Chapter 13
The Fourier Transform

For a bounded domain $\Omega \subset \mathbb{R}^n$, Theorem 11.7 shows that we can effectively "diagonalize" the Laplacian by choosing an orthonormal basis for $L^2(\Omega)$ consisting of eigenfunctions. Such a result is not possible on $\mathbb{R}^n$ itself; the Laplacian has no eigenfunctions in $L^2(\mathbb{R}^n)$.

The closest analog to eigenfunctions on $\mathbb{R}^n$ are the spatial components of the plane wave solutions introduced in Exercise 4.8,

$$\phi_{\boldsymbol{\xi}}(\boldsymbol{x}) := e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}},$$

associated to a frequency vector $\boldsymbol{\xi} \in \mathbb{R}^n$. These functions satisfy a convenient differentiation formula,

$$D^\alpha \phi_{\boldsymbol{\xi}} = (i\boldsymbol{\xi})^\alpha \phi_{\boldsymbol{\xi}},$$

and in particular

$$-\Delta e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} = |\boldsymbol{\xi}|^2 e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}.$$

The appropriate generalization of the Fourier series to $L^2(\mathbb{R}^n)$ is an integral transform based on these plane waves. Although the technical details are quite different from Fourier series, the transform serves a similar purpose in that it exchanges the roles of differentiation and multiplication.

13.1 Fourier Transform

The *Fourier transform* of a function in $f \in L^1(\mathbb{R}^n)$ is a function of the frequency $\boldsymbol{\xi} \in \mathbb{R}^n$ defined by

$$\hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} f(\boldsymbol{x})\, d^n\boldsymbol{x}. \tag{13.1}$$

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Note that the integral is well defined by the integrability of f, and in fact

$$\left|\hat{f}(\boldsymbol{\xi})\right| \le \|f\|_{L^1} \tag{13.2}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$. As a map the transform is denoted by

$$\mathcal{F} : f \mapsto \hat{f}.$$

To develop the properties of the Fourier transform, it proves convenient to introduce a particular class of test functions, called *Schwartz functions*. The space $\mathcal{S}$ consists of smooth functions which, along with all derivatives, decay *rapidly* at infinity. The precise meaning of "rapid" is "faster than any power of r." An alternate form of this definition is

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n);\ \left\|\boldsymbol{x}^\alpha D^\beta f\right\|_\infty < \infty \text{ for all } \alpha, \beta \right\}, \tag{13.3}$$

where $\|\cdot\|_\infty$ is the sup norm introduced in Sect. 7.3.

A basic example of a Schwartz function is a *Gaussian function* of the form

$$f(\boldsymbol{x}) = e^{-a|\boldsymbol{x}|^2},$$

with $a > 0$. We also have

$$C^\infty_{\text{cpt}}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n),$$

because compactly supported functions obviously satisfy the decay requirement.

As in the discrete case, the Fourier transform interchanges the operations of differentiation and multiplication in a convenient way. For this statement, we let $D^\alpha_{\boldsymbol{x}}$ and $D^\alpha_{\boldsymbol{\xi}}$ denote partial derivatives with respect to $\boldsymbol{x}$ or $\boldsymbol{\xi}$, respectively.

Lemma 13.1 *For $\psi \in \mathcal{S}(\mathbb{R}^n)$,*

$$\mathcal{F}[D^\alpha_{\boldsymbol{x}}\psi](\boldsymbol{\xi}) = (i\boldsymbol{\xi})^\alpha \hat{\psi}(\boldsymbol{\xi}), \tag{13.4}$$

and

$$\mathcal{F}[\boldsymbol{x}^\alpha\psi](\boldsymbol{\xi}) = (iD_{\boldsymbol{\xi}})^\alpha \hat{\psi}(\boldsymbol{\xi}). \tag{13.5}$$

Proof The first identity follows from integration by parts,

$$\begin{aligned}
\mathcal{F}[D^\alpha\psi](\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} D^\alpha_{\boldsymbol{x}}\psi(\boldsymbol{x})\, d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} \psi(\boldsymbol{x})(iD_{\boldsymbol{\xi}})^\alpha(e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}})\, d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} \psi(\boldsymbol{x})(i\boldsymbol{\xi})^\alpha e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \\
&= (i\boldsymbol{\xi})^\alpha \hat{\psi}(\boldsymbol{\xi}).
\end{aligned}$$

