Chapter 10
Weak Solutions

In Sect. 1.2 we observed that d'Alembert's formula for a solution of the wave equation makes sense even when the initial data are not differentiable. This concept of a *weak solution* that is not actually required to solve the equation literally has come up in other contexts as well, for example in the discussion of the traffic equation in Sect. 3.4. In this chapter we will discuss the mathematical formulation of this generalized notion of solution.

Weak solutions first appeared in physical applications as idealized, limiting cases of true solutions. For example, one might replace a smooth density function by a simpler piecewise linear approximation, as illustrated in Fig. 10.1, in order to simplify computations. (We used this idea in Example 3.9.)

Up until the late 19th century, the limiting process by which weak solutions were obtained was understood rather loosely, and justified mainly by physical intuition. Weak solutions proved to be extremely useful, and eventually a consistent mathematical framework was developed.

10.1 Test Functions and Weak Derivatives

Consider a linear equation of the form $Lu = f$, where L is a differential operator on a domain $\Omega \subset \mathbb{R}^n$. Suppose that u represents a physical quantity such as temperature or density. Direct observation of such quantities at a single point is a practical impossibility. Even the most sensitive instrument will only be able to measure the weighted average over some small region.

To formalize this notion of a local average, we use the concept of a *test function* $\psi \in C^\infty_{\text{cpt}}(\Omega)$. The test function defines a local measurement of a quantity u through the integral

$$\int_\Omega u\psi \, d^n\boldsymbol{x}. \tag{10.1}$$

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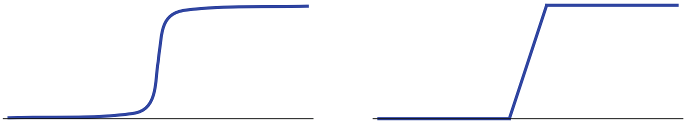


Fig. 10.1 A smooth function and its piecewise linear approximation

The function ψ plays the role of a experimental probe that takes a particular sample of the values of u.

Let us consider how we would "detect" a derivative using test functions. Suppose for the moment that $u \in C^1(\mathbb{R})$, with $u' = f$. If we measure this derivative associated using the test function $\psi \in C^\infty_{\text{cpt}}(\mathbb{R})$, the result is

$$\int_{\mathbb{R}} u'\psi \, dx = \int_{\mathbb{R}} f\psi \, dx. \tag{10.2}$$

The fact that $u' = f$ is equivalent to the statement that (10.2) holds for all $\psi \in C^\infty_{\text{cpt}}(\mathbb{R})$.

Note that the left-hand side could be integrated by parts, since ψ has compact support, yielding

$$-\int_{\mathbb{R}} u\psi' \, dx = \int_{\mathbb{R}} f\psi \, dx. \tag{10.3}$$

This condition now makes sense even when u fails to be differentiable. The only requirement is that u and f be integrable on compact sets, a property we refer to as *local integrability*. We can say that locally integrable functions satisfy $u' = f$ in the weak sense provided (10.3) holds for all $\psi \in C^\infty_{\text{cpt}}(\mathbb{R})$.

To generalize this definition to a domain $\Omega \subset \mathbb{R}^n$, let us define the space of locally integrable functions,

$$L^1_{\text{loc}}(\Omega) := \left\{ f : \Omega \to \mathbb{C};\ \ f|_K \in L^1(K) \text{ for all compact } K \subset \Omega \right\}.$$

The same equivalence relation (7.6) used for L^p spaces applies to L^1_{loc}, i.e., functions that differ on a set of measure zero are considered to be the same.

Inspired by (10.3), for u and $f \in L^1_{\text{loc}}(\Omega)$ we say that

$$\frac{\partial u}{\partial x_j} = f$$

as a *weak derivative* if

$$\int_\Omega u \frac{\partial \psi}{\partial x_j} \, dx = -\int_\Omega f\psi \, dx \tag{10.4}$$

for all $\psi \in C^\infty_{\text{cpt}}(\mathbb{R})$. The condition (10.4) determines f uniquely as an element of $L^1_{\text{loc}}(\Omega)$, by the following:

Lemma 10.1 *If $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfies*

$$\int_\Omega f\psi \, d^n\boldsymbol{x} = 0$$

for all $\psi \in C^\infty_{\mathrm{cpt}}(\Omega)$, then $f \equiv 0$.

Proof It suffices to consider the case when Ω is bounded, since a larger domain could be subdivided into bounded pieces. For bounded Ω the local integrability of f implies that $f \in L^2(\Omega)$. By Theorem 7.5 we can choose a sequence ψ_k in $C^\infty_{\mathrm{cpt}}(\Omega)$ such that $\psi_k \to f$ in $L^2(\Omega)$. This implies that

$$\lim_{k\to\infty} \langle f, \psi_k \rangle = \|f\|_2.$$

The inner products $\langle f, \psi_k \rangle$ are zero by hypothesis, so we conclude that $f \equiv 0$. □

Example 10.2 In $\mathbb{R}$, consider the piecewise linear function

$$g(x) := \begin{cases} 0, & x < 0, \\ x, & 0 \le x \le 1, \\ 1, & x > 0. \end{cases}$$

If we ignore the points where g is not differentiable, then we would expect that the derivative of g is given by

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

These functions are illustrated in Fig. 10.2.

Let us check that this works in the sense of weak derivatives. We take $\psi \in C^\infty_{\mathrm{cpt}}(\mathbb{R})$ and compute

$$\int_{-\infty}^{\infty} g\psi' \, dx = \int_0^1 x\psi'(x) \, dx + \int_1^\infty \psi' \, dx.$$

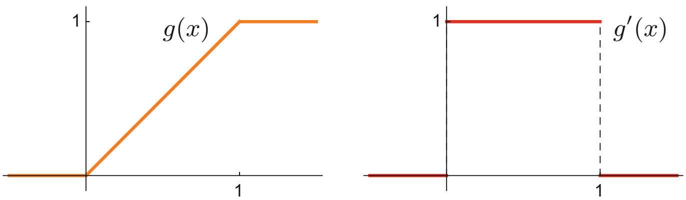


Fig. 10.2 A piecewise linear function and its weak derivative

Using integration by parts on the first term and evaluating the second gives

$$\begin{aligned}\int_{-\infty}^{\infty} g\psi'\,dx &= \psi(1) - \int_0^1 \psi\,dx - \psi(1)\\ &= -\int_{\infty}^{\infty} f\psi\,dx.\end{aligned}$$

This verifies that $g' = f$ in the weak sense. ♢

Example 10.3 For $t \in \mathbb{R}$, define $w \in L^1_{\text{loc}}(\Omega)$ by

$$w(t) = \begin{cases} w_-(t), & t < 0\\ w_+(t), & t \ge 0,\end{cases} \tag{10.5}$$

where $w_\pm \in C^1(\mathbb{R})$. For $\psi \in C^\infty_{\text{cpt}}(\mathbb{R})$,

$$\begin{aligned}-\int_{-\infty}^{\infty} w\psi'\,dt &= -\int_{-\infty}^{0} w_-\psi'\,dt - \int_0^{\infty} w_+\psi'\,dt\\ &= [w_+(0) - w_-(0)]\psi(0) + \int_{-\infty}^{0} w_-'\psi\,dt + \int_0^{\infty} w_+'\psi\,dt.\end{aligned}$$

The term proportional to $\psi(0)$ could not possibly come from the integral of ψ against a locally integrable function, because the value of the integrand at a single point does not affect the integral. Hence w admits a weak derivative only under the matching condition

$$w_-(0) = w_+(0).$$

If this is satisfied, then the derivative is

$$w'(t) = \begin{cases} w_-'(t), & t < 0,\\ w_+'(t), & t > 0.\end{cases}$$

♢

Weak derivatives of higher order are defined by an extension of (10.4). To write the corresponding formulas, it is helpful to have a simplified notation for higher partials. For each *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$, we define the differential operator on $\mathbb{R}^n$,

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}. \tag{10.6}$$

The order of this operator is denoted by

$$|\alpha| := \alpha_1 + \cdots + \alpha_n.$$

Repeated integration by parts introduces a minus sign for each derivative. Therefore, a function $u \in L^1_{\mathrm{loc}}(\Omega)$ admits a weak derivative $D^\alpha u \in L^1_{\mathrm{loc}}(\Omega)$ if

$$\int_\Omega (D^\alpha u)\psi \, d^n \boldsymbol{x} = (-1)^{|\alpha|} \int_\Omega u D^\alpha \psi \, d^n \boldsymbol{x} \tag{10.7}$$

for all $\psi \in C^\infty_{\mathrm{cpt}}(\Omega)$.

