Chapter 4
The Wave Equation

As we noted in Sect. 1.2, d'Alembert's derivation of the wave equation in the 18th century was an early milestone in the development of PDE theory. In this chapter we will develop this equation as a model for the vibrating string problem, and derive d'Alembert's explicit solution in one dimension using the method of characteristics introduced in Chap. 3.

In higher dimensions the wave equation is used to model electromagnetic or acoustic waves. We will discussion the derivation of the acoustic model later in Sect. 4.5. A clever reduction trick allows the solution formula for $\mathbb{R}^n$ to be deduced from the one-dimensional case. The resulting integral formula yields insight into the propagation of waves in different dimensions.

The chapter concludes with a discussion of the energy of a solution, based on the physical principles of kinetic and potential energy.

4.1 Model Problem: Vibrating String

Consider a flexible string that is stretched tight between two points, like the strings on a violin or guitar. The stretching of the string creates a tension force T that pulls in both directions at each point along its length. For simplicity, let us assume that any other forces acting on the string, including gravity, are negligible compared to the tension. The linear density of mass ρ is taken to be constant along the string.

For a violin string it is also reasonable to assume that the displacement of the string is extremely small relative to its length. This assumption justifies taking T to be a fixed constant, ignoring the additional stretching that occurs when the string is displaced. It also allows us to treat horizontal and vertical components of the displacement independently, so we can restrict our attention to the vertical.

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Let the string be parametrized by $x \in [0, \ell]$. The vertical displacement as a function of time is denoted by $u(t, x)$. To develop an equation for u, we first discretize the model by subdividing the total length ℓ into segments of length $\Delta x = \ell/n$ for some large n. Each segment has a mass $\rho \Delta x$ and is subject to the tension forces pulling in the direction of its neighbors on either side.

For $j = 0, \ldots n$, let $x_j := j\Delta x$ be the position of the jth segment along the string. The segments $j = 0$ and $j = n$ represent the fixed endpoints, with $j = 1, \ldots, n-1$ in the interior. Let $u(t, x_j)$ denote the vertical displacement of the jth segment as a function of time. Figure 4.1 illustrates this discretization (with displacements greatly exaggerated).

To develop an equation for the string, we apply Newton's laws of motion to the segments of the discretization, as if they were single particles. The jth particle is being pulled by its neighbors with a force T on each side. Unless the string is straight, these forces are not quite aligned.

In terms of the angles labeled in Fig. 4.2, the net vertical force on a single segment is

$$\Delta F(t, x_j) = T \sin \alpha_j + T \sin \beta_j.$$

We have assumed that the relative displacements are extremely small, so the angles α_j, β_j will be very small also. To leading order, we can replace the sines by tangents, which are linear in u,

$$\sin \alpha_j \approx \frac{u(t, x_{j-1}) - u(t, x_j)}{\Delta x}, \quad \sin \beta_j \approx \frac{u(t, x_{j+1}) - u(t, x_j)}{\Delta x}.$$

With this linear approximation, the net vertical force at the point x_j becomes

$$\Delta F(t, x_j) = \frac{T}{\Delta x}\big[u(t, x_{j+1}) + u(t, x_{j-1}) - 2u(t, x_j)\big]. \tag{4.1}$$

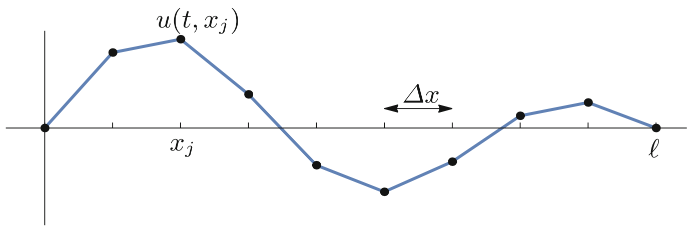

Fig. 4.1 Discrete model for the displacement of the string

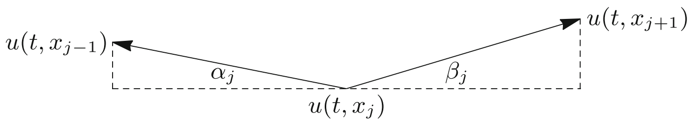

Fig. 4.2 Discrete model for the displacement of the string

The equation of motion for the jth segment now comes from Newton's law: mass times acceleration equals force. At the point x_j this translates to

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2}(t, x_j) = \Delta F(t, x_j). \tag{4.2}$$

Using (4.1) on the right then gives

$$\frac{\partial^2 u}{\partial t^2}(t, x_j) = \frac{T}{\rho} \frac{u(t, x_{j+1}) + u(t, x_{j-1}) - 2u(t, x_j)}{(\Delta x)^2}. \tag{4.3}$$

The final step is to take the continuum limit $n \to \infty$ and $\Delta x \to 0$. Assuming that u is twice continuously differentiable as a function of x, we can deduce from the quadratic Taylor approximation of $u(t, x)$ that

$$\lim_{\Delta x \to 0} \frac{u(t, x + \Delta x) + u(t, x - \Delta x) - 2u(t, x)}{(\Delta x)^2} = \frac{\partial^2 u}{\partial x^2}(t, x).$$

Hence, taking $\Delta x \to 0$ in (4.3) gives

$$\frac{\partial^2 u}{\partial t^2} - \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2} = 0. \tag{4.4}$$

This is the one-dimensional *wave equation*. The fixed ends of the string correspond to Dirichlet boundary conditions,

$$u(t, 0) = u(t, \ell) = 0.$$

4.2 Characteristics

For convenience, set $c^2 := T/\rho$ in (4.4), assuming $c > 0$, and rewrite the equation as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{4.5}$$

The constant c is called the *propagation speed*, for reasons that will become apparent as we analyze the equation.

Let the physical domain be $x \in \mathbb{R}$ for the moment; we will discuss boundary conditions later. The key to applying the method of characteristics to (4.5) is that the differential operator appearing in the equation factors as a product of two first-order operators, i.e.,

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right). \tag{4.6}$$

Individually, these operators have characteristic lines $t \mapsto x_0 \pm ct$. Both sets of characteristics will play an important role here.

Theorem 4.1 *Under the initial conditions*

$$u(0,x) = g(x), \qquad \frac{\partial u}{\partial t}(0,x) = h(x), \tag{4.7}$$

for $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, the wave equation (4.5) admits a unique solution

$$u(t,x) = \frac{1}{2}\Big[g(x+ct) + g(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau. \tag{4.8}$$

Proof Consider the auxiliary function $w(t,x)$ defined by

$$w := \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}. \tag{4.9}$$

By (4.6), w satisfies the linear conservation equation

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0.$$

The characteristics for this equation are given by $x_+(t) = x_0 + ct$. By Theorem 3.2 the unique solution with an initial condition $w(0,x) = w_0(x)$ is

$$w(t,x) = w_0(x-ct). \tag{4.10}$$

We will relate w_0 back to the initial conditions g and h in a moment.

