Second-order linear differential equations

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2.1 Algebraic properties of solutions

A second-order differential equation is an equation of the form

$$\frac{d^2y}{dt^2}=f\left(t,y,\frac{dy}{dt}\right). \tag{1}$$

For example, the equation

$$\frac{d^2y}{dt^2}=\sin t+3y+\left(\frac{dy}{dt}\right)^2$$

is a second-order differential equation. A function $y=y(t)$ is a solution of (1) if $y(t)$ satisfies the differential equation; that is

$$\frac{d^2y(t)}{dt^2}=f\left(t,y(t),\frac{dy(t)}{dt}\right).$$

Thus, the function $y(t)=\cos t$ is a solution of the second-order equation $d^2y/dt^2=-y$ since $d^2(\cos t)/dt^2=-\cos t$.

Second-order differential equations arise quite often in applications. The most famous second-order differential equation is Newton's second law of motion (see Section 1.7)

$$m\frac{d^2y}{dt^2}=F\left(t,y,\frac{dy}{dt}\right)$$

which governs the motion of a particle of mass m moving under the influence of a force F. In this equation, m is the mass of the particle, $y=y(t)$ is its position at time t, dy/dt is its velocity, and F is the total force acting on the particle. As the notation suggests, the force F may depend on the position and velocity of the particle, as well as on time.

In addition to the differential equation (1), we will often impose initial conditions on $y(t)$ of the form

$$y(t_0)=y_0, \qquad y'(t_0)=y'_0. \tag{1'}$$

The differential equation (1) together with the initial conditions (1′) is referred to as an initial-value problem. For example, let $y(t)$* denote the position at time t of a particle moving under the influence of gravity. Then, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2}=-g; \qquad y(t_0)=y_0, \quad y'(t_0)=y'_0,$$

where y_0 is the initial position of the particle and y'_0 is the initial velocity of the particle.

Second-order differential equations are extremely difficult to solve. This should not come as a great surprise to us after our experience with first-order equations. We will only succeed in solving the special differential equation

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=g(t). \tag{2}$$

Fortunately, though, many of the second-order equations that arise in applications are of this form.

The differential equation (2) is called a second-order linear differential equation. We single out this equation and call it linear because both y and dy/dt appear by themselves. For example, the differential equations

$$\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+(\sin t)y=e^t$$

and

$$\frac{d^2y}{dt^2}+e^t\frac{dy}{dt}+2y=1$$

are linear, while the differential equations

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+\sin y=t^3$$

and

$$\frac{d^2y}{dt^2}+\left(\frac{dy}{dt}\right)^2=1$$

are both nonlinear, due to the presence of the $\sin y$ and $(dy/dt)^2$ terms, respectively.

We consider first the second-order linear homogeneous equation

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{3}$$

*The positive direction of y is taken upwards.

which is obtained from (2) by setting $g(t)=0$. It is certainly not obvious at this point how to find all the solutions of (3), or how to solve the initial-value problem

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0; \qquad y(t_0)=y_0, \quad y'(t_0)=y_0'. \tag{4}$$

Therefore, before trying to develop any elaborate procedures for solving (4), we should first determine whether it actually has a solution. This information is contained in the following theorem, whose proof will be indicated in Chapter 4.

Theorem 1. (Existence–uniqueness Theorem). *Let the functions $p(t)$ and $q(t)$ be continuous in the open interval $\alpha<t<\beta$. Then, there exists one, and only one function $y(t)$ satisfying the differential equation* (3) *on the entire interval $\alpha<t<\beta$, and the prescribed initial conditions $y(t_0)=y_0$, $y'(t_0)=y_0'$. In particular, any solution $y=y(t)$ of* (3) *which satisfies $y(t_0)=0$ and $y'(t_0)=0$ at some time $t=t_0$ must be identically zero.*

Theorem 1 is an extremely important theorem for us. On the one hand, it is our hunting license to find the unique solution $y(t)$ of (4). And, on the other hand, we will actually use Theorem 1 to help us find all the solutions of (3).

We begin our analysis of Equation (3) with the important observation that the left-hand side

$$y''+p(t)y'+q(t)y \qquad \left(y'=\frac{dy}{dt},\, y''=\frac{d^2y}{dt^2}\right)$$

of the differential equation can be viewed as defining a "function of a function": with each function y having two derivatives, we associate another function, which we'll call $L[y]$, by the relation

$$L[y](t)=y''(t)+p(t)y'(t)+q(t)y(t).$$

In mathematical terminology, L is an operator which operates on functions; that is, there is a prescribed recipe for associating with each function y a new function $L[y]$.