The second is also a direct computation,

$$\begin{aligned}\mathcal{F}[\boldsymbol{x}^\alpha f](\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\boldsymbol{x}^\alpha f(\boldsymbol{x})\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} (iD_{\boldsymbol{\xi}})^\alpha (e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}) f(\boldsymbol{x})\, d^n\boldsymbol{x} \\ &= (iD_{\boldsymbol{\xi}})^\alpha \hat{f}(\boldsymbol{\xi}).\end{aligned}$$

Pulling the differentiation outside the integral in the final step is justified by the smoothness and decay assumptions on ψ. □

In Lemma 13.1 we can see Schwartz's motivation for the definition of $\mathcal{S}$. Under the Fourier transform, smoothness translates to rapid decay, and vice versa. These properties are balanced in the definition of $\mathcal{S}$, which leads to the following result.

Lemma 13.2 *The Fourier transform* $\mathcal{F}$ *maps* $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

Proof Suppose that $f \in \mathcal{S}$. In order to show that $\hat{f}$ is Schwartz, we need to produce a bound on the function $\boldsymbol{\xi}^\beta D^\alpha \hat{f}$ for each α, β. By (13.4) and (13.5),

$$\boldsymbol{\xi}^\beta D^\alpha_{\boldsymbol{\xi}} \hat{f}(\boldsymbol{\xi}) = i^{|\alpha|+|\beta|} \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\boldsymbol{x}^\alpha D^\beta_{\boldsymbol{x}} f(\boldsymbol{x})\, d^n\boldsymbol{x}. \tag{13.6}$$

To estimate, we set

$$M_{N,\alpha,\beta} := \left\| (1+|\boldsymbol{x}|^2)^N \boldsymbol{x}^\alpha D^\beta_{\boldsymbol{x}} f \right\|_\infty,$$

which is finite by the definition (13.3). Because $(1+|x|^2)^{-N}$ is integrable for N sufficiently large, we can estimate (13.6) by

$$\left| \boldsymbol{\xi}^\beta D^\alpha \hat{f}(\boldsymbol{\xi}) \right| \le M_{N,\alpha,\beta} \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^N}\, d^n\boldsymbol{x}.$$

The right-hand side is independent of $\boldsymbol{\xi}$, so this yields the required estimate. □

Example 13.3 Consider the one-dimensional Gaussian function

$$\varphi(x) := e^{-ax^2}$$

for $a > 0$. Note that φ satisfies the ODE

$$\frac{d\varphi}{dx} = -2ax\varphi.$$

Taking the Fourier transform of both sides and applying Lemma 13.1 gives

$$i\xi\hat{\varphi} = -2ai\frac{d\hat{\varphi}}{d\xi},$$

which reduces to

$$\frac{d\hat{\varphi}}{d\xi} = -\frac{\xi}{2a}\hat{\varphi}.$$

Separating variables and integrating yields the solution

$$\hat{\varphi}(\xi) = \hat{\varphi}(0)e^{-\xi^2/4a}.$$

To fix the constant, we can use (2.19) with $n = 1$ to compute

$$\begin{aligned}\hat{\varphi}(0) &= \int_{-\infty}^{\infty} e^{-ax^2}\, dx \\ &= \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-x^2}\, dx \\ &= \sqrt{\frac{\pi}{a}}.\end{aligned} \tag{13.7}$$

Thus,

$$\hat{\varphi}(\xi) = \sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}.$$

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The computation from Example 13.3 can be generalized to $\mathbb{R}^n$ by factoring the integrals,

$$\begin{aligned}\mathcal{F}\left[e^{-a|\boldsymbol{x}|^2}\right](\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}-a|\boldsymbol{x}|^2}\, d^n\boldsymbol{x} \\ &= \prod_{j=1}^{n}\left(\int_{-\infty}^{\infty} e^{-ix_j\xi_j-ax_j^2}\, dx_j\right) \\ &= \prod_{j=1}^{n}\sqrt{\frac{\pi}{a}}e^{-\xi_j^2/4a}.\end{aligned}$$

Thus

$$\mathcal{F}\left[e^{-a|\boldsymbol{x}|^2}\right](\boldsymbol{\xi}) = \left(\frac{\pi}{a}\right)^{\frac{n}{2}} e^{-|\boldsymbol{\xi}|^2/4a} \tag{13.8}$$

for $a > 0$.