With w given by (4.10), the definition (4.9) can be regarded as a linear conservation equation for u,

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w, \tag{4.11}$$

where w acts as a forcing term as described in Exercise 3.2. The characteristics of (4.11) are $x_-(t) = x_0 - ct$. By Theorem 3.2, we can thus reduce the equation to the form

$$\frac{d}{dt}u(t, x_0 - ct) = w(t, x_0 - ct). \tag{4.12}$$

The unique solution to (4.12) under the initial condition $u(0,x) = g(x)$ is given by direct integration with respect to time:

$$u(t, x_0 - ct) = g(x_0) + \int_0^t w(s, x_0 - cs)\,ds.$$

Setting $x = x_0 - ct$ then gives

$$u(t,x) = g(x+ct) + \int_0^t w(s, x - c(s-t))\, ds.$$

Using the formula (4.10) for the solution w, we obtain

$$u(t,x) = g(x+ct) + \int_0^t w_0(x - 2cs + ct)\, ds.$$

As a final step, the substitution $\tau := x + ct - 2cs$ gives

$$u(t,x) = g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\, d\tau. \tag{4.13}$$

The function w_0 can be computed from the initial conditions (4.7),

$$\begin{aligned} w_0(x) &:= \frac{\partial u}{\partial t}(0,x) - c\frac{\partial u}{\partial x}(0,x) \\ &= h(x) - c\frac{\partial g}{\partial x}(x). \end{aligned}$$

The w_0 contribution to (4.13) is then given by

$$\begin{aligned} \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\, d\tau &= \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\, d\tau - \frac{1}{2}\int_{x-ct}^{x+ct} \frac{\partial g}{\partial x}(\tau)\, d\tau \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\, d\tau - \frac{1}{2}\big[g(x+ct) - g(x-ct)\big]. \end{aligned}$$

Substituting back into (4.13) now gives the formula (4.8). □

To highlight the role played by the characteristic lines in the solution of Theorem 4.1, consider the functions

$$u_\pm(x) := \frac{1}{2}g(x) \mp \frac{1}{2c}\int_0^x h(\tau)\, d\tau.$$

In terms of $u_\pm$, the solution (4.8) simplifies to

$$u(x,t) = u_+(x-ct) + u_-(x+ct), \tag{4.14}$$

matching the form of the solution stated in (1.3). The subscripts in $u_\pm$ indicate the propagation direction, i.e., u_+ propagates to the right and u_- to the left. In either direction the speed of propagation is the parameter c.

Example 4.2 Consider the wave equation (4.5) with the initial conditions $h(x) = 0$ and

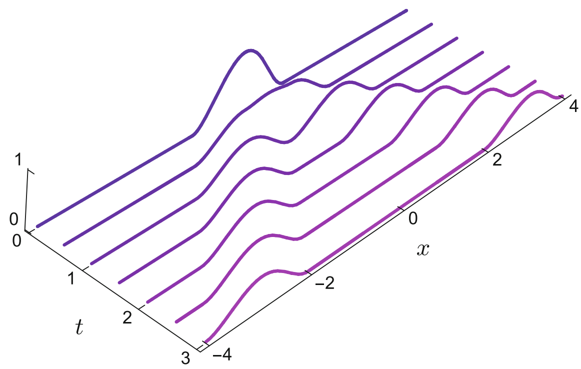

Fig. 4.3 Evolution of a solution to the wave equation

$$g(x) = \begin{cases} (1-x^2)^2, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

By (4.8) the solution is the superposition of two localized bumps, which propagate in opposite directions as illustrated in Fig. 4.3. ◊

In Example 4.2 the initial condition was supported in $[-1, 1]$, and we can see in Fig. 4.3 that the resulting solution has support in a V-shaped region. This region could be identified as the span of the characteristic lines emerging from the initial support interval.

This restriction of the support of a solution is closely related to *Huygens' principle*, an empirical law for propagation of light waves published by Christiaan Huygens in 1678. The one-dimensional wave equation exhibits a special, strict form of this principle:

Theorem 4.3 (**Huygens' principle in dimension one**) *Suppose u solves the wave equation* (4.5) *for $t \geq 0$, $x \in \mathbb{R}$, with initial data given by* (4.7). *If the functions g, h are supported in a bounded interval $[a, b]$, then*

$$\operatorname{supp} u \subset \Big\{(t, x) \in \mathbb{R}^+ \times \mathbb{R};\ x \in [a - ct, b + ct]\Big\}.$$

Proof Consider the components of the solution (4.8). The g term will vanish unless $x \pm ct \in [a, b]$. The support of this term is thus restricted to $x \in [a - ct, b - ct]$ or $x \in [a + ct, b + ct]$. As for the h term, the integral over τ will vanish unless the interval $[x - ct, x + ct]$ intersects $[a, b]$, which occurs only when $x \in [a - ct, b + ct]$. □

The restriction of support described in Theorem 4.3 is illustrated in Fig. 4.4. The term g contributes only in the regions shown in blue, but the h term may contribute throughout the full support region. However, the solution is constant (equal to $\int_a^b h(\tau)d\tau$ when $[a, b]$ is contained in $[x - ct, x + ct]$. This constant region shown in purple in Fig. 4.4.

Example 4.4 Suppose the initial data from Example 4.2 are altered to include a singularity at $x = 0$. For example,

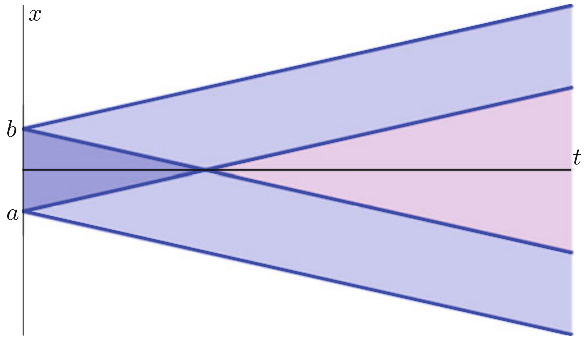

Fig. 4.4 Support of a wave solution with initial data in a bounded interval

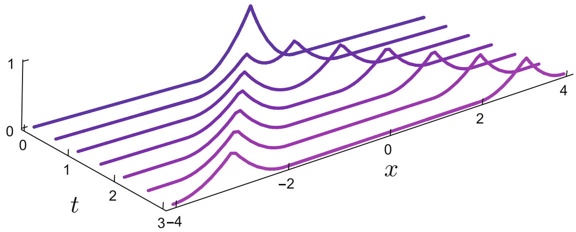

Fig. 4.5 Propagation of singularities of the wave equation along characteristic lines

$$g(x) = \begin{cases} (1-|x|)^2, & |x| \le 1, \\ 0, & |x| \ge 1. \end{cases}$$

Then (4.8) still gives a formula for the solution even though g is not differentiable. (This is a weak solution in the sense we will describe in Chap. 10). A set of solutions at different points in time is plotted in Fig. 4.5. Observe that the original singularity splits into two singularities, which propagate outward along the two characteristic lines emanating from $x = 0$. ♢

4.3 Boundary Problems

In the string model of Sect. 4.1 the domain of the wave equation (4.5) was restricted to $x \in [0, \ell]$, with Dirichlet boundary conditions

$$u(t, 0) = u(t, \ell) = 0, \quad \text{for all } t \ge 0. \tag{4.15}$$

Suppose the initial data are given for $x \in [0, \ell]$ by

$$u(0, x) = g(x), \qquad \frac{\partial u}{\partial t}(0, x) = h(x), \tag{4.16}$$

with $g \in C^2[0,\ell]$, $h \in C^1[0,\ell]$. Both g and h are assumed to vanish at the endpoints of $[0,\ell]$.

The solution of the wave equation on $\mathbb{R}$ provided in Theorem 4.1 can be adapted to the boundary conditions (4.15). The idea is to extend g, h to $\mathbb{R}$ in such a way that the formula (4.8) gives a solution satisfying the boundary conditions for all t.

Theorem 4.5 *The wave equation* (4.5) *on* $[0,\ell]$*, with Dirichlet boundary conditions and satisfying the initial conditions* (4.16)*, admits a solution of the form* (4.8)*, only if the initial data extensions to* $\mathbb{R}$ *as odd,* 2ℓ*-periodic functions, with* $g \in C^2(\mathbb{R})$ *and* $h \in C^1(\mathbb{R})$.

Proof By linearity we can consider the g and h terms independently. Assume that the g term,

$$\frac{1}{2}\big[g(x+ct)+g(x-ct)\big], \tag{4.17}$$

is defined for all t and x and satisfies the boundary conditions on $[0,\ell]$ for all values of t. At $x=0$ the condition $u(t,0)=0$ will be satisfied if and only if

$$g(ct)+g(-ct)=0, \quad \text{for all } t \geq 0.$$

In other words, $u(t,0)=0$ if and only if g is odd. At $x=\ell$ the condition is

$$g(\ell+ct)+g(\ell-ct)=0, \quad \text{for all } t \geq 0.$$

This is equivalent to the condition that g is odd with respect to reflection at the point $x=\ell$.

The composition of the reflections about 0 and ℓ gives translation by 2ℓ. Hence the expression (4.17) satisfies the boundary conditions if and only if g is odd and 2ℓ-periodic.

