Chapter 2
Preliminaries

In this chapter we set the stage for the study of PDE with a review of some core background material.

2.1 Real Numbers

The real number system $\mathbb{R}$ is constructed as the "completion" of the field of rational numbers. This means that in addition to the algebraic axioms for addition and multiplication, $\mathbb{R}$ satisfies an additional axiom related to the existence of limits. To state this axiom we use the concept of the *supremum* (or "least upper bound") of subset $A \subset \mathbb{R}$. The supremum is a number $\sup(A) \in \mathbb{R}$ such that (1) all elements of A are less than or equal to $\sup(A)$; and (2) no number strictly less than $\sup(A)$ has this property. The *completeness* axiom says that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum. An equivalent statement is that a nonempty subset that is bounded below has an *infimum* ("greatest lower bound"), which is denoted $\inf(A)$.

It is convenient to extend these definitions to unbounded sets by defining $\sup(A) := \infty$ when A is not bounded above, and $\inf(A) := -\infty$ when the set is not bounded below. We also set $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) := +\infty$. With these extensions, sup and inf are defined for all subsets of $\mathbb{R}$.

To illustrate the definition, we present a simple result that will prove useful in the construction of approximating sequences for solutions of PDE.

Lemma 2.1 *For a nonempty set $A \subset \mathbb{R}$, there exists a sequence of points $x_k \in A$ such that*

$$\lim_{k \to \infty} x_k = \sup A,$$

and similarly for $\inf A$.

Proof If A is not bounded above then there exists a sequence of $x_k \in A$ with $x_k \to \infty$. Therefore the claim holds when $\sup A = \infty$.

Now suppose that $\sup A = \alpha \in \mathbb{R}$. By the definition of the supremum, $\alpha - 1/k$ is not an upper bound of A for $k \in \mathbb{N}$. Therefore, for each k there exists $x_k \in A$ such that $\alpha - 1/k < x_k \le \alpha$. This yields a sequence such that $x_k \to \alpha$. $\qquad\square$

There is an important distinction between supremum and infimum and the related concepts of *maximum* and *minimum*. The latter are required to be elements of the set and thus may not exist. For example, the interval $(0, 1)$ has $\sup = 1$ and $\inf = 0$, but has neither max nor min.

2.2 Complex Numbers

The complex number system $\mathbb{C}$ consists of numbers of the form $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 := -1$. The numbers x and y are called the real and imaginary parts of z. The *conjugate* of z is

$$\bar{z} := x - iy,$$

so that

$$\operatorname{Re} z := \frac{z + \bar{z}}{2}, \qquad \operatorname{Im} z := \frac{z - \bar{z}}{2i}.$$

A nonzero complex number has a multiplicative inverse, given by

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.$$

The absolute value on $\mathbb{C}$ is the vector absolute value from Euclidean $\mathbb{R}^2$,

$$|z| := \sqrt{x^2 + y^2}.$$

This can be written in terms of conjugation,

$$|z| = \sqrt{z\bar{z}},$$

which shows in particular that the absolute value is multiplicative,

$$|zw| := |z||w|$$

for $z, w \in \mathbb{C}$.

The basic theory of sequences and series carries over from $\mathbb{R}$ to $\mathbb{C}$ with only minor changes. A sequence $\{z_k\}$ in $\mathbb{C}$ converges to z if

$$\lim_{k \to \infty} |z_k - z| = 0,$$

and a series $\sum a_k$ converges if the sequence of partial sums $\sum_{k=1}^{n} a_k$ is convergent. The series converges *absolutely* if

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

It follows from the completeness axiom of $\mathbb{R}$ that absolute convergence of a series in $\mathbb{C}$ implies convergence.

The exponential series,

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}, \tag{2.1}$$

converges absolutely for all $z \in \mathbb{C}$. The special case where z is purely imaginary gives an important relation called *Euler's formula*:

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots\right)$$
$$= \cos\theta + i\sin\theta. \tag{2.2}$$

Leonhard Euler, arguably the most influential mathematician of the 18th century, published this identity in 1748. It yields a natural polar-coordinate representation of complex numbers,

$$z = re^{i\theta},$$

where $r = |z|$ and θ is the angle between z and the positive real axis.

The product rule for complex exponentials,

$$e^z e^w = e^{z+w}, \tag{2.3}$$

follows from the power series definition just as in the real case. In combination with (2.2) this allows for a very convenient manipulation of trigonometric functions. For example, setting $z = i\alpha$ and $w = i\beta$ in (2.3) and taking the real and imaginary parts recovers the identities

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

The calculus rules for differentiating and integrating exponentials are derived from the power series expansion, and thus extend to the complex case. In particular,

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

for $a \in \mathbb{C}$.

