Chapter 5
Separation of Variables

Some PDE can be split into pieces that involve distinct variables. For example, the equation

$$\frac{\partial u}{\partial t} - a(t)b(\boldsymbol{x})\Delta u = 0$$

could be written as

$$\frac{1}{a(t)}\frac{\partial u}{\partial t} = b(\boldsymbol{x})\Delta u,$$

provided $a(t) \neq 0$. This puts all of the t derivatives and t-dependent coefficients on the left and all of the terms involving $\boldsymbol{x}$ on the right.

Splitting an equation this way is called *separation of variables*. For PDE that admit separation, it is natural to look for product solutions whose factors depend on the separate variables, e.g., $u(t, \boldsymbol{x}) = v(t)\phi(\boldsymbol{x})$. The full PDE then reduces to a pair of equations for the factors. In some cases, one or both of the reduced equations is an ODE that can be solved explicitly.

This idea is most commonly applied to evolution equations such as the heat or wave equations. The classical versions of these PDE have constant coefficients, and separation of variables can thus be used to split the time variable from the spatial variables. This reduces the evolution equation to a simple temporal ODE and a spatial PDE problem.

Separation among the spatial variables is sometimes possible as well, but this requires symmetry in the equation that is also shared by the domain. For example, we can separate variables for the Laplacian on rectangular or circular domains in $\mathbb{R}^2$. But if the domain is irregular or the differential operator has variable coefficients, then separation is generally not possible.

Despite these limitations, separation of variables plays a significant role the development of PDE theory. Explicit solutions can still yield valuable information even if they are very special cases.

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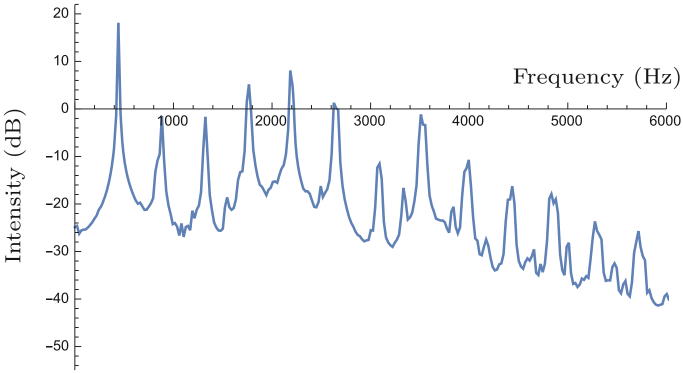

Fig. 5.1 Frequency decomposition for the sound of a violin string

5.1 Model Problem: Overtones

In 1636 the mathematician Marin Mersenne published his observation that a vibrating string produces multiple pitches simultaneously. The most audible pitch corresponds to the lowest frequency of vibration, called the fundamental tone of the string. Mersenne also detected higher pitches, at integer multiples of the fundamental frequency. (The relationship between frequency and pitch is logarithmic; doubling the frequency raises the pitch by one octave.)

The higher multiples of the fundamental frequency are called *overtones* of the string. Figure 5.1 shows the frequency decomposition for a sound sample of a bowed violin string, with a fundamental frequency of 440 Hz. The overtones appear as peaks in the intensity plot at multiples of 440.

At the time of Mersenne's observations, there was no theoretical model for string vibration that would explain the overtones. The wave equation that d'Alembert subsequently developed (a century later) gave the first theoretical justification. However, this connection is not apparent in the explicit solution formula developed in Sect. 4.3. To understand how the overtones are predicted by the wave equation, we need to organize the solutions in terms of frequency.

5.2 Helmholtz Equation

The classical evolution equations on $\mathbb{R}^n$ have the form

$$P_t u - \Delta u = 0, \tag{5.1}$$

where P_t is a first- or second-order differential operator involving only the time variable. Examples include the wave equation ($P_t = \partial^2/\partial t^2$), heat equation ($P_t = \partial/\partial t$), and Schrödinger equation ($P_t = -i\partial/\partial t$).

Lemma 5.1 *If u is a classical solution of (5.1) of the form*

$$u(t,\boldsymbol{x}) = v(t)\phi(\boldsymbol{x}),$$

for $t \in \mathbb{R}$ and $\boldsymbol{x} \in \Omega \subset \mathbb{R}^n$, then in any region where u is nonzero there is a constant κ such that the components solve the equations

$$P_t v = \kappa v, \qquad \Delta\phi = \kappa\phi. \tag{5.2}$$

Proof Substituting $u = v\phi$ into (5.1) gives

$$\phi P_t v - v\Delta\phi = 0.$$

Assuming that u is nonzero, we can divide by u to obtain

$$\frac{1}{v}P_t v = \frac{1}{\phi}\Delta\phi.$$

The left hand-side is independent of $\boldsymbol{x}$ and the right is independent of t. We conclude that both sides must be equal to some constant κ. □

The two differential equations in (5.2) are analogous to eigenvalue equations from linear algebra, with the role of the linear operator or matrix taken by the differential operators P_t or Δ.

Let us first focus on the spatial problem, which is usually written in the form

$$-\Delta\phi = \lambda\phi. \tag{5.3}$$

This is called the *Helmholtz equation*, after the 19th century physicist Hermann von Helmholtz. The minus sign is included so that $\lambda \geq 0$ for the most common types of boundary conditions. Adapting the linear algebra terminology, we refer to the number λ in (5.3) as an *eigenvalue* and the corresponding solution ϕ as an *eigenfunction*. The Helmholtz equation is sometimes called the Laplacian eigenvalue equation.

We will present a general analysis of the Helmholtz problem on any bounded domain in $\mathbb{R}^n$ in Chap. 11, and later in this chapter we will consider some two- or three-dimensional cases for which further spatial separation is possible. For the remainder of this section we restrict our attention to problems in one spatial dimensional, for which (5.3) is an ODE.

