fact the convergence is uniform on $\{r \le R\}$ for $R < 1$.

We can write $u(r, \theta)$ more explicitly by substituting the definition of $c_k[g]$ into the integral,

$$u(r, \theta) = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi} \int_0^{2\pi} r^{|k|} e^{ik(\theta-\eta)} g(\eta)\, d\eta.$$

Uniform convergence in θ for $r < 1$ allows us to move the sum inside the integral, yielding the formula

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \eta) g(\eta)\, d\eta, \tag{9.4}$$

where

$$P_r(\theta) := \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik\theta}. \tag{9.5}$$

This function is called the *Poisson kernel*. Its behavior as $r \to 1$ is illustrated in Fig. 9.1.

Summing by geometric series gives the formula

$$\begin{aligned} P_r(\theta) &= 1 + \sum_{k=1}^{\infty} (re^{i\theta})^k + \sum_{k=1}^{\infty} (re^{-i\theta})^k \\ &= 1 + \frac{re^{i\theta}}{1 - re^{i\theta}} + \frac{re^{-i\theta}}{1 - re^{-i\theta}} \\ &= \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \end{aligned} \tag{9.6}$$

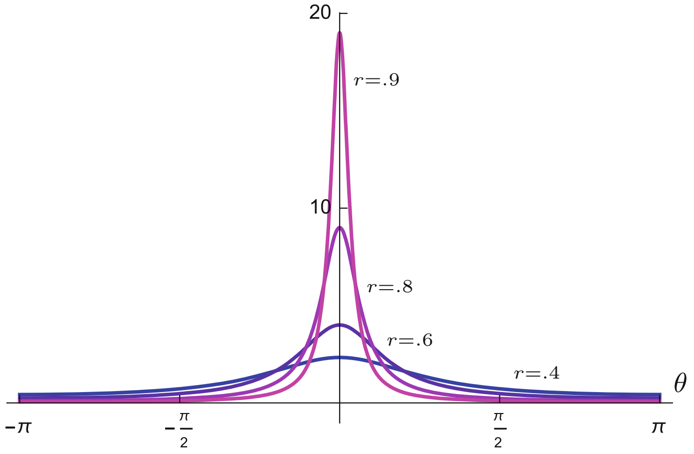

Fig. 9.1 The Poisson kernel $P_r(\theta)$ for a succession of radii

From the series formula (9.5) we can also deduce directly that

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)\,d\theta = 1, \tag{9.7}$$

since the only nonzero contribution comes from the term $k = 0$.

By periodicity, a change of variables $\eta \to \theta - \eta$ in (9.4) gives the alternate form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\eta) g(\theta - \eta)\,d\eta. \tag{9.8}$$

In view of (9.7), this could be interpreted as a weighted average of f with a weight function that depends on r. As $r \to 1^-$ this weight function becomes concentrated at 0, as Fig. 9.1 demonstrates. This is the mechanism by which we expect to have $u(r,\theta) \to g(\theta)$ as $r \to 1^-$.

Theorem 9.1 *For $f \in C^0(\partial\mathbb{D})$, the Laplace equation,*

$$\Delta u = 0 \text{ in } \mathbb{D}, \qquad u|_{\partial\mathbb{D}} = g,$$

admits a classical solution $u \in C^\infty(\mathbb{D}) \cap C^0(\bar{\mathbb{D}})$ given by the Poisson integral (9.4).

Proof The function $P_r(\theta)$ is smooth for $r < 1$, and it follows from (9.5) that

$$\Delta P_r(\theta) = 0,$$

where $P_r(\theta)$ is interpreted as a function on $\mathbb{D}$ written in polar coordinates. By passing derivatives inside the integral, we can deduce from (9.4) that $u \in C^\infty(\mathbb{D})$ and

$$\Delta u = 0.$$

To complete the proof we need to check that

$$\lim_{r\to 1^-} u(r,\theta) = g(\theta) \tag{9.9}$$

for every $\theta \in \partial\mathbb{D}$, which will also show that $u \in C^0(\bar{\mathbb{D}})$. Note that (9.9) is not the same as claiming that the Fourier series for g converges, which is not necessarily true. The difference lies in the order of the limits. In (9.9) we take the limit of the Fourier series first for $r < 1$, and then the limit $r \to 1^-$. This limit exists, as we will see below, but if we first set $r = 1$ in (9.3) then the sum over k may diverge.

By (9.7) and (9.8) we can write

$$u(r,\theta) - g(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\eta)\Big[g(\theta-\eta) - g(\theta)\Big]d\eta. \tag{9.10}$$

The goal is to estimate the left-hand side for r close to 1. Fix $\theta \in \mathbb{D}$ and let $\varepsilon > 0$. Since g is continuous, there exists $\delta > 0$ so that

$$|g(\theta-\eta) - g(\theta)| < \varepsilon \tag{9.11}$$

for $|\eta| < \delta$. For $|\eta| \geq \delta$ we can estimate

$$\max_{\delta\leq|\eta|\leq\pi} P_r(\eta) = P_r(\delta). \tag{9.12}$$

Thus, splitting the integral (9.10) at $|\eta| = \delta$ gives

$$\begin{aligned}
|u(r,\theta) - g(\theta)| &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\eta)\Big|g(\theta-\eta) - g(\theta)\Big|\,d\eta \\
&\quad + \frac{1}{2\pi}\int_{\delta<|\eta|<\pi} P_r(\eta)\Big|g(\theta-\eta) - g(\theta)\Big|\,d\eta \\
&\leq \frac{\varepsilon}{2\pi}\int_{-\delta}^{\delta} P_r(\eta)d\eta + \frac{P_r(\delta)}{2\pi}\int_{\delta<|\eta|<\pi}\Big|g(\theta-\eta) - g(\theta)\Big|\,d\eta.
\end{aligned}$$