For $f, g \in \mathcal{S}(\mathbb{R}^n)$, consider the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\boldsymbol{x})e^{-i\boldsymbol{x}\cdot\boldsymbol{y}}g(\boldsymbol{y})\, d^n\boldsymbol{x}\, d^n\boldsymbol{y}. \tag{13.9}$$

The integrals over $\boldsymbol{x}$ and $\boldsymbol{y}$ can be taken in either order, yielding the useful identity:

$$\int_{\mathbb{R}^n} f\hat{g}\, d^n\boldsymbol{x} = \int_{\mathbb{R}^n} \hat{f} g\, d^n\boldsymbol{y} \tag{13.10}$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

Theorem 13.4 *The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ has an inverse $\mathcal{F}^{-1}$ given by*

$$f(\boldsymbol{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \hat{f}(\boldsymbol{\xi})\, d^n\boldsymbol{\xi}. \tag{13.11}$$

Proof In (13.10) let us set $g = e^{-a\boldsymbol{x}^2}$ for $a > 0$, By (13.8), this implies

$$\left(\frac{\pi}{a}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-\boldsymbol{x}^2/4a}\, d^n\boldsymbol{x} = \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{y}) e^{-a\boldsymbol{y}^2}\, d^n\boldsymbol{y}. \tag{13.12}$$

On the left-hand side we can use the same argument as in the proof of Lemma 12.1 to show that

$$\begin{aligned}\lim_{a\to 0}\left[\left(\frac{\pi}{a}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-\boldsymbol{x}^2/4a}\, d^n\boldsymbol{x}\right] &= \pi^{\frac{n}{2}} \lim_{a\to 0} \int_{\mathbb{R}^n} f(\sqrt{a}\boldsymbol{x}) e^{-\boldsymbol{x}^2/4}\, d^n\boldsymbol{x} \\ &= (2\pi)^n f(0).\end{aligned}$$

We claim that the corresponding limit on the right-hand side of (13.12) is

$$\lim_{a\to 0} \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{y}) e^{-a\boldsymbol{y}^2}\, d^n\boldsymbol{y} = \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{y})\, d^n\boldsymbol{y}. \tag{13.13}$$

Because the convergence is not uniform, we will check this carefully. The difference of the two sides can be estimated by

$$\left|\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{y}) e^{-a\boldsymbol{y}^2}\, d^n\boldsymbol{y} - \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{y})\, d^n\boldsymbol{y}\right| \le \int_{\mathbb{R}^n} \left|\hat{f}(\boldsymbol{y})\right| \left(1 - e^{-a\boldsymbol{y}^2}\right)\, d^n\boldsymbol{y}.$$

Given $\varepsilon > 0$ we can choose R large enough that

$$\int_{|\boldsymbol{x}|>R} \left|\hat{f}(\boldsymbol{y})\right|\, d^n\boldsymbol{y} < \varepsilon,$$

since $\hat{f}$ is integrable. Splitting the integral at $|\boldsymbol{x}| = R$ gives the estimate

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\hat{f}(\boldsymbol{y})\right| \left(1 - e^{-a\boldsymbol{y}^2}\right)\, d^n\boldsymbol{y} &\le \varepsilon + \int_{B(0;R)} \left|\hat{f}(\boldsymbol{y})\right| \left(1 - e^{-a\boldsymbol{y}^2}\right)\, d^n\boldsymbol{y} \\ &\le \varepsilon + \left(1 - e^{-aR^2}\right) \|\hat{f}\|_1.\end{aligned}$$

The second term approaches zero as $a \to 0$, so that

$$\int_{\mathbb{R}^n} \left|\hat{f}(\boldsymbol{y})\right| \left(1 - e^{-ay^2}\right) \, d^n\boldsymbol{y} < 2\varepsilon,$$

for a sufficiently small. This establishes (13.13).

By these calculations, the limit of (13.12) as $a \to 0$ yields

$$(2\pi)^n f(0) = \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{x}) \, d^n\boldsymbol{x}. \tag{13.14}$$

This is a special case of the desired formula.