2.3 Domains in $\mathbb{R}^n$

For points in $\mathbb{R}^n$ we will use the vector notation $x = (x_1, \ldots, x_n)$. The Euclidean dot product is denoted $x \cdot y$, and the Euclidean length of a vector is written

$$|x| := \sqrt{x \cdot x}.$$

The Euclidean distance is used to define limits: $\lim_{k \to \infty} x_k = w$ means that

$$\lim_{k \to \infty} |x_k - w| = 0.$$

The *ball* of radius $R > 0$ centered at a point $x_0 \in \mathbb{R}^n$ is

$$B(x_0; R) := \left\{ x \in \mathbb{R}^n; \ |x - x_0| < R \right\}.$$

A small ball centered at x_0 is called a *neighborhood* of x_0. If $x \in A$ has a neighborhood contained in A then x is called an *interior point*.

A subset $U \subset \mathbb{R}^n$ is *open* if all of its points are interior. This generalizes the notion of an open interval in one dimension. The ball $B(x_0; R)$ is open, for example, as is $\mathbb{R}^n$ itself. The empty set is open by default.

A *boundary point* of $A \subset \mathbb{R}^n$ is a point $x \in \mathbb{R}^n$ such that every neighborhood of x intersects both A and its complement. The distinction between interior and boundary points is illustrated in Fig. 2.1. Note that boundary points may or may not be included in the set itself. The boundary of A is denoted
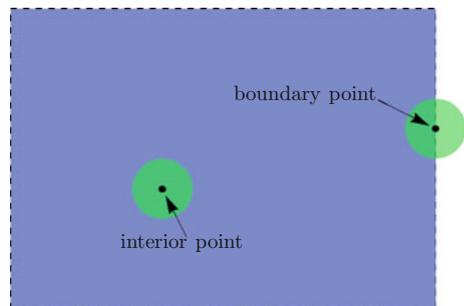
$$\partial A := \{\text{boundary points of } A\}.$$

For example, the boundary of the ball $B(x_0; R)$ is the sphere

$$\partial B(x_0; R) = \left\{ x \in \mathbb{R}^n; \ |x - x_0| = R \right\}.$$

A set is open if and only if it contains no boundary points.

Fig. 2.1 Open rectangle in $\mathbb{R}^2$

A subset of $\mathbb{R}^n$ is *connected* if any two points in the set can be joined by a continuous path within the set. For an open set U this is equivalent to the condition that U cannot be written as the disjoint union of two nonempty open sets.

We will refer to a connected open subset $\Omega \subset \mathbb{R}^n$ as a *domain*, and reserve the notation Ω for this usage. For some problems we assume the domain is *bounded*, meaning that $\Omega \subset B(0; R)$ for sufficiently large R.

The concept of a closed interval can also be generalized to higher dimension. A subset $F \subset \mathbb{R}^n$ is *closed* if it contains all of its boundary points, i.e.,

$$\partial F \subset F.$$

The union of a subset $A \subset \mathbb{R}^n$ with its boundary is called the *closure* of A and denoted

$$\overline{A} := A \cup \partial A.$$

For example,

$$\overline{B(\boldsymbol{x}_0; R)} := \left\{ \boldsymbol{x} \in \mathbb{R}^n; \; |\boldsymbol{x} - \boldsymbol{x}_0| \leq R \right\}.$$

It is potentially confusing that an overline is used for set closure and complex conjugation, but these notations are standard. Note that closure applies only to sets and not to numbers or functions.

A closed set $F \in \mathbb{R}^n$ contains the limits of all sequences in F that converge in $\mathbb{R}^n$. This is because the limit of a sequence contained in a set must either be a point in the set or a boundary point.

Closed and open sets are related in the sense that the complement of an open set is closed, and vice versa. However, the terms are not mutually exclusive, and a set might not have either property. The interval $(a, b] \subset \mathbb{R}$ is neither open nor closed, for example. The only subsets of $\mathbb{R}^n$ with both properties are $\mathbb{R}^n$ itself and $\emptyset$.

2.4 Differentiability

The space of continuous, complex-valued functions on a domain $\Omega \subset \mathbb{R}^n$ which admit continuous partial derivatives up to order m is denoted by $C^m(\Omega)$. The assumption of continuity for derivatives insures that mixed partials are independent of the order of differentiation. A *smooth* function has continuous derivatives to all orders; the corresponding space is written $C^\infty(\Omega)$.