Example 1. Let $p(t)=0$ and $q(t)=t$. Then,

$$L[y](t)=y''(t)+ty(t).$$

If $y(t)=\cos t$, then

$$L[y](t)=(\cos t)''+t\cos t=(t-1)\cos t,$$

and if $y(t)=t^3$, then

$$L[y](t)=(t^3)''+t(t^3)=t^4+6t.$$

Thus, the operator L assigns the function $(t-1)\cos t$ to the function $\cos t$, and the function $6t+t^4$ to the function t^3.

The concept of an operator acting on functions, or a "function of a function" is analogous to that of a function of a single variable t. Recall the definition of a function f on an interval I: with each number t in I we associate a new number called $f(t)$. In an exactly analogous manner, we associate with each function y having two derivatives a new function called $L[y]$. This is an extremely sophisticated mathematical concept, because in a certain sense, we are treating a function exactly as we do a point. Admittedly, this is quite difficult to grasp. It's not surprising, therefore, that the concept of a "function of a function" was not developed till the beginning of this century, and that many of the "high powered" theorems of mathematical analysis were proved only after this concept was mastered.

We now derive several important properties of the operator L, which we will use to great advantage shortly.

Property 1. $L[cy]=cL[y]$, *for any constant* c.

PROOF.
$$\begin{aligned}L[cy](t)&=(cy)''(t)+p(t)(cy)'(t)+q(t)(cy)(t)\\&=cy''(t)+cp(t)y'(t)+cq(t)y(t)\\&=c\left[y''(t)+p(t)y'(t)+q(t)y(t)\right]\\&=cL[y](t).\end{aligned}$$ □

The meaning of Property 1 is that the operator L assigns to the function (cy) c times the function it assigns to y. For example, let

$$L[y](t)=y''(t)+6y'(t)-2y(t).$$

This operator L assigns the function

$$(t^2)''+6(t^2)'-2(t^2)=2+12t-2t^2$$

to the function t^2. Hence, L must assign the function $5(2+12t-2t^2)$ to the function $5t^2$.

Property 2. $L[y_1+y_2]=L[y_1]+L[y_2]$.

PROOF.

$$\begin{aligned}L[y_1+y_2](t)&=(y_1+y_2)''(t)+p(t)(y_1+y_2)'(t)+q(t)(y_1+y_2)(t)\\&=y_1''(t)+y_2''(t)+p(t)y_1'(t)+p(t)y_2'(t)+q(t)y_1(t)+q(t)y_2(t)\\&=\left[y_1''(t)+p(t)y_1'(t)+q(t)y_1(t)\right]+\left[y_2''(t)+p(t)y_2'(t)+q(t)y_2(t)\right]\\&=L[y_1](t)+L[y_2](t).\end{aligned}$$ □

The meaning of Property 2 is that the operator L assigns to the function y_1+y_2 the sum of the functions it assigns to y_1 and y_2. For example, let

$$L[y](t)=y''(t)+2y'(t)-y(t).$$

This operator L assigns the function

$$(\cos t)'' + 2(\cos t)' - \cos t = -2\cos t - 2\sin t$$

to the function $\cos t$, and the function

$$(\sin t)'' + 2(\sin t)' - \sin t = 2\cos t - 2\sin t$$

to the function $\sin t$. Hence, L assigns the function

$$(-2\cos t - 2\sin t) + 2\cos t - 2\sin t = -4\sin t$$

to the function $\sin t + \cos t$.

Definition. An operator L which assigns functions to functions and which satisfies Properties 1 and 2 is called a linear operator. All other operators are nonlinear. An example of a nonlinear operator is

$$L[y](t) = y''(t) - 2t[y(t)]^4.$$

This operator assigns the function

$$\left(\frac{1}{t}\right)'' - 2t\left(\frac{1}{t}\right)^4 = \frac{2}{t^3} - \frac{2}{t^3} = 0$$

to the function $1/t$, and the function

$$\left(\frac{c}{t}\right)'' - 2t\left(\frac{c}{t}\right)^4 = \frac{2c}{t^3} - \frac{2c^4}{t^3} = \frac{2c(1-c^3)}{t^3}$$

to the function c/t. Hence, for $c \neq 0, 1$, and $y(t) = 1/t$, we see that $L[cy] \neq cL[y]$.

The usefulness of Properties 1 and 2 lies in the observation that the solutions $y(t)$ of the differential equation (3) are exactly those functions y for which

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t) = 0.$$

In other words, the solutions $y(t)$ of (3) are exactly those functions y to which the operator L assigns the zero function.* Hence, if $y(t)$ is a solution of (3) then so is $cy(t)$, since

$$L[cy](t) = cL[y](t) = 0.$$

If $y_1(t)$ and $y_2(t)$ are solutions of (3), then $y_1(t) + y_2(t)$ is also a solution of (3), since

$$L[y_1 + y_2](t) = L[y_1](t) + L[y_2](t) = 0 + 0 = 0.$$

Combining Properties 1 and 2, we see that all linear combinations

$$c_1 y_1(t) + c_2 y_2(t)$$

of solutions of (3) are again solutions of (3).