It might seem that we should distinguish between classical and weak derivatives in the notation. This is made unnecessary by the following:

Theorem 10.4 (Consistency of weak derivatives) *If $u \in C^m(\Omega)$ then u is weakly differentiable to order k and the weak derivatives equal the classical derivatives.*

Conversely, if $u \in L^1_{\mathrm{loc}}(\Omega)$ admits weak derivatives $D^\alpha u$ for $|\alpha| \le m$, and each $D^\alpha u$ can be represented by a continuous function, then u is equivalent to a function in $C^m(\Omega)$ whose classical derivatives match the weak derivatives.

In one direction the argument is straightfoward. Classical derivatives satisfy the criterion (10.7) by integration by parts, so they automatically qualify as weak derivatives. Lemma 10.1 shows that weak derivatives are uniquely defined.

The argument for the converse statement, that continuity of the weak derivatives $D^\alpha u$ implies classical differentiability, is much more technical and we will not be able to give the details. The basic idea is to show that one can approximate u by a sequence $\psi_k \in C^\infty_{\mathrm{cpt}}(\Omega)$ such that $D^\alpha \psi_k \to D^\alpha u$ uniformly on every compact subset of Ω for $|\alpha| \le m$. The fact that the weak derivatives $D^\alpha u$ are continuous makes this possible. Uniform convergence then allows the classical derivatives of u to be computed as limits of the functions $D^\alpha \psi_k$.

10.2 Weak Solutions of Continuity Equations

Consider the continuity equation on $\mathbb{R}$ introduced in Sect. 3.1,

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad u|_{t=0} = g. \tag{10.8}$$

The flux q could depend on u as well as t and x. To allow for the nonlinear case, we will assume that u is real-valued here.

Suppose for the moment that q is differentiable and u is a classical solution of (10.8). Let ψ be a test function in $C^\infty_{\mathrm{cpt}}([0,\infty) \times \mathbb{R})$. Use of the closed interval $[0,\infty)$ means that ψ and its derivatives are not necessarily zero at $t = 0$. Pairing $\frac{\partial u}{\partial t}$ with ψ and integrating by parts thus generates a boundary term,

$$\int_0^\infty \frac{\partial u}{\partial t} \psi \, dt = -u\psi\big|_{t=0} - \int_0^\infty u \frac{\partial \psi}{\partial t} \, dt.$$

On the other hand, the spatial integration parts has no boundary term,

$$\int_{-\infty}^{\infty} \frac{\partial q}{\partial x} \psi \, dx = -\int_{-\infty}^{\infty} q \frac{\partial \psi}{\partial x} \, dx.$$

When the left-hand side of (10.8) is paired with ψ and integrated over both t and x, the result is thus

$$\int_0^{\infty} \int_{-\infty}^{\infty} \Big[\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x}\Big] \psi \, dx \, dt = -\int_0^{\infty} \int_{-\infty}^{\infty} \Big[u \frac{\partial \psi}{\partial t} + q \frac{\partial \psi}{\partial x}\Big] \psi \, dx \, dt \\ - \int_{-\infty}^{\infty} u\psi|_{t=0} \, dx.$$

If u is a classical solution of (10.8), then

$$\int_0^{\infty} \int_{-\infty}^{\infty} \Big[u \frac{\partial \psi}{\partial t} + q \frac{\partial \psi}{\partial x}\Big] dx \, dt + \int_{-\infty}^{\infty} g\psi|_{t=0} \, dx = 0 \tag{10.9}$$

for all $\psi \in C^{\infty}_{\text{cpt}}([0, \infty) \times \mathbb{R})$.

The $t = 0$ integral in (10.9) makes sense for $g \in L^1_{\text{loc}}(\mathbb{R})$. Under this assumption, we define $u \in L^1_{\text{loc}}((0, \infty) \times \mathbb{R}; \mathbb{R})$ to be a weak solution of (10.8) provided $q \in L^1_{\text{loc}}((0, \infty) \times \mathbb{R}; \mathbb{R})$ and (10.9) holds for all test functions.

Example 10.5 Consider the linear conservation equation with constant velocity, which means $q = cu$ in (10.8). By the method of characteristics (Theorem 3.2), the solution is

$$u(t, x) = g(x - ct).$$

Let us check that this defines a weak solution for $g \in L^1_{\text{loc}}(\mathbb{R})$.

For $\psi \in C^{\infty}_{\text{cpt}}(\mathbb{R})$ the first term in (10.9) is

$$\int_0^{\infty} \int_{-\infty}^{\infty} g(x - ct) \Big[\frac{\partial \psi}{\partial t}(t, x) - c \frac{\partial \psi}{\partial x}(t, x)\Big] dx \, dt. \tag{10.10}$$

To evaluate the integral, introduce the variables

$$\tau = t, \quad y = x - ct,$$

and define

$$\tilde{\psi}(\tau, y) := \psi(\tau, y + c\tau).$$

By the chain rule,

$$\frac{\partial \tilde{\psi}}{\partial \tau} = \frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x}. \tag{10.11}$$

The Jacobian determinant of the transformation $(\tau, y) \mapsto (t, x)$ is 1, so that

$$\int_0^\infty \int_{-\infty}^\infty u\Big[\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x}\Big]dx\,dt = \int_0^\infty \int_{-\infty}^\infty g(y)\frac{\partial \tilde{\psi}}{\partial \tau}\,dy\,d\tau.$$

The τ integration can now be done directly,

$$\int_0^\infty \frac{\partial \tilde{\psi}}{\partial \tau}(\tau, y)\,d\tau = -\tilde{\psi}(0, y).$$

This gives

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty u\Big[\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x}\Big]dx\,dt &= -\int_{-\infty}^\infty g(y)\tilde{\psi}(0, y)\,dy \\ &= -\int_{-\infty}^\infty g(y)\psi(0, y)\,dy,\end{aligned}$$

which verifies (10.9). ♢

We saw in Example 10.3 that in one dimension a jump discontinuity precludes the existence of a weak derivative. Example 10.5 shows that this is not the case in higher dimension. For $g \in L^1_{\text{loc}}(\mathbb{R})$, the solution $u(t, x) = g(x - ct)$ could be highly discontinuous. The direction of the derivative is crucial here; regularity is required only along the characteristics.

As an application of the weak formulation (10.9), let us return to an issue that arose in the traffic model in Sect. 3.4. For certain initial conditions the characteristic lines crossed each other, ruling out a classical solution of the PDE. We will see that weak solutions can still exist in this case.

Consider a one-dimensional quasilinear equation of the form

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}q(u) = 0 \tag{10.12}$$

with the flux $q(u)$ a smooth function of u which is independent of t and x. As we saw in Sect. 3.4, the characteristics are straight lines whose slope depends on the initial conditions. Let us study the situation pictured in Fig. 3.11, where a shock forms as characteristic lines cross at some point. For simplicity we assume that the initial crossing occurs at the origin.

One possible way to resolve the issue of crossing characteristics is to subdivide the (t, x) plane into two regions by drawing a *shock curve* C, as illustrated in Fig. 10.3. Suppose that classical solutions $u_\pm$ are derived by the method of characteristics above and below this curve. We will show that this combination yields a weak solution provided a certain jump condition is satisfied along C. The jump condition was discovered in the 19th century by engineers William Rankine and Pierre Hugoniot, who developed the first theories of shock waves in the context of gas dynamics.