A similar argument works for the h term,

$$u(t,x)=\frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau. \tag{4.18}$$

The requirement at $x=0$ is

$$\int_{-ct}^{ct} h(\tau)\,d\tau=0, \quad \text{for all } t \geq 0. \tag{4.19}$$

Differentiation with respect to t, using the fundamental theorem of calculus, shows that (4.19) is satisfied if and only if h is odd with respect to reflection at 0. Similarly, the condition

$$\int_{\ell-ct}^{\ell+ct} h(\tau)\,d\tau=0, \quad \text{for all } t \geq 0$$

requires odd symmetry with respect to reflection at $x=\ell$. □

Example 4.6 Consider the vibrating string problem with $c = 1$ and $\ell = 1$. Suppose that the solution initially has the form (4.14) with $u_+ = 0$ and the left-propagating solution given by the function u_- shown in Fig. 4.6. For small $t > 0$ the solution is

$$u(t, x) = u_-(x + t), \tag{4.20}$$

but eventually the bump hits the boundary at $x = 0$, and we would like to understand what happens then.

To apply Theorem 4.5, we must first solve for g and h in terms of u_+. By (4.20) we set $g(x) = u_+(x)$ and

$$h(x) = \frac{\partial}{\partial t} u_-(x + t)\Big|_{t=0} = \frac{du_-}{dx}(x).$$

The resulting functions g and h, extended to odd functions on $\mathbb{R}$, are shown in Fig. 4.7.

According to Theorem 4.5 we can compute the solution from (4.8) using these odd periodic extensions of g and h. The results are shown in Fig. 4.8. The bump temporarily disappears at $t = 0.3$ and then reemerges as an inverted bump traveling in the opposite direction. ♢

4.4 Forcing Terms

The derivation of the string model in Sect. 4.1 assumed that no external forces act on the string. Additional forces could be incorporated by adding extra terms to the expression (4.1) for the force on a segment. In the continuum limit this yields a

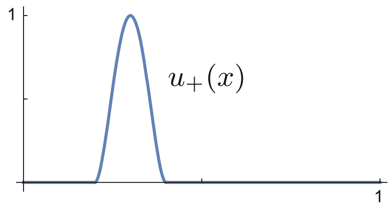

Fig. 4.6 The initial waveform u_+

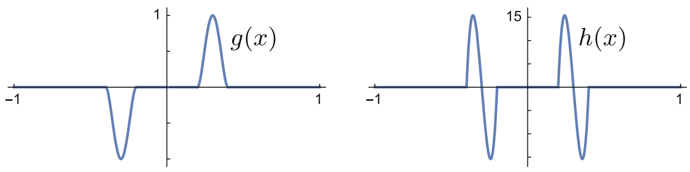

Fig. 4.7 The odd extensions of the initial conditions g and h

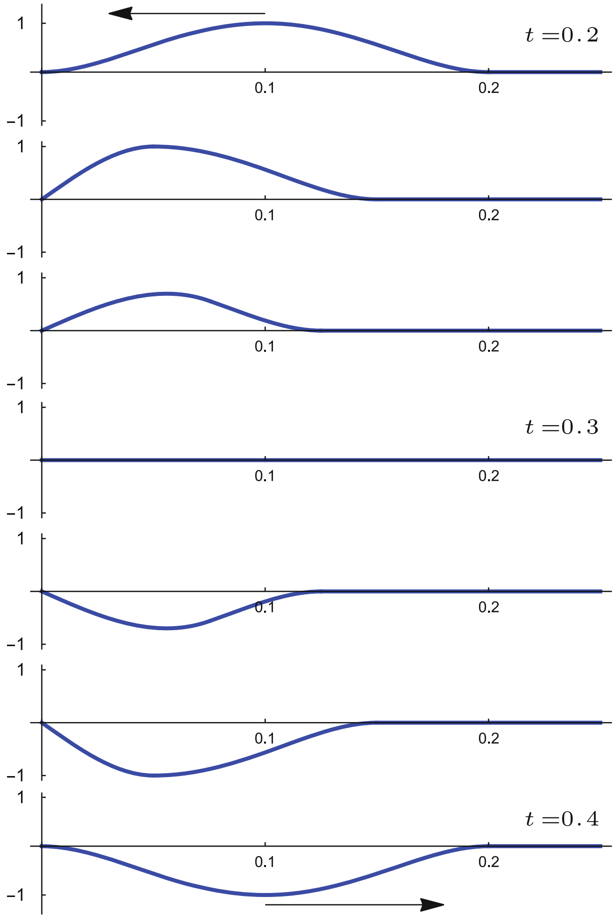

Fig. 4.8 Reflection of a propagating bump at the endpoint of the string

forcing term on the right-hand side of (4.5):

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f, \tag{4.21}$$

where $f = f(t, x)$. The forcing term could be used to model plucking or bowing of the string, for example.

In this section we introduce a technique, called *Duhamel's method*, that allows us to adapt solution methods for evolution equations to include a forcing term. The idea, which is closely related to a standard ODE technique called *variation of parameters*, is to reformulate the forcing term as an initial condition. This technique is named for the 19th century French mathematician and physicist Jean-Marie Duhamel, who developed the idea in a study of the heat equation.

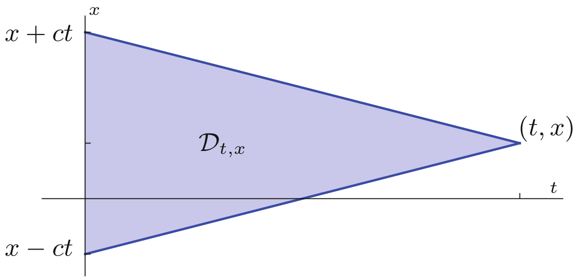

Fig. 4.9 Domain of dependence for the point (t, x)

To focus our attention on the driving term, let us consider (4.21) on the domain $x \in \mathbb{R}$ with the initial conditions set to zero. For a given c, define the *domain of dependence* of a point (t, x) with $t > 0$ and $x \in \mathbb{R}$ by

$$\mathcal{D}_{t,x} := \left\{(s, x') \in \mathbb{R}_+ \times \mathbb{R} : \; x - c(t-s) \le x' \le x + c(t-s)\right\}.$$

This is a triangular region, as pictured in Fig. 4.9. The terminology refers to the fact that the solution $u(t, x)$ is influenced only by the values of f within $\mathcal{D}_{t,x}$, as the next result shows.

Theorem 4.7 *For $f \in C^1(\mathbb{R})$, the unique solution of (4.21) satisfying the initial conditions*

$$u(0, x) = 0, \quad \frac{\partial u}{\partial t}(0, x) = 0,$$

is given by

$$u(t, x) = \frac{1}{2c} \int_{\mathcal{D}_{t,x}} f(s, x')\, dx'\, ds. \tag{4.22}$$

Proof For each $s \ge 0$, let $\eta_s(t, x)$ be the solution of the homogeneous wave equation (4.5) for $t \ge s$, subject to the initial conditions

$$\eta_s(t, x)\big|_{t=s} = 0, \qquad \frac{\partial \eta_s}{\partial t}(t, x)\big|_{t=s} = f(s, x). \tag{4.23}$$

This function can be written explicitly by shifting t to $t - s$ in (4.8),

$$\eta_s(t, x) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(s, x')\, dx'. \tag{4.24}$$

We claim that the solution of (4.21) is given by the integral

$$u(t, x) := \int_0^t \eta_s(t, x)\, ds. \tag{4.25}$$

Note that the integration variable here is s rather than t.