By (9.7) and the fact that $P_r > 0$,

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\eta)\,d\eta \leq 1.$$

Furthermore, since g is continuous, $|g(\theta-\eta) - g(\theta)|$ is bounded by some constant M for all θ and η. This reduces the bound to

$$|u(r,\theta) - g(\theta)| \leq \varepsilon + M P_r(\delta).$$

We can now use the fact that

$$\lim_{r\to 1^-} P_r(\delta) = 0$$

to choose $R < 1$ so that

$$M P_r(\delta) \le \varepsilon$$

for $R < r < 1$. We conclude that

$$|u(r, \theta) - g(\theta)| \le 2\varepsilon$$

for $R < r < 1$. Since ε was arbitrary, this shows

$$\lim_{r\to 1^-} |u(r, \theta) - g(\theta)| = 0.$$

□

For students who know some complex analysis, we note that the formula (9.4) could be deduced from the Cauchy integral formula, because any harmonic function on $\mathbb{D}$ is the real part of a holomorphic function.

Example 9.2 For $0 < a < \pi$, suppose the boundary function is given by

$$g(\theta) = \begin{cases} 1 - \frac{|\theta|}{a}, & |\theta| \le a, \\ 0, & a < |\theta| \le \pi, \end{cases}$$

as shown in Fig. 9.2.

This boundary condition could represent, for example, a hot spot at one point on the edge of a metal plate. The corresponding equilibrium temperature distribution within the plate is given by calculating the Fourier coefficients of g and substituting into (9.3). The resulting solution,

$$u(r, \theta) = \frac{a}{2\pi} + \frac{2}{\pi a} \sum_{k=1}^{\infty} \frac{1 - \cos(ka)}{k^2} \cos(k\theta),$$

is illustrated in Fig. 9.3. ◊

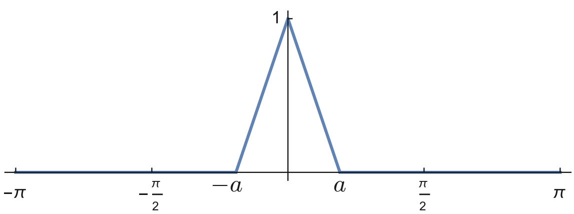

Fig. 9.2 Boundary function with a triangular peak

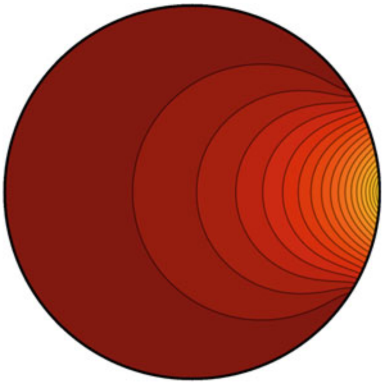

Fig. 9.3 Contour plot of the harmonic function from Example 9.2

9.2 Mean Value Formula

Setting $r = 0$ in the Poisson formula (9.4) gives

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\, d\theta, \tag{9.13}$$

because $P_0(\theta) = 1$. In other words, the value of a harmonic function at the center of the disk is equal to its average value on the boundary. This phenomenon is illustrated in Fig. 9.4. In this section we will extend (9.13) to an averaging formula that works in any dimension.

The $(n-1)$-dimensional volume of a sphere of radius r is

$$\mathrm{vol}[\partial B(\boldsymbol{x}_0; r)] = A_n r^{n-1}, \tag{9.14}$$

where A_n denotes the volume of the unit sphere in $\mathbb{R}^n$, as defined in (2.13). It follows from the radial integral formula (2.10) that

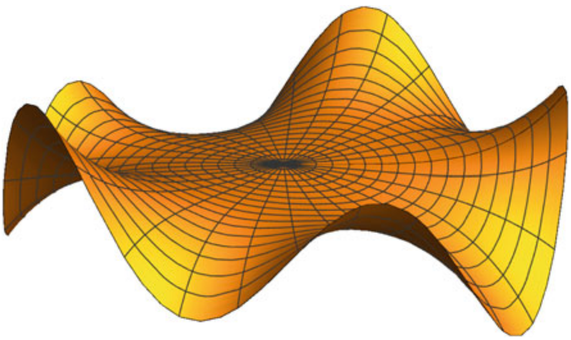

Fig. 9.4 Mean value property of a harmonic function

$$\mathrm{vol}[B(\boldsymbol{x}_0; r)] = \frac{A_n r^n}{n}. \tag{9.15}$$

To state the mean value formula for a ball of radius R, we introduce the family of radial functions,

$$G_R(\boldsymbol{x}) := \begin{cases} \frac{1}{2\pi} \ln(\frac{r}{R}), & n = 2, \\ \frac{1}{(n-2)A_n}\left[\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}\right], & n \geq 3. \end{cases} \tag{9.16}$$

The function G_R is the unique solution of the equations

$$\frac{\partial G_R}{\partial r} = \frac{1}{A_n r^{n-1}}, \quad G_R\big|_{r=R} = 0. \tag{9.17}$$

Note that G_R is integrable on $B(0; R)$, despite the singularity at the origin, because the radial volume element is $A_n r^{n-1}\, dr$ by (2.10).