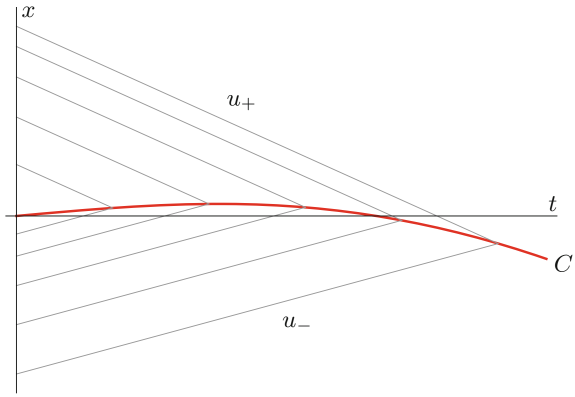


Fig. 10.3 Shock curve with solutions $u_\pm$ on either side

Theorem 10.6 (Rankine-Hugoniot condition) *Let C be a curve parametrized as $x = \sigma(t)$ with $\sigma \in C^1[0, \infty)$. Suppose that u is a weak solution of* (10.12) *given by*

$$u(t, x) = \begin{cases} u_-(t, x), & x < \sigma(t), \\ u_+(t, x), & x > \sigma(t), \end{cases}$$

where $u_\pm$ are classical solutions. Then, at each point of C,

$$q(u_+) - q(u_-) = (u_+ - u_-)\sigma'. \tag{10.13}$$

Proof Since we are not concerned with the boundary conditions, we consider a test function $\psi \in C^\infty_{\text{cpt}}((0, \infty) \times \mathbb{R})$, for which (10.9) specializes to

$$\int_0^\infty \int_{-\infty}^\infty \left[u \frac{\partial \psi}{\partial t} + q(u)\frac{\partial \psi}{\partial x}\right] dx\, dt = 0. \tag{10.14}$$

Since the solutions $u_\pm$ are classical and σ is C^1, we can separate the integral (10.14) at the shock curve and integrate by parts on either side.

Consider first the u_- side. For the term involving the x derivative the integration by parts is straightforward:

$$\begin{aligned}\int_0^\infty \int_{-\infty}^{\sigma(t)} q(u_-)\frac{\partial \psi}{\partial x}\, dx\, dt = &-\int_0^\infty \int_{-\infty}^{\sigma(t)} \psi \frac{\partial}{\partial x} q(u_-)\, dx\, dt \\ &+ \int_0^\infty \psi q(u_-)\big|_{x=\sigma(t)}\, dt.\end{aligned}$$

For the t-derivative term we start by using the fundamental theorem of calculus to derive

$$\frac{d}{dt}\int_{-\infty}^{\sigma(t)} \psi u_- \, dx = \int_{-\infty}^{\sigma(t)} \left[\frac{\partial \psi}{\partial t} u_- + \psi \frac{\partial u_-}{\partial t}\right] dx + \sigma'(t)\psi u_-\big|_{x=\sigma(t)}.$$

By the compact support of ψ, the integral over t of the left-hand side vanishes, yielding

$$\begin{aligned}\int_0^\infty \int_{-\infty}^{\sigma(t)} \frac{\partial \psi}{\partial t} u_- \, dx\, dt = &-\int_0^\infty \int_{-\infty}^{\sigma(t)} \psi \frac{\partial u_-}{\partial t} \, dx\, dt \\ &- \int_0^\infty \sigma'(t)\psi u_-\big|_{x=\sigma(t)} \, dt.\end{aligned}$$

Combining these integration by parts formulas gives

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^{\sigma(t)} \left[u_- \frac{\partial \psi}{\partial t} + q(u_-)\frac{\partial \psi}{\partial x}\right] dx\, dt \\ &\quad = -\int_0^\infty \int_{-\infty}^{\sigma(t)} \psi \left[\frac{\partial u_-}{\partial t} - \frac{\partial}{\partial x} q(u_-)\right] dx\, dt \\ &\qquad - \int_0^\infty \left[\sigma'(t) u_- - q(u_-)\right]\psi\Big|_{x=\sigma(t)} \, dt.\end{aligned}$$

The first term on the right vanishes by the assumption that u_- is a classical solution, leaving

$$\int_0^\infty \int_{-\infty}^{\sigma(t)} \left[u_- \frac{\partial \psi}{\partial t} + q(u_-)\frac{\partial \psi}{\partial x}\right] dx\, dt = -\int_0^\infty \left[\sigma' u_- - q(u_-)\right]\psi\Big|_{x=\sigma(t)} \, dt.$$

The corresponding calculation on the u_+ side yields

$$\int_0^\infty \int_{\sigma(t)}^{\infty} \left[u_+ \frac{\partial \psi}{\partial t} + q(u_+)\frac{\partial \psi}{\partial x}\right] dx\, dt = \int_0^\infty \left[\sigma' u_+ - q(u_+)\right]\psi\Big|_{x=\sigma(t)} \, dt.$$

By (10.14) the sum of the u_- and u_+ integrals is zero, which implies

$$\int_0^\infty \left[(u_+ - u_-)\sigma' - (q(u_+) - q(u_-))\right]\psi\Big|_{x=\sigma(t)} \, dt.$$

Since this holds for all $\psi \in C^\infty((0,\infty)\times\mathbb{R})$, we conclude that

$$\left[(u_+ - u_-)\sigma' - (q(u_+) - q(u_-))\right]\Big|_{x=\sigma(t)} = 0$$

for all $t > 0$. □

Example 10.7 Consider the traffic equation introduced in Sect. 3.4,

$$\frac{\partial u}{\partial t} + (1 - 2u)\frac{\partial u}{\partial x} = 0,$$

for which $q(u) = u - u^2$. For the initial condition take a step function,

$$g(x) := \begin{cases} a, & x < 0, \\ b, & x > 0. \end{cases} \tag{10.15}$$

From 3.28 the characteristic lines are given by

$$x(t) = \begin{cases} x_0 + (1 - 2a)t, & x_0 < 0, \\ x_0 + (1 - 2b)t, & x_0 > 0. \end{cases}$$

These intersect to form a shock provided $a < b$.

The solutions above and below the shock line are given by constants,

$$u_-(t, x) = a, \quad u_+(t, x) = b.$$

The Rankine-Hugoniot condition (10.13) thus reduces to

$$q(b) - q(a) = (b - a)\sigma'.$$

Substituting with $q(u) = u - u^2$ reduces this condition to

$$\sigma' = 1 - b - a.$$

Since the discontinuity starts at the origin, the shock curve is thus given by

$$\sigma(t) = (1 - b - a)t.$$

Hence the weak solution is

$$u(t, x) = \begin{cases} a, & x < (1 - b - a)t, \\ b, & x > (1 - b - a)t. \end{cases}$$

Some cases are illustrated in Fig. 10.4. In the plot on the left, the shock wave propagates backwards. ♢

For certain initial conditions, the definition (10.9) of a weak solution is not sufficient to determine the solution uniquely. For example, if we had taken $a > b$ in (10.15), then instead of overlapping the characteristics originating from $t = 0$ would separate, leaving a triangular region with no characteristic lines. An additional physical condition is required to specify the solution uniquely in this case. We will discuss this further in the exercises.

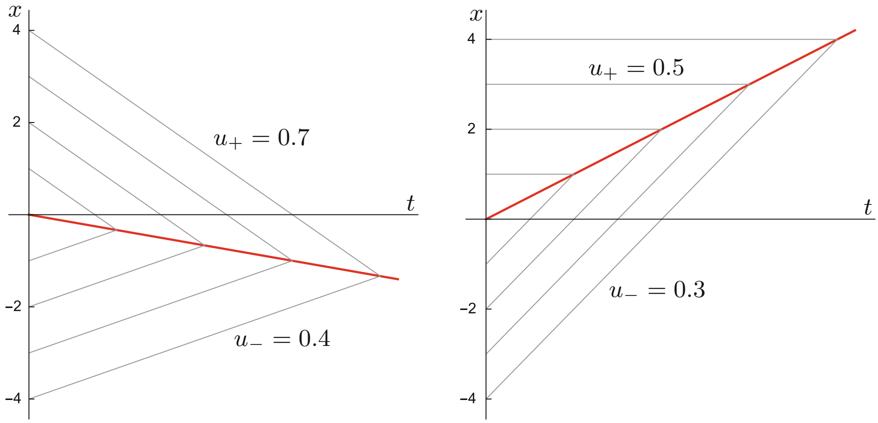


Fig. 10.4 Characteristic lines meeting at the shock wave

10.3 Sobolev Spaces

Boundary values are not well defined for locally integrable functions. We were able to avoid this issue in the discussion of the continuity equation in Sect. 10.2, because solutions were required to be constant along characteristics. In general, the formulation of boundary or initial conditions requires a class of functions with greater regularity.

The most obvious class to consider consists of functions that admit weak higher partial derivatives. However, it proves to be very helpful to strengthen the integrability requirements as well. Such function spaces were introduced by Sergei Sobolev in the mid 20th century and have since become fundamental tools of analysis.

The Sobolev spaces based on L^2 are defined by

$$H^m(\Omega) := \left\{u \in L^2(\Omega);\ D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \le m\right\},$$

for $m \in \mathbb{N}_0$, with derivatives interpreted in the weak sense. An extended family of Sobolev spaces $W^{m,p}$ is given by replacing L^2 with L^p in the definition. The extended family is important in the analysis of nonlinear PDE, but our focus will be limited to linear applications involving H^m.