The general inverse formula can be deduced from (13.14) by a simple translation argument. For $\boldsymbol{w} \in \mathbb{R}^n$, define the translation operator $T_{\boldsymbol{w}}$ on $\mathcal{S}(\mathbb{R}^n)$ by

$$T_{\boldsymbol{w}} f(\boldsymbol{y}) := f(\boldsymbol{y} + \boldsymbol{w}).$$

A change of variables shows that

$$\begin{aligned} \widehat{T_{\boldsymbol{w}} f}(\boldsymbol{x}) &= \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y} + \boldsymbol{w}) \, d^n\boldsymbol{y} \\ &= \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot(\boldsymbol{y}-\boldsymbol{w})} f(\boldsymbol{y}) \, d^n\boldsymbol{y} \\ &= e^{i\boldsymbol{x}\cdot\boldsymbol{w}} \hat{f}(\boldsymbol{x}). \end{aligned}$$

Since $T_{\boldsymbol{w}} f(0) = f(\boldsymbol{w})$, plugging $T_{\boldsymbol{w}} f$ into (13.14) gives

$$(2\pi)^n f(\boldsymbol{w}) = \int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{w}} \hat{f}(\boldsymbol{x}) \, d^n\boldsymbol{x}.$$

□

The pairing formula (13.10) suggests that the L^2 inner product will behave naturally under the Fourier transform. Indeed, by Theorem 13.4 we can compute

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi})\overline{\hat{g}(\boldsymbol{\xi})} \, d^n\boldsymbol{\xi} &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \, d^n\boldsymbol{x} \right) \overline{\hat{g}(\boldsymbol{\xi})} \, d^n\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^n} \overline{\left(\int_{\mathbb{R}^n} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} \hat{g}(\boldsymbol{\xi}) \, d^n\boldsymbol{\xi} \right)} f(\boldsymbol{x}) \, d^n\boldsymbol{x} \\ &= (2\pi)^n \int_{\mathbb{R}^n} f(\boldsymbol{x})\overline{g(\boldsymbol{x})} \, d^n\boldsymbol{x}, \end{aligned}$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$. In other words,

$$\langle \hat{f}, \hat{g} \rangle = (2\pi)^n \langle f, g \rangle. \tag{13.15}$$

The integral (13.1) defining the Fourier transform does not necessarily converge for $f \in L^2(\mathbb{R}^n)$, but the identity (13.15) makes it possible to define transforms on L^2 by taking limits.

Theorem 13.5 (Plancherel's theorem) *The Fourier transform extends from $\mathcal{S}(\mathbb{R}^n)$ to an invertible map on $L^2(\mathbb{R}^n)$, such that (13.15) holds for all $f, g \in L^2(\mathbb{R}^n)$.*

Proof First note that Theorem 7.5 implies that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ because it includes the compactly supported smooth functions. Hence for $f \in L^2(\mathbb{R}^n)$ there exists a sequence of Schwartz functions $\phi_k \to f$ in L^2. As a convergent sequence, $\{\phi_k\}$ is automatically Cauchy, i.e.,

$$\lim_{k,m\to\infty} \|\phi_k - \phi_m\|_2 = 0.$$

By (13.15),

$$\left\| \hat{\phi}_k - \hat{\phi}_m \right\|_2 = (2\pi)^{n/2} \|\phi_k - \phi_m\|_2 ,$$

implying that $\{\hat{\phi}_k\}$ is also Cauchy in $L^2(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n)$ is complete by Theorem 7.7, this implies convergence, and we can then define

$$\hat{f} := \lim_{k\to\infty} \hat{\phi}_k,$$

with the limit taken in the L^2 sense.

To show that (13.15) extends to L^2, suppose for $f, g \in L^2$ that $\phi_k \to f$ and $\psi_m \to g$ are approximating sequences of Schwartz functions. By the property (13.15),

$$\langle \hat{\phi}_k, \hat{\psi}_m \rangle = (2\pi)^n \langle \phi_k, \psi_m \rangle.$$

Taking the limit $k, m \to \infty$ then gives

$$\langle \hat{f}, \hat{g} \rangle = (2\pi)^n \langle f, g \rangle.$$

The same argument can be used to show that $\hat{f}$ is independent of the choice of approximating sequence. □

13.2 Tempered Distributions

Since $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ to itself, to extend the Fourier transform to distributions it is natural replace $C^\infty_{\text{cpt}}(\mathbb{R}^n)$ by $\mathcal{S}(\mathbb{R}^n)$ as the space of test functions. The result is the space of *tempered distributions*

$$\mathcal{S}'(\mathbb{R}^n) := \left\{ \text{continuous linear functionals } \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} \right\}.$$

Here the word "tempered" refers to a restriction on the growth at infinity. Because the Schwartz functions decay rapidly, a locally integrable function is essentially required to have a polynomial growth rate at infinity in order to define an element of $\mathcal{S}'(\mathbb{R}^n)$.