*The zero function is the function whose value at any time t is zero.

The preceding argument shows that we can use our knowledge of two solutions $y_1(t)$ and $y_2(t)$ of (3) to generate infinitely many other solutions. This statement has some very interesting implications. Consider, for example, the differential equation

$$\frac{d^2y}{dt^2}+y=0. \tag{5}$$

Two solutions of (5) are $y_1(t)=\cos t$ and $y_2(t)=\sin t$. Hence,

$$y(t)=c_1\cos t+c_2\sin t \tag{6}$$

is also a solution of (5), for every choice of constants c_1 and c_2. Now, Equation (6) contains two arbitrary constants. It is natural to suspect, therefore, that this expression represents the general solution of (5); that is, every solution $y(t)$ of (5) must be of the form (6). This is indeed the case, as we now show. Let $y(t)$ be any solution of (5). By the existence-uniqueness theorem, $y(t)$ exists for all t. Let $y(0)=y_0$, $y'(0)=y_0'$, and consider the function

$$\phi(t)=y_0\cos t+y_0'\sin t.$$

This function is a solution of (5) since it is a linear combination of solutions of (5). Moreover, $\phi(0)=y_0$ and $\phi'(0)=y_0'$. Thus, $y(t)$ and $\phi(t)$ satisfy the same second-order linear homogeneous equation and the same initial conditions. Therefore, by the uniqueness part of Theorem 1, $y(t)$ must be identically equal to $\phi(t)$, so that

$$y(t)=y_0\cos t+y_0'\sin t.$$

Thus, Equation (6) is indeed the general solution of (5).

Let us return now to the general linear equation (3). Suppose, in some manner, that we manage to find two solutions $y_1(t)$ and $y_2(t)$ of (3). Then, every function

$$y(t)=c_1y_1(t)+c_2y_2(t) \tag{7}$$

is again a solution of (3). Does the expression (7) represent the general solution of (3)? That is to say, does every solution $y(t)$ of (3) have the form (7)? The following theorem answers this question.

Theorem 2. *Let $y_1(t)$ and $y_2(t)$ be two solutions of* (3) *on the interval $\alpha<t<\beta$, with*

$$y_1(t)y_2'(t)-y_1'(t)y_2(t)$$

unequal to zero in this interval. Then,

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

is the general solution of (3).

Proof. Let $y(t)$ be any solution of (3). We must find constants c_1 and c_2 such that $y(t)=c_1y_1(t)+c_2y_2(t)$. To this end, pick a time t_0 in the interval

(α,β) and let y_0 and y'_0 denote the values of y and y' at $t=t_0$. The constants c_1 and c_2, if they exist, must satisfy the two equations

$$c_1y_1(t_0)+c_2y_2(t_0)=y_0$$
$$c_1y'_1(t_0)+c_2y'_2(t_0)=y'_0.$$

Multiplying the first equation by $y'_2(t_0)$, the second equation by $y_2(t_0)$ and subtracting gives

$$c_1\left[y_1(t_0)y'_2(t_0)-y'_1(t_0)y_2(t_0)\right]=y_0y'_2(t_0)-y'_0y_2(t_0).$$

Similarly, multiplying the first equation by $y'_1(t_0)$, the second equation by $y_1(t_0)$ and subtracting gives

$$c_2\left[y'_1(t_0)y_2(t_0)-y_1(t_0)y'_2(t_0)\right]=y_0y'_1(t_0)-y'_0y_1(t_0).$$

Hence,

$$c_1=\frac{y_0y'_2(t_0)-y'_0y_2(t_0)}{y_1(t_0)y'_2(t_0)-y'_1(t_0)y_2(t_0)}$$

and

$$c_2=\frac{y'_0y_1(t_0)-y_0y'_1(t_0)}{y_1(t_0)y'_2(t_0)-y'_1(t_0)y_2(t_0)}$$

if $y_1(t_0)y'_2(t_0)-y'_1(t_0)y_2(t_0)\neq 0$. Now, let

$$\phi(t)=c_1y_1(t)+c_2y_2(t)$$

for this choice of constants c_1,c_2. We know that $\phi(t)$ satisfies (3), since it is a linear combination of solutions of (3). Moreover, by construction, $\phi(t_0)=y_0$ and $\phi'(t_0)=y'_0$. Thus, $y(t)$ and $\phi(t)$ satisfy the same second-order linear homogeneous equation and the same initial conditions. Therefore, by the uniqueness part of Theorem 1, $y(t)$ must be identically equal to $\phi(t)$; that is,

$$y(t)=c_1y_1(t)+c_2y_2(t),\qquad \alpha<t<\beta.$$ □

Theorem 2 is an extremely useful theorem since it reduces the problem of finding all solutions of (3), of which there are infinitely many, to the much simpler problem of finding just two solutions $y_1(t),y_2(t)$. The only condition imposed on the solutions $y_1(t)$ and $y_2(t)$ is that the quantity $y_1(t)y'_2(t)-y'_1(t)y_2(t)$ be unequal to zero for $\alpha<t<\beta$. When this is the case, we say that $y_1(t)$ and $y_2(t)$ are a *fundamental* set of solutions of (3), since all other solutions of (3) can be obtained by taking linear combinations of $y_1(t)$ and $y_2(t)$.