Sobolev spaces are useful as theoretical tools, but they also have a practical side. For a bounded domain Ω, the space $H^1(\Omega)$ includes the continuous *piecewise linear* functions. A function is called piecewise linear if the domain can be decomposed into a finite number of polygonal subdomains, on which the function is linear. Figure 10.5 shows a two-dimensional example. Sobolev spaces provide a natural framework for the approximation of solutions by computationally simple classes of functions.

The space $H^m(\Omega)$ carries a natural inner product,

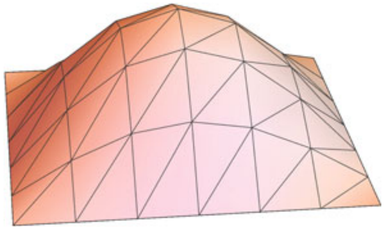


Fig. 10.5 Graph of a piecewise linear H^1 function on the unit square

$$\langle u, v\rangle_{H^m} := \sum_{|\alpha|\le m} \langle D^\alpha u, D^\alpha v\rangle. \tag{10.16}$$

(Our convention will be that a bracket without subscript denotes the L^2 inner product.) The corresponding norm is

$$\|u\|_{H^m} := \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|_2^2\Big)^{\frac{1}{2}}. \tag{10.17}$$

Lebesgue integration theory gives us the following completeness result, analogous to Theorem 7.7.

Theorem 10.8 *For $\Omega \subset \mathbb{R}^n$ and $m \in \mathbb{N}_0$, $H^m(\Omega)$ is a Hilbert space.*

Recall that Theorem 7.5 says that $C^\infty_{\text{cpt}}(\Omega)$ is a dense subspace of $L^2(\Omega)$. This means that the closure of $C^\infty_{\text{cpt}}(\Omega)$ with respect to the L^2 norm is $L^2(\Omega)$. This result no longer holds for the Sobolev space $H^m(\Omega)$ with $m \ge 1$. In particular, the closure of $C^\infty_{\text{cpt}}(\Omega)$ with respect to the H^1 norm defines a subspace

$$H^1_0(\Omega) = \left\{u \in H^m(\Omega);\ \lim_{k\to\infty} \|u - \psi_k\|_{H^1} = 0 \text{ for } \psi_k \in C^\infty_{\text{cpt}}(\Omega)\right\}. \tag{10.18}$$

By Lemma 7.8 $H^1_0(\Omega)$ is also a Hilbert space with respect to the H^1 norm.

If $\partial\Omega$ is piecewise C^1, then for functions in $H^1(\Omega)$ it is possible to define boundary restrictions in $L^2(\partial\Omega)$ that generalize the boundary restriction of a continuous function. In this case, $H^1_0(\Omega)$ consists precisely of the functions whose boundary restriction vanishes. Thus the space $H^1_0(\Omega)$ can be interpreted as the class of H^1 functions satisfying Dirichlet boundary conditions on $\partial\Omega$.

The theory of boundary restrictions is too technical for us to cover here, but we can at least show how this works in the one-dimensional case.

Theorem 10.9 *If $u \in H^1_0(a, b)$ then u is continuous on $[a, b]$ and equal to zero at the endpoints.*

Proof Suppose $u \in H^1_0(a, b)$. By definition, there exists a sequence of $C^\infty_{\text{cpt}}(a, b)$ such that

$$\lim_{k\to\infty} \|\psi_k - u\|_{H^1} = 0.$$

For $x \in [a, b]$,

$$\psi_j(x) - \psi_k(x) = \int_a^x [\psi_j'(t) - \psi_k'(t)]\, dt$$

The integral on the right could be expressed as an inner product on $\mathbb{R}$,

$$\int_a^x [\psi_j'(t) - \psi_k'(t)]\, dt = \langle \psi_j' - \psi_k', \chi_{[a,x]} \rangle,$$

where χ_I denotes the characteristic function of the interval I. Thus, by the Cauchy-Schwarz inequality (Theorem 7.1),

$$|\psi_j(x) - \psi_k(x)| \le \sqrt{x-a}\, \|\psi_j' - \psi_k'\|_2.$$

In view of the definition of the H^1 norm, this implies the uniform bound

$$\|\psi_j - \psi_k\|_\infty \le \sqrt{b-a}\, \|\psi_j - \psi_k\|_{H^1} \tag{10.19}$$

Since $\{\psi_k\}$ converges and is therefore Cauchy with respect to the H^1 norm, it follows from (10.19) implies that the sequence $\{\psi_k\}$ is also Cauchy in the uniform sense. By the completeness of $L^\infty(a, b)$ (Theorem 7.7) and Lemma 8.4, this implies that $\psi_k \to g$ uniformly for some $g \in C^0[a, b]$.

At this point we have $\psi_k \to u$ in H^1 and $\psi_k \to g$ uniformly. Uniform convergence on a bounded interval implies convergence in L^2, by a simple integral estimate. Therefore $\psi_k \to g$ in L^2 also, implying that $u = g$ in L^2. Hence $u \in C^0[a, b]$.

To show that u vanishes at the endpoints, note that

$$\max\{|u(a)|, |u(b)|\} \le \sup_{[a,b]} |\psi_k - u|. \tag{10.20}$$

because $\psi_k(a) = \psi_k(b) = 0$. By uniform convergence, the left-hand side of (10.20) approaches zero as $k \to \infty$, showing that

$$u(a) = u(b) = 0.$$

□

In higher dimensions, functions in H^1 are not necessarily continuous. However, H^m does imply continuity if m is sufficiently large relative to the dimension. We will develop this regularity theory in Sect. 10.4.

We conclude this section with an extension property that will prove useful in Chap. 11.

Lemma 10.10 *For $\Omega \subset \widetilde{\Omega} \subset \mathbb{R}^n$, the extension by zero of an element of $H^1_0(\Omega)$ gives an element of $H^1_0(\widetilde{\Omega})$.*

Proof For $u \in H^1_0(\Omega)$, let $\tilde{u}$ denote the extension by zero to $\widetilde{\Omega}$. The weak gradient $\nabla u \in L^2(\Omega; \mathbb{R}^n)$ can also be extended by zero to $\widetilde{\nabla u} \in L^2(\widetilde{\Omega}; \mathbb{R}^n)$. We need to show that $\widetilde{\nabla u}$ is the weak gradient of $\tilde{u}$. This is the condition that

$$\int_{\widetilde{\Omega}} \phi \widetilde{\nabla u}\, d^n \boldsymbol{x} = - \int_{\widetilde{\Omega}} \tilde{u} \nabla \phi \, d^n \boldsymbol{x}, \tag{10.21}$$

for all $\phi \in C^\infty_{\mathrm{cpt}}(\widetilde{\Omega})$.

By the definition of $H^1_0(\Omega)$, there exists a sequence of $\psi_k \in C^\infty_{\mathrm{cpt}}(\Omega)$ such that $\psi_k \to u$ in the H^1 norm. Since ψ_k has compact support within Ω, integration by parts gives

$$\int_{\Omega} \phi \nabla \psi_k \, d^n \boldsymbol{x} = - \int_{\Omega} \psi_k \nabla \phi \, d^n \boldsymbol{x}. \tag{10.22}$$

By the H^1 convergence $\psi_k \to u$, we can take the limit $k \to \infty$ on both sides of (10.22) to obtain

$$\int_{\Omega} \phi \nabla u \, d^n \boldsymbol{x} = - \int_{\Omega} u \nabla \phi \, d^n \boldsymbol{x}.$$

Since $\tilde{u}$ and $\widetilde{\nabla u}$ are equal to u and ∇u on Ω and vanish on $\widetilde{\Omega} - \Omega$, this is equivalent to (10.21). □

10.4 Sobolev Regularity

In this section we will consider the relationship between weak regularity, defined in terms of Sobolev spaces, and regularity in the classical sense. This connection plays a central role in the application of Sobolev spaces to PDE.

Theorem 10.11 (Sobolev embedding theorem) *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain. If $m > k + \frac{n}{2}$, then*

$$H^m(\Omega) \subset C^k(\Omega).$$

This result can be sharpened and extended in various ways. One important variant includes differentiability up to the boundary under certain conditions on $\partial\Omega$. For example, if the boundary $\partial\Omega$ is piecewise C^1 then it is possible to show that

$$H^m(\Omega) \subset C^k(\overline{\Omega}).$$

These boundary results are quite important but too technically difficult for us to include here.