The definition of continuity of a functional on $\mathcal{S}(\mathbb{R}^n)$ depends on a notion of convergence for Schwartz functions. A sequence $\{\psi_k\} \subset \mathcal{S}(\mathbb{R}^n)$ converges if the sequences $\left\{\boldsymbol{x}^\alpha D^\beta \psi_k\right\}$ converge uniformly for each α, β. To say that $u \in \mathcal{S}'(\mathbb{R}^n)$ is continuous means that $(u, \psi_k) \to (u, \psi)$ whenever $\psi_k \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$.

The delta function $\delta_{\boldsymbol{x}}$ and its derivatives are clearly tempered distributions. We claim also that

$$L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n),$$

for $p \in [1, \infty]$. This follows fairly directly from the fact that $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $p \in [1, \infty]$.

The pairing formula (13.10) gives the prescription for extending $\mathcal{F}$ to the tempered distributions. For $u \in \mathcal{S}'(\mathbb{R}^n)$, we define $\hat{u}$ by

$$(\hat{u}, \phi) := (u, \hat{\phi}) \tag{13.16}$$

for $\phi \in \mathcal{S}(\mathbb{R}^n)$. To justify this definition one needs to check that the Fourier transform is continuous as a map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. This essentially follows from the calculations in the proof of Lemma 13.2.

As an example, consider the function $u = 1$ as an element of $\mathcal{S}'(\mathbb{R}^n)$. For $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}(\hat{1}, \psi) &:= (1, \hat{\psi}) \\ &= \int_{\mathbb{R}^n} \hat{\psi}(\boldsymbol{x})\, d^n\boldsymbol{x}.\end{aligned}$$

According to the inverse Fourier transform formula (13.11),

$$\int_{\mathbb{R}^n} \hat{\psi}(\boldsymbol{x})\, d^n\boldsymbol{x} = (2\pi)^n \psi(0).$$

Therefore

$$\hat{1} = (2\pi)^n \delta. \tag{13.17}$$

Physicists often express this fact by writing

$$\delta(\boldsymbol{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}\, d^n\boldsymbol{\xi},$$

with the understanding that the integral on the right is not to be taken literally.

The Fourier transform of δ is a similar calculation. For $\psi \in \mathcal{S}$,

$$\begin{aligned}(\hat{\delta}, \psi) &:= (\delta, \hat{\psi}) \\ &= \hat{\psi}(0) \\ &= \int_{\mathbb{R}^n} \psi(\boldsymbol{x})\, d^n\boldsymbol{x}.\end{aligned} \tag{13.18}$$

Therefore

$$\hat{\delta} = 1. \tag{13.19}$$

Because differentiation and multiplication by polynomials are continuous operations on $\mathcal{S}(\mathbb{R}^n)$, they extend to tempered distributions. From Lemma 13.1 we immediately derive the following:

Lemma 13.6 *For $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\mathcal{F}[D_{\boldsymbol{x}}^\alpha u] = (i\boldsymbol{\xi})^\alpha \hat{u},$$

and

$$\mathcal{F}[\boldsymbol{x}^\alpha u] = (iD)_{\boldsymbol{\xi}}^\alpha \hat{u}.$$

The Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ is particularly useful in the construction of fundamental solutions. Consider the constant coefficient operator

$$L = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

with $a_\alpha \in \mathbb{C}$. According to Lemma 13.6 and (13.19), the Fourier transform of the equation

$$L\Phi = \delta$$

is

$$P(\boldsymbol{\xi})\hat{\Phi} = 1,$$

where

$$P(\boldsymbol{\xi}) := \sum_{|\alpha| \le m} a_\alpha (i\boldsymbol{\xi})^\alpha.$$

If the reciprocal of $P(\boldsymbol{\xi})$ makes sense as a tempered distribution then we can set $\hat{\Phi}(\boldsymbol{\xi}) = 1/P(\boldsymbol{\xi})$ and take the inverse Fourier transform to construct a fundamental solution Φ as an element of $\mathcal{S}'(\mathbb{R}^n)$.