Theorem 9.3 (Mean value formula) *Assume that $u \in C^2(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. For $R > 0$ such that $\overline{B(\boldsymbol{x}_0; R)} \subset \Omega$,*

$$u(\boldsymbol{x}_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(\boldsymbol{x}_0; R)} u(\boldsymbol{x}) dS + \int_{B(\boldsymbol{x}_0; R)} G_R(\boldsymbol{x} - \boldsymbol{x}_0) \Delta u(\boldsymbol{x}) d^n \boldsymbol{x}.$$

Proof By a change of variables, it suffices to consider the case $\boldsymbol{x}_0 = 0$. The formula (2.15) for the radial component of the Laplacian implies that

$$\Delta G_R(\boldsymbol{x}) = 0$$

for $\boldsymbol{x} \neq 0$. For $\varepsilon > 0$, we can therefore apply Green's second identity (Theorem 2.11) on the domain $\{\varepsilon < r < R\}$ to obtain

$$\begin{aligned} \int_{\{\varepsilon < r < R\}} G_R \Delta u \, d^n \boldsymbol{x} = & \int_{\{r=R\}} \left(G_R \frac{\partial u}{\partial r} - u \frac{\partial G_R}{\partial r} \right) dS \\ & - \int_{\{r=\varepsilon\}} \left(G_R \frac{\partial u}{\partial r} - u \frac{\partial G_R}{\partial r} \right) dS. \end{aligned} \tag{9.18}$$

Because G_R is integrable on $B(0; R)$, on the left-hand side of (9.18) we can take $\varepsilon \to 0$ to obtain

$$\lim_{\varepsilon \to 0} \int_{\{\varepsilon < r < R\}} G_R \Delta u \, d^n \boldsymbol{x} = \int_{B(0; R)} G_R \Delta u \, d^n \boldsymbol{x}. \tag{9.19}$$

By (9.17), the first term on the right in (9.18) reduces to

$$\int_{\{r=R\}} \left(G_R \frac{\partial u}{\partial r} - u \frac{\partial G_R}{\partial r} \right) dS = -\frac{1}{A_n R^{n-1}} \int_{\{r=R\}} u \, dS. \tag{9.20}$$

The second term on the right in (9.18) is

$$\begin{aligned}\int_{\{r=\varepsilon\}} &\left(G_R \frac{\partial u}{\partial r} - u \frac{\partial G_R}{\partial r}\right) dS \\ &= G_R(\varepsilon) \int_{\{r=\varepsilon\}} \frac{\partial u}{\partial r}\, dS + \frac{1}{A_n \varepsilon^{n-1}} \int_{\{r=\varepsilon\}} u\, dS.\end{aligned}$$

The first of these integrals can be estimated by noting that $\partial u/\partial r$ is a directional derivative and thus bounded by the magnitude of $|\nabla u|$. By the assumption that $u \in C^2(\Omega)$, $|\partial u/\partial r|$ is therefore bounded by a constant C for $r \le R$, yielding the estimate

$$\left|\int_{\{r=\varepsilon\}} \frac{\partial u}{\partial r}\, dS\right| \le C A_n \varepsilon^{n-1}.$$

Since the divergent term in $G_R(\varepsilon)$ as $\varepsilon \to 0$ is proportional to ε^{2-n} for $n \ge 3$ and $\log \varepsilon$ for $n = 2$, this implies

$$\lim_{\varepsilon \to 0} \left[G_R(\varepsilon) \int_{\{r=\varepsilon\}} \frac{\partial u}{\partial r}\, dS\right] = 0.$$

Hence

$$\lim_{\varepsilon \to 0} \int_{\{r=\varepsilon\}} \left(G_R \frac{\partial u}{\partial r} - u \frac{\partial G_R}{\partial r}\right) dS = \lim_{\varepsilon \to 0} \left[\frac{1}{A_n \varepsilon^{n-1}} \int_{\{r=\varepsilon\}} u\, dS\right].$$

The term in brackets is the average of u over a sphere of radius ε. Since u is continuous, this average approaches $u(0)$ as $\varepsilon \to 0$, so that

$$\lim_{\varepsilon \to 0} \int_{\{r=\varepsilon\}} \left(G_R \frac{\partial u}{\partial r} - u \frac{\partial G_R}{\partial r}\right) dS = u(0). \tag{9.21}$$

Applying (9.19), (9.20), and (9.21) to (9.18) gives

$$\int_{B(0;R)} G_R \Delta u\, d^n\boldsymbol{x} = u(0) - \frac{1}{A_n R^{n-1}} \int_{\partial B(0;R)} u\, dS,$$

which completes the proof. □

For harmonic functions, Theorem 9.3 gives a generalization of the circle formula (Theorem 9.3) to spherical averages in higher dimensions. As we will now show, the mean value property can be stated in a equivalent form in terms of averages over a ball.

Corollary 9.4 (Mean value for harmonic functions) *Suppose $\Omega \subset \mathbb{R}^n$ for $n \ge 2$. For $u \in C^2(\Omega)$ the following properties are equivalent:*

(A) The function u is harmonic on Ω.
(B) For $\overline{B(\boldsymbol{x}_0; R)} \subset \Omega$,

$$u(\boldsymbol{x}_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(\boldsymbol{x}_0; R)} u \, dS.$$

(C) For $\overline{B(\boldsymbol{x}_0; R)} \subset \Omega$,

$$u(\boldsymbol{x}_0) = \frac{n}{A_n R^n} \int_{B(\boldsymbol{x}_0; R)} u \, d^n\boldsymbol{x}.$$

Proof The fact that (A) implies (B) follows immediately by setting $\Delta u = 0$ in the formula of Theorem 9.3.

To see that (B) and (C) are equivalent, fix some $\boldsymbol{x}_0 \in \Omega$ and define

$$h(r) := \int_{B(\boldsymbol{x}_0; r)} u \, d^n\boldsymbol{x},$$

for $r \geq 0$ such that $\overline{B(\boldsymbol{x}_0; r)} \subset \Omega$. As we saw in Exercise 2.4, the derivative of $h(r)$ is given by a surface integral

$$h'(r) = \int_{\partial B(\boldsymbol{x}_0; r)} u \, dS.$$

Hence property (B) says that

$$h'(r) = A_n r^{n-1} u(\boldsymbol{x}_0),$$

while property (C) says that

$$h(r) = \frac{A_n r^n}{n} u(\boldsymbol{x}_0).$$

Since $h(0) = 0$ by definition, these are two statements are equivalent.