We will first check that this definition of u satisfies the initial conditions. For $t = 0$ the integral in (4.25) clearly vanishes, so that $u(0, x) = 0$ is satisfied. By the fundamental theorem of calculus, differentiating (4.25) with respect to t gives

$$\frac{\partial u}{\partial t}(t, x) = \eta_s(t, x)\Big|_{s=t} + \int_0^t \frac{\partial \eta_s}{\partial t}(t, x)\, ds.$$

The first term vanishes for all t by the initial condition (4.23), leaving

$$\frac{\partial u}{\partial t}(t, x) = \int_0^t \frac{\partial \eta_s}{\partial t}(t, x)\, ds, \tag{4.26}$$

for all $t \geq 0$. Setting $t = 0$ gives

$$\frac{\partial u}{\partial t}(0, x) = 0.$$

Now let us check that the u defined in (4.25) solves (4.21). Differentiating (4.26) once more gives

$$\frac{\partial^2 u}{\partial t^2}(t, x) = \frac{\partial \eta_s}{\partial t}(t, x)\Big|_{s=t} + \int_0^t \frac{\partial^2 \eta_s}{\partial t^2}(t, x)\, ds. \tag{4.27}$$

By (4.23) the first term on the right is equal to $f(t, x)$. To simplify the second term, we use the fact that η_s solves (4.5) and the definition of u to compute

$$\begin{aligned}\int_0^t \frac{\partial^2 \eta_s}{\partial t^2}(t, x)\, ds &= c^2 \int_0^t \frac{\partial^2 \eta_s}{\partial x^2}(t, x)\, ds \\ &= c^2 \frac{\partial^2 u}{\partial x^2}.\end{aligned}$$

Therefore, (4.27) reduces to

$$\frac{\partial^2 u}{\partial t^2} = f + c^2 \frac{\partial^2 u}{\partial x^2},$$

proving that u solves (4.21).

Combining (4.24) and (4.25), we can write the formula for u as

$$u(t, x) = \frac{1}{2c} \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(s, x')\, dx' \right) ds,$$

which is equivalent to (4.22).

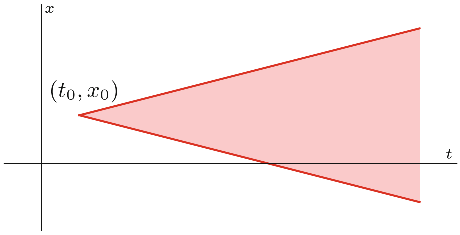

Fig. 4.10 Range of influence of the point (t_0, x_0)

To prove uniqueness, suppose u_1 and u_2 are solutions of (4.21). Then $u_2 - u_1$ is a solution of (4.5). Since $u_2 - u_1$ also has vanishing initial conditions, Theorem 4.1 implies that $u_2 - u_1 = 0$. Hence the solution is unique. □

By the superposition principle, Theorem 4.7 is easily extended to the case of nonzero initial conditions, by setting $u = v + w$ where v is a solution of the form (4.22) and w is a solution of the form (4.8).

The concept of domain of dependence still applies when the initial conditions are nonzero. In the solution formula (4.8), $u(t, x)$ depends only on the values of g and h at the base of the triangle, $\mathcal{D}_{t,x} \cap \{t = 0\}$. Thus it is still the case that the solution $u(t, x)$ depends only on the data within $\mathcal{D}_{t,x}$.

The existence of the domain of dependence is a limitation imposed by the propagation speed c. For systems governed by the wave equations (4.5) or (4.21), no information can travel at a speed faster than c.

The region of the space-time plane in which solutions can be affected by the data at a particular point (t_0, x_0) is called the *range of influence* of this point. By the definition of the domain of dependence, the range of influence consists of the points (t, x) such that $(t_0, x_0) \in \mathcal{D}_{t,x}$. This region is a triangle with vertex (t_0, x_0) and sides given by the characteristics $(t, x_0 \pm c(t - t_0))$, as shown in Fig. 4.10.

Duhamel's method applies also to the case of a vibrating string with fixed ends. Assuming that $f(t, x)$ satisfies the boundary conditions at $x = 0$ and $x = \ell$, we extend f to an odd 2ℓ-periodic function on $\mathbb{R}$, just as in Theorem 4.5. This extension guarantees that the intermediate solution η_s defined by (4.24) will satisfy the boundary conditions also. And then so will the solution $u(t, x)$ defined by (4.25).

Example 4.8 Consider a string of length ℓ with propagation speed $c = 1$. Suppose the forcing term is given by

$$f(t, x) = \cos(\omega t) \sin(\omega_0 x), \tag{4.28}$$

where $\omega_0 := \pi/\ell$ and $\omega > 0$ is the driving frequency. Since $\sin(\omega_0 x)$ is odd and 2π-periodic, the extension required by Theorem 4.5 is automatic. As in Theorem 4.7, let us set the initial conditions $g = h = 0$ to focus on the forcing term.

Substituting (4.28) into (4.22) gives

$$\begin{aligned}u(t,x) &= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \cos(\omega s)\sin(\omega_0 x')\,dx'\,ds \\ &= \frac{1}{2\omega_0}\int_0^t [\cos(\omega_0(x-t+s)) - \cos(\omega_0(x+t-s))]\cos(\omega s)\,ds\end{aligned}$$

A trigonometric identity reduces this to

$$u(t,x) = \frac{\sin(\omega_0 x)}{\omega_0}\int_0^t \sin(\omega_0(t-s))\cos(\omega s)\,ds. \tag{4.29}$$

For $\omega \neq \omega_0$ we obtain

$$u(t,x) = \frac{\sin(\omega_0 x)}{\omega_0^2 - \omega^2}\,[\cos(\omega t) - \cos(\omega_0 t)]\,.$$

Note that the x dependence of the solution matches that of the forcing term. The interesting part of this solution is the oscillation, which includes both frequencies ω and ω_0. Figure 4.11 illustrates the behavior of the amplitude as a function of time, in a case where $\omega \ll \omega_0$. The large-scale oscillation has a period $1/\omega$, corresponding to the low driving frequency. The solution also exhibits fast oscillations at the frequency ω_0 which depends only on ℓ.

For $\omega = \omega_0$ the formula (4.29) gives the solution

$$u(t,x) = \frac{t}{2\omega_0}\sin(\omega_0 x)\sin(\omega_0 t).$$

The resulting amplitude grows linearly, as shown in Fig. 4.12. ◊

The physical phenomenon illustrated by Example 4.8 is called *resonance*. If the string is driven at its natural frequency ω_0 then it will continually absorb energy from the driving force. Of course, there is a limit to how much energy a physical string could absorb before it breaks. Once the displacement amplitude becomes sufficiently large, the linear wave equation (4.5) no longer serves as an appropriate model.

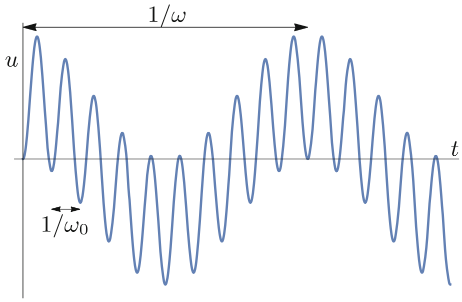

Fig. 4.11 Oscillation pattern with a driving frequency $\omega = \omega_0/10$

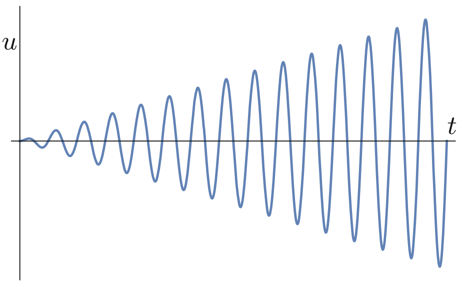

Fig. 4.12 Growth of the amplitude at the resonance frequency $\omega = \omega_0$

4.5 Model Problem: Acoustic Waves

The vibration of a drumhead can be modeled on a bounded domain $\Omega \subset \mathbb{R}^2$, with a function $u(t, \boldsymbol{x})$ representing the vertical displacement of the membrane at time t and position $\boldsymbol{x} \in \Omega$. With arguments similar to those in Sect. 4.1, one can derive the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \tag{4.30}$$

where Δ is the Laplacian operator (1.7). The wave equation (4.30) appears in many other contexts as well, including the propagation of light and all other forms of electromagnetic radiation. In all these cases the constant c represents the speed of propagation.