Theorem 5.2 *For $\phi \in C^2[0,\ell]$ the equation*

$$-\frac{d^2\phi}{dx^2} = \lambda\phi, \quad \phi(0) = \phi(\ell) = 0, \tag{5.4}$$

has nonzero solutions only if

$$\lambda_n := \frac{\pi^2 n^2}{\ell^2}$$

for $n \in \mathbb{N}$. *Up to a constant multiple, the corresponding solutions are*

$$\phi_n(x) := \sin(\sqrt{\lambda_n}x). \tag{5.5}$$

Proof Note that (5.4) implies

$$\lambda \int_0^\ell |\phi|^2 \, dx = -\int_0^\ell \frac{d^2\phi}{dx^2}\,\bar{\phi}\, dx.$$

Using the Dirichlet boundary conditions we can integrate by parts on the right without any boundary term, yielding

$$\lambda \int_0^\ell |\phi|^2 \, dx = \int_0^\ell \left|\frac{d\phi}{dx}\right|^2 \, dx. \tag{5.6}$$

Assuming that ϕ is not identically zero, this shows that $\lambda \geq 0$. Furthermore, $\lambda = 0$ implies $d\phi/dx = 0$, which gives a constant solution. The only constant solution is the trivial case $\phi = 0$, because of the boundary conditions.

It therefore suffices to consider the case $\lambda > 0$, for which the ODE in (5.4) reduces to the *harmonic oscillator* equation, with independent solutions given by $\sin(\sqrt{\lambda}x)$ and $\cos(\sqrt{\lambda}x)$. Only sine satisfies the condition $\phi(0) = 0$, so the possible solutions have the form

$$\phi(x) = \sin(\sqrt{\lambda}x).$$

To satisfy the condition $\phi(\ell) = 0$ we must have

$$\sin(\sqrt{\lambda}\ell) = 0.$$

For a nonzero solution this imposes the restriction that $\sqrt{\lambda}\ell \in \pi\mathbb{N}$, which gives the claimed set of solutions. □

Some of the eigenfunctions obtained in Theorem 5.2 are illustrated in Fig. 5.2. For the sake of application to our original string model, let us reinstate the propagation speed $c := \sqrt{T/\rho}$ and write the string equation as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \quad u(t,0) = u(0,\ell) = 0. \tag{5.7}$$

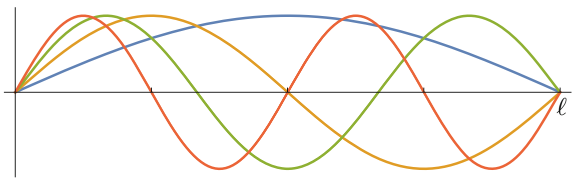

Fig. 5.2 The first four eigenfunctions for a vibrating string with fixed ends

With the spatial solution given by the eigenfunction associated to λ_n, the corresponding temporal eigenvalue equation is also a harmonic oscillator ODE,

$$-\frac{d^2v}{dt^2} = c^2\lambda_n v.$$

The solutions could be written in terms of sines and cosines, but for the temporal component it is usually more convenient to use the complex exponential form. The general complex-valued solution is

$$v_n(t) = a_n e^{i\omega_n t} + b_n e^{-i\omega_n t},$$

with $a_n, b_n \in \mathbb{C}$ and

$$\omega_n := c\sqrt{\lambda_n} = \frac{c\pi n}{\ell}. \tag{5.8}$$

For real-valued solutions, the coefficients are restricted by $b_n = \overline{a_n}$.

Combining the temporal and spatial components gives a set of solutions for the vibrating string problem:

$$u_n(t,x) = \left[a_n e^{i\omega_n t} + b_n e^{-i\omega_n t}\right]\sin(\sqrt{\lambda_n}x), \tag{5.9}$$

for $n \in \mathbb{N}$.

The functions (5.9) are referred to as "pure-tone" solutions, because they model oscillation at a single frequency ω_n. In the case of visible light waves, the frequency corresponds directly to color. For this reason the set of frequencies $\{\omega_n\}$ is called the *spectrum*. By association, the term spectrum is also used for sets of eigenvalues appearing in more general problems. For example, the set $\{\lambda_n\}$ of eigenvalues for which the Helmholtz problem has a nontrivial solution is called the spectrum of the Laplacian, even though λ_n is proportional to the square of the frequency ω_n.

From (5.8) we can deduce the fundamental tone of the string, as predicted by d'Alembert's wave equation model. To convert frequency to the standard unit of Hz (cycles per second), we divide ω_1 by 2π to obtain the formula

$$\frac{\omega_1}{2\pi} = \frac{1}{2\ell}\sqrt{\frac{T}{\rho}}. \tag{5.10}$$

This is known as *Mersenne's law*, published in 1637.

The wave equation model also predicts the higher frequencies $\omega_n = n\omega_1$, corresponding to the sequence of overtones noted illustrated in Fig. 5.1. The fact that each overtone is associated with a particular spatial eigenfunction is significant. The waveforms for higher overtones have *nodes*, meaning points where the string is stationary. As we can see in Fig. 5.2, the nodes associated to the frequency ω_n subdivide the string into n equal segments. Touching the string lightly at one of these nodes will knock out the lower frequencies, a practice string players refer to as playing a "harmonic".

As this discussion illustrates, the "spectral analysis" of the wave equation is more directly connected to experimental observation than the explicit solution formula (4.8). The displacement of a vibrating string is technically difficult to observe directly because the motion is both rapid and of small amplitude. Such observations were first achieved by Hermann von Helmholtz in the mid-19th century.

Example 5.3 The one-dimensional wave equation can be used to model for the fluctuations of air pressure inside a clarinet. The interior of a clarinet is essentially a cylindrical column, and for simplicity we can assume that the pressure is constant on cross-sections of the cylinder, so that the variations in pressure are described by a function $u(t, x)$ with $x \in [0, \ell]$, where ℓ is the length of the instrument. Pressure fluctuations are measured relative to the fixed atmospheric background, with $u = 0$ for atmospheric pressure.