The strategy we will use for Theorem 10.11 is based on the connection established in Sect. 8.6 between regularity and the decay of Fourier coefficients. Recall the definition $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ introduced in Sect. 8.2. To extend Fourier series to higher dimensions we introduce the corresponding space

$$\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n.$$

A function on $\mathbb{T}^n$ is a function on $\mathbb{R}^n$ which is 2π-periodic in each coordinate.

The periodic Fourier series theory from can be carried over to $\mathbb{T}^n$ directly. For $f \in L^2(\mathbb{T}^n)$ and $\boldsymbol{k} \in \mathbb{Z}^n$ we define the coefficients

$$c_{\boldsymbol{k}}[f] := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\, d^n\boldsymbol{x}. \tag{10.23}$$

The integral over $\mathbb{T}^n$ can be taken over $[-\pi, \pi]^n$, or any translate of this cube. The argument from Theorem 8.6 can be adapted, with minor notational changes, to prove the following:

Theorem 10.12 *For $f \in L^2(\mathbb{T}^n)$, the series*

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^n} c_{\boldsymbol{k}}[f]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

converges to f in the L^2 norm.

As a corollary, we obtain the generalization of the Parseval identity (8.36),

$$\langle f, g\rangle = (2\pi)^n \sum_{\boldsymbol{k}\in\mathbb{Z}^n} c_{\boldsymbol{k}}[f]\overline{c_{\boldsymbol{k}}[g]} \tag{10.24}$$

for $f, g \in L^2(\mathbb{T}^n)$.

Because of the periodic structure of $\mathbb{T}^n$, it is not necessary to assume that test functions have compact support. For $f \in L^1_{loc}(\mathbb{T})$ the weak derivative $D^\alpha f \in L^1_{loc}(\mathbb{T})$ is defined by the condition that

$$\int_0^{2\pi} \psi D^\alpha f\, dx = (-1)^{|\alpha|}\int_0^{2\pi} f D^\alpha \psi\, dx \tag{10.25}$$

for all $\psi \in C^\infty(\mathbb{T})$. The space $H^m(\mathbb{T}^n)$ consists of functions in $L^2(\mathbb{T}^n)$ which have weak partial derivatives up to order m contained in $L^2(\mathbb{T}^n)$.

It is convenient notate powers of the components of $\boldsymbol{k}$ by analogy with D^α,

$$\boldsymbol{k}^\alpha := k_1^{\alpha_1}\cdots k_n^{\alpha_n},$$

for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$. A simple computation shows that

$$D^\alpha e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = (i\boldsymbol{k})^\alpha e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Thus, for $f \in H^m(\mathbb{T})$, substituting $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ into (10.25) gives

$$c_{\boldsymbol{k}}[D^\alpha f] = (i\boldsymbol{k})^\alpha c_{\boldsymbol{k}}[f] \tag{10.26}$$

for $|\alpha| \le m$. This generalizes the integration by parts formula (8.30).

Theorem 10.13 *A function $f \in L^2(\mathbb{T})$ lies in $H^m(\mathbb{T})$ for $m \in \mathbb{N}$ if and only if*

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\boldsymbol{k}|^{2m} \, |c_{\boldsymbol{k}}[f]|^2 < \infty. \tag{10.27}$$

Proof By (10.26) and Bessel's inequality (Proposition 7.9), the condition that $D^\alpha f \in L^2(\mathbb{T}^n)$ implies that

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^n} \left|\boldsymbol{k}^\alpha c_{\boldsymbol{k}}[f]\right|^2 < \infty. \tag{10.28}$$

This holds for all $|\alpha| \le m$, implying (10.27).

Conversely, if $f \in L^2(\mathbb{T}^n)$ satisfies (10.27), then (10.28) holds for $|\alpha| \le m$. We can therefore define functions $g_\alpha \in L^2(\mathbb{T}^n)$ by the Fourier series

$$g_\alpha(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^n} (i\boldsymbol{k})^\alpha c_{\boldsymbol{k}}[f] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

By Parseval's identity (10.24), the inner product of g_α with $\psi \in C^\infty_{\text{cpt}}(\mathbb{T}^n)$ gives

$$\begin{aligned}
\langle g_\alpha, \psi\rangle &= (2\pi)^n \sum_{\boldsymbol{k}\in\mathbb{Z}^n} (i\boldsymbol{k})^\alpha c_{\boldsymbol{k}}[f]\overline{c_{\boldsymbol{k}}[\psi]} \\
&= (-1)^{|\alpha|} 2\pi \sum_{k\in\mathbb{Z}} c_{\boldsymbol{k}}[f]\overline{c_{\boldsymbol{k}}[D^\alpha\psi]} \\
&= (-1)^{|\alpha|}\langle f, D^\alpha\psi\rangle.
\end{aligned}$$

This shows that the weak derivative $D^\alpha f$ exists and is equal to g_α. □

Theorem 10.13 makes the connection between Sobolev regularity and decay of Fourier coefficients. Our task is now to translate this back into classical regularity.

Theorem 10.14 (Periodic Sobolev embedding) *If $m > q + \frac{n}{2}$, then*

$$H^m(\mathbb{T}^n) \subset C^q(\mathbb{T}^n).$$

Proof Using the notation for discrete spaces introduced in Sect. 7.4, the space $\ell^2(\mathbb{Z}^n)$ is defined as the Hilbert space of functions $\mathbb{Z}^n \to \mathbb{C}$, equipped with the inner product

$$\langle \beta, \gamma \rangle_{\ell^2} := \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \beta(\boldsymbol{k}) \overline{\gamma(\boldsymbol{k})}.$$

Consider the function

$$\beta(\boldsymbol{k}) := (1 + |\boldsymbol{k}|)^{-m}.$$

The ℓ^2 norm of β can be estimated with an integral,

$$\begin{aligned}\|\beta\|_{\ell^2}^2 &:= \sum_{\boldsymbol{k} \in \mathbb{Z}^n} (1 + |\boldsymbol{k}|)^{-2m} \\ &\le \int_{\mathbb{R}^n} (1 + |\boldsymbol{x}|)^{-2m} \, d^n \boldsymbol{x} \\ &= A_n \int_0^\infty (1 + r)^{-2m} r^{n-1} \, dr.\end{aligned}$$

The integral is finite if $2m > n$, implying that $\beta \in \ell^2(\mathbb{Z}^n)$ for $m > \frac{n}{2}$.

By Theorem 10.13, for $f \in H^m(\mathbb{T}^n)$ we can also define an element of $\ell^2(\mathbb{Z}^n)$ by

$$\gamma(\boldsymbol{k}) := (1 + |\boldsymbol{k}|)^m |c_{\boldsymbol{k}}[f]|,$$

so that

$$\langle \beta, \gamma \rangle_{\ell^2} = \sum_{\boldsymbol{k} \in \mathbb{Z}^n} |c_{\boldsymbol{k}}[f]|.$$

It then follows from the Cauchy-Schwarz inequality on ℓ^2 that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n} |c_{\boldsymbol{k}}[f]| \le \|\beta\|_{\ell^2} \|\gamma\|_{\ell^2}, \tag{10.29}$$

which is finite for $m > \frac{n}{2}$.

Since $|e^{i\boldsymbol{k}\cdot\boldsymbol{x}}| = 1$, the estimate (10.29) implies that the Fourier series for f converges uniformly. By Lemma 8.4 the limit of this series is continuous. Thus, after possible replacement by an equivalent function in L^2, f is continuous.

This argument shows that

$$H^m(\mathbb{T}^n) \subset C^0(\mathbb{T}^n) \tag{10.30}$$

for $m > \frac{n}{2}$. To apply it to higher derivatives we note that if $f \in H^m(\mathbb{T}^n)$ for $m > q + \frac{n}{2}$ then for $|\alpha| \le q$ the weak derivatives $D^\alpha f$ will lie in $H^{m-q}(\mathbb{T}^N)$. For $m > q + \frac{n}{2}$ it follows from (10.30) that these derivatives are continuous. By Theorem 10.4, this shows that $u \in C^q(\mathbb{T}^n)$. □

We are now prepared to derive the Sobolev embedding result for a bounded domain as a consequence of Theorem 10.14.