In this section we will derive the three-dimensional wave equation as a model for acoustic waves traveling through the air. Acoustic waves consist of fluctuations of pressure which propagate through a gas. To analyze them, we must consider the relationships between the pressure P, the velocity field $\boldsymbol{v}$, and the density ρ. For a gas in motion these are all functions of both time and position.

Because acoustic waves involve minute pressure fluctuations with very little heat transfer, the relationship between pressure and density is given by the *adiabatic gas law*

$$P = C\rho^\gamma, \tag{4.31}$$

where C and γ are physical constants. We will fix background atmospheric values of the pressure P_0 and density ρ_0 and focus on the deviations

$$u := P - P_0, \qquad \sigma := \rho - \rho_0.$$

Applying (4.31) to P/P_0 gives the equation

$$1 + \frac{u}{P_0} = \left(1 + \frac{\sigma}{\rho_0}\right)^\gamma.$$

Since σ/ρ_0 is assumed to be very small, we can linearize by taking a first-order Taylor approximation on the right side. This yields

$$u = \frac{\gamma P_0}{\rho_0}\sigma. \tag{4.32}$$

The dynamics of the gas are modeled with two conservation laws. The first is conservation of mass (3.20), which yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

Since σ and $\boldsymbol{v}$ are both assumed to be very small, for the leading approximation we can replace ρ by ρ_0 to obtain

$$\frac{\partial \sigma}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v} = 0. \tag{4.33}$$

The second dynamical law is conservation of momentum. This is encapsulated in a fluid equation derived by Euler in 1757, called *Euler's force equation*:

$$-\nabla P = \rho \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) \boldsymbol{v}.$$

Euler's equation is an aggregate form of Newton's second law (force equals mass times acceleration). Note that the "acceleration" term on the right is the Lagrangian derivative of the velocity field $\boldsymbol{v}$. As above, we substitute $P = P_0 + u$ and $\rho = \rho_0 + \sigma$ and keep only the first order terms to derive the linearization

$$-\nabla u = \rho_0 \frac{\partial \boldsymbol{v}}{\partial t}. \tag{4.34}$$

The final step is to eliminate the velocity field from the equation. Substituting (4.32) into (4.33) and differentiating with respect to time gives

$$\frac{\partial^2 u}{\partial t^2} = -\gamma P_0 \frac{\partial}{\partial t} (\nabla \cdot \boldsymbol{v}). \tag{4.35}$$

On the other hand, by (4.34),

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \cdot \boldsymbol{v}) &= \nabla \cdot \frac{\partial \boldsymbol{v}}{\partial t} \\ &= \nabla \cdot \left(-\frac{\nabla u}{\rho_0} \right) \\ &= -\frac{1}{\rho_0} \Delta u. \end{aligned}$$

Substituting this in (4.35) yields the *acoustic wave equation*,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\gamma P_0}{\rho_0} \Delta u = 0.$$

As with our previous derivations, many approximations are required to produce a linear equation. These simplifications are well justified for sound waves at ordinary volume levels, but more dramatic pressure fluctuations would require a nonlinear equation.

4.6 Integral Solution Formulas

Let us consider the wave equation (4.30) on $\mathbb{R}^3$ with $c = 1$,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0.$$

This problem can be reduced to the one-dimensional case by a clever averaging trick.

For $f \in C^0(\mathbb{R}^3)$, define

$$\tilde{f}(\boldsymbol{x}; \rho) := \frac{1}{4\pi\rho} \int_{\partial B(\boldsymbol{x};\rho)} f(\boldsymbol{w})\, dS(\boldsymbol{w}), \tag{4.36}$$

where $\boldsymbol{x} \in \mathbb{R}^3$ and $\rho > 0$. The surface area of $\partial B(\boldsymbol{x}; \rho)$ is $4\pi\rho^2$, so $\tilde{f}$ is ρ times the spherical average of f. By continuity the spherical average approaches the value of the function at the center point as $\rho \to 0$, so that

$$\lim_{\rho\to 0} \frac{\tilde{f}(\boldsymbol{x}; \rho)}{\rho} = f(\boldsymbol{x}). \tag{4.37}$$

The dimensional reduction of the wave equation is based on the following formula of Jean-Gaston Darboux.

Lemma 4.9 (Darboux's formula) *For* $f \in C^2(\mathbb{R}^3)$,

$$\frac{\partial^2}{\partial \rho^2} \tilde{f}(\boldsymbol{x}; \rho) = \Delta_{\boldsymbol{x}} \tilde{f}(\boldsymbol{x}; \rho).$$

Proof To compute the radial derivative of the spherical average, it is helpful to change coordinates by setting $\boldsymbol{w} = \boldsymbol{x} + \rho\boldsymbol{y}$, so that the domain of $\boldsymbol{y}$ is the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$,

$$\frac{1}{4\pi\rho^2} \int_{\partial B(\boldsymbol{x};\rho)} f(\boldsymbol{w})\, dS(\boldsymbol{w}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(\boldsymbol{x} + \rho\boldsymbol{y})\, dS(\boldsymbol{y}).$$

Differentiation under the integral gives,

$$\frac{\partial}{\partial \rho} \int_{\mathbb{S}^2} f(\boldsymbol{x} + \rho \boldsymbol{y})\, dS(\boldsymbol{y}) \frac{1}{4\pi} = \int_{\mathbb{S}^2} \nabla f(\boldsymbol{x} + \rho \boldsymbol{y}) \cdot \boldsymbol{y}\, dS(\boldsymbol{y}).$$

In the original coordinates this implies

$$\begin{aligned} &\frac{\partial}{\partial \rho} \left[\frac{1}{4\pi\rho^2} \int_{\partial B(\boldsymbol{x};\rho)} f(\boldsymbol{w})\, dS(\boldsymbol{w}) \right] \\ &\qquad = \frac{1}{4\pi\rho^2} \int_{\partial B(\boldsymbol{x};\rho)} \nabla f(\boldsymbol{w}) \cdot \left(\frac{\boldsymbol{w} - \boldsymbol{x}}{\rho} \right) dS(\boldsymbol{w}). \end{aligned} \tag{4.38}$$

Since $(\boldsymbol{w} - \boldsymbol{x})/\rho$ is the outward unit normal to $\partial B(\boldsymbol{x}; \rho)$,

$$\nabla f(\boldsymbol{w}) \cdot \left(\frac{\boldsymbol{w} - \boldsymbol{x}}{\rho} \right) = \frac{\partial f}{\partial \nu}(\boldsymbol{w}).$$

Furthermore, by Corollary 2.8,

$$\int_{\partial B(\boldsymbol{x};\rho)} \frac{\partial f}{\partial \nu}(\boldsymbol{w})\, dS(\boldsymbol{w}) = \int_{B(\boldsymbol{x};\rho)} \Delta f(\boldsymbol{w})\, d^3\boldsymbol{w}.$$

Applying this to the right-hand side of (4.38) gives

$$\frac{\partial}{\partial \rho} \left[\frac{1}{4\pi\rho^2} \int_{\partial B(\boldsymbol{x};\rho)} f(\boldsymbol{w})\, dS(\boldsymbol{w}) \right] = \frac{1}{4\pi\rho^2} \int_{B(\boldsymbol{x};\rho)} \Delta f(\boldsymbol{w})\, d^3\boldsymbol{w}. \tag{4.39}$$

Substituting the definition of $\tilde{f}$ in (4.39) yields

$$\frac{\partial}{\partial \rho} \tilde{f}(\boldsymbol{x}; \rho) = \frac{1}{4\pi\rho^2} \int_{\partial B(\boldsymbol{x};\rho)} f(\boldsymbol{w})\, dS(\boldsymbol{w}) + \frac{1}{4\pi\rho} \int_{B(\boldsymbol{x};\rho)} \Delta f(\boldsymbol{w})\, d^3\boldsymbol{w}$$