We use the notation $C^m(\Omega; \mathbb{R})$ to specify real-valued functions, and similarly $C^m(\Omega; \mathbb{R}^n)$ denotes the space of vector-valued functions. It is common to use C^m as an adjective, short for "m-times continuously differentiable".

The definition of $C^m(\Omega)$ makes no conditions on the behavior of functions as the boundary is approached. To impose such restrictions, we use the notation $C^m(\overline{\Omega})$ to denote the space of functions that admit C^m extensions across the boundary. For example, the function $\sqrt{x} \in C^\infty(0, 1)$ is an element of $C^0[0, 1]$, but not $C^1[0, 1]$.

The *support* of $f \in C^0(\Omega)$ is defined as

$$\operatorname{supp} f := \overline{\{x \in \Omega;\ f(x) \neq 0\}}. \tag{2.4}$$

Note that the definition includes a closure. This means that the support does not exclude points where the function merely "passes through" zero. For example, the support of $\sin(x)$ is $\mathbb{R}$ rather than $\mathbb{R} \setminus \pi \mathbb{Z}$.

A closed and bounded subset of $\mathbb{R}^n$ is said to be *compact*. We denote by $C_{\mathrm{cpt}}^m(\Omega)$ the space of functions on Ω that have *compact support*, meaning that supp f is a compact subset of Ω. Since Ω is open and the support is closed, this requires in particular that supp f be a strict subset of Ω. For example, $1 - x^2$ vanishes at the boundary of $(-1, 1)$, but does not have compact support in this domain because its support is $[-1, 1]$.

Example 2.2 To demonstrate the existence of compactly supported smooth functions, consider

$$h(x) = \begin{cases} e^{-1/(1-x^2)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

which has support $[-1, 1]$. As illustrated in Fig. 2.2, the function becomes extremely flat as $x \to \pm 1$.

To show that h is in fact smooth, we note that

$$h^{(m)}(x) = \begin{cases} \frac{q_m(x)}{(1-x^2)^m} e^{-1/(1-x^2)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where q_m denotes a polynomial of degree m. As $x \to \pm 1$, the term $(1 - x^2)^{-m}$ blows up while the exponential term tends rapidly to zero. Using l'Hôpital's rule one can check that the exponential dominates this limit, so that all derivatives of h vanish as $x \to \pm 1$ from $|x| \leq 1$. This shows that $h \in C_{\mathrm{cpt}}^\infty(\mathbb{R})$.

The function h can be integrated to produce a smooth function that is constant for $|x| \geq 1$. By translating and rescaling, this construction gives, for $a < b$, a function $\varphi \in C^\infty(\mathbb{R})$ satisfying

$$\varphi(x) = \begin{cases} 0, & x \leq a, \\ 1, & x \geq b. \end{cases}$$

Fig. 2.2 Compactly supported smooth function
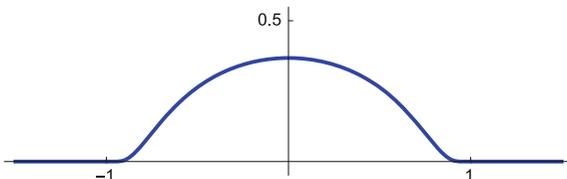
These "smooth step functions" are useful as building blocks for pasting together smooth functions on different domains. ◇

Certain problems require regularity of the boundary of Ω. For example, many theorems in vector calculus require the existence of a normal vector on the boundary, which does not exist for a general domain. A standard hypothesis for such theorems is that $\partial\Omega$ is piecewise C^1. This means that $\partial\Omega$ consists of a finite number of components which admit *regular coordinate parametrization*. A coordinate parametrization is a map $\boldsymbol{\sigma} \in C^1(\overline{U}; \mathbb{R}^n)$ where U is a domain in $\mathbb{R}^{n-1}$. To say the parametrization is regular means that the tangent vectors defined by

$$\frac{\partial\boldsymbol{\sigma}}{\partial w_j} := \left(\frac{\partial\sigma_1}{\partial w_j}, \ldots, \frac{\partial\sigma_n}{\partial w_j} \right),$$

$j = 1, \ldots, n-1$, are linearly independent at each point of $\partial\Omega$.

The piecewise C^1 boundary assumption guarantees that the points of each boundary component have well-defined tangent spaces and normal directions. As an example, the unit cube in $\mathbb{R}^3$ has piecewise C^1 boundary, consisting of 6 planar components with normal directions parallel to the coordinate axes.