Definition. The quantity $y_1(t)y'_2(t)-y'_1(t)y_2(t)$ is called the *Wronskian* of y_1 and y_2, and is denoted by $W(t)=W[y_1,y_2](t)$.

Theorem 2 requires that $W[y_1,y_2](t)$ be unequal to zero at all points in the interval (α,β). In actual fact, the Wronskian of any two solutions

$y_1(t), y_2(t)$ of (3) is either identically zero, or is never zero, as we now show.

Theorem 3. *Let $p(t)$ and $q(t)$ be continuous in the interval $\alpha < t < \beta$, and let $y_1(t)$ and $y_2(t)$ be two solutions of (3). Then, $W[y_1,y_2](t)$ is either identically zero, or is never zero, on the interval $\alpha < t < \beta$.*

We prove Theorem 3 with the aid of the following lemma.

Lemma 1. *Let $y_1(t)$ and $y_2(t)$ be two solutions of the linear differential equation $y'' + p(t)y' + q(t)y = 0$. Then, their Wronskian*

$$W(t) = W[y_1,y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

satisfies the first-order differential equation

$$W' + p(t)W = 0.$$

PROOF. Observe that

$$\begin{aligned} W'(t) &= \frac{d}{dt}(y_1y_2' - y_1'y_2) \\ &= y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2 \\ &= y_1y_2'' - y_1''y_2. \end{aligned}$$

Since y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$, we know that

$$y_2'' = -p(t)y_2' - q(t)y_2$$

and

$$y_1'' = -p(t)y_1' - q(t)y_1.$$

Hence,

$$\begin{aligned} W'(t) &= y_1[-p(t)y_2' - q(t)y_2] - y_2[-p(t)y_1' - q(t)y_1] \\ &= -p(t)[y_1y_2' - y_1'y_2] \\ &= -p(t)W(t). \end{aligned}$$

□

We can now give a very simple proof of Theorem 3.

PROOF OF THEOREM 3. Pick any t_0 in the interval (α, β). From Lemma 1,

$$W[y_1,y_2](t) = W[y_1,y_2](t_0)\exp\left(-\int_{t_0}^{t} p(s)\,ds\right).$$

Now, $\exp\left(-\int_{t_0}^{t} p(s)\,ds\right)$ is unequal to zero for $\alpha < t < \beta$. Therefore, $W[y_1,y_2](t)$ is either identically zero, or is never zero. □

The simplest situation where the Wronskian of two functions $y_1(t), y_2(t)$ vanishes identically is when one of the functions is identically zero. More generally, the Wronskian of two functions $y_1(t), y_2(t)$ vanishes identically if one of the functions is a constant multiple of the other. If $y_2 = cy_1$, say, then

$$W[y_1, y_2](t) = y_1(cy_1)' - y_1'(cy_1) = 0.$$

Conversely, suppose that the Wronskian of two *solutions* $y_1(t), y_2(t)$ of (3) vanishes identically. Then, one of these solutions must be a constant multiple of the other, as we now show.

Theorem 4. *Let $y_1(t)$ and $y_2(t)$ be two solutions of* (3) *on the interval $\alpha < t < \beta$, and suppose that $W[y_1, y_2](t_0) = 0$ for some t_0 in this interval. Then, one of these solutions is a constant multiple of the other.*

PROOF #1. Suppose that $W[y_1, y_2](t_0) = 0$. Then, the equations

$$c_1 y_1(t_0) + c_2 y_2(t_0) = 0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = 0$$

have a nontrivial solution c_1, c_2; that is, a solution c_1, c_2 with $|c_1| + |c_2| \neq 0$. Let $y(t) = c_1 y_1(t) + c_2 y_2(t)$, for this choice of constants c_1, c_2. We know that $y(t)$ is a solution of (3), since it is a linear combination of $y_1(t)$ and $y_2(t)$. Moreover, by construction, $y(t_0) = 0$ and $y'(t_0) = 0$. Therefore, by Theorem 1, $y(t)$ is identically zero, so that

$$c_1 y_1(t) + c_2 y_2(t) = 0, \qquad \alpha < t < \beta,$$

If $c_1 \neq 0$, then $y_1(t) = -(c_2/c_1) y_2(t)$, and if $c_2 \neq 0$, then $y_2(t) = -(c_1/c_2) y_1(t)$. In either case, one of these solutions is a constant multiple of the other. □