Finally, we need to show that (B) implies (A). Assuming that (B) holds, Theorem 9.3 gives

$$\int_{B(\boldsymbol{x}_0; R)} G_R(\boldsymbol{x} - \boldsymbol{x}_0) \Delta u(\boldsymbol{x}) d^n\boldsymbol{x} = 0 \tag{9.22}$$

provided $\overline{B(\boldsymbol{x}_0; R)} \subset \Omega$. Suppose $\Delta u(\boldsymbol{x}_0) < 0$ for some $\boldsymbol{x}_0 \in \Omega$. Then by continuity there exists some $\varepsilon > 0$ and $\delta > 0$ such that $\Delta u \leq -\varepsilon$ on $B(\boldsymbol{x}_0; \delta)$. Since G_R is strictly negative and decreasing as $r \to 0$, this implies

$$\int_{B(\boldsymbol{x}_0; \delta)} G_R(\boldsymbol{x} - \boldsymbol{x}_0) \Delta u(\boldsymbol{x}) d^n\boldsymbol{x} > -\varepsilon G_R\big|_{r=\delta} > 0,$$

which contradicts (9.22). The same argument applies if $\Delta u(\boldsymbol{x}_0) > 0$. We thus conclude that (B) implies $\Delta u \equiv 0$. □

9.3 Strong Principle for Subharmonic Functions

A real-valued C^2 function that satisfies

$$-\Delta u \leq 0$$

is called *subharmonic*. The case $-\Delta u \geq 0$ is similarly called *superharmonic*. We will focus on the subharmonic case. The results can easily be translated to the superharmonic case by replacing u by $-u$.

The "weak" maximum principle says that for a subharmonic function the maximum value occurs at a boundary point. We will prove here a "strong" version of this principle, which says furthermore that if that the global maximum occurs at an interior point then the function is constant.

Theorem 9.5 (Strong maximum principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega})$ is subharmonic then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

The maximum is attained at an interior point only if u is a constant function.

Proof By the extreme value theorem (Theorem A.2), u achieves a global maximum at some point $\boldsymbol{x}_0 \in \overline{\Omega}$. If $\boldsymbol{x}_0 \in \partial\Omega$ then the claimed equality clearly holds. The goal is thus to show that $\boldsymbol{x}_0 \in \Omega$ implies that u is constant.

Because Ω is open, an interior point $\boldsymbol{x}_0$ has a neighborhood contained in Ω. We may thus assume that $\overline{B(\boldsymbol{x}_0; R)} \subset \Omega$ for some $R > 0$. Applying Theorem 9.3 to this ball gives

$$u(\boldsymbol{x}_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(\boldsymbol{x}_0; R)} u(\boldsymbol{x}) dS + \int_{B(\boldsymbol{x}_0; R)} G_R(\boldsymbol{x} - \boldsymbol{x}_0) \Delta u(\boldsymbol{x}) d^n\boldsymbol{x}.$$

By the definition (9.16), $G_R \leq 0$ for $0 < r \leq R$. Therefore, since $\Delta u \geq 0$ by assumption,

$$u(\boldsymbol{x}_0) \leq \frac{1}{A_n R^{n-1}} \int_{\partial B(\boldsymbol{x}_0; R)} u(\boldsymbol{x}) dS. \tag{9.23}$$

Using (9.14), we can subtract $u(\boldsymbol{x}_0)$ from both sides to obtain

$$\frac{1}{A_n R^{n-1}} \int_{\partial B(\boldsymbol{x}_0; R)} [u(\boldsymbol{x}) - u(\boldsymbol{x}_0)]\, dS \geq 0. \tag{9.24}$$

By assumption $u(\boldsymbol{x}_0)$ is the global maximum of u, implying that the integrand of (9.24) is nonpositive. The inequality therefore shows that the integrand vanishes, and we conclude that $u(\boldsymbol{x}) = u(\boldsymbol{x}_0)$ on $\partial B(\boldsymbol{x}_0; R)$.

Note that the same argument works for every radius $r < R$, so this argument shows that $u \equiv u(\boldsymbol{x}_0)$ on all of $B(\boldsymbol{x}_0; R)$.

To extend the conclusion to the full domain, let M denote the maximum value of u on $\overline{\Omega}$. We can write Ω as a disjoint union $E \cup F$, where

$$E := \{\boldsymbol{x} \in \Omega;\ u(\boldsymbol{x}) < M\},$$
$$F := \{\boldsymbol{x} \in \Omega;\ u(\boldsymbol{x}) = M\}.$$

By the argument given above, a point $\boldsymbol{x} \in F$ has a neighborhood $B(\boldsymbol{x}; R) \subset \Omega$ on which u is equal to M. Hence F is open.

On the other hand, for $\boldsymbol{x} \in E$ we can set $\varepsilon = M - u(\boldsymbol{x})$ and use the continuity of u to find a $\delta > 0$ such that

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| < \varepsilon$$

for $\boldsymbol{y} \in B(\boldsymbol{x}; \delta)$. This implies in particular that $u(\boldsymbol{y}) < M$, so that $B(\boldsymbol{x}; \delta) \in E$. Thus E is open also.

Recall from Sect. 2.3 that the fact that Ω is connected means that the domain cannot be written as a disjoint union of nonempty open sets. Since $\Omega = E \cup F$ with E and F both open, one of the two sets is empty. If E is empty then u is constant on Ω, while if F is empty then the maximum of u is not attained in the interior. □

For a superharmonic function $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega})$, reversing the sign yields a minimum principle,

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Both principles apply to a harmonic function u, which therefore satisfies

$$\min_{\partial\Omega} u \le u(\boldsymbol{x}) \le \max_{\partial\Omega} u$$

for all $\boldsymbol{x} \in \Omega$.