Example 13.7 The polynomial corresponding to $-\Delta$ on $\mathbb{R}^n$ is $P(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2$. For $n \ge 3$ the function $|\boldsymbol{\xi}|^{-2}$ is locally integrable and decays at infinity, so this defines a tempered distribution. Hence we should be able to compute Φ as the inverse Fourier transform of $|\boldsymbol{\xi}|^{-2}$.

Because $|\boldsymbol{\xi}|^{-2}$ is not globally integrable, we cannot apply the formula (13.11) directly. A trick to get around this is based on the fact that

$$\int_0^\infty e^{-\alpha t}\,dt = \alpha^{-1}$$

for $\alpha > 0$. Setting $\alpha = |\boldsymbol{\xi}|^2$ gives

$$|\boldsymbol{\xi}|^{-2} = \int_0^\infty e^{-t|\xi|^2}\,dt$$

for $\boldsymbol{\xi} \neq 0$. We can pair both sides with a Schwartz function $\psi(\boldsymbol{\xi})$ and integrate to show that

$$\mathcal{F}^{-1}\left[|\boldsymbol{\xi}|^{-2}\right] = \int_0^\infty \mathcal{F}^{-1}\left[e^{-t|\boldsymbol{\xi}|^2}\right]dt. \tag{13.20}$$

Setting $a = 1/(4t)$ in (13.8) gives

$$\mathcal{F}^{-1}\left[e^{-t|\xi|^2}\right](\boldsymbol{x}) = (4\pi t)^{-\frac{n}{2}} e^{-|\boldsymbol{x}|^2/4t},$$

so that (13.20) reduces to

$$\mathcal{F}^{-1}\left[|\boldsymbol{\xi}|^{-2}\right] = \int_0^\infty (4\pi t)^{-\frac{n}{2}} e^{-|\boldsymbol{x}|^2/4t} dt.$$

To evaluate the integral we substitute $s = |\boldsymbol{x}|^2/4t$ to obtain

$$\begin{aligned}\mathcal{F}^{-1}\left[|\boldsymbol{\xi}|^{-2}\right] &= \int_0^\infty \left(\frac{\pi\,|\boldsymbol{x}|^2}{s}\right)^{-\frac{n}{2}} e^{-s}\,\frac{|\boldsymbol{x}|^2}{4s^2} ds\\ &= \frac{1}{4}\pi^{-\frac{n}{2}}\,|\boldsymbol{x}|^{2-n}\int_0^\infty s^{\frac{n}{2}-2}e^{-s}\,ds.\end{aligned}$$

In terms of the gamma function (2.17) this calculation gives the fundamental solution

$$\Phi(\boldsymbol{x}) = \frac{1}{4}\pi^{-\frac{n}{2}}\Gamma(\tfrac{n}{2}-1)\,|\boldsymbol{x}|^{2-n}. \tag{13.21}$$

This agrees with the formula for Φ from Theorem 12.8, because

$$A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

and

$$\Gamma(\tfrac{n}{2}-1) = \frac{\Gamma(\frac{n}{2})}{\frac{n}{2}-1}.$$

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13.3 The Wave Kernel

Following the discussion in Sect. 12.6, we can try to define the wave kernel W_t on $\mathbb{R}^n$ by solving the distributional equations

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) W_t = 0, \quad W_0 = 0, \quad \left.\frac{\partial W_t}{\partial t}\right|_{t=0} = \delta. \tag{13.22}$$

If we assume that $W_t \in \mathcal{S}'(\mathbb{R}^n)$, then the spatial Fourier transform allows us to analyze this equation by turning it into a simple ODE.

For each t define $\hat{W}_t \in \mathcal{S}'(\mathbb{R}^n)$ by the (spatial) distributional transform (13.16). By Lemma 13.6 and (13.19), (13.22) transforms to

$$\left(\frac{\partial^2}{\partial t^2} + |\boldsymbol{\xi}|^2\right) \hat{W}_t = 0, \quad \hat{W}_0 = 0, \quad \left.\frac{\partial \hat{W}_t}{\partial t}\right|_{t=0} = 1.$$

The unique solution to this ODE is

$$\hat{W}_t(\boldsymbol{\xi}) = \begin{cases} \frac{\sin(t|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}, & \boldsymbol{\xi} \neq 0, \\ t, & \boldsymbol{\xi} = 0. \end{cases} \tag{13.23}$$