PROOF #2. Suppose that $W[y_1, y_2](t_0) = 0$. Then, by Theorem 3, $W[y_1, y_2](t)$ is identically zero. Assume that $y_1(t) y_2(t) \neq 0$ for $\alpha < t < \beta$. Then, dividing both sides of the equation

$$y_1(t) y_2'(t) - y_1'(t) y_2(t) = 0$$

by $y_1(t) y_2(t)$ gives

$$\frac{y_2'(t)}{y_2(t)} - \frac{y_1'(t)}{y_1(t)} = 0.$$

This equation implies that $y_1(t) = c y_2(t)$ for some constant c.

Next, suppose that $y_1(t) y_2(t)$ is zero at some point $t = t^*$ in the interval $\alpha < t < \beta$. Without loss of generality, we may assume that $y_1(t^*) = 0$, since otherwise we can relabel y_1 and y_2. In this case it is simple to show (see Exercise 19) that either $y_1(t) \equiv 0$, or $y_2(t) = [y_2'(t^*)/y_1'(t^*)] y_1(t)$. This completes the proof of Theorem 4. □

Definition. The functions $y_1(t)$ and $y_2(t)$ are said to be *linearly dependent* on an interval I if one of these functions is a constant multiple of the other on I. The functions $y_1(t)$ and $y_2(t)$ are said to be *linearly independent* on an interval I if they are not linearly dependent on I.

Corollary to Theorem 4. *Two solutions $y_1(t)$ and $y_2(t)$ of* (3) *are linearly independent on the interval $\alpha<t<\beta$ if, and only if, their Wronskian is unequal to zero on this interval. Thus, two solutions $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of* (3) *on the interval $\alpha<t<\beta$ if, and only if, they are linearly independent on this interval.*

Exercises

1. Let $L[y](t)=y''(t)-3ty'(t)+3y(t)$. Compute
(a) $L[e^t]$, (b) $L[\cos\sqrt{3}\,t]$, (c) $L[2e^t+4\cos\sqrt{3}\,t]$,
(d) $L[t^2]$, (e) $L[5t^2]$, (f) $L[t]$, (g) $L[t^2+3t]$.

2. Let $L[y](t)=y''(t)-6y'(t)+5y(t)$. Compute
(a) $L[e^t]$, (b) $L[e^{2t}]$, (c) $L[e^{3t}]$, (d) $L[e^{rt}]$,
(e) $L[t]$, (f) $L[t^2]$, (g) $L[t^2+2t]$.

3. Show that the operator L defined by

$$L[y](t)=\int_a^t s^2 y(s)\,ds$$

is linear; that is, $L[cy]=cL[y]$ and $L[y_1+y_2]=L[y_1]+L[y_2]$.

4. Let $L[y](t)=y''(t)+p(t)y'(t)+q(t)y(t)$, and suppose that $L[t^2]=t+1$ and $L[t]=2t+2$. Show that $y(t)=t-2t^2$ is a solution of $y''+p(t)y'+q(t)y=0$.

5. (a) Show that $y_1(t)=\sqrt{t}$ and $y_2(t)=1/t$ are solutions of the differential equation

$$2t^2y''+3ty'-y=0 \qquad (*)$$

on the interval $0<t<\infty$.
(b) Compute $W[y_1,y_2](t)$. What happens as t approaches zero?
(c) Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of (*) on the interval $0<t<\infty$.
(d) Solve the initial-value problem $2t^2y''+3ty'-y=0$; $y(1)=2$, $y'(1)=1$.

6. (a) Show that $y_1(t)=e^{-t^2/2}$ and $y_2(t)=e^{-t^2/2}\int_0^t e^{s^2/2}ds$ are solutions of

$$y''+ty'+y=0 \qquad (*)$$

on the interval $-\infty<t<\infty$.
(b) Compute $W[y_1,y_2](t)$.
(c) Show that y_1 and y_2 form a fundamental set of solutions of (*) on the interval $-\infty<t<\infty$.
(d) Solve the initial-value problem $y''+ty'+y=0$; $y(0)=0$, $y'(0)=1$.

7. Compute the Wronskian of the following pairs of functions.

(a) $\sin at, \cos bt$ (b) $\sin^2 t, 1-\cos 2t$

(c) e^{at}, e^{bt} (d) e^{at}, te^{at}

(e) $t, t\ln t$ (f) $e^{at}\sin bt, e^{at}\cos bt$

8. Let $y_1(t)$ and $y_2(t)$ be solutions of (3) on the interval $-\infty<t<\infty$ with $y_1(0)=3$, $y_1'(0)=1$, $y_2(0)=1$, and $y_2'(0)=\frac{1}{3}$. Show that $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $-\infty<t<\infty$.