The maximum principle implies the following stability result for the Laplace equation.

Corollary 9.6 *Suppose that* $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *are solutions of the Laplace equation* $\Delta u = 0$ *with boundary values*

$$u_1|_{\partial\Omega} = g_1, \qquad u_2|_{\partial\Omega} = g_2,$$

for $g_1, g_2 \in C^0(\partial\Omega)$*. Then*

$$\max_{\overline{\Omega}} |u_2 - u_1| \le \max_{\partial\Omega} |g_2 - g_1|. \tag{9.25}$$

In particular, a solution the Laplace equation is uniquely determined by its boundary data.

Proof By superposition, $u_2 - u_1$ is a harmonic function with boundary data $g_2 - g_1$. Theorem 9.5 applies to $\pm\operatorname{Re}(u_2 - u_1)$ as well as $\pm\operatorname{Im}(u_2 - u_1)$. Combining these estimates yields the inequality (9.25). □

Note that uniqueness of solutions of the Laplace equation also follows directly from Green's first identity (Theorem 2.10), in the case where Ω has piecewise C^1 boundary and $u \in C^2(\Omega; \mathbb{R})$. If $\Delta u = 0$, then setting $v = u$ in Green's formula gives

$$\int_\Omega \|\nabla u\|^2 \, d^n\boldsymbol{x} = \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \, dS.$$

Thus if $u = 0$ on $\partial\Omega$, then

$$\int_\Omega \|\nabla u\|^2 \, d^n\boldsymbol{x} = 0.$$

Since the integrand is positive, this implies $\nabla u \equiv 0$, so that u is constant. The assumption $u|_{\partial\Omega} = 0$ then gives $u \equiv 0$ on the full domain. (This is the "energy method" argument, as introduced in Sect. 4.7.)

One advantage that maximum principle has over the energy method is the explicit stability formula (9.25). In terms of the L^∞ norm introduced in Sect. 7.3, this inequality could be written

$$\|u_2 - u_1\|_\infty \le \|g_2 - g_1\|_\infty .$$

This is an explicit formulation of the continuity requirement for well-posedness: a small change in boundary data results in a correspondingly small change in the solution.

9.4 Weak Principle for Elliptic Equations

Although the mean value formula gives a direct proof of the strong maximum principle, this approach applies only to the Laplacian itself. In this section we will present an alternative approach that generalizes quite easily to operators with variable coefficients.

On a domain $\Omega \subset \mathbb{R}^n$ let us consider a second order elliptic operator of the form

$$L = -\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(\boldsymbol{x}) \frac{\partial}{\partial x_j}, \tag{9.26}$$

where the coefficients a_{ij} and b_j are continuous functions on Ω. As defined in Sect. 1.3, ellipticity means that the symmetric matrix $[a_{ij}]$ is positive definite at each point.

For the maximum principle we need a stronger assumption, called *uniform ellipticity*, that says that for some fixed constant $\kappa > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) v_i v_j \geq \kappa \|\boldsymbol{v}\|^2 \tag{9.27}$$

for all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{v} \in \mathbb{R}^n$. An equivalent way to say this is that the smallest eigenvalue of $[a_{ij}]$ is bounded below by κ at each point $\boldsymbol{x}$.

Theorem 9.7 (Weak maximum principle) *Suppose $\Omega \subset \mathbb{R}^n$ is bounded, and L is an operator of the form* (9.26) *satisfying the uniform ellipticity condition* (9.27)*. If $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega})$ satisfies*

$$Lu \leq 0$$

in Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof For the moment let u be a general function in $C^2(\Omega; \mathbb{R})$. Suppose that u has a local maximum at $\boldsymbol{x}_0 \in \Omega$. The first partial derivatives of u vanish at a local maximum, so that

$$Lu(\boldsymbol{x}_0) = -\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}_0). \tag{9.28}$$

Furthermore, we claim that the matrix of second partials of u is negative definite at $\boldsymbol{x}_0$, meaning

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}_0) v_i v_j \leq 0$$

for $\boldsymbol{v} \in \mathbb{R}^n$. To see this, set $h(t) := u(\boldsymbol{x}_0 + t\boldsymbol{v})$ and note that h has a local maximum at $t = 0$, implying $h''(0) \leq 0$. Evaluating $h''(0)$ yields the inequality stated above.

The right-hand side of (9.28) could be written as $\operatorname{tr}(AB)$ where A and B are the positive symmetric matrices

$$A = \left[a_{ij}(\boldsymbol{x}_0)\right], \quad B = \left[-\frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}_0)\right].$$

By switching to a basis in which A is diagonal, $\operatorname{tr}(AB)$ can be written in terms of the eigenvalues $\{\lambda_j\}$ of A as

$$\operatorname{tr}(AB) = \sum_{j=1}^{n} \lambda_j b_{jj}.$$

If the eigenvalues are ordered $\lambda_1 \le \cdots \le \lambda_n$, then

$$\operatorname{tr}(AB) \ge \lambda_1 \operatorname{tr} B.$$

The positivity of B implies $\operatorname{tr} B \ge 0$, so we conclude that

$$Lu(\boldsymbol{x}_0) \ge 0.$$

Thus argument shows that $Lu(\boldsymbol{x}_0) \ge 0$ for $\boldsymbol{x}_0$ a local interior maximum. Therefore, the strict inequality $Lu < 0$ implies that u cannot have a local interior maximum and that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u. \tag{9.29}$$