In this text we will focus on relatively simple domains with straightforward boundary parametrizations.

2.5   Ordinary Differential Equations

Our development of PDE theory will not rely on any advanced techniques for the solution of ODE, but it will be useful to recall some basic material.

First-order ODE can often be solved directly by methods from calculus. The easiest cases are equations of the form

$$\frac{dy}{dt} = g(y)h(t),$$

where the variables can be separated to yield an integral formula

$$\int \frac{dy}{g(y)} = \int \frac{dt}{h(t)}.$$

Integrating both sides yields a family of solutions with one undetermined constant.

Example 2.3 Consider the equation

$$\frac{dy}{dt} = ay, \quad y(0) = y_0.$$

for $a \neq 0$. This is called the growth or decay equation, depending on the sign of a. Separating the variables gives

$$\int \frac{dy}{y} = \int a \, dt,$$

which integrates to $\ln y = at + C$. Solving for y and using the initial condition gives

$$y(t) = y_0 e^{at}.$$

$\diamond$

Higher-order ODE are generally analyzed by reducing to a system of first-order equations. To reduce the nth-order equation

$$y^{(n)}(t) = F\left(t, y, y', \ldots, y^{(n-1)}\right), \tag{2.5}$$

we define the vector-valued function $\boldsymbol{w} = (y, y', \ldots, y^{(n-1)})$. This satisfies the first-order system

$$\frac{d}{dt} \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{pmatrix} = \begin{pmatrix} w_2 \\ \vdots \\ w_n \\ F(t, \boldsymbol{w}) \end{pmatrix}.$$

First-order systems can be solved generally by the strategy of Picard iteration, named for mathematician Émile Picard. The first step is to write the vector equation,

$$\frac{d\boldsymbol{w}}{dt} = \boldsymbol{F}(t, \boldsymbol{w}), \qquad \boldsymbol{w}(t_0) = \boldsymbol{w}_0, \tag{2.6}$$

in an equivalent form as a recursive integral equation,

$$\boldsymbol{w}(t) = \boldsymbol{w}_0 + \int_{t_0}^{t} \boldsymbol{F}(s, \boldsymbol{w}(s)) \, ds.$$

For the construction, we set $\boldsymbol{u}_0(t) := \boldsymbol{w}_0$ and define a sequence of functions by

$$\boldsymbol{u}_k(t) = \boldsymbol{w}_0 + \int_{t_0}^{t} \boldsymbol{F}(s, \boldsymbol{u}_{k-1}(s)) \, ds \tag{2.7}$$

for $k = 1, 2, \ldots$. It can be shown that the limit of this sequence exists and solves (2.6) under some general assumptions on $\boldsymbol{F}$, which leads to a proof of the following result.

Theorem 2.4 (Picard iteration) *Suppose that $\boldsymbol{F}$ is a continuous function on $I \times \Omega$ where I is an open interval containing t_0 and Ω is a domain in $\mathbb{R}^n$ containing $\boldsymbol{w}_0$, and that $\boldsymbol{F}$ is continuously differentiable with respect to $\boldsymbol{w}$. Then (2.6) admits a unique solution on some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ with $\varepsilon > 0$.*

Applying Theorem 2.4 to (2.5) shows that an nth order ODE satisfying the regularity assumptions has a unique local solution specified by the initial values of the function and its first $n - 1$ derivatives.

The C^1 hypothesis on $\boldsymbol{F}$ is stronger than necessary, but this version will suffice for our purposes. The point we would like to stress here is the relative ease with which ODE can be analyzed under very general conditions. This is very different from the PDE theory, where no such general results are possible.

Example 2.5 The *harmonic ODE* is the equation

$$\frac{d^2 y}{dt^2} = -\kappa^2 y,$$

for $\kappa > 0$. In view of the solution to the growth/decay equation in Example 2.3, it is reasonable to start with an exponential solution as a guess. Substituting $e^{\alpha t}$ into the equation yields $\alpha^2 = -\kappa^2$. From $\alpha = \pm i\kappa$ we obtain the general solution,

$$y(t) = c_1 e^{i\kappa t} + c_2 e^{-i\kappa t}.$$

To see how this relates to the Picard iteration method described above, consider the corresponding system (2.6) for $\boldsymbol{w} = (y, y')$:

$$\frac{d\boldsymbol{w}}{dt} = \begin{pmatrix} 0 & 1 \\ -\kappa^2 & 0 \end{pmatrix} \boldsymbol{w}.$$