9. (a) Let $y_1(t)$ and $y_2(t)$ be solutions of (3) on the interval $\alpha<t<\beta$, with $y_1(t_0)=1$, $y_1'(t_0)=0$, $y_2(t_0)=0$, and $y_2'(t_0)=1$. Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of (3) on the interval $\alpha<t<\beta$.

(b) Show that $y(t)=y_0y_1(t)+y_0'y_2(t)$ is the solution of (3) satisfying $y(t_0)=y_0$ and $y'(t_0)=y_0'$.

10. Show that $y(t)=t^2$ can never be a solution of (3) if the functions $p(t)$ and $q(t)$ are continuous at $t=0$.

11. Let $y_1(t)=t^2$ and $y_2(t)=t|t|$.

(a) Show that y_1 and y_2 are linearly dependent on the interval $0\leqslant t\leqslant 1$.

(b) Show that y_1 and y_2 are linearly independent on the interval $-1\leqslant t\leqslant 1$.

(c) Show that $W[y_1,y_2](t)$ is identically zero.

(d) Show that y_1 and y_2 can never be two solutions of (3) on the interval $-1<t<1$ if both p and q are continuous in this interval.

12. Suppose that y_1 and y_2 are linearly independent on an interval I. Prove that $z_1=y_1+y_2$ and $z_2=y_1-y_2$ are also linearly independent on I.

13. Let y_1 and y_2 be solutions of Bessel's equation

$$t^2y''+ty'+(t^2-n^2)y=0$$

on the interval $0<t<\infty$, with $y_1(1)=1$, $y_1'(1)=0$, $y_2(1)=0$, and $y_2'(1)=1$. Compute $W[y_1,y_2](t)$.

14. Suppose that the Wronskian of any two solutions of (3) is constant in time. Prove that $p(t)=0$.

In Problems 15–18, assume that p and q are continuous, and that the functions y_1 and y_2 are solutions of the differential equation

$$y''+p(t)y'+q(t)y=0$$

on the interval $\alpha<t<\beta$.

15. Prove that if y_1 and y_2 vanish at the same point in the interval $\alpha<t<\beta$, then they cannot form a fundamental set of solutions on this interval.

16. Prove that if y_1 and y_2 achieve a maximum or minimum at the same point in the interval $\alpha<t<\beta$, then they cannot form a fundamental set of solutions on this interval.

17. Prove that if y_1 and y_2 are a fundamental set of solutions, then they cannot have a common point of inflection in $\alpha<t<\beta$ unless p and q vanish simultaneously there.

18. Suppose that y_1 and y_2 are a fundamental set of solutions on the interval $-\infty < t < \infty$. Show that there is one and only one zero of y_1 between consecutive zeros of y_2. *Hint*: Differentiate the quantity y_2/y_1 and use Rolle's Theorem.

19. Suppose that $W[y_1,y_2](t^*)=0$, and, in addition, $y_1(t^*)=0$. Prove that either $y_1(t)\equiv 0$ or $y_2(t)=[y_2'(t^*)/y_1'(t^*)]y_1(t)$. *Hint*: If $W[y_1,y_2](t^*)=0$ and $y_1(t^*)=0$, then $y_2(t^*)y_1'(t^*)=0$.

2.2 Linear equations with constant coefficients

We consider now the homogeneous linear second-order equation with constant coefficients

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0 \tag{1}$$

where a, b, and c are constants, with $a\neq 0$. Theorem 2 of Section 2.1 tells us that we need only find two independent solutions y_1 and y_2 of (1); all other solutions of (1) are then obtained by taking linear combinations of y_1 and y_2. Unfortunately, Theorem 2 doesn't tell us how to find two solutions of (1). Therefore, we will try an educated guess. To this end, observe that a function $y(t)$ is a solution of (1) if a constant times its second derivative, plus another constant times its first derivative, plus a third constant times itself is identically zero. In other words, the three terms ay'', by', and cy must cancel each other. In general, this can only occur if the three functions $y(t)$, $y'(t)$, and $y''(t)$ are of the "same type". For example, the function $y(t)=t^5$ can never be a solution of (1) since the three terms $20at^3$, $5bt^4$, and ct^5 are polynomials in t of different degree, and therefore cannot cancel each other. On the other hand, the function $y(t)=e^{rt}$, r constant, has the property that both $y'(t)$ and $y''(t)$ are multiples of $y(t)$. This suggests that we try $y(t)=e^{rt}$ as a solution of (1). Computing

$$\begin{aligned} L[e^{rt}] &= a(e^{rt})''+b(e^{rt})'+c(e^{rt}) \\ &= (ar^2+br+c)e^{rt}, \end{aligned}$$

we see that $y(t)=e^{rt}$ is a solution of (1) if, and only if

$$ar^2+br+c=0. \tag{2}$$

Equation (2) is called the *characteristic equation* of (1). It has two roots r_1, r_2 given by the quadratic formula