To complete the proof, we must relax the hypothesis to $Lu \le 0$. The strategy is to perturb u slightly to reduce to the previous case. For $M > 0$, let

$$h(\boldsymbol{x}) := e^{Mx_1}.$$

By the definition of L,

$$Lh = \left[-a_{11}M^2 + b_1 M\right] h.$$

The ellipticity condition (9.27) implies that $a_{11} \ge \kappa$, so by choosing

$$M > \frac{1}{\kappa} \max_{\overline{\Omega}} b_1,$$

we can guarantee that

$$Lh < 0.$$

If we now assume now that u satisfies the hypothesis $Lu \le 0$, then

$$L(u + \varepsilon h) < 0$$

for $\varepsilon > 0$. Applying (9.29) to $u + \varepsilon h$ gives

$$\max_{\overline{\Omega}}(u + \varepsilon h) = \max_{\partial\Omega}(u + \varepsilon h). \tag{9.30}$$

Since $h \ge 0$, clearly

$$\max_{\overline{\Omega}} u \le \max_{\overline{\Omega}}(u + \varepsilon h).$$

On the other hand, since Ω is bounded, we may assume $x_1 < R$ in $\overline{\Omega}$, for some R sufficiently large. This implies

$$h \le e^{MR},$$

so that

$$\max_{\partial\Omega}(u + \varepsilon h) \le \max_{\partial\Omega} u + \varepsilon e^{MR}.$$

From (9.30) we therefore conclude that

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + \varepsilon e^{MR}$$

for all $\varepsilon > 0$. Since M and R are independent of ε, we can take $\varepsilon \to 0$ to conclude that

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u,$$

and the result follows. □

The maximum principle implies in particular the only solution of $Lu = 0$ with $u|_{\partial\Omega} = 0$ is $u \equiv 0$. Hence a solution of the equation

$$Lu = f, \quad u|_{\partial\Omega} = g,$$

is uniquely determined by f and g if it exists.

9.5 Application to the Heat Equation

Fourier's law of heat conduction, as introduced in Sect. 6.1, suggests a maximum principle for solutions of the heat equation. Because heat flows away from a spatial maximum of the temperature, a local spatial maximum of the temperature should be impossible at time $t > 0$. The global maximum of the temperature therefore must occur either at $t = 0$ or on the boundary.

Although it is possible to prove the maximum principle via a mean value formula as in the proof of Theorem 9.5, in this section we will follow the more direct approach from Sect. 9.4, which has the advantage of generalizing to operators with variable coefficients.

Because the heat equation is second order with respect to spatial variables and first order in the time variable, it makes sense to define a domain for classical solutions that takes this structure into account. For $\Omega \subset \mathbb{R}^n$, define

$$C^{\text{heat}}(\Omega) := \left\{u \in C^0\left([0,\infty) \times \overline{\Omega}; \mathbb{R}\right);\ u(\cdot, \boldsymbol{x}) \in C^1(0,\infty), u(t,\cdot) \in C^2(\Omega)\right\}.$$

Note that this definition includes only real-valued functions.

Theorem 9.8 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u \in C^{\text{heat}}(\Omega)$ satisfies*

$$\frac{\partial u}{\partial t} - \Delta u \le 0, \tag{9.31}$$

on $(0, T) \times \Omega$. *Then the maximum value of* u *within* $[0, T] \times \overline{\Omega}$ *occurs at a point* $(t_0, \boldsymbol{x}_0)$ *with either* $t_0 = 0$ *or* $\boldsymbol{x}_0 \in \partial\Omega$.

Proof Suppose that u attains a maximum at $(t_0, \boldsymbol{x}_0) \subset (0, T) \times \Omega$. By the same calculus argument used in Sect. 9.4, this implies

$$\frac{\partial u}{\partial t}(t_0, \boldsymbol{x}_0) = 0, \tag{9.32}$$

as well as

$$\frac{\partial u}{\partial x_j}(t_0, \boldsymbol{x}_0) = 0, \quad \frac{\partial^2 u}{\partial x_j^2}(t_0, \boldsymbol{x}_0) \leq 0. \tag{9.33}$$

In particular,

$$\left[\frac{\partial u}{\partial t} - \Delta u\right](t_0, \boldsymbol{x}_0) \geq 0. \tag{9.34}$$

If (9.31) were a strict inequality this would complete the proof.

To proceed we use a perturbation strategy as in the proof of Theorem 9.7. For $\varepsilon > 0$, set

$$u_\varepsilon := u + \varepsilon \left|\boldsymbol{x}\right|^2 .$$

Because $\Delta \left|\boldsymbol{x}\right|^2 = 2n$, the hypothesis on u gives

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \frac{\partial u}{\partial t} - \Delta u - 2n\varepsilon < 0. \tag{9.35}$$

The existence of a local maximum for u_ε within $(0, T) \times \Omega$ is ruled out by (9.34). We conclude that u_ε attains a global maximum at a boundary point of $[0, T] \times \overline{\Omega}$.

Let us label this point $(t_\varepsilon, \boldsymbol{x}_\varepsilon)$, so that

$$\max_{[0,T]\times\overline{\Omega}} u_\varepsilon = u_\varepsilon(t_\varepsilon, \boldsymbol{x}_\varepsilon). \tag{9.36}$$

Since $(t_\varepsilon, \boldsymbol{x}_\varepsilon)$ is on the boundary, either $t_\varepsilon = 0$, $t_\varepsilon = T$, or $\boldsymbol{x}_\varepsilon \in \partial\Omega$.