The maximum pressure fluctuation occurs at the mouthpiece at $x = 0$, where a reed vibrates as the player blows air into the instrument. Since a local maximum of the pressure corresponds to a critical point of $u(t, \cdot)$, the appropriate boundary condition is

$$\frac{\partial u}{\partial x}(t, 0) = 0. \tag{5.11}$$

At the opposite end the air column is open to the atmosphere, so the pressure does not fluctuate,

$$u(t, \ell) = 0. \tag{5.12}$$

The evolution of u as a function of t is governed by the wave equation (4.5), with c equal to the speed of sound. The corresponding Helmholtz problem is

$$-\frac{d^2\phi}{dx^2} = \lambda\phi, \quad \phi'(0) = 0, \quad \phi(\ell) = 0. \tag{5.13}$$

The boundary condition at $x = 0$ implies that

$$\phi(x) = \cos(\sqrt{\lambda}x),$$

and the condition at $x = \ell$ then requires

$$\cos(\sqrt{\lambda}\ell) = 0.$$

This means that the eigenvalues are given by

$$\lambda_n := \frac{\pi^2}{\ell^2}(n - \tfrac{1}{2})^2,$$

for $n \in \mathbb{N}$. Some of the resulting eigenfunctions are shown in Fig. 5.3.

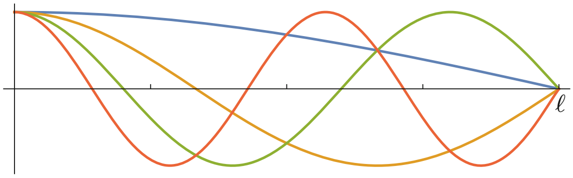

Fig. 5.3 The first four eigenfunctions for pressure fluctuations in a clarinet

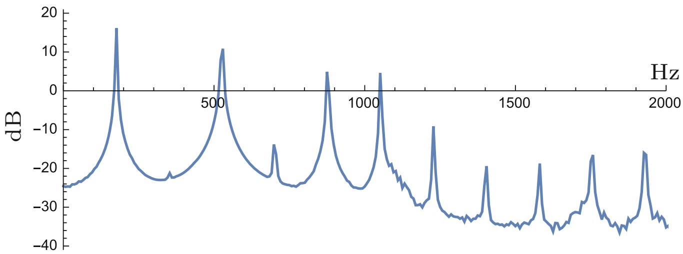

Fig. 5.4 Clarinet frequency decomposition

The corresponding oscillation frequencies are given by

$$\omega_n = \frac{c\pi}{\ell}(n - \tfrac{1}{2}).$$

In contrast to the string, the model predicts that the clarinet's spectrum will contain only odd multiples of the fundamental frequency ω_1. Figure 5.4 shows the frequency decomposition for a clarinet sound sample. The prediction holds true for the first few modes, but the simple model appears to break down at higher frequencies. ♢

5.3 Circular Symmetry

In dimension greater than one, spatial separation of variables is essentially the only way to compute explicit solutions of the Helmholtz equation (5.3), and this only works for very special cases. The most straightforward example is a rectangular domain in $\mathbb{R}^n$, which we will discuss in the exercises.

In this section we consider the simplest non-rectangular case, based on polar coordinates (r, θ) in $\mathbb{R}^2$. Separation in polar coordinates allows us to compute eigenfunctions and eigenvalues on a disk in $\mathbb{R}^2$, for example.

With $\boldsymbol{x} = (x_1, x_2)$ in $\mathbb{R}^2$, polar coordinates are defined by

$$(x_1, x_2) = (r\cos\theta, r\sin\theta).$$

The polar form of the Laplacian is computed by writing

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

and then converting the partials with respect to x_1 and x_2 into r and θ derivatives using the chain rule. The result is

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{5.14}$$

Note that there are no mixed partials involving both r and θ, and that the coefficients do not depend on θ. This allows separation of r and θ, provided the domain is defined by specifying ranges of r and θ.

To solve the radial eigenvalue equation, we will use *Bessel functions*, named for the astronomer Friedrich Bessel. Bessel's equation is the ODE:

$$z^2 f''(z) + zf'(z) + (z^2 - k^2) f(z) = 0, \tag{5.15}$$

with $k \in \mathbb{C}$ in general. For our application k will be an integer. The standard pair of linearly independent solutions is given by the Bessel functions $J_k(z)$ and $Y_k(z)$.

The Bessel J-functions, a few of which are pictured in Fig. 5.5, satisfy

$$J_{-k}(z) = (-1)^k J_k(z), \tag{5.16}$$

for all $k \in \mathbb{Z}$. Bessel represented these solutions as integrals:

$$J_k(z) := \frac{1}{\pi}\int_0^\pi \cos\big(z\sin\theta - k\theta\big)\, d\theta.$$

One can also write J_k as a power series $k \in \mathbb{N}_0$,

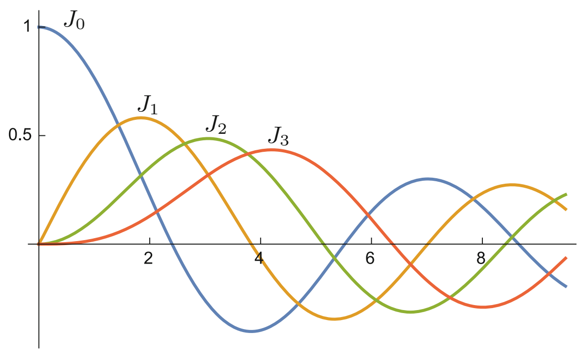

Fig. 5.5 The first four Bessel J-functions

$$J_k(z) = \left(\frac{z}{2}\right)^k \sum_{l=0}^{\infty} \frac{1}{l!(k+l)!} \left(-\frac{z^2}{4}\right)^l. \tag{5.17}$$

Together with (5.16), this shows that $J_k(z) \sim c_k z^{|k|}$ as $z \to 0$ for any $k \in \mathbb{Z}$. In contrast, the Bessel Y-function satisfies $Y_k(z) \sim c_k z^{-|k|}$ as $z \to 0$.