A further differentiation using (4.39) and the radial derivative formula from Exercise 2.4 then gives

$$\begin{aligned} \frac{\partial^2}{\partial \rho^2} \tilde{f}(\boldsymbol{x}; \rho) &= \frac{1}{4\pi\rho} \frac{\partial}{\partial \rho} \int_{B(\boldsymbol{x};\rho)} \Delta f(\boldsymbol{w})\, d^3\boldsymbol{w} \\ &= \frac{1}{4\pi\rho} \int_{\partial B(\boldsymbol{x};\rho)} \Delta f(\boldsymbol{w})\, dS(\boldsymbol{w}). \end{aligned} \tag{4.40}$$

On the other hand

$$\begin{aligned}\Delta_{\boldsymbol{x}} \tilde{f}(\boldsymbol{x}; \rho) &= \Delta_{\boldsymbol{x}} \left[\frac{1}{4\pi\rho} \int_{\mathbb{S}^2} f(\boldsymbol{x} + \rho \boldsymbol{y})\, dS(\boldsymbol{y}) \right] \\ &= \frac{1}{4\pi\rho} \int_{\mathbb{S}^2} \Delta f(\boldsymbol{x} + \rho \boldsymbol{y})\, dS(\boldsymbol{y}) \\ &= \frac{1}{4\pi\rho} \int_{\partial B(\boldsymbol{x};\rho)} \Delta f(\boldsymbol{w})\, dS(\boldsymbol{w}).\end{aligned} \tag{4.41}$$

The claim thus follows from (4.40). □

Lemma 4.9 allows us to relate the three-dimensional wave equation in variables $(t, \boldsymbol{x})$ to a one-dimensional equation in variables (t, ρ). The result is a solution formula first derived in 1883 by the physicist Gustav Kirchhoff.

Theorem 4.10 (Kirchhoff's integral formula) *For $u \in C^2([0, \infty) \times \mathbb{R}^3)$, suppose that*

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

under the initial conditions

$$u|_{t=0} = g, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = h.$$

Then

$$u(t, \boldsymbol{x}) = \frac{\partial}{\partial t} \tilde{g}(\boldsymbol{x}; t) + \tilde{h}(\boldsymbol{x}; t),$$

with $\tilde{g}$ and $\tilde{h}$ defined as in (4.36).

Proof Define

$$\tilde{u}(t, \boldsymbol{x}; \rho) := \frac{1}{4\pi\rho} \int_{\partial B(\boldsymbol{x};\rho)} u(t, \boldsymbol{w})\, dS(\boldsymbol{w}).$$

Since u satisfies the wave equation, differentiating under the integral gives

$$\frac{\partial^2}{\partial t^2} \tilde{u}(t, \boldsymbol{x}; \rho) = \frac{1}{4\pi\rho} \int_{\partial B(\boldsymbol{x};\rho)} \Delta u(t, \boldsymbol{w})\, dS(\boldsymbol{w}).$$

By the calculation (4.41) this is equivalent to

$$\frac{\partial^2}{\partial t^2} \tilde{u}(t, \boldsymbol{x}; \rho) = \Delta_{\boldsymbol{x}} \tilde{u}(t, \boldsymbol{x}; \rho).$$

Lemma 4.9 then shows that

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \rho^2} \right) \tilde{u}(t, \boldsymbol{x}; \rho) = 0. \tag{4.42}$$

The initial conditions for $\tilde{u}$ follow from the initial conditions for u,

$$\tilde{u}(0, \boldsymbol{x}; \rho) = \tilde{g}(\boldsymbol{x}; \rho), \quad \frac{\partial}{\partial t}\tilde{u}(0, \boldsymbol{x}; \rho) = \tilde{h}(\boldsymbol{x}; \rho).$$

By (4.37) we also have a boundary condition at $\rho = 0$,

$$\tilde{u}(t, \boldsymbol{x}; 0) = 0.$$

Using Theorem 4.1 and the reflection argument from Theorem 4.5, we conclude that the unique solution of (4.42) under these conditions is given by extending $\tilde{g}(\boldsymbol{x}; \rho)$ and $\tilde{h}(\boldsymbol{x}; \rho)$ to $\rho \in \mathbb{R}$ with odd symmetry and then using the d'Alembert formula,

$$\tilde{u}(t, \boldsymbol{x}; \rho) = \frac{1}{2}\left[\tilde{g}(\boldsymbol{x}; \rho + t) + \tilde{g}(\boldsymbol{x}; \rho - t)\right] + \frac{1}{2}\int_{\rho-t}^{\rho+t} \tilde{h}(\boldsymbol{x}; \tau)\, d\tau. \tag{4.43}$$

By (4.37), we can recover u from this formula by setting

$$u(t, \boldsymbol{x}) = \lim_{\rho \to 0} \frac{\tilde{u}(t, \boldsymbol{x}; \rho)}{\rho}. \tag{4.44}$$

To evaluate this limit, first note that for $0 \le \rho \le t$ the odd symmetry of $\tilde{g}$ and $\tilde{h}$ with respect to ρ can be used to rewrite (4.43) as

$$\tilde{u}(t, \boldsymbol{x}; \rho) = \frac{1}{2}\left[\tilde{g}(\boldsymbol{x}; t + \rho) - \tilde{g}(\boldsymbol{x}; t - \rho)\right] + \frac{1}{2}\int_{t-\rho}^{t+\rho} \tilde{h}(\boldsymbol{x}; \tau)\, d\tau.$$

The computations are now straightforward:

$$\lim_{\rho \to 0} \frac{1}{2\rho}\left[\tilde{g}(\boldsymbol{x}; t + \rho) - \tilde{g}(\boldsymbol{x}; t - \rho)\right] = \frac{\partial}{\partial t}\tilde{g}(\boldsymbol{x}; t),$$

and

$$\lim_{\rho \to 0} \frac{1}{2\rho}\int_{t-\rho}^{t+\rho} \tilde{h}(\boldsymbol{x}; \tau)\, d\tau = \tilde{h}(\boldsymbol{x}; t).$$

The claimed solution formula thus follows from (4.44). □

One interesting consequence of the Kirchhoff formula is the fact that three-dimensional wave propagation exhibits a strict form of the Huygens' principle. Theorem 4.10 shows that the range of influence of the point $(t_0, \boldsymbol{x}_0)$ is the *forward light cone*,

$$\Gamma_+(t_0, \boldsymbol{x}_0) := \{(t, \boldsymbol{x});\ t > t_0,\ |\boldsymbol{x} - \boldsymbol{x}_0| = t - t_0\}.$$

This matches the result of Theorem 4.3 for the one-dimensional wave equation. The strict Huygens phenomenon is readily observable for acoustic waves, in the fact that

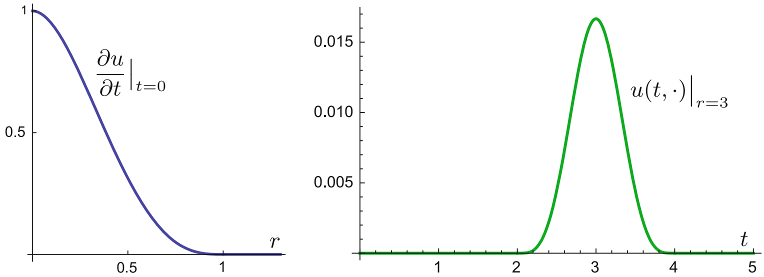

Fig. 4.13 The plot on the right shows the observed waveform at a distance 3 from the origin, caused by the initial radial impulse shown on the left

a sudden sound like a clap propagates as a sharp wavefront that is heard as a single discrete event, without aftereffects unless there are reflective surfaces to cause an echo. Figure 4.13 illustrates this effect; an observer located away from the origin experiences a waveform of duration equal to the diameter of the initial impulse. The strict Huygens' principle holds in every odd dimension greater than 1, but fails in even dimensions, as we will illustrate below.

The spherical averaging trick used for Theorem 4.10 also works in higher odd dimensions, although the solution formulas become more complicated. For even dimensions, solution formulas can be derived from the odd-dimensional case by a technique called the *method of descent*.