Suppose that $t_\varepsilon = T$ and $\boldsymbol{x}_\varepsilon \in \Omega$. Then $u_\varepsilon(t, x_\varepsilon) \leq u_\varepsilon(T, x_\varepsilon)$ for $t \in [0, T]$, implying that

$$\frac{\partial u_\varepsilon}{\partial t}(T, \boldsymbol{x}_\varepsilon) \geq 0.$$

By (9.35), this implies also that $\Delta u_\varepsilon(T, \boldsymbol{x}_\varepsilon) > 0$, which is ruled out by (9.33). Hence $t_\varepsilon \neq T$ if $\boldsymbol{x}_\varepsilon \in \Omega$.

Therefore $(t_\varepsilon, \boldsymbol{x}_\varepsilon)$ lies in the set

$$\Gamma := (\{0\} \times \Omega) \cap ([0, T] \times \partial\Omega) . \tag{9.37}$$

Let R be sufficiently large so that $\Omega \subset B(0; R)$. This means that $|\boldsymbol{x}| \leq R$ on $\overline{\Omega}$, so the inequality

$$u \leq u_\varepsilon \leq u + \varepsilon R^2$$

holds at every point in $[0, T] \times \overline{\Omega}$. From (9.36) we can thus conclude that

$$\begin{aligned} \max_{[0,T]\times\overline{\Omega}} u &\leq u_\varepsilon(t_\varepsilon, \boldsymbol{x}_\varepsilon) \\ &\leq u(t_\varepsilon, \boldsymbol{x}_\varepsilon) + \varepsilon R^2. \end{aligned} \tag{9.38}$$

This implies that

$$\max_{[0,T]\times\overline{\Omega}} u \leq \max_{\Gamma} u + \varepsilon R^2,$$

because $(t_\varepsilon, \boldsymbol{x}_\varepsilon) \in \Gamma$. Since this inequality holds for every $\varepsilon > 0$, this proves

$$\max_{[0,T]\times\overline{\Omega}} u \leq \max_{\Gamma} u.$$

□

For a solution of the heat equation, both $\pm u$ satisfy the hypothesis of Theorem 9.8, which implies that

$$\min_{\Gamma} u \leq u(t, \boldsymbol{x}) \leq \max_{\Gamma} u,$$

for $(t, \boldsymbol{x}) \in (0, T) \times \Omega$, where Γ is defined by (9.37). In particular this yields the following:

Corollary 9.9 *Let $\Omega \in \mathbb{R}^n$ be a bounded domain. A solution of the heat equation $u \in C^{\text{heat}}(\Omega)$ is uniquely determined by $u|_{\partial\Omega}$ and $u|_{t=0}$.*

The same arguments could be applied to the more general parabolic equation

$$\frac{\partial u}{\partial t} - Lu = 0,$$

where L is a uniformly elliptic operator as defined in (9.27).

In Sect. 6.3, we stated without proof a uniqueness result for solutions of the heat equation on $\mathbb{R}^n$. We now have the means to prove this, by establishing a maximum principle for $\mathbb{R}^n$ as a corollary of Theorem 9.8.

Corollary 9.10 *Suppose that u is a classical solution of the heat equation*

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad u|_{t=0} = g, \tag{9.39}$$

on $[0, \infty) \times \mathbb{R}^n$, and that u is bounded on $[0, T] \times \mathbb{R}^n$ for $T > 0$. Then

$$\max_{[0,\infty)\times\mathbb{R}^n} u \leq \max_{\mathbb{R}^n} g. \tag{9.40}$$

Proof Assume that u satisfies (9.39) and also

$$u(t, \boldsymbol{x}) \leq M$$

for $t \in [0, T]$ and $\boldsymbol{x} \in \mathbb{R}^n$. For $\boldsymbol{y} \in \mathbb{R}^n$ and $\varepsilon > 0$ set

$$v(t, \boldsymbol{x}) := u(t, \boldsymbol{x}) - \varepsilon(T - t)^{-\frac{n}{2}} e^{\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4(T-t)}}.$$

The ε term resembles the heat kernel defined by (6.16), except that the sign in the exponential is reversed. Direct differentiation shows that this expression satisfies the heat equation on $(0, T) \times \mathbb{R}^n$, and hence v does also.

For $R > 0$, let us apply the maximum principle of Theorem 9.8 to v on the domain $(0, T) \times B(\boldsymbol{y}; R)$. By construction,

$$v(0, \boldsymbol{x}) \leq g(\boldsymbol{x}),$$

and for $\boldsymbol{x} \in \partial B(\boldsymbol{y}; R)$,

$$\begin{aligned} v(t, \boldsymbol{x}) &\leq M - \varepsilon(T - t)^{-\frac{n}{2}} e^{\frac{R^2}{4(T-t)}} \\ &\leq M - \varepsilon T^{-\frac{n}{2}} e^{R^2/4T}. \end{aligned}$$

With T fixed, the right-hand side of this second inequality is arbitrarily negative for large R. Therefore, for sufficiently large R, Theorem 9.8 implies that

$$\max_{[0,T]\times B(\boldsymbol{y};R)} v \leq \max_{B(\boldsymbol{y};R)} g.$$

In particular, setting $\boldsymbol{x} = \boldsymbol{y}$ in this inequality gives

$$v(t, \boldsymbol{y}) \leq \max_{\mathbb{R}^n} g.$$

for $t \in [0, T]$ and $\boldsymbol{y} \in \mathbb{R}^n$. By the definition of v, this implies that

$$u(t, \boldsymbol{y}) \leq \max_{\mathbb{R}^n} g + \varepsilon(T - t)^{-\frac{n}{2}}.$$

We can now take $\varepsilon \to 0$ and $T \to \infty$ to conclude that

$$u(t, \boldsymbol{y}) \leq \max_{\mathbb{R}^n} g$$

for all $t \in [0, \infty)$ and $\boldsymbol{y} \in \mathbb{R}^n$. □

The argument given here can be refined to show that conclusion (9.40) holds under the weaker growth condition

$$u(t, \boldsymbol{x}) \leq M e^{c|\boldsymbol{x}|^2}$$

for $t \in [0, T]$.