A change of sign in (5.15) gives the equation

$$z^2 f''(z) + z f'(z) + (z^2 + k^2) f(z) = 0. \tag{5.18}$$

Its standard solutions are the *modified Bessel functions* $I_k(z)$ and $K_k(z)$. As $z \to 0$ these satisfy the asymptotics $I_k(z) \sim c_k z^{|k|}$, as illustrated in Fig. 5.6, and $K_k(z) \sim c_k z^{-|k|}$.

Lemma 5.4 *Suppose $\phi \in C^2(\mathbb{R}^2)$ is a solution of*

$$-\Delta\phi = \lambda\phi,$$

that factors as a product $h(r)w(\theta)$. Then, up to a multiplicative constant, ϕ has the form

$$\phi_{\lambda,k}(r,\theta) := h_k(r) e^{ik\theta}, \tag{5.19}$$

for some $k \in \mathbb{Z}$, with

$$h_k(r) := \begin{cases} r^{|k|}, & \lambda = 0, \\ J_k(\sqrt{\lambda} r), & \lambda > 0, \\ I_k(\sqrt{-\lambda} r), & \lambda < 0. \end{cases}$$

Proof Under the assumption $\phi = hw$, the Helmholtz equation reduces by (5.14) to

$$\frac{w}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{h}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \lambda h w = 0.$$

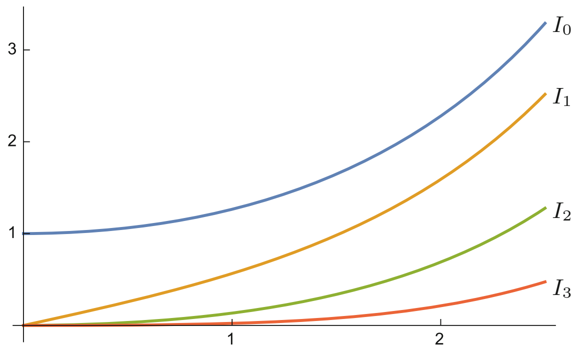

Fig. 5.6 The first four modified Bessel I-functions

With some rearranging, we can separate the r and θ variables,

$$\frac{1}{h}\left(r\frac{\partial}{\partial r}\right)^2 h + \lambda^2 r^2 = -\frac{1}{w}\frac{\partial^2 w}{\partial \theta^2}, \tag{5.20}$$

provided h and w are nonzero.

As in Lemma 5.1, we conclude that both sides must be equal to some constant κ. The θ equation is

$$-\frac{\partial^2 w}{\partial \theta^2} = \kappa w. \tag{5.21}$$

The function $w(\theta)$ is assumed to be 2π-periodic. By the arguments used in Theorem 5.2, a nontrivial solution is possible only if $\kappa = k^2$ where k is an integer. A full set of 2π-periodic solutions of (5.21) is given by

$$w_k(\theta) := e^{ik\theta}, \quad k \in \mathbb{Z}.$$

Before examining the radial equation, let us note that the assumption that ϕ is C^2 imposes a boundary condition at $r = 0$. To see this, first note that the function $r = \sqrt{x_1^2 + x_2^2}$ is continuous at $(0, 0)$ but not differentiable. For $r > 0$,

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r},$$

which does not have a limit as $r \to 0$. On the other hand, the functions

$$re^{\pm i\theta} = x_1 \pm ix_2$$

are C^∞. Similarly, for $k \in \mathbb{Z}$ we have

$$r^{|k|}e^{ik\theta} = \begin{cases} (x_1 + ix_2)^k, & k \in \mathbb{N}_0, \\ (x_1 - ix_2)^{-k}, & -k \in \mathbb{N}. \end{cases} \tag{5.22}$$

These functions are polynomial and hence C^∞. We will see below that the solutions of the radial equation corresponding to $\kappa = k^2$ satisfy $h(r) \sim ar^{\pm k}$ as $r \to 0$, for some constant a. The differentiability of ϕ at the origin will require the asymptotic condition

$$h_k(r) \sim ar^{|k|} \tag{5.23}$$

as $r \to 0$.

For $w_k(\theta) = e^{ik\theta}$, the radial component of (5.20) is

$$\left(r\frac{\partial}{\partial r}\right)^2 h_k + (\lambda r^2 - k^2)h_k = 0. \tag{5.24}$$

The case $\lambda = 0$ is relatively straightforward to analyze. In this case (5.24) is homogeneous in the r variable (meaning invariant under scaling). Such equations are solved by monomials of the form $h_k(r) = r^\alpha$ with $\alpha \in \mathbb{R}$. If we substitute this guess into (5.24) with $\lambda = 0$, the equation reduces to

$$\alpha^2 - k^2 = 0,$$

with solutions $\alpha = \pm k$. Since a second order ODE has exactly two independent solutions, the functions $r^{\pm k}$ give a full set of solutions for $k \neq 0$. For $k = 0$ the two possibilities are 1 and $\ln r$. By the condition (5.23), the solutions $\ln r$ and $r^{-|k|}$ must be ruled out. The only possible solutions for $\lambda = 0$ are thus

$$h_k(r) = r^{|k|}.$$

Note that the resulting solutions,

$$\phi_{0,k}(r, \theta) := r^{|k|} e^{ik\theta},$$

are precisely the polynomials (5.22).