Proof of Theorem 10.11 Suppose $u \in H^m(\Omega)$ for $\Omega \in \mathbb{R}^n$, let $\boldsymbol{x}_0 \in \Omega$. Because Ω is open, we can choose $\varepsilon > 0$ small enough that

$$B(\boldsymbol{x}_0; \varepsilon) \subset \Omega.$$

Suppose that $\psi \in C^\infty_{\text{cpt}}(\Omega)$ has support contained in $B(\boldsymbol{x}_0; \varepsilon)$ and is equal to 1 inside $B(\boldsymbol{x}_0; \varepsilon/2)$. (Such a function can be constructed as in Example 2.2.) Since ψ is smooth, $u\psi \in H^m(\Omega)$ also. Thus, assuming $\varepsilon < 2\pi$, we can extend $u\psi$ by periodicity to a function in $H^m(\mathbb{T}^n)$. Theorem 10.14 then shows that $u\psi \in C^k(\mathbb{T}^n)$ if $m > k + n/2$. Since $u\psi$ and u agree in a neighborhood of $\boldsymbol{x}_0$, this shows that u is k-times continuously differentiable at $\boldsymbol{x}_0$. This argument applies at every interior point of Ω, so we conclude that $u \in C^k(\Omega)$. □

10.5 Weak Formulation of Elliptic Equations

The Laplace equation introduced in Sect. 9.1 is the prototypical elliptic equation. Another classic example is the Poisson equation $-\Delta u = f$, which we will discuss in more detail in Sect. 11.1.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, then for $u, \psi \in C^\infty_{\text{cpt}}(\Omega)$, Green's first identity (Theorem 2.10) gives

$$\int_\Omega \psi \Delta u \, d^n\boldsymbol{x} = -\int_\Omega \nabla u \cdot \nabla \psi \, d^n\boldsymbol{x}. \tag{10.31}$$

On the other hand, the H^1 inner product on Ω is given by

$$\langle u, \psi \rangle_{H^1} := \langle u, \psi \rangle + \int_\Omega \nabla u \cdot \nabla \psi \, d^n\boldsymbol{x}. \tag{10.32}$$

The right-hand side of (10.31) is thus well-defined for $u \in H^1_0(\Omega)$.

To account for applications to the Helmholtz equation as well as the Laplace equation, let us consider the PDE

$$-\Delta u = \lambda u + f, \qquad u|_{\partial\Omega} = 0. \tag{10.33}$$

For $f \in L^2(\Omega)$, we say that $u \in H^1_0(\Omega)$ constitutes a weak solution of (10.33) if

$$\int_\Omega \Big[\nabla u \cdot \nabla \psi - \lambda u \psi - f\psi\Big] d^n\boldsymbol{x} = 0 \tag{10.34}$$

for every $\psi \in C^\infty_{\text{cpt}}(\Omega)$. This definition could be extended to more general elliptic equations of the form $Lu = f$, but for simplicity we restrict our attention to the case of the Laplacian. We will study the existence and regularity of solutions of (10.34) extensively in Chap. 11. For now, we consider a simple one-dimensional case to illustrate how the definition works.

Example 10.15 On the interval [0, 2], consider the equation

$$u'' = f, \quad u(0) = u(2) = 0,$$

with

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ -1, & 1 \le x \le 2. \end{cases}$$

Since f is piecewise linear, it makes sense to try using classical solutions on the two subintervals. Imposing the boundary and continuity requirements gives a family of possible solutions

$$u(x) = \begin{cases} \frac{1}{6}x^3 - ax, & 0 \le x \le 1, \\ -\frac{1}{2}x^2 + (a + \frac{4}{3})x - 2a - \frac{2}{3}, & 1 \le x \le 2. \end{cases}$$

To determine a we apply the weak solution condition,

$$\int_0^2 \Big[u'\psi' + f\psi \Big] dx = 0, \tag{10.35}$$

for $\psi \in C^\infty_{\text{cpt}}(0, 2)$. Using integration by parts, the first term evaluates to

$$\int_0^2 u'\psi'\,dx = \int_0^1 (\tfrac{1}{2}x^2 - a)\psi'(x)\,dx + \int_1^2 (-x + a + \tfrac{4}{3})\psi'(x)\,dx$$
$$\begin{aligned} \int_0^2 u'\psi'\,dx &= (\tfrac{1}{2} - a)\psi(1) - \int_0^1 x\psi(x)\,dx - (a + \tfrac{1}{3})\psi(1) + \int_1^2 \psi(x)\,dx \\ &= (\tfrac{1}{6} - 2a)\psi(1) - \int_0^2 f\psi\,dx. \end{aligned}$$

The weak solution condition (10.35) requires $a = \frac{1}{12}$. This gives

$$u(x) = \begin{cases} \frac{1}{6}x^3 - \frac{1}{12}x, & 0 \le x \le 1, \\ -\frac{1}{2}x^2 + \frac{17}{12}x - \frac{5}{6}, & 1 \le x \le 2. \end{cases}$$

This result is illustrated in Fig. 10.6. Note that the condition on a corresponds to a matching of the first derivatives at $x = 1$, so that $u \in C^1[0, 2]$. ♢

We will show in Sect. 11.3 that solutions of (10.34) are unique, so the function obtained in Example 10.15 is the only possible solution. The matching of derivatives required for this solution is indicative of a more general regularity property for solutions of elliptic equation, which we will discuss in detail in Sect. 11.4.

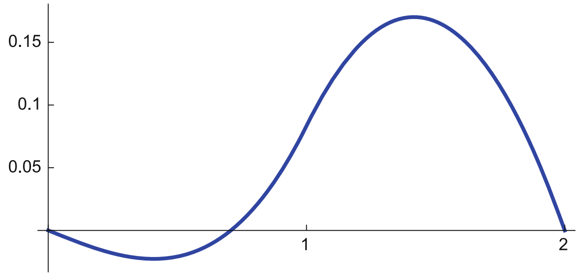


Fig. 10.6 One-dimensional weak solution

10.6 Weak Formulation of Evolution Equations

The heat and wave equations are the primary examples of linear evolution equations. Weak solutions for these equations can be defined by essentially the same strategy used in Sect. 10.5. Starting from a classical solution, we pair with a test function and use integration by parts to find the corresponding integral equation. Unfortunately, the time dependence creates some technicalities in the definition that we are not equipped to fully resolve here, but we can at least illustrate the basic philosophy by working through some examples.

Consider first the wave equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \quad u|_{\boldsymbol{x} \in \partial\Omega} = 0, \tag{10.36}$$

subject to the initial conditions

$$u|_{t=0} = g, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = h.$$

Assuming u is a classical solution, pairing the wave equation for u with a test function $\psi \in C^\infty_{\text{cpt}}([0, \infty) \times \Omega)$ gives

$$\int_0^\infty \int_\Omega \left[\psi \frac{\partial^2 u}{\partial t^2} - \psi \Delta u\right] d^n\boldsymbol{x}\, dt = 0.$$

Integration by parts for the Δu term works just as in (10.31), yielding

$$\int_\Omega \psi \Delta u\, d^n\boldsymbol{x} = -\int_\Omega \nabla u \cdot \nabla \psi\, d^n\boldsymbol{x}.$$

In the t variable, we integrate by parts twice and pick up a boundary term each time because the test function is not assumed to vanish at $t = 0$. The result is

$$\begin{aligned}\int_0^\infty \psi \frac{\partial^2 u}{\partial t^2}\,dt &= -g\frac{\partial \psi}{\partial t}\Big|_{t=0} - \int_0^\infty \frac{\partial \psi}{\partial t}\frac{\partial u}{\partial t}\,dt \\ &= -g\frac{\partial \psi}{\partial t}\Big|_{t=0} + h\psi|_{t=0} + \int_0^\infty u\frac{\partial^2 \psi}{\partial t^2}\,dt.\end{aligned}$$

Combining this with the spatial integral yields

$$\begin{aligned}&\int_0^\infty \int_\Omega \Big[u\frac{\partial^2\psi}{\partial t^2} + \nabla u \cdot \nabla \psi\Big] d^n\boldsymbol{x}\,dt \\ &\qquad = -\int_\Omega g\frac{\partial \psi}{\partial t}\Big|_{t=0} d^n\boldsymbol{x} + \int_\Omega h\psi|_{t=0}\,d^n\boldsymbol{x}.\end{aligned} \tag{10.37}$$

As in Sect. 10.5, the Dirichlet boundary condition is imposed by assuming that $u(t,\cdot) \in H_0^1(\Omega)$ for all t. To make sense of the boundary and initial terms we need to assume at least that $g \in L^1_{\text{loc}}(\Omega)$ and $h \in L^1_{\text{loc}}[0,\infty)$. To interpret (10.37) we also need to require that the spatial pairing of ∇u with $\nabla\psi$ is integrable over t. This condition is more technical. It turns out to be sufficient to assume that $\|u(t,\cdot)\|_{H^1}$ is integrable as a function of t, but we will not attempt to justify this here. Instead we will limit our discussion to examples for which the existence of the integrals in (10.37) is clear.