Corollary 9.10 implies Theorem 6.3 by the argument used in Corollary 9.9. That is, if u_1 and u_2 are bounded solutions of (6.19), then $\pm(u_1 - u_2)$ solves (6.19) with $g = 0$. It then follows from (9.40) that $u_1 = u_2$.

9.6 Exercises

9.1 Suppose that $u, \phi \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega})$ on a bounded domain $\Omega \subset \mathbb{R}^n$. Assume that u subharmonic and ϕ harmonic, with matching boundary values:

$$u|_{\partial\Omega} = \phi|_{\partial\Omega}.$$

Show that

$$u \leq \phi$$

at all points of Ω. (This is the motivation for the term "subharmonic".)

9.2 *Liouville's theorem* says that a bounded harmonic function on $\mathbb{R}^n$ is constant. To show this, assume $u \in C^2(\mathbb{R}^n)$ is harmonic and satisfies

$$|u(\boldsymbol{x})| \leq M$$

for all $\boldsymbol{x} \in \mathbb{R}^n$.

(a) For $\boldsymbol{x}_0 \in \mathbb{R}^n$, set $r_0 = |\boldsymbol{x}_0|$. Use Corollary 9.4 at the centers 0 and $\boldsymbol{x}_0$ to show that

$$u(0) - u(\boldsymbol{x}_0) = \frac{n}{A_n R^n} \left[\int_{B(0;R)} u \, d^n\boldsymbol{x} - \int_{B(\boldsymbol{x}_0;R)} u \, d^n\boldsymbol{x} \right] \tag{9.41}$$

for $R > 0$. Note that the integrals cancel on the intersection of the two balls.

(b) Show that

$$\begin{aligned} \operatorname{vol}\left[B(0; R)\backslash B(\boldsymbol{x}_0; R)\right] &\leq \operatorname{vol}\left[B(0; R)\backslash B(\tfrac{\boldsymbol{x}_0}{2}; R - \tfrac{r_0}{2})\right] \\ &= \frac{A_n}{n}\left[R^n - \left(R - \tfrac{r_0}{2}\right)^n\right], \end{aligned}$$

and the same for $B(\boldsymbol{x}_0; R)\backslash B(0; R)$.

(c) Apply the volume estimates and the fact that $|u| \leq M$ to (9.41) to estimate

$$|u(0) - u(\boldsymbol{x}_0)| \leq 2M \left[\frac{R^n - (R - \frac{r_0}{2})^n}{R^n} \right].$$

Take the limit $R \to \infty$ to show that $u(\boldsymbol{x}_0) = u(0)$.

9.3 Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, with $\Omega \subset B(0; R)$, and assume that $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega})$ satisfies

$$-\Delta u = f, \quad u|_{\partial\Omega} = 0,$$

for $f \in C^0(\overline{\Omega})$.

(a) Find a constant $c > 0$ (depending on f and R), such that $u + c\,|\boldsymbol{x}|^2$ is subharmonic on Ω.
(b) For this value of c, apply the maximum principle to $u + c\,|\boldsymbol{x}|^2$ to deduce that

$$\max_{\overline{\Omega}} |u| \leq C \max_{\overline{\Omega}} |f|\,,$$

where C depends only on R.

9.4 Suppose u is a harmonic function on a domain that includes $\overline{B(0; 4R)}$ for some $R > 0$, and assume $u \geq 0$. Show that

$$\max_{B(0;R)} u \leq 3^n \min_{B(0;R)} u.$$

Hint: For $\boldsymbol{x} \in B(0; R)$, apply the maximum principle to write $u(\boldsymbol{x})$ as an integral over the balls $B(\boldsymbol{x}; R)$ and $B(\boldsymbol{x}; 3R)$. Then show that

$$B(\boldsymbol{x}; R) \subset B(0; 2R) \subset B(\boldsymbol{x}; 3R),$$

and use this to estimate the integrals.

9.5 Suppose $u \in C^2(B; R) \cap C^0(\overline{B})$ is a nonconstant subharmonic function and assume that the maximum of u on B is attained at the point $\boldsymbol{x}_0 \in \partial B$. This automatically implies that $\frac{\partial u}{\partial r}(\boldsymbol{x}_0) \geq 0$. *Hopf's lemma* says that this inequality is strict,

$$\frac{\partial u}{\partial r}(\boldsymbol{x}_0) > 0.$$

To show this, let $B := B(0; R) \subset \mathbb{R}^n$ for some $R > 0$, and set

$$A := \{R/2 < |\boldsymbol{x}| < R\}\,.$$

(a) Consider the function

$$h(\boldsymbol{x}) := e^{-2n|\boldsymbol{x}|^2/R^2} - e^{-2n}.$$

Compute Δh and show that h is subharmonic on A.

(b) Set

$$m = \max_{\{r=R/2\}} u, \quad M = \max_{\{r=R\}} u,$$

and show that $m < M$.

(c) For $\varepsilon > 0$ set

$$u_\varepsilon := u + \varepsilon h,$$

and show that by taking ε sufficiently small we may assume that

$$\max_{\partial A} u_\varepsilon \leq M.$$

(d) Show that $u_\varepsilon(\boldsymbol{x}) \leq M$ for $\boldsymbol{x} \in A$, and hence that

$$\frac{\partial u_\varepsilon}{\partial r}(\boldsymbol{x}_0) \geq 0.$$

(e) By computing $\partial u_\varepsilon/\partial r$ and taking $\varepsilon \to 0$, conclude that

$$\frac{\partial u}{\partial r}(\boldsymbol{x}_0) > 0.$$