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

If b^2-4ac is positive, then r_1 and r_2 are real and distinct. In this case, $y_1(t)=e^{r_1t}$ and $y_2(t)=e^{r_2t}$ are two distinct solutions of (1). These solutions are clearly linearly independent (on any interval I), since e^{r_2t} is obviously not a constant multiple of e^{r_1t} for $r_2\neq r_1$. (If the reader is unconvinced of this he can compute

$$W\left[e^{r_1t}, e^{r_2t}\right] = (r_2 - r_1)e^{(r_1+r_2)t},$$

and observe that W is never zero. Hence, e^{r_1t} and e^{r_2t} are linearly independent on any interval I.)

Example 1. Find the general solution of the equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0. \tag{3}$$

Solution. The characteristic equation $r^2+5r+4=(r+4)(r+1)=0$ has two distinct roots $r_1=-4$ and $r_2=-1$. Thus, $y_1(t)=e^{-4t}$ and $y_2(t)=e^{-t}$ form a fundamental set of solutions of (3), and every solution $y(t)$ of (3) is of the form

$$y(t) = c_1e^{-4t} + c_2e^{-t}$$

for some choice of constants c_1, c_2.

Example 2. Find the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 2y = 0; \qquad y(0) = 1, \quad y'(0) = 2.$$

Solution. The characteristic equation $r^2+4r-2=0$ has 2 roots

$$r_1 = \frac{-4+\sqrt{16+8}}{2} = -2+\sqrt{6}$$

and

$$r_2 = \frac{-4-\sqrt{16+8}}{2} = -2-\sqrt{6}\ .$$

Hence, $y_1(t)=e^{r_1t}$ and $y_2(t)=e^{r_2t}$ are a fundamental set of solutions of $y''+4y'-2y=0$, so that

$$y(t) = c_1e^{(-2+\sqrt{6})t} + c_2e^{(-2-\sqrt{6})t}$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$c_1 + c_2 = 1 \quad \text{and} \quad (-2+\sqrt{6})c_1 + (-2-\sqrt{6})c_2 = 2.$$

From the first equation, $c_2 = 1 - c_1$. Substituting this value of c_2 into the second equation gives

$$(-2+\sqrt{6})c_1 - (2+\sqrt{6})(1-c_1) = 2, \quad \text{or} \quad 2\sqrt{6}\,c_1 = 4+\sqrt{6}\ .$$

Therefore, $c_1 = 2/\sqrt{6} + \frac{1}{2}$, $c_2 = 1 - c_1 = \frac{1}{2} - 2/\sqrt{6}$, and

$$y(t) = \left(\frac{1}{2} + \frac{2}{\sqrt{6}}\right)e^{(-2+\sqrt{6})t} + \left(\frac{1}{2} - \frac{2}{\sqrt{6}}\right)e^{-(2+\sqrt{6})t}.$$

Exercises

Find the general solution of each of the following equations.

1. $\dfrac{d^2y}{dt^2}-y=0$

2. $6\dfrac{d^2y}{dt^2}-7\dfrac{dy}{dt}+y=0$

3. $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}+y=0$

4. $3\dfrac{d^2y}{dt^2}+6\dfrac{dy}{dt}+2y=0$

Solve each of the following initial-value problems.

5. $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}-4y=0;\quad y(0)=1,\ y'(0)=0$

6. $2\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}-10y=0;\quad y(1)=5,\ y'(1)=2$

7. $5\dfrac{d^2y}{dt^2}+5\dfrac{dy}{dt}-y=0;\quad y(0)=0,\ y'(0)=1$

8. $\dfrac{d^2y}{dt^2}-6\dfrac{dy}{dt}+y=0;\quad y(2)=1,\ y'(2)=1$

Remark. In doing Problems 6 and 8, observe that $e^{r(t-t_0)}$ is also a solution of the differential equation $ay''+by'+cy=0$ if $ar^2+br+c=0$. Thus, to find the solution $y(t)$ of the initial-value problem $ay''+by'+cy=0$; $y(t_0)=y_0, y'(t_0)=y_0'$, we would write $y(t)=c_1e^{r_1(t-t_0)}+c_2e^{r_2(t-t_0)}$ and solve for c_1 and c_2 from the initial conditions.

9. Let $y(t)$ be the solution of the initial-value problem

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0;\qquad y(0)=1,\quad y'(0)=V.$$

For what values of V does $y(t)$ remain nonnegative for all $t\geqslant 0$?