The function $\hat{W}_t$ is smooth and bounded, and therefore defines a tempered distribution on $\mathbb{R}^n$. The inverse Fourier transform $W_t \in \mathcal{S}'(\mathbb{R}^n)$ thus yields a general solution formula for the wave equation on $\mathbb{R}^n$. For initial conditions $g, h \in \mathcal{S}(\mathbb{R}^n)$,

$$u(t, \cdot) = \frac{\partial W_t}{\partial t} * g + W_t * h. \tag{13.24}$$

The direct computation of the inverse Fourier transform of (13.23) is rather tricky, but we can check this formula against the results we already know. For $n = 1$ we have $W_t = \frac{1}{2}\chi_{[-t,t]}$ from the d'Alembert formula. Since this is integrable the Fourier transform can be computed directly:

$$\begin{aligned} \hat{W}_t(\xi) &= \int_{-\infty}^{\infty} \frac{1}{2}\chi_{[-t,t]}(x)\, e^{-ix\xi}\, dx \\ &= \frac{1}{2}\int_{-t}^{t} e^{-ix\xi}\, dx \\ &= \begin{cases} \frac{\sin(t\xi)}{\xi}, & \xi \neq 0, \\ t, & \xi = 0. \end{cases} \end{aligned}$$

For $n = 3$, the Kirchhoff formula from Theorem 4.10 shows that the wave kernel is the distribution defined by

$$(W_t, \psi) := \frac{1}{4\pi t} \int_{\partial B(0;t)} \psi \, dS,$$

for $\psi \in \mathcal{S}(\mathbb{R}^3)$. By definition, the Fourier transform is given by

$$\begin{aligned}\left(\hat{W}_t, \psi\right) &:= \frac{1}{4\pi t} \int_{\partial B(0;t)} \hat{\psi}(\boldsymbol{x}) \, dS(\boldsymbol{x}) \\ &= \frac{1}{4\pi t} \int_{\partial B(0;t)} \left(\int_{\mathbb{R}^3} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \psi(\boldsymbol{\xi}) \, d^3\boldsymbol{\xi} \right) dS(\boldsymbol{x}).\end{aligned}$$

Since $\psi(\boldsymbol{y})$ has rapid decay as $\boldsymbol{y} \to \infty$ and the $\boldsymbol{x}$ integral is restricted to a sphere, we can switch the order of integration and conclude that

$$\hat{W}_t(\boldsymbol{\xi}) = \frac{1}{4\pi t} \int_{\partial B(0;t)} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \, dS(\boldsymbol{x}).$$

To compute this surface integral, note that we could rotate the $\boldsymbol{x}$ coordinate without changing the result of the integration. It therefore suffices to consider the case where $\boldsymbol{\xi}$ is parallel to the x_3 axis. If we then use the spherical coordinates (r, θ, ϕ) for the $\boldsymbol{x}$ variables, this gives

$$\boldsymbol{x} \cdot \boldsymbol{\xi} = |\boldsymbol{\xi}| \, r \cos\phi.$$

For the surface integral at radius $r = t$,

$$dS(\boldsymbol{x}) = t^2 \sin\phi \, d\phi \, d\theta.$$

The Fourier transform is thus

$$\begin{aligned}\hat{W}_t(\boldsymbol{\xi}) &= \frac{1}{4\pi t} \int_0^{2\pi} \int_0^{\pi} e^{-it|\boldsymbol{\xi}|\cos\theta} \, t^2 \sin\phi \, d\phi \, d\theta \\ &= \frac{t}{2} \int_0^{\pi} e^{-it|\boldsymbol{\xi}|\cos\theta} \, \sin\phi \, d\phi.\end{aligned}$$

With the substitution $u = \cos\phi$ this becomes

$$\begin{aligned}\hat{W}_t(\boldsymbol{\xi}) &:= \frac{t}{2} \int_{-1}^{1} e^{-it|\boldsymbol{\xi}|u} \, du \\ &= \begin{cases} \frac{\sin(t|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}, & \boldsymbol{\xi} \neq 0, \\ t, & \boldsymbol{\xi} = 0. \end{cases}\end{aligned}$$

Hence the Kirchhoff formula agrees with the transform solution (13.23).