We will work this out for the two-dimensional case. Suppose $u \in C^2([0,\infty)\times\mathbb{R}^2)$ solves the wave equation with initial conditions

$$u|_{t=0} = g, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = h,$$

with g, h functions on $\mathbb{R}^2$. If we extend g and h to $\mathbb{R}^3$ as functions that are independent of x_3, then Kirchhoff's formula gives a solution to the three dimensional problem. Since this solution is also independent of x_3, it "descends" to a solution in $\mathbb{R}^2$. The resulting formula was first worked out by Siméon Poisson in the early 19th century (well before Kirchhoff's three-dimensional formula).

Corollary 4.11 (Poisson's integral formula) *For $u \in C^2([0,\infty)\times\mathbb{R}^2)$, suppose that*

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

under the initial conditions

$$u|_{t=0} = g, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = h.$$

Then

$$u(t, \boldsymbol{x}) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{\mathbb{D}} \frac{g(\boldsymbol{x} - t\boldsymbol{y})}{\sqrt{1 - |\boldsymbol{y}|^2}}\, d^2\boldsymbol{y}\right) + \frac{t}{2\pi}\int_{\mathbb{D}} \frac{h(\boldsymbol{x} - t\boldsymbol{y})}{\sqrt{1 - |\boldsymbol{y}|^2}}\, d^2\boldsymbol{y}.$$

Proof Following the procedure described above, we extend g and h to functions on $\mathbb{R}^3$ independent of x_3. In this case the integral (4.36) becomes

$$\tilde{g}(\boldsymbol{x}; \rho) = \frac{\rho}{4\pi}\int_{\mathbb{S}^2} g(x_1 + \rho y_1, x_2 + \rho y_2)\, dS(\boldsymbol{y}) \tag{4.45}$$

for $\boldsymbol{x} \in \mathbb{R}^2$. By symmetry we can restrict our attention to the upper hemisphere, parametrized in polar coordinates by

$$\boldsymbol{y} = \left(r\cos\theta, r\sin\theta, \sqrt{1 - r^2}\right).$$

The surface area element is

$$dS = \frac{r}{\sqrt{1 - r^2}}\, dr\, d\theta,$$

so that (4.45) becomes

$$\begin{aligned}
\tilde{g}(\boldsymbol{x}; \rho) &= \frac{\rho}{2\pi}\int_0^{2\pi}\int_0^r \frac{g(x_1 + \rho r\cos\theta, x_2 + \rho r\sin\theta)}{\sqrt{1 - r^2}}\, r\, dr\, d\theta \\
&= \frac{\rho}{2\pi}\int_{\mathbb{D}} \frac{g(\boldsymbol{x} + \rho\boldsymbol{y})}{\sqrt{1 - |\boldsymbol{y}|^2}}\, d^2\boldsymbol{y}.
\end{aligned}$$

The claimed two-dimensional solution follows by substituting this formula for $\tilde{g}$ and the corresponding result for $\tilde{h}$ in the Kirchhoff formula from Theorem 4.10. □

Corollary 4.11 shows that the range of influence of $(t_0, \boldsymbol{x}_0)$ for the two-dimensional wave equation is the solid region bounded by the forward light cone $\Gamma_+(t_0, \boldsymbol{x}_0)$, not just the surface. Thus, in $\mathbb{R}^2$ the wave caused by a sudden disturbance has a lingering "tail" after the initial wavefront has passed, as illustrated in Fig. 4.14.

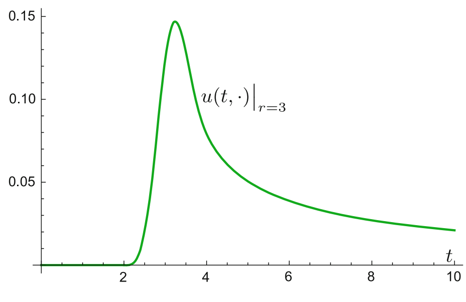

Fig. 4.14 Two-dimensional waveform observed at a distance 3 from the origin, corresponding to the radial impulse shown on the left in Fig. 4.13

In all dimensions, solutions of the wave equation exhibit a phenomenon known as *finite propagation speed*. If the constant c is reinstated as in (4.30), then the range of influence is restricted to spacetime points that are reachable at a speed less than or equal to c.

4.7 Energy and Uniqueness

In Theorem 4.1, uniqueness of solutions was a consequence of the method of characteristics. In this section we will present an alternative approach, which allows us to deduce uniqueness directly from the equation without requiring any knowledge of the solution other than differentiability. This argument is based on the concept of *energy* of a solution, which proves to be a powerful tool for analyzing many different types of PDE.

To motivate the definition, let us specialize again to the case of a string of length ℓ with fixed ends. Assume that $u \in C^2([0,\infty] \times [0,\ell])$ satisfies the string wave equation (4.4) with Dirichlet boundary conditions. In the discrete model of the string used for the derivation of the equation, the segment of length Δx located at x_j had mass $\rho\Delta x$ and velocity $\frac{\partial u}{\partial t}(x_j)$. By the standard expression for the kinetic energy of a moving particle, $\frac{1}{2}(\text{mass})\times(\text{velocity})^2$, the kinetic energy of this segment is therefore

$$\frac{1}{2}\rho\Delta x\left[\frac{\partial u}{\partial t}(x_j)\right]^2.$$

Summing over the segments and passing the continuum limit gives a formula for the total kinetic energy of the string:

$$\mathcal{E}_K := \frac{\rho}{2}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 dx.$$

The *potential energy* of the solution can be calculated as the energy required to move the string from zero displacement into the configuration described by $u(t,\cdot)$. Let us represent this process by scaling the displacement to $su(t,\cdot)$ for $s \in [0,1]$. By (4.1) the opposing force generated by the tension also scales proportionally to s. The work required to shift the segment at x_j from s to $s+\Delta s$ is therefore $s\Delta F(t,x_j)u(t,x_j)\Delta s$. The potential energy associated with this segment is

$$\begin{aligned}\Delta\mathcal{E}_P(t,x_j) &:= -\int_0^1 s\Delta F(t,x_j)u(t,x_j)\,ds \\ &= -\frac{1}{2}u(t,x_j)\Delta F(t,x_j) \\ &\approx -\frac{T}{2}u(t,x_j)\frac{\partial^2 u}{\partial x^2}(t,x_j)\Delta x,\end{aligned}$$

with a minus sign because the displacement and force are in opposing directions.

Summing over the segments and taking the continuum limit gives the total potential energy,

$$\mathcal{E}_P(t) := -\frac{T}{2}\int_0^{\ell} u\,\frac{\partial^2 u}{\partial x^2}\,dx.$$

For comparison to the kinetic term, it is convenient to integrate by parts and rewrite this in the form

$$\mathcal{E}_P(t) = \frac{T}{2}\int_0^{\ell}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

The total energy of the one-dimensional string at time t is given by

$$\mathcal{E} = \mathcal{E}_K + \mathcal{E}_P.$$

For the higher-dimensional wave equation (4.30) on a domain $\Omega \subset \mathbb{R}^n$ the corresponding definition is

$$\mathcal{E}[u](t) := \frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2\right] d^n\boldsymbol{x} \tag{4.46}$$

This is well-defined for $u \in C^2([0,\infty)\times\overline{\Omega})$, provided Ω is bounded.

Theorem 4.12 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary. If $u \in C^2([0,\infty)\times\overline{\Omega})$ is a solution of (4.30) with $u|_{\partial\Omega} = 0$, then the energy $\mathcal{E}[u]$ defined by (4.46) is independent of t.*

Proof The assumptions on u justify differentiating under the integral, so that

$$\frac{d}{dt}\mathcal{E}[u] = \int_{\Omega}\left[\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2\nabla\left(\frac{\partial u}{\partial t}\right)\cdot\nabla u\right] d^n\boldsymbol{x}.$$

Under the condition $u|_{\partial\Omega} = 0$, Green's first identity (Theorem 2.10) applies to the second term to give

$$\int_{\Omega}\nabla\left(\frac{\partial u}{\partial t}\right)\cdot\nabla u\,d^n\boldsymbol{x} = -\int_{\Omega}\frac{\partial u}{\partial t}\Delta u\,d^n\boldsymbol{x}. \tag{4.47}$$

Thus

$$\frac{d}{dt}\mathcal{E}[u] = \int_{\Omega}\left[\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - c^2\Delta u\right)\right] d^n\boldsymbol{x},$$

and (4.30) implies that $\mathcal{E}$ is constant. □

Corollary 4.13 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary. A solution $u \in C^2(\mathbb{R}_+ \times \overline{\Omega})$ of the equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f, \quad u|_{\partial\Omega} = 0,$$

$$u|_{t=0} = g, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = h,$$

is uniquely determined by the functions f, g, h.