With $\boldsymbol{w}_0 = (a, b)$, the recursive formula (2.7) yields the sequence of functions

$$\boldsymbol{u}_k(t) = \sum_{j=0}^{k} \frac{t^j}{j!} \begin{pmatrix} 0 & 1 \\ -\kappa^2 & 0 \end{pmatrix}^j \begin{pmatrix} a \\ b \end{pmatrix}$$

for $k \in \mathbb{N}$. In the limit $k \to \infty$ this gives

$$\boldsymbol{w}(t) = \left[1 - \frac{(\kappa t)^2}{2!} + \frac{(\kappa t)^4}{4!} - \cdots \right] \begin{pmatrix} a \\ b \end{pmatrix}$$
$$+ \frac{1}{\kappa} \left[\kappa t - \frac{(\kappa t)^3}{3!} + \frac{(\kappa t)^5}{5!} - \cdots \right] \begin{pmatrix} b \\ -\kappa^2 a \end{pmatrix}.$$

Reading off y as the first component of $\boldsymbol{w}$ gives the familiar trigonometric solution,

$$y(t) = a\cos(\kappa t) + \frac{b}{\kappa}\sin(\kappa t).$$

The trigonometric and complex exponential solutions are related by Euler's formula (2.2). ◊

2.6   Vector Calculus

The classical theorems of vector calculus were motivated by PDE problems arising in physics. For our purposes the most important of these results is the divergence theorem. We assume that the reader is familiar with the divergence theorem in the context of $\mathbb{R}^2$ or $\mathbb{R}^3$. In this section we will cover the basic definitions needed to state the result in $\mathbb{R}^n$ and develop its corollaries.

As noted in Sect. 2.3, we always take a domain $\Omega \subset \mathbb{R}^n$ to be connected and open. The *gradient* of $f \in C^1(\Omega)$ is the vector-valued function

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

For a vector-valued function $\boldsymbol{v} \in C^1(\Omega; \mathbb{R}^n)$ with components $(v_1, \ldots, v_n)$, the *divergence* is

$$\nabla \cdot \boldsymbol{v} := \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}.$$

The Laplacian operator introduced in (1.7) is the divergence of the gradient

$$\Delta u := \nabla \cdot (\nabla u).$$

For this reason Δ is sometimes written ∇^2.

If Ω is bounded then the Riemannian integral of $f \in C^0(\overline{\Omega})$ exists and is denoted by

$$\int_\Omega f(\boldsymbol{x})\, d^n\boldsymbol{x},$$

where $d^n\boldsymbol{x}$ is a shorthand for $dx_1 \cdots dx_n$. The integral can be extended to unbounded domains if the appropriate limits exist. We will discuss a further generalization of the Riemann definition in Chap. 7.

One issue we will come across frequently is differentiation under the integral. If $\Omega \subset \mathbb{R}^n$ is a bounded domain and u and $\partial u/\partial t$ are continuous functions on $(a, b) \times \overline{\Omega}$, then the Leibniz integral rule says that

$$\frac{d}{dt} \int_{\Omega} u(t, \boldsymbol{x}) \, d^n \boldsymbol{x} = \int_{\Omega} \frac{\partial u}{\partial t}(t, \boldsymbol{x}) \, d^n \boldsymbol{x}.$$

Differentiation under the integral may still work when the integrals are improper, but this requires greater care.

To set up boundary integrals for a domain with piecewise C^1 boundary, we need to define the surface integral over a regular coordinate patch $\boldsymbol{\sigma} : U \subset \mathbb{R}^{n-1} \to \partial\Omega$. Let $\boldsymbol{v} : U \to \mathbb{R}^n$ denote the unit normal vector pointing outwards from the domain. The surface integral for such a patch is defined by

$$\int_{\boldsymbol{\sigma}(U)} f \, dS := \int_U f(\boldsymbol{\sigma}(\boldsymbol{w})) \left| \det \left[\frac{\partial \boldsymbol{\sigma}}{\partial w_1}, \dots, \frac{\partial \boldsymbol{\sigma}}{\partial w_{n-1}}, \boldsymbol{v} \right] \right| d^{n-1} \boldsymbol{w}, \qquad (2.8)$$

where $\det[\dots]$ denotes the determinant of a matrix of column vectors. The full surface integral over $\partial\Omega$ is defined by summing over the boundary coordinate patches. For simplicity, we notate this as a single integral,

$$\int_{\partial\Omega} f \, dS.$$

In $\mathbb{R}^2$, a boundary parametrization will be a curve $\boldsymbol{\sigma}(t)$ and (2.8) reduces to the arclength integral

$$\int_{\boldsymbol{\sigma}(U)} f \, dS := \int_U f(\boldsymbol{\sigma}(t)) \left| \frac{d\boldsymbol{\sigma}}{dt} \right| dt.$$