13.4 The Heat Kernel

By analogy with (13.22), the *heat kernel* H_t is defined as the solution of the distributional equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) H_t = 0, \qquad H_0 = \delta. \tag{13.25}$$

Assuming $H_t \in \mathcal{S}'(\mathbb{R}^n)$, let $\hat{H}_t$ denote the spatial Fourier transform of H_t. By Lemma 13.6 and (13.19), (13.25) transforms to

$$\left(\frac{\partial}{\partial t} + |\boldsymbol{\xi}|^2\right) \hat{H}_t = 0, \qquad \hat{H}_0 = 1.$$

This simple ODE has the unique solution

$$\hat{H}_t(\boldsymbol{\xi}) = e^{-t|\boldsymbol{\xi}|^2}. \tag{13.26}$$

Because $\hat{H}_t$ is a Schwartz function for $t > 0$, we can compute the inverse Fourier transform by the direct integral formula (13.11), which gives

$$H_t(\boldsymbol{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} e^{-t|\boldsymbol{\xi}|^2}\, d^n\boldsymbol{\xi}.$$

According to (13.8), this inverse transform is

$$H_t(\boldsymbol{x}) = (4\pi t)^{-\frac{n}{2}} e^{-|\boldsymbol{x}|^2/4t}. \tag{13.27}$$

In Sect. 6.3 we guessed this formula from a calculation in the one-dimensional case. The Fourier transform allows for a systematic derivation.

13.5 Exercises

13.1 Let $\mathbb{H} \subset \mathbb{R}^2$ denote the upper half space $\{x_2 > 0\}$. The Poisson kernel on $\mathbb{H}$ is the distributional solution of the equation

$$\Delta P = 0, \qquad P|_{x_2=0} = \delta.$$

(a) Let $\hat{P}(\xi, x_2)$ denote the distributional Fourier transform of P with respect to the x_1 variable. Find the corresponding equation for $\hat{P}$.
(b) Show that the unique solution of the ODE with $\hat{P}(\cdot, x_2) \in \mathcal{S}'(\mathbb{R})$ is the function

$$\hat{P}(\xi, x_2) = e^{-x_2|\xi|}.$$

(c) Compute the inverse transform to show that

$$P(\boldsymbol{x}) = \frac{x_2}{\pi(x_1^2 + x_2^2)}.$$

(d) For $f \in \mathcal{S}(\mathbb{R})$, use P to write an integral formula for the solution of the Laplace problem on $\mathbb{H}$:

$$\Delta u = 0, \qquad u|_{x_2=0} = f.$$

13.2 For $\psi \in \mathcal{S}(\mathbb{R})$, the *Poisson summation formula* says that

$$\sum_{k=-\infty}^{\infty} \psi(k) = \sum_{m=-\infty}^{\infty} \hat{\psi}(2\pi m).$$

Derive this formula using the steps below.

(a) Define a periodic function $f \in C^\infty(\mathbb{T})$ (where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ as in Sect. 8.2) by averaging $\hat{\psi}$,

$$f(x) := \sum_{m=-\infty}^{\infty} \hat{\psi}(x + 2\pi m).$$

Show that

$$c_k[f] = \psi(-k).$$

(b) Obtain the summation formula by comparing f to its Fourier series expansion at $x = 0$.

13.3 Recall that the heat equation on $\mathbb{T}$ was solved by Fourier series in Theorem 8.13.

(a) Use the solution formula (8.44) to show that the heat kernel on $\mathbb{T}$ is given by the series

$$h_t(x) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-k^2 t + ikx}.$$

(b) Use the Poisson summation formula from Exercise 13.2 to show that the periodic heat kernel h_t and the heat kernel H_t on $\mathbb{R}$ are related by averaging

$$h_t(x) = \sum_{m=-\infty}^{\infty} H_t(x + 2\pi m)$$

for $t > 0$. (Note that this shows $h_t(x) > 0$ for all $x \in \mathbb{T}$, $t > 0$, which is not clear in the formula from (a).)

13.4 The Schrödinger equation on $\mathbb{R}^n$,

$$-i\frac{\partial u}{\partial t} - \Delta u = 0, \qquad u|_{t=0} = g,$$

was introduced in Exercise 4.7.

(a) Assuming that $g \in \mathcal{S}(\mathbb{R}^n)$, find a formula for the spatial Fourier transform $\hat{u}(t,\xi)$.
(b) Show that the result from Exercise 4.7,

$$\int_{\mathbb{R}^n} |u(t,\boldsymbol{x})|^2\, d^n\boldsymbol{x} = \int_{\mathbb{R}^n} |g|^2\, d^n\boldsymbol{x}$$

for all $t \geq 0$, follows from the Plancherel theorem (Theorem 13.5).