For $\lambda > 0$ (5.24) can be reduced to the Bessel form (5.15) by the change of variables $z = \sqrt{\lambda} r$. The possible solutions $Y_k(\sqrt{\lambda} r)$ are ruled out because they diverge at $r = 0$. On the other hand, the power series (5.17) shows that the function $h_k(r) = J_k(\sqrt{\lambda} r)$ satisfies the condition (5.23). Thus for $\lambda > 0$ the possible eigenfunction with $k \in \mathbb{Z}$ is

$$\phi_{\lambda,k}(r, \theta) := J_k(\sqrt{\lambda} r) e^{ik\theta}.$$

We should check that this function is at least C^2 at the origin. In fact, it follows from the power series expansion (5.17) that $\phi_{\lambda,k}$ is C^∞ on $\mathbb{R}^2$.

Similar considerations apply for $\lambda < 0$, except that this time the substitution $z = \sqrt{-\lambda} r$ reduces (5.24) to (5.18). The condition (5.23) is satisfied only for the solution $I_k(\sqrt{-\lambda} r)$. □

Example 5.5 The linear model for the vibration of a drumhead is the wave equation (4.30). For a circular drum we can take the spatial domain to be the unit disk $\mathbb{D} := \{r < 1\} \subset \mathbb{R}^2$. Lemma 5.1 reduces the problem of determining the frequencies of the drum to the Helmholtz equation,

$$-\Delta \phi = \lambda \phi, \qquad \phi|_{\partial \mathbb{D}} = 0. \tag{5.25}$$

The possible product solutions are given by Lemma 5.4, subject to the boundary condition $h_k(1) = 0$. This rules out $\lambda \leq 0$, because in that case $h_k(r)$ has no zeros for $r > 0$.

For $\lambda > 0$, we have $h_k(r) = J_k(\sqrt{\lambda} r)$, and the boundary condition takes the form

$$J_k(\sqrt{\lambda}) = 0.$$

Table 5.1 Zeros of the Bessel function J_k. For each k, the spacing between zeros approaches π as $m \to \infty$

k	$j_{k,1}$	$j_{k,2}$	$j_{k,3}$	$j_{k,4}$
0	2.405	5.520	8.654	11.792
1	3.832	7.016	10.174	13.324
2	5.136	8.417	11.620	14.796
3	6.380	9.761	13.015	16.223
4	7.588	11.065	14.373	17.616

(This is analogous to the condition $\sin(\sqrt{\lambda}\ell) = 0$ from the one-dimensional string problem.) Although J_k is not a periodic function, it does have an infinite sequence of positive zeros with roughly evenly spacing. It is customary to write these zeros in increasing order as

$$0 < j_{k,1} < j_{k,2} < \dots.$$

By the symmetry (5.16),

$$j_{-k,m} = j_{k,m}.$$

Table 5.1 lists some of these zeros.

Restricting $\sqrt{\lambda}$ to the set of Bessel zeros gives the set of eigenvalues

$$\lambda_{k,m} = j_{k,m}^2,$$

indexed by $k \in \mathbb{Z}$, $m \in \mathbb{N}$. The corresponding eigenfunctions are

$$\phi_{k,m}(r,\theta) := J_k(j_{k,m}r)e^{ik\theta}. \tag{5.26}$$

The first set of these are illustrated in Fig. 5.7.

The collection of functions (5.26) yields a complete list of eigenfunctions and eigenvalues for $\mathbb{D}$, although that is not something we can prove here. ♢

The eigenvalues calculated in Example 5.5 correspond to vibrational frequencies

$$\omega_{k,m} := cj_{k,m},$$

for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. The propagation speed c depends on physical properties such as tension and density. The relative size of the frequencies helps to explain the lack of definite pitch in the sound of a drum. The ratios of overtones above the fundamental $\omega_{0,1}$ are shown in Table 5.2. In contrast to the vibrating string case, where the corresponding ratios were integers $1, 2, 3, \dots$, or the clarinet model of Example 5.3 with ratios $1, 3, 5, \dots$, the frequencies of the drum are closely spaced with no evident pattern.

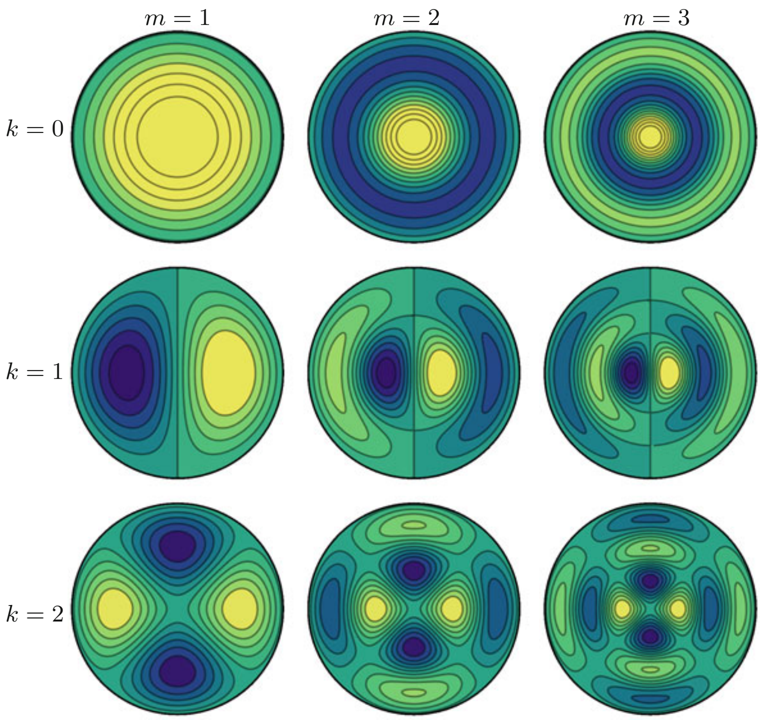

Fig. 5.7 Contour plots of the spatial component of the eigenfunctions of $\mathbb{D}$

Table 5.2 Frequency ratios for a circular drumhead

k	m	$\omega_{k,m}/\omega_{0,1}$
0	1	1
1	1	1.593
2	1	2.136
0	2	2.295
3	1	2.653
1	2	2.917
4	1	3.155

5.4 Spherical Symmetry

Another special case that allows separation of spatial variables is spherical symmetry in $\mathbb{R}^3$. Spherical coordinates (r, φ, θ) are defined through the relation

$$(x_1, x_2, x_3) = (r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi).$$

Note that θ is the azimuthal angle here and φ the polar angle, consistent with the notation from Sect. 5.3. (This convention is standard in mathematics; in physics the roles are often reversed.)