Proof If u_1 and u_2 are solutions of the equation with the same initial conditions, then $w := u_1 - u_2$ satisfies (4.30) with the initial conditions

$$w(0, x) = 0, \quad \frac{\partial w}{\partial t}(0, x) = 0.$$

At time $t = 0$ this gives $\mathcal{E}[w] = 0$, and Theorem 4.12 then implies that $\mathcal{E}[w] = 0$ for all t. Since the terms in the integrand of $\mathcal{E}[w]$ are non-negative, they must each vanish. This shows that w is constant, and hence $w = 0$ by the initial conditions. Therefore $u_1 = u_2$. □

4.8 Exercises

4.1 Suppose $u(t, x)$ is a solution of the wave equation (4.5) for $x \in \mathbb{R}$. Let $\mathcal{P}$ be a parallelogram in the (t, x) plane whose sides are characteristic lines. Show that the value of u at each vertex of $\mathcal{P}$ is determined by the values at the other three vertices.

4.2 The wave equation (4.5) is an appropriate model for the longitudinal vibrations of a spring. In this application $u(t, x)$ represents displacement parallel to the spring. Suppose that spring has length ℓ and is free at the ends. This corresponds to the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, \ell) = 0, \quad \text{for all } t \geq 0.$$

Assume the initial conditions are g and h as in (4.16), which also satisfy Neumann boundary conditions on $[0, \ell]$. Determine the appropriate extensions of g and h from $[0, \ell]$ to $\mathbb{R}$ so that the solution $u(t, x)$ given by (4.8) will satisfy the Neumann boundary problem for all t.

4.3 In the derivation in Sect. 4.1, suppose we include the effect of gravity by adding a term $-\rho g \Delta x$ to the discrete equation of motion (4.2), where $g > 0$ is the constant of gravitational acceleration. The wave equation is then modified to

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -g. \tag{4.48}$$

Assume that $x \in [0, \ell]$, with u satisfying Dirichlet boundary conditions at the endpoints.

(a) Find an *equilibrium* solution $u_0(x)$ for (4.48), that satisfies the boundary conditions but does not depend on time.

(b) Show that if u_1 is a solution of the original wave equation (4.5), also with Dirichlet boundary conditions, then $u = u_0 + u_1$ solves (4.48).

(c) Given the initial conditions $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = 0$, find the corresponding initial conditions for u_1. Then apply Theorem 4.5 to find u_1 and hence solve for u.

4.4 In Example 4.8, let the forcing term be

$$f(t, x) = \cos(\omega t)\sin(\omega_k x),$$

with $\omega > 0$ and

$$\omega_k := \frac{k\pi}{\ell}.$$

Find the solution $u(t, x)$ given initial conditions $g = h = 0$. Include both cases $\omega \neq \omega_k$ and $\omega = \omega_k$.

4.5 The *telegraph equation* is a variant of the wave equation that describes the propagation of electrical signals in a one-dimensional cable:

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu - c^2\frac{\partial^2 u}{\partial x^2} = 0,$$

where $u(t, x)$ is the line voltage, c is the propagation speed, and $a, b > 0$ are determined by electrical properties of the cable (resistance, inductance, etc.). Show that the substitution

$$u(t, x) = e^{-at/2}w(t, x)$$

reduces the telegraph equation to an ordinary wave equation for w, provided a and b satisfy a certain condition. Find the general solution in this case. (This result has important practical applications, in that the electrical properties of long cables can be "tuned" to eliminate distortion.)

4.6 An alternative approach to the one-dimensional wave equation is to recast the PDE as a pair of ODE. Consider the wave equation with forcing term,

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f.$$

(a) Define a vector-valued function $\boldsymbol{v} = (v_1, v_2)$ with components

$$v_1 := \frac{\partial u}{\partial t}, \qquad v_2 := \frac{\partial u}{\partial x}.$$

Show that $\boldsymbol{v}$ satisfies a vector equation

$$\frac{\partial \boldsymbol{v}}{\partial t} - A \cdot \frac{\partial \boldsymbol{v}}{\partial x} = \boldsymbol{b}. \tag{4.49}$$

where $\boldsymbol{b} := (f, 0)$ and A is the matrix

$$A := \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix}.$$

(b) The vector equation (4.49) can be solved by diagonalizing A. Check that if we set

$$T := \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix},$$

then

$$TAT^{-1} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}.$$

Then show under that the substitution

$$\boldsymbol{w} := T\boldsymbol{v},$$

(4.49) reduces to a pair of linear conservation equations for the components of $\boldsymbol{w}$:

$$\begin{cases} \frac{\partial w_1}{\partial t} - c\frac{\partial w_1}{\partial x} = f, \\ \frac{\partial w_2}{\partial t} + c\frac{\partial w_2}{\partial x} = f. \end{cases} \tag{4.50}$$

(c) Translate the initial conditions

$$u(0, x) = g(x), \qquad \frac{\partial u}{\partial t}(0, x) = h(x),$$

into initial conditions for w_1 and w_2, and then solve (4.50) using the method of characteristics.

(d) Combine the solutions for w_1 and w_2 to compute $v_1 = \partial u/\partial t$, and then integrate to solve for u. Your answer should be a combination of of the d'Alembert formula (4.8) and the Duhamel formula (4.22).

4.7 The evolution of a quantum-mechanical wave function $u(t, \boldsymbol{x})$ is governed by the *Schrödinger equation*:

$$\frac{\partial u}{\partial t} - i\Delta u = 0 \tag{4.51}$$

(ignoring the physical constants). Suppose that $u(t, \boldsymbol{x})$ is a solution of (4.51) for $t \in [0, \infty)$ and $\boldsymbol{x} \in \mathbb{R}^n$, with initial condition

$$u(0, x) = g(x).$$

Assume that

$$\int_{\mathbb{R}^n} |g|^2 \, d^n\boldsymbol{x} < \infty.$$

(a) Show that for all $t \geq 0$,

$$\int_{\mathbb{R}^n} |u(t, \boldsymbol{x})|^2 \, d^n\boldsymbol{x} = \int_{\mathbb{R}^n} |g|^2 \, d^n\boldsymbol{x}.$$

(In quantum mechanics $|u|^2$ is interpreted as a probability density, so this identity is conservation of total probability.)

(b) Show that a solution of Schrödinger's equation is uniquely determined by the initial condition g.

4.8 In $\mathbb{R}^n$ consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = 0. \tag{4.52}$$

The *plane wave* solutions have the form

$$u(t, x) = e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{4.53}$$

where $\omega \in \mathbb{R}$ and $\boldsymbol{k} \in \mathbb{R}^n$ are constants.

(a) Find the condition on $\omega = \omega(\boldsymbol{k})$ for which u solves (4.52).

(b) For fixed $t, \theta \in \mathbb{R}$, show that $\left\{\boldsymbol{x} \in \mathbb{R}^n;\ u(t, \boldsymbol{x}) = e^{i\theta}\right\}$ is a set of planes perpendicular to $\boldsymbol{k}$. Show that these planes propagate, as t increases, in a direction parallel to $\boldsymbol{k}$ with speed given by c. (Hence the term "plane" wave.)

4.9 The *Klein-Gordon equation* in $\mathbb{R}^n$ is a variant of the wave equation that appears in relativistic quantum mechanics,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u = 0, \tag{4.54}$$

where m is the mass of a particle.

(a) Find a formula for $\omega = \omega(\boldsymbol{k}, m)$ under which this equation admits plane wave solutions of the form (4.53).

(b) Show that we can define a conserved energy $\mathcal{E}$ for this equation by adding a term proportional to u^2 to the integrand in (4.46).