Example 10.16 Consider the piecewise linear d'Alembert solution for the wave equation introduced in Sect. 1.2. On $[0,2]$ we take the initial conditions $h = 0$ and

$$g(x) := \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2. \end{cases}$$

By Theorem 4.5, d'Alembert's solution is given by extending g to an odd periodic function on $\mathbb{R}$ with period 4, and then setting

$$u(t,x) = \frac{1}{2}\Big[g(x+t) + g(x-t)\Big].$$

The linear components of the resulting solution are shown in Fig. 10.7. Because u is piecewise linear and vanishes at $x = 0$ and 2, it is clear that $u(t,\cdot) \in H_0^1(0,2)$ for each t.

For this case the weak solution condition (10.37) specializes to

$$\int_0^\infty \int_0^2 \Big[u\frac{\partial^2\psi}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x}\Big] dx\,dt = -\int_0^2 g\frac{\partial\psi}{\partial t}\Big|_{t=0} dx. \tag{10.38}$$

Checking this is essentially a matter of integration by parts, but the integrals must be broken into many pieces for large t. As a sample case, let us assume that ψ has support in $[0,1) \times (0,2)$.

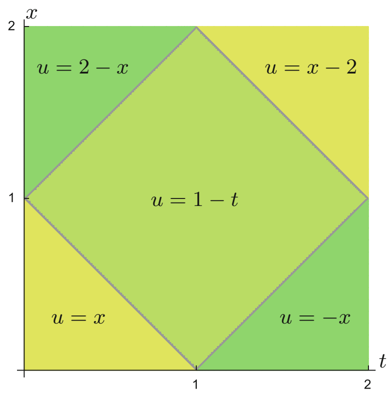


Fig. 10.7 Piecewise linear wave solution $u(t, x)$

The first integral in (10.38) becomes

$$\begin{aligned}\int_0^1 \int_0^2 u\frac{\partial^2\psi}{\partial t^2}\,dx\,dt &= \int_0^1 \Big[\int_0^{1-x} x\frac{\partial^2\psi}{\partial t^2}\,dt + \int_{1-x}^1 (1-t)\frac{\partial^2\psi}{\partial t^2}\,dt\Big]dx \\ &\quad \int_1^2 \Big[\int_{x-1}^1 (1-t)\frac{\partial^2\psi}{\partial t^2}\,dt + \int_0^{x-1} (2-x)\frac{\partial^2\psi}{\partial t^2}\,dt\Big]dx \\ &= \int_0^1 \Big[-x\frac{\partial\psi}{\partial t}(0,x) - \psi(1-x,x)\Big]dx \\ &\quad \int_1^2 \Big[-(2-x)\frac{\partial\psi}{\partial t}(0,x) - \psi(x-1,x)\Big]dx\end{aligned}$$

Similarly, the second term in (10.38) evaluates to

$$\begin{aligned}\int_0^\infty \int_0^2 \frac{\partial u}{\partial x}\frac{\partial\psi}{\partial x}\,dx\,dt &= \int_0^1 \int_0^{1-t} \frac{\partial\psi}{\partial x}\,dx\,dt - \int_0^1 \int_{1+t}^2 \frac{\partial\psi}{\partial x}\,dx\,dt \\ &= \int_0^1 \psi(t, 1-t)\,dt + \int_0^1 \psi(t, 1+t)\,dt \\ &= \int_0^1 \psi(1-x, x)\,dx + \int_1^2 \psi(x-1, x)\,dx.\end{aligned}$$

Adding these pieces together gives

$$\begin{aligned}\int_0^\infty \int_0^2 \Big[u\frac{\partial^2\psi}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial\psi}{\partial x}\Big]dx\,dt &= -\int_0^1 x\frac{\partial\psi}{\partial t}(0,x)\,dx - \int_1^2 (2-x)\frac{\partial\psi}{\partial t}(0,x)\,dx \\ &= -\int_0^2 g(x)\frac{\partial\psi}{\partial t}(0,x)\,dx,\end{aligned}$$

which verifies (10.38) for this case. ◊

Now let us consider the weak formulation of the heat equation with Dirichlet boundary conditions,

$$\frac{\partial u}{\partial t} - \Delta u = 0, \quad u|_{\boldsymbol{x}\in\partial\Omega} = 0, \quad u|_{t=0} = h. \tag{10.39}$$

Derivation of the integral equation works just as for the wave equation, except that there is only a single integration by parts in the time variable. Assuming that $u(t, \cdot) \in H_0^1(\Omega)$ for each $t > 0$ and $h \in L^1_{\mathrm{loc}}(\Omega)$, the weak solution condition is

$$\int_0^\infty \int_\Omega \Big[-u\frac{\partial\psi}{\partial t} + \nabla u \cdot \nabla\psi\Big] d^n\boldsymbol{x}\, dt = \int_\Omega h\psi|_{t=0}\, d^n\boldsymbol{x} \tag{10.40}$$

for all $\psi \in C^\infty_{\mathrm{cpt}}([0,\infty) \times \Omega)$.

Example 10.17 Consider the heat equation on the interval $(0, \pi)$, with initial condition $h \in L^2(0, \pi; \mathbb{R})$. In view of the Dirichlet boundary conditions, we use the orthonormal basis for $L^2(0, \pi)$ developed in Exercise 8.4, given by the sine functions

$$\phi_k(x) := \sqrt{\frac{2}{\pi}}\sin(kx)$$

for $k \in \mathbb{N}$. The coefficients associated to h are

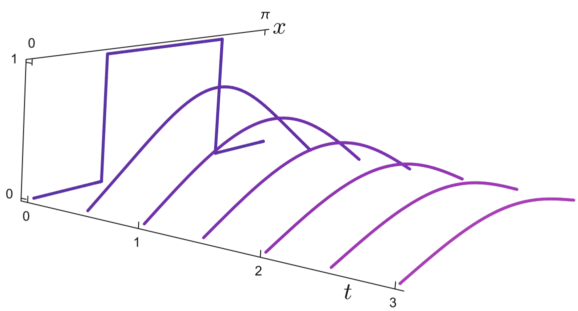
$$a_k := \int_0^\pi h(x)\phi_k(x)\, dx,$$

and $\sum a_k\phi_k$ converges to h in the L^2 sense by Theorem 8.6. The corresponding heat solution is

$$u(t, x) = \sum_{k=1}^\infty a_k e^{-k^2 t}\phi_k(x). \tag{10.41}$$

For example, solution corresponding to a step function h is illustrated in Fig. 10.8.

Fig. 10.8 Heat solution with L^2 initial data



As noted in Corollary 8.14, this yields a classical solution if h is continuous. If h is not continuous then u might not have a well-defined limit as $t \to 0$. Nevertheless, we can check that the weak solution condition is satisfied. Given a (real-valued) test function $\psi \in C^\infty_{\mathrm{cpt}}([0,\infty)\times(0,\pi);\mathbb{R})$, define the time-dependent Fourier coefficients

$$b_k(t) := \int_0^\pi \psi(t,x)\phi_k(x)\,dx.$$

By the smoothness of ψ, Theorem 8.10 implies that the coefficients satisfy $b_k(t) = O(k^{-\infty})$, uniformly in t, and so the series

$$\psi(t,x) = \sum_{k=1}^\infty b_k(t)\phi_k(x)$$

converges uniformly as well as in L^2. By the same principle, the series

$$\frac{\partial\psi}{\partial t}(t,x) = \sum_{k=1}^\infty b_k'(t)\phi_k(x)$$

is also uniformly convergent. Since $\{\phi_k\}$ is an orthonormal basis, we deduce from Parseval's identity (8.36) that

$$\int_0^\pi u\frac{\partial\psi}{\partial t}\,dt = \sum_{k=1}^\infty a_k e^{-k^2 t} b_k'(t) \tag{10.42}$$

for $t \geq 0$.