As in the circular case, we can use the chain rule to translate the three-dimensional Laplacian into spherical variables:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}. \tag{5.27}$$

It is not immediately clear that this operator admits separation, because the coefficients depend on both r and φ. Note, however, that we can factor r^{-2} out of the angular derivative terms, to write (5.27) as

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^2}, \tag{5.28}$$

where

$$\Delta_{\mathbb{S}^2} := \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}. \tag{5.29}$$

Here $\mathbb{S}^2$ stands for the unit sphere $\{r = 1\} \subset \mathbb{R}^3$, and $\Delta_{\mathbb{S}^2}$ is called the *spherical Laplacian*.

The expression (5.29) may look awkward at first glance, but $\Delta_{\mathbb{S}^2}$ is a very natural operator geometrically. From the fact that Δ is invariant under rotations of $\mathbb{R}^3$ about the origin, we can deduce that $\Delta_{\mathbb{S}^2}$ is also invariant under rotations of the sphere. It is possible to show that $\Delta_{\mathbb{S}^2}$ is the only second-order operator with this property, up to a multiplicative constant. The operator $\Delta_{\mathbb{S}^2}$ is thus as symmetric as possible, and the reason that (5.29) looks so complicated is that the standard coordinate system (θ, φ) does not reflect the full symmetry of the sphere.

We will discuss the radial component of (5.28) in an example below. For now let us focus on the Helmholtz problem on the sphere, which allows further separation of the θ and φ variables.

The classical ODE that arises from separation of the angle variables is the *associated Legendre equation*:

$$(1 - z^2)f''(z) - 2zf'(z) + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)f(z) = 0, \tag{5.30}$$

with parameters $\mu, \nu \in \mathbb{C}$. A pair of linearly independent solutions is given by the *Legendre functions* $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$.

In the special case where ν is replaced by $l \in \mathbb{N}_0$ and μ by a number $m \in \{-l, \dots, l\}$, respectively, the Legendre P-functions are given by a relatively simple formula:

$$P_l^m(z) = \frac{(-1)^m}{2^l l!}(1 - z^2)^{m/2}\frac{d^{l+m}}{dz^{l+m}}(z^2 - 1)^l. \tag{5.31}$$

Associated to this set of Legendre functions are functions of the angle variables called *spherical harmonics*. These are defined by

$$Y_l^m(\varphi, \theta) := c_{l,m} e^{im\theta} P_l^m(\cos\varphi), \tag{5.32}$$

where $c_{l,m}$ is a normalization constant whose value is not important for us.

From (5.31), using $z = \cos\varphi$ and $1 - z^2 = \sin^2\varphi$, we can see that Y_l^m is a polynomial of degree l in $\sin\varphi$ and $\cos\varphi$. This makes it relatively straightforward to check that each $Y_l^m(\varphi, \theta)$ is a smooth function on $\mathbb{S}^2$.

Lemma 5.6 *Suppose $u \in C^2(\mathbb{S}^2)$ is a solution of the equation*

$$-\Delta_{\mathbb{S}^2} u = \lambda u \tag{5.33}$$

that factors as $u(\varphi, \theta) = v(\varphi)w(\theta)$. Then up to a multiplicative constant, u is equal to a spherical harmonic Y_l^m for $l \in \mathbb{N}_0$ and $m \in \{-l, \dots, l\}$. The corresponding eigenvalues depend only on l,

$$\lambda_l = l(l+1),$$

and each has multiplicity $2l + 1$.

Proof By (5.29), the substitution $u = vw$ leads to the separated equation

$$\frac{\sin\varphi}{v}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial v}{\partial\varphi}\right) + \lambda = -\frac{1}{w}\frac{\partial^2 w}{\partial\theta^2}.$$

The continuity of u requires that w be 2π-periodic. Hence, for the θ equation

$$-\frac{\partial^2 w}{\partial\theta^2} = \kappa w,$$

the full set of solutions is represented by $w(\theta) = e^{im\theta}$ with $\kappa = m^2$ for $m \in \mathbb{Z}$.

With $u(\theta, \varphi) = v_m(\varphi)e^{im\theta}$, the eigenvalue equation (5.33) reduces to

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{dv_m}{d\varphi}\right) + \left(\lambda - \frac{m^2}{\sin^2\varphi}\right)v_m = 0.$$

Under the substitutions $z = \cos\varphi$ and $v_m(\varphi) = f(\cos\varphi)$, this becomes

$$(1 - z^2)f'' - 2zf' + \left(\lambda - \frac{m^2}{1 - z^2}\right)f = 0,$$

which is recognizable as the Legendre equation (5.30) with parameters $m = \mu$ and $\lambda = \nu(\nu + 1)$.

Although $\mathbb{S}^2$ does not have a boundary, use of the coordinate $\varphi \in [0, \pi]$ creates artificial boundaries at the endpoints, i.e., at the poles of the sphere. This is analogous to the boundary at $r = 0$ in Lemma 5.4. We need to find solutions which will be smooth at the poles.

It turns out that for $m \in \mathbb{Z}$, the function $Q_\nu^m(z)$ diverges as $z \to 1$ for any $\nu \in \mathbb{C}$. Similarly, the functions $P_\nu^m(z)$ diverge as $z \to -1$ except for the special cases $P_l^m(z)$ given by (5.31). In other words, up to a multiplicative constant $v_m(\varphi)$ must be equal to $P_l^m(\cos\varphi)$ for some $l \in \mathbb{N}_0$ with $l \geq |m|$. The corresponding solution u is proportional to the spherical harmonic Y_l^m.