In $\mathbb{R}^3$, the unit normal for a surface patch can be computed from the cross product of the tangent vectors. This leads to the surface integral formula

$$\int_{\boldsymbol{\sigma}(U)} f \, dS := \int_U f(\boldsymbol{\sigma}(\boldsymbol{w})) \left| \frac{\partial \boldsymbol{\sigma}}{\partial w_1} \times \frac{\partial \boldsymbol{\sigma}}{\partial w_2} \right| d^2 \boldsymbol{w}.$$

Even in low dimensions surface integrals can be rather complicated. We will make explicit use of these formulas only in relatively simple cases, such as rectangular regions and spheres.

We can use (2.8) to decompose integrals into radial and spherical components. This is particularly useful when the domain is a ball. Let $r := |\boldsymbol{x}|$ be the radial coordinate, and define the unit sphere

$$\mathbb{S}^{n-1} := \{r = 1\} \subset \mathbb{R}^n.$$

A point $\boldsymbol{x} \neq 0$ can be written uniquely as $r\boldsymbol{\omega}$ for $\boldsymbol{\omega} \in \mathbb{S}^{n-1}$ and $r > 0$. Let $\boldsymbol{\omega}(\boldsymbol{y})$ be a parametrization of $\mathbb{S}^{n-1}$ by coordinates $\boldsymbol{y} \in U \subset \mathbb{R}^{n-1}$. For the change of variables $(r, \boldsymbol{y}) \mapsto \boldsymbol{x} = r\boldsymbol{\omega}(\boldsymbol{y})$, the Jacobian formula gives

$$d^n x = \left| \det \left[\frac{\partial x}{\partial y_1}, \ldots, \frac{\partial x}{\partial y_{n-1}}, \frac{\partial x}{\partial r} \right] \right| dr\, dy_1 \cdots dy_{n-1} \qquad (2.9)$$

$$= \left| \det \left[\frac{\partial \boldsymbol{\omega}}{\partial y_1}, \ldots, \frac{\partial \boldsymbol{\omega}}{\partial y_{n-1}}, \boldsymbol{\omega} \right] \right| r^{n-1}\, dr\, dy_1 \cdots dy_{n-1}.$$

On the unit sphere, the outward unit normal $\boldsymbol{v}$ is equal to $\boldsymbol{\omega}$. Thus (2.9) reduces to

$$d^n x = r^{n-1}\, dr\, dS(\boldsymbol{y}).$$

For an integral over the ball this yields the radial integral formula,

$$\int_{B(0;R)} f(\boldsymbol{x})\, d^n x = \int_{\mathbb{S}^{n-1}} \int_0^R f(r\boldsymbol{\omega}(\boldsymbol{y}))\, r^{n-1} dr\, dS(\boldsymbol{y}). \qquad (2.10)$$

With these definitions in place, we turn to the divergence theorem, which relates the flux of a vector field through a closed surface to the divergence of the field in the interior. This result is generally attributed to Carl Friedrich Gauss, who published a version in 1813 in conjunction with his work on electrostatics.

Theorem 2.6 (Divergence theorem) *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary. For a vector field $\boldsymbol{F} \in C^1(\overline{\Omega}; \mathbb{R}^n)$,*

$$\int_\Omega \nabla \cdot \boldsymbol{F}\, d^n x = \int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{v}\, dS,$$

where $\boldsymbol{v}$ is the outward unit normal to $\partial\Omega$.

A full proof can be found in advanced calculus texts. To illustrate the idea, we will show how the argument works for a spherical domain in $\mathbb{R}^3$.

Example 2.7 Let $\mathbb{B}^3 = \{r < 1\} \subset \mathbb{R}^3$. Because a vector field can be decomposed into components, it suffices to consider a field parallel to one of the coordinate axes, say $\boldsymbol{F} = (0, 0, f)$. The divergence is then

$$\nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x_3}.$$

In cylindrical coordinates, $\boldsymbol{x} = (\rho \cos\theta, \rho \sin\theta, z)$, the volume element is

$$d^3 x = \rho\, d\rho\, d\phi\, dz,$$

so the left side of the divergence formula becomes

$$\int_{\mathbb{B}^3} \nabla \cdot \boldsymbol{F}\, d^3\boldsymbol{x} = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-\rho^2}}^{\sqrt{1-\rho^2}} \frac{\partial f}{\partial z}\, \rho\, dz\, d\rho\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[f\left(\rho\cos\theta, \rho\sin\theta, \sqrt{1-\rho^2}\right) \right. \qquad (2.11)$$
$$\left. - f\left(\rho\cos\theta, \rho\sin\theta, -\sqrt{1-\rho^2}\right) \right] \rho\, d\rho\, d\theta.$$

Note that $z = \pm\sqrt{1-\rho^2}$ gives the restriction to the upper and lower hemispheres, respectively.