Similarly, for $t \geq 0$ we have L^2 convergent series

$$\begin{aligned}\frac{\partial\psi}{\partial x}(t,x) &= \sum_{k=1}^\infty b_k(t)k\sqrt{\frac{2}{\pi}}\cos(kx),\\ \frac{\partial u}{\partial x}(t,x) &= \sum_{k=1}^\infty a_k(t)e^{-k^2 t}k\sqrt{\frac{2}{\pi}}\cos(kx).\end{aligned}$$

By the Parseval identity for the cosine basis, this gives

$$\int_0^\pi \frac{\partial u}{\partial x}\frac{\partial\psi}{\partial x}\,dx = \sum_{k=1}^\infty k^2 a_k e^{-k^2 t} b_k(t). \tag{10.43}$$

Applying (10.42) and (10.43) to the left-hand side of (10.40) yields

$$\int_0^\infty \int_0^\pi \Big[-u\frac{\partial\psi}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial\psi}{\partial x}\psi\Big] d^n\boldsymbol{x}\, dt \\ = \int_0^\infty \Big[\sum_{k=1}^\infty a_k e^{-k^2 t}\Big(k^2 b_k(t) - b_k'(t)\Big)\Big] dt. \tag{10.44}$$

Switching the order of the summation and integration is justified if series converges uniformly on the domain of the integral, but that is not necessarily the case here. To check this carefully, we break the sum at some value $k = N$. For the finite sum there is no convergence issue, so that

$$\begin{aligned}\int_0^\infty \Big[\sum_{k=1}^N a_k e^{-k^2 t}\Big(k^2 b_k(t) - b_k'(t)\Big)\Big] dt &= \sum_{k=1}^N \int_0^\infty a_k \frac{d}{dt}\Big(-e^{-k^2 t} b_k(t)\Big) dt \\ &= \sum_{k=1}^N a_k b_k(0).\end{aligned}$$

To estimate the tail of the sum, note that the sequence $\{a_k\}$ is bounded because $\sum |a_k|^2 < \infty$. For $b_k(t)$, we apply repeated integration by parts to deduce

$$\begin{aligned}\int_0^\pi \phi_k(x)\Big(\frac{\partial}{\partial x}\Big)^{2m}\psi(t,x)\, dx &= \int_0^\pi \psi(t,x)\Big(\frac{\partial}{\partial x}\Big)^{2m}\phi_k(x)\, dx \\ &= (-1)^m k^{2m}\int_0^\pi \psi(t,x)\phi_k(x)\, dx \\ &= (-1)^m k^{2m} b_k(t).\end{aligned}$$

for $m \in \mathbb{N}$. Since $\psi \in C^\infty_{\mathrm{cpt}}([0,\infty)\times(0,\pi))$, this gives an estimate

$$|b_k(t)| \le C_m k^{-2m},$$

where C_m is independent of t. The same reasoning applies to $b_k'(t)$. Combining the $m = 2$ estimate for b_k with the $m = 1$ case for b_k' gives

$$\left| a_k e^{-k^2 t}\Big(k^2 b_k(t) - b_k'(t)\Big)\right| \le C k^{-2}.$$

This shows that

$$\left|\sum_{k=N+1}^\infty a_k e^{-k^2 t}\Big(k^2 b_k(t) - b_k'(t)\Big)\right| \le C N^{-1}, \tag{10.45}$$

independently of t.

Now fix $M > 0$ so the support of ψ is contained in $[0, M]$. Applying (10.45) to the integral gives

$$\left| \int_0^M \left[\sum_{k=N+1}^{\infty} a_k e^{-k^2 t} \Big(k^2 b_k(t) - b_k'(t) \Big) \right] dt \right| \le CMN^{-1}.$$

Returning to (10.44), our analysis of the sum over k now gives

$$\int_0^{\infty} \int_0^{\pi} \left[-u \frac{\partial \psi}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial \psi}{\partial x} \psi \right] d^n\boldsymbol{x}\, dt = \sum_{k=1}^{N} a_k b_k(0) + O(N^{-1}).$$

By taking $N \to \infty$, we deduce

$$\int_0^{\infty} \int_0^{\pi} \left[-u \frac{\partial \psi}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial \psi}{\partial x} \psi \right] d^n\boldsymbol{x}\, dt = \sum_{k=1}^{\infty} a_k b_k(0).$$

On the other hand Parseval's identity gives

$$\int_0^{\pi} h(x)\psi(0, x)\, dx = \sum_{k=1}^{\infty} a_k b_k(0),$$

so the weak solution condition (10.40) is satisfied. ♢

10.7 Exercises

10.1 On $\mathbb{R}$ consider the ordinary differential equation

$$x \frac{du}{dx} = 1.$$

(a) Develop a weak formulation of this ODE in terms of pairing with a test function $\psi \in C^{\infty}_{\text{cpt}}(\mathbb{R})$

(b) Show that $u(x) = \log |x|$ is locally integrable and solves the equation in the weak sense.

10.2 In Exercise 3.6 we studied Burger's equation,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$

with the initial condition

$$u(0,x) = \begin{cases} a, & x \le 0, \\ a(1-x) + bx, & 0 < x < 1, \\ b, & x \ge 1. \end{cases}$$

For $a > b$ a shock forms at some positive time. Assuming u is a weak solution, find the equation of the shock curve starting from this point.

10.3 Consider the traffic equation

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0$$

with initial data

$$g(x) = \begin{cases} 0, & x > 0, \\ 1, & x < 0. \end{cases}$$

(a) Sketch the characteristic lines for this initial condition, and show that they leave a triangular region uncovered.

(b) Show that the constant solution $u(t,x) = g(x)$ satisfies the Rankine-Hugoniot condition for the shock curve $\sigma(t) = 0$ and thus gives a weak-solution of the traffic equation. (This solution is considered unphysical because characteristic lines emerge from the shock line to fill the triangular region.)

(c) The physical solution is specified by an *entropy* condition that says that characteristic lines may only intersect when followed forwards in time. Show that the continuous function

$$u(t,x) = \begin{cases} 0, & x > t, \\ \frac{1}{2} - \frac{x}{2t}, & -t < x < t, \\ 1, & x < -t, \end{cases}$$

satisfies the weak solution condition (10.9) (with $q = u - u^2$), as well as this entropy condition. (This type of solution is called a *rarefaction wave*.)

10.4 Define $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$f(\boldsymbol{x}) = \begin{cases} f_+(\boldsymbol{x}), & x_n > 0, \\ f_-(\boldsymbol{x}), & x_n < 0. \end{cases}$$

with $f_\pm \in C^1(\mathbb{R}^n)$.

(a) For $j = 1, \dots, n-1$, show that f has weak partial derivatives given by $\frac{\partial f_\pm}{\partial x_j}$ for $\pm x_n > 0$.

(b) Show that the weak partial $\frac{\partial f}{\partial x_n}$ exists and is given by $\frac{\partial f_\pm}{\partial x_n}$ for $\pm x_n > 0$ only if f extends to a continuous function at $x_n = 0$.

10.5 Let $\mathbb{D} \subset \mathbb{R}^2$ be the unit disk $\{r < 1\}$ with $r := |\boldsymbol{x}|$. Consider the function $u(\boldsymbol{x}) = r^\alpha$ with $\alpha \in \mathbb{R}$ constant.

(a) Compute the ordinary partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, 2$, for $r \neq 0$.

(b) Show that for $\alpha > -1$ these partials lie in $L^1(\mathbb{D})$ and define weak derivatives.

(c) For what values of α is $u \in H^1(\mathbb{D})$?

10.6 In $\mathbb{R}^3$ consider the equation

$$\Delta u = \begin{cases} 1, & r \leq a, \\ 0, & r > a, \end{cases}$$

with $r := |\boldsymbol{x}|$ and $a > 0$. (With appropriate physical constants this is the equation for the gravitational potential of a spherical planet of radius a.)

(a) Assuming that u depends only on r, formulate a weak solution condition in terms of pairing with a test function $\psi(r)$ with $\psi \in C^\infty_{\text{cpt}}[0, \infty)$.

(b) Find the unique solution which is smooth at $r = 0$ and for which $u(r) \to 0$ as $r \to \infty$.

10.7 Let $\Omega \subset \mathbb{R}^n$ be bounded with $\partial\Omega$ piecewise C^1. For $u \in C^2(\overline{\Omega})$ and $f \in C^0(\overline{\Omega})$, suppose that

$$\int_\Omega \Big[\nabla u \cdot \nabla \psi - f\psi\Big]\, d^n\boldsymbol{x} = 0. \tag{10.46}$$

for all $\psi \in C^\infty(\overline{\Omega})$. Show that u satisfies the Poisson equation with Neumann boundary condition,

$$-\Delta u = f, \qquad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{10.47}$$

(Thus (10.46) allows a weak formulation of (10.47) for $u \in H^1(\Omega)$.)