By the identification $\nu = l$, the eigenvalue is given by $\lambda = l(l+1)$. The corresponding multiplicity is the number of possible choices of $m \in \{-l, \ldots, l\}$, namely $2l + 1$. □

The spherical harmonics appearing in Lemma 5.6 give a complete set of eigenfunctions for $\Delta_{\mathbb{S}^2}$, in the sense that the only possible eigenvalues are $l(l+1)$ for $l \in \mathbb{N}_0$ and an eigenfunction with eigenvalue $l(l+1)$ is a linear combination of the Y_l^m for $m \in \{-l, \ldots, l\}$. To prove this requires more advanced methods than we have available here.

Example 5.7 In 1925, Erwin Schrödinger developed a quantum model for the hydrogen atom in which the electron energy levels are given by the eigenvalues of the equation

$$\left(-\Delta - \frac{1}{r}\right)\phi = \lambda\phi \tag{5.34}$$

on $\mathbb{R}^3$. (We have omitted the physical constants.) The eigenfunctions ϕ are assumed to be bounded near $r = 0$ and decaying to zero as $r \to \infty$.

Since the term $1/r$ is radial, separation of the radial and angular variables is possible in (5.34). By Lemma 5.6, the angular components are given by spherical harmonics. A corresponding full solution has the form

$$\phi(r, \varphi, \theta) = h(r) Y_l^m(\varphi, \theta). \tag{5.35}$$

Substituting this into (5.34) and using the spherical form of the Laplacian (5.28) gives the radial equation

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{l(l+1)}{r^2} - \frac{1}{r}\right] h(r) = \lambda h(r). \tag{5.36}$$

One strategy used to analyze an ODE such as (5.36) is to first consider the asymptotic behavior of solutions as $r \to 0$ or ∞.

Suppose we assume $h(r) \sim r^\alpha$ as $r \to 0$. Plugging this into (5.36) and comparing the two sides gives a leading term

$$-\alpha(\alpha+1)r^{\alpha-2} + l(l+1)r^{\alpha-2}$$

on the left side, with all other terms of order $r^{\alpha-1}$ or less. This shows that $h(r) \sim r^\alpha$ as $r \to 0$ is possible only if

$$\alpha(\alpha + 1) = l(l + 1).$$

The two solutions are $\alpha = l$ or $\alpha = -l - 1$. Taking $\alpha < 0$ would cause $h(r)$ to diverge as $r \to 0$. Therefore to obtain a solution bounded at the origin, we will assume that

$$h(r) \sim r^l$$

as $r \to 0$.

As $r \to \infty$, if we consider the terms in (5.36) with coefficients of order r^0 and drop the rest, the equation becomes

$$-h''(r) \sim \lambda h(r). \tag{5.37}$$

If $\lambda \geq 0$ then this shows that $h(r)$ could not possibly decay at infinity. Hence we assume that $\lambda < 0$ and set

$$\sigma^2 := -\lambda,$$

with $\sigma \in \mathbb{R}$. The asymptotic equation (5.37) implies the behavior

$$h(r) \sim ce^{-\sigma r}$$

as $r \to \infty$.

Determining these asymptotics allows us to make an educated guess for the form of the solution. For an as yet undetermined function $q(r)$, we set

$$h(r) = q(r)r^l e^{-\sigma r}, \tag{5.38}$$

with the conditions that $q(0) = 1$ and $q(r)$ has subexponential growth as $r \to \infty$. The goal of setting up the solution this way is that the equation for $q(r)$ will simplify. Substituting (5.38) into (5.36) leads to the equation

$$rq'' + 2(1 + l - r\sigma)q' + (1 - 2\sigma(l + 1))q = 0. \tag{5.39}$$

To find solutions, we suppose $q(r)$ is given by a power series

$$q(r) = \sum_{k=0}^{\infty} a_k r^k,$$

with $a_0 = 1$. Plugging this into (5.39) gives

$$0 = \sum_{k=0}^{\infty} \Big[k(k-1)a_k r^{k-1} + 2(1 + l - r\sigma)ka_k r^{k-1} + (1 - 2\sigma(l+1))a_k r^k \Big].$$

Equating the coefficient of r^k to zero then gives a recursive relation

$$a_{k+1} = \frac{2\sigma(k+l+1) - 1}{(k+1)(k+2l+2)} a_k. \tag{5.40}$$

If we assume that the numerator of (5.40) never vanishes, then the recursion relation implies that

$$a_k \sim \frac{(2\sigma)^k}{k!}$$

as $k \to \infty$. This would give $q(r) \sim ce^{2\sigma r}$ as $r \to \infty$, making $h(r)$ also grow exponentially as $r \to \infty$.

The only way to avoid this exponential growth is for the sequence of a_k to terminate at some point, so that q is a polynomial. The numerator on the right side of (5.40) will eventually vanish if and only if

$$\sigma = \frac{1}{2n},$$

for some integer $n \geq l + 1$. Under this assumption the sequence a_k terminates at $k = n - l - 1$. Since $\lambda = -\sigma^2$, this restriction on σ gives the set of eigenvalues

$$\lambda_n := -\frac{1}{4n^2}, \quad n \in \mathbb{N}.$$

This is in fact the complete set of eigenvalues for this problem, given the conditions we have imposed at $r = 0$ and $r \to \infty$. With this eigenvalue calculation, Schrödinger was able to give the first theoretical explanation of the emission spectrum of hydrogen gas (i.e., the set of wavelengths observed when the gas is excited electrically). The origin of these emission lines had been a mystery since their discovery by Anders Jonas Ångström in the mid-19th century.