We denote the two hemispheres $\mathbb{S}_\pm^2 \subset \mathbb{S}^2$ and parametrize them as

$$\boldsymbol{\omega}_\pm(\rho, \theta) = \left(\rho\cos\theta, \rho\sin\theta, \pm\sqrt{1-\rho^2}\right).$$

The corresponding surface area elements are given by

$$dS = \left| \frac{\partial \boldsymbol{\omega}_\pm}{\partial \rho} \times \frac{\partial \boldsymbol{\omega}_\pm}{\partial \theta} \right| d\rho\, d\theta$$

$$= \frac{\rho}{\sqrt{1-\rho^2}}\, d\rho\, d\theta.$$

Thus,

$$\int_0^{2\pi} \int_0^1 f\left(\rho\cos\theta, \rho\sin\theta, \pm\sqrt{1-\rho^2}\right) \rho\, d\rho\, d\theta = \int_{\mathbb{S}_\pm^2} f\sqrt{1-\rho^2}\, dS.$$

On $\mathbb{S}_\pm^2$ we have $\boldsymbol{F} \cdot \boldsymbol{v} = \pm f\sqrt{1-\rho^2}$, so that

$$\int_{\mathbb{S}_\pm^2} f\sqrt{1-\rho^2}\, dS = \pm \int_{\mathbb{S}_\pm^2} \boldsymbol{F} \cdot \boldsymbol{v}\, dS.$$

Applying this to (2.11) reduces the equation to

$$\int_{\mathbb{B}^3} \nabla \cdot \boldsymbol{F}\, d^3\boldsymbol{x} = \int_{\mathbb{S}^2} \boldsymbol{F} \cdot \boldsymbol{v}\, dS,$$

verifying the divergence theorem in this special case.   ◇

Theorem 2.6 can be used to evaluate integrals of the Laplacian of a function by substituting $\boldsymbol{F} = \nabla u$ for the vector field. Inside the volume integral this yields the integrand

$$\nabla \cdot \boldsymbol{F} = \Delta u.$$

On the surface side, the integrand becomes the directional derivative with respect to the outward unit normal, which is denoted

$$\frac{\partial u}{\partial v} := \boldsymbol{v} \cdot \nabla u\big|_{\partial\Omega}.$$

Corollary 2.8 *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary, and $u \in C^2(\overline{\Omega})$, then*

$$\int_{\Omega} \Delta u \, d^n \boldsymbol{x} = \int_{\partial\Omega} \frac{\partial u}{\partial v} \, dS.$$

The application we will encounter most frequently is to the ball $B(0; R) \in \mathbb{R}^n$. The outward unit normal is parallel to the position vector, so that

$$v = \frac{\boldsymbol{x}}{R}.$$

It follows from the chain rule that

$$\frac{\partial u}{\partial v} = \frac{\partial u}{\partial r}. \tag{2.12}$$

Example 2.9 Consider a radial function $g(r)$ where $r := |\boldsymbol{x}|$ for $\boldsymbol{x} \in \mathbb{R}^n$. For the ball $B(0; a)$, the radial integral formula (2.10) gives

$$\int_{B(0;a)} \Delta g \, d^n \boldsymbol{x} = A_n \int_0^a \Delta g(r) r^{n-1} \, dr,$$

where

$$A_n := \text{vol}(\mathbb{S}^{n-1}). \tag{2.13}$$

By (2.12),

$$\int_{\partial B(0;a)} \frac{\partial g}{\partial v} \, dS = \int_{\partial B(0;a)} \frac{\partial g}{\partial r}(a) \, dS$$
$$= A_n a^{n-1} \frac{\partial g}{\partial r}(a).$$

The formula from Corollary 2.8 reduces in this case to

$$\int_0^a \Delta g(r) r^{n-1} \, dr = a^{n-1} \frac{\partial g}{\partial r}(a). \tag{2.14}$$