Each value of n corresponds to a family of eigenfunctions given by

$$\phi_{n,l,m}(r, \varphi, \theta) = r^l q_{n,l}(r) e^{-\frac{r}{2n}} Y_l^m(\varphi, \theta),$$

for $l \in \{0, \dots, n-1\}$, $m \in \{-l, \dots, l\}$. Here $q_{n,l}(r)$ denotes the polynomial of degree $n - l - 1$ with coefficients specified by (5.40). To compute the multiplicity of λ_n, we count $n - 1$ choices for l and then $2l + 1$ choices of m for each l. The total multiplicity is

$$\sum_{l=0}^{n-1} (2l + 1) = n^2.$$

◊

5.5 Exercises

5.1 On the half-strip $\Omega = (0, 1) \times (0, \infty) \subset \mathbb{R}^2$, find the solutions of

$$\Delta u = 0$$

that factor as a product $u(x_1, x_2) = g(x_1)h(x_2)$, under the boundary conditions

$$u(0, x_2) = u(1, x_2) = u(x_1, 0) = 0.$$

5.2 The linear model for vibrations of a rectangular drumhead is the wave equation (4.30) with Dirichlet boundary conditions on a rectangle $\mathcal{R} := [0, \ell_1] \times [0, \ell_2] \subset \mathbb{R}^2$. Separation of variables leads to the corresponding Helmholtz problem

$$-\Delta\phi = \lambda\phi, \qquad \phi|_{\partial\mathcal{R}} = 0.$$

Find the eigenfunctions of product type, $\phi(x_1, x_2) = \phi_1(x_1)\phi_2(x_2)$, and the associated frequencies of vibration. For $\ell_1 = \ell_2$, compare the ratios of these frequencies to Table 5.2. Would a square drum do a better job of producing a definite pitch?

5.3 The one-dimensional *heat equation* for the temperature $u(t, x)$ of a metal bar of length ℓ is

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

for $t \geq 0$ and $x \in (0, \ell)$. (We will derive this in Sect. 6.1.) If the ends of the bar are insulated, then u should satisfy Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, \ell) = 0.$$

Find the product solutions $u(t, x) = v(t)\phi(x)$.

5.4 The *damped wave equation* on $\Omega \subset \mathbb{R}^n$ is

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} - \Delta u = 0, \tag{5.41}$$

where $u \in C^2([0, \infty) \times \Omega)$ and $\gamma \geq 0$ is a constant called the coefficient of friction. Suppose that $\phi \in C^2(\Omega)$ satisfies the Helmholtz equation (5.3) on Ω with eigenvalue $\lambda > 0$, for some appropriate choice of boundary conditions. Show that that (5.41) has solutions of the form

$$u(t, \boldsymbol{x}) = \phi(\boldsymbol{x})e^{i\omega t},$$

and find the set of possible values of ω. In particular, show that Im $\omega > 0$ if $\gamma > 0$, which implies that the solutions decay exponentially in time. Does this decay rate depend on the oscillation frequency?

5.5 Consider this example of a nonlinear diffusion equation:

$$\frac{\partial u}{\partial t} - \Delta(u^2) = 0,$$

for $t \geq 0$, $\boldsymbol{x} \in \mathbb{R}^n$.

(a) Assuming a product solution of the form $u(t, x) = v(t)\phi(\boldsymbol{x})$, separate variables and find the equations for $v(t)$ and $\phi(\boldsymbol{x})$.
(b) Show that $\phi(\boldsymbol{x}) = |\boldsymbol{x}|^2$ solves the spatial equation, and find the corresponding function $v(t)$ given the initial condition $v(0) = a > 0$. (Observe that the solution "blows up" at a finite time that depends on a.)

5.6 In polar coordinates for $\mathbb{R}^2$, define the domain

$$\Omega = \left\{(r, \theta);\ 0 < r < 1,\ 0 < \theta < \pi/3\right\},$$

which is a sector within the unit disk. Find the eigenvalues of Δ on Ω with Dirichlet boundary conditions.

5.7 The quantum energy levels of a harmonic oscillator in $\mathbb{R}^n$ are the eigenvalues of the equation

$$\left(-\Delta + |\boldsymbol{x}|^2\right)\phi = \lambda\phi, \tag{5.42}$$

under the condition that $\phi \in C^2(\mathbb{R}^n)$ and $\phi \to 0$ at infinity.

(a) First consider the case $n = 1$:

$$\left(-\frac{\partial^2}{\partial x^2} + x^2\right)\phi = \kappa\phi, \tag{5.43}$$

Substitute

$$\phi(x) = q(x)e^{-x^2/2}$$

into (5.43) and find the corresponding ODE for q.
(b) Assume that the function q from (a) is given by a power series in x,

$$q(x) = \sum_{k=0}^{\infty} a_k x^k,$$

and find a recursive equation for a_{k+2} in terms of a_k.
(c) Find the values of κ for which the power series for q from (b) truncates to a polynomial. (The resulting functions q are called *Hermite polynomials*.)

(d) Returning to the original problem, by reducing (5.42) to n copies of the case (5.43), deduce the set of eigenvalues λ.

5.8 Let $\mathbb{B}^3 \subset \mathbb{R}^3$ be the unit ball $\{r < 1\}$. Consider the Helmholtz problem

$$-\Delta\phi = \lambda\phi,$$

with Dirichlet boundary conditions at $r = 1$.

(a) Assume that

$$\phi(r, \varphi, \theta) = h_l(r) Y_l^m(\varphi, \theta),$$

where Y_l^m is the spherical harmonic introduced in Sect. 5.4. Find the radial equation for $h_l(r)$.

(b) For $l = 0$ show that the radial equation is solved by

$$h_0(r) = \frac{\sin(\sqrt{\lambda} r)}{r}.$$

What set of eigenvalues λ does this give?

(c) Show that the substitution,

$$h_l(r) = r^{-\frac{1}{2}} f_l(\sqrt{\lambda} r),$$

reduces the equation from (a) to a Bessel equation (5.15) for $f_l(z)$, with a fractional value of k. Use this to write the solution $h_l(r)$ in terms of J_k.

(d) Express the eigenvalues λ in terms of Bessel zeros with fractional values of k.