Differentiating (2.14) with respect to a gives, by the fundamental theorem of calculus,

$$a^{n-1} \Delta g(a) = \frac{\partial}{\partial a} \left[a^{n-1} \frac{\partial g}{\partial r}(a) \right].$$

This holds for all $a > 0$, so evidently the Laplacian of a radial function is given by

$$\Delta g = r^{1-n} \frac{\partial}{\partial r} \left[r^{n-1} \frac{\partial}{\partial r} \right] g. \tag{2.15}$$

In principle one could derive this formula directly from the chain rule, but the direct computation is difficult in high dimensions. $\diamond$

There are two other direct corollaries of the divergence theorem that will be used frequently. These are named for the mathematical physicist George Green, who used them to develop solution formulas for some classical PDE.

The first result is a generalization of Corollary 2.8, obtained from Theorem 2.6 by the substitution $\boldsymbol{F} = v\nabla u$ for a pair of functions u, v. The product rule for differentiation gives

$$\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v\Delta u, \tag{2.16}$$

which can easily be checked by writing out the components of the gradient.

Theorem 2.10 (Green's first identity) *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary, then for $u \in C^2(\overline{\Omega})$, and $v \in C^1(\overline{\Omega})$,*

$$\int_{\Omega} [\nabla v \cdot \nabla u + v\Delta u] \, d^n\boldsymbol{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, dS.$$

The second identity follows from the first by interchanging u with v and then subtracting the result.

Theorem 2.11 (Green's second identity) *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary, then for $u, v \in C^2(\overline{\Omega})$,*

$$\int_{\Omega} [v\Delta u - u\Delta v] \, d^n\boldsymbol{x} = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) dS.$$

2.7   Exercises

2.1 For $r := |\boldsymbol{x}|$ in $\mathbb{R}^n$, and $\alpha \in \mathbb{R}$, compute $\nabla(r^\alpha)$ and $\Delta(r^\alpha)$.

2.2 Polar coordinates (r, θ) in $\mathbb{R}^2$ are related to Cartesian coordinates (x_1, x_2) by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta.$$

(a) Use the chain rule to compute $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$.
(b) Find the expression for Δ in the (r, θ) coordinates. (The radial part should agree with (2.15).)

2.3 In $\mathbb{R}^n$ let Ω be the unit cube $(0, 1)^n$. Define

$$\boldsymbol{w}(\boldsymbol{x}) = f(\boldsymbol{x})e_j,$$

where $f \in C^\infty(\overline{\Omega})$ and e_j is the jth coordinate vector, $e_j := (0, \ldots, 1, \ldots, 0)$. Compute both sides of the formula from Theorem 2.6 in this case, and show that the result reduces to an application of the fundamental theorem of calculus in the x_j variable.

2.4 For $f \in C^0(\mathbb{R}^n)$ set

$$h(t) := \int_{B(0;t)} f(\boldsymbol{x}) \, d^n \boldsymbol{x}.$$

for $t \geq 0$. Use the radial decomposition formula (2.10) to show that

$$\frac{dh}{dt} = \int_{\partial B(0;t)} f(\boldsymbol{w}) \, dS(\boldsymbol{w}).$$

2.5 The gamma function is defined for $z > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, dt. \tag{2.17}$$

Note that $\Gamma(1) = 1$ and integration by parts gives the recursion relation $\Gamma(z + 1) = z\Gamma(z)$. In this problem we will show that the volume of the unit sphere in $\mathbb{R}^n$ is given by

$$A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{2.18}$$

(a) Use the radial formula (2.10) and the substitution $u := r^2$ to compute that

$$\int_{\mathbb{R}^n} e^{-r^2} \, d^n \boldsymbol{x} = \frac{1}{2} A_n \Gamma(\tfrac{n}{2}). \tag{2.19}$$

(b) Observe that we can rewrite

$$\int_{\mathbb{R}^n} e^{-r^2} \, d^n \boldsymbol{x} = \int_{\mathbb{R}^n} e^{-(x_1^2 + \cdots + x_n^2)} \, d^n \boldsymbol{x}$$
$$= \left[2 \int_0^\infty e^{-x^2} \, dx \right]^n.$$

Substitute $t = x^2$ to evaluate the one-dimensional integral in terms of $\Gamma(\tfrac{1}{2})$.
(c) Compare (a) to (b) to obtain a formula for A_n.
(d) Use (c) and the fact that $A_2 = 2\pi$ to compute $\Gamma(\tfrac{1}{2})$ and reduce the formula to (2.18).