10. The differential equation

$$L[y]=t^2y''+\alpha ty'+\beta y=0 \tag{*}$$

is known as Euler's equation. Observe that t^2y'', ty', and y are all multiples of t^r if $y=t^r$. This suggests that we try $y=t^r$ as a solution of (*). Show that $y=t^r$ is a solution of (*) if $r^2+(\alpha-1)r+\beta=0$.

11. Find the general solution of the equation

$$t^2y''+5ty'-5y=0,\qquad t>0$$

12. Solve the initial-value problem

$$t^2y''-ty'-2y=0;\qquad y(1)=0,\quad y'(1)=1$$

on the interval $0<t<\infty$.

2.2.1 Complex roots

If b^2-4ac is negative, then the characteristic equation $ar^2+br+c=0$ has complex roots

$$r_1=\frac{-b+i\sqrt{4ac-b^2}}{2a} \quad\text{and}\quad r_2=\frac{-b-i\sqrt{4ac-b^2}}{2a}.$$

We would like to say that e^{r_1t} and e^{r_2t} are solutions of the differential equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0. \tag{1}$$

However, this presents us with two serious difficulties. On the one hand, the function e^{rt} is not defined, as yet, for r complex. And on the other hand, even if we succeed in defining e^{r_1t} and e^{r_2t} as complex-valued solutions of (1), we are still faced with the problem of finding two *real-valued* solutions of (1).

We begin by resolving the second difficulty, since otherwise there's no sense tackling the first problem. Assume that $y(t)=u(t)+iv(t)$ is a complex-valued solution of (1). This means, of course, that

$$a\left[u''(t)+iv''(t)\right]+b\left[u'(t)+iv'(t)\right]+c\left[u(t)+iv(t)\right]=0. \tag{2}$$

This complex-valued solution of (1) gives rise to *two* real-valued solutions, as we now show.

Lemma 1. *Let $y(t)=u(t)+iv(t)$ be a complex-valued solution of* (1), *with a, b, and c real. Then, $y_1(t)=u(t)$ and $y_2(t)=v(t)$ are two real-valued solutions of* (1). *In other words, both the real and imaginary parts of a complex-valued solution of* (1) *are solutions of* (1). (*The imaginary part of the complex number $\alpha+i\beta$ is β. Similarly, the imaginary part of the function $u(t)+iv(t)$ is $v(t)$.*)

PROOF. From Equation (2),

$$\left[au''(t)+bu'(t)+cu(t)\right]+i\left[av''(t)+bv'(t)+cv(t)\right]=0. \tag{3}$$

Now, if a complex number is zero, then both its real and imaginary parts must be zero. Consequently,

$$au''(t)+bu'(t)+cu(t)=0 \quad\text{and}\quad av''(t)+bv'(t)+cv(t)=0,$$

and this proves Lemma 1. □

The problem of defining e^{rt} for r complex can also be resolved quite easily. Let $r=\alpha+i\beta$. By the law of exponents,

$$e^{rt}=e^{\alpha t}e^{i\beta t}. \tag{4}$$

Thus, we need only define the quantity $e^{i\beta t}$, for β real. To this end, recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{5}$$

Equation (5) makes sense, formally, even for x complex. This suggests that we set

$$e^{i\beta t} = 1 + i\beta t + \frac{(i\beta t)^2}{2!} + \frac{(i\beta t)^3}{3!} + \dots .$$

Next, observe that

$$\begin{aligned} 1 + i\beta t + \frac{(i\beta t)^2}{2!} + \dots &= 1 + i\beta t - \frac{\beta^2 t^2}{2!} - \frac{i\beta^3 t^3}{3!} + \frac{\beta^4 t^4}{4!} + \frac{i\beta^5 t^5}{5!} + \dots \\ &= \left[1 - \frac{\beta^2 t^2}{2!} + \frac{\beta^4 t^4}{4!} + \dots\right] + i\left[\beta t - \frac{\beta^3 t^3}{3!} + \frac{\beta^5 t^5}{5!} + \dots\right] \\ &= \cos\beta t + i \sin\beta t. \end{aligned}$$

Hence,

$$e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t). \tag{6}$$

Returning to the differential equation (1), we see that

$$\begin{aligned} y(t) &= e^{[-b + i\sqrt{4ac - b^2}\,]t/2a} \\ &= e^{-bt/2a}\left[\cos\sqrt{4ac - b^2}\; t/2a + i\sin\sqrt{4ac - b^2}\; t/2a\right] \end{aligned}$$

is a complex-valued solution of (1) if $b^2 - 4ac$ is negative. Therefore, by Lemma 1,

$$y_1(t) = e^{-bt/2a}\cos\beta t \quad \text{and} \quad y_2(t) = e^{-bt/2a}\sin\beta t, \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

are two real-valued solutions of (1). These two functions are linearly independent on any interval I, since their Wronskian (see Exercise 10) is never zero. Consequently, the general solution of (1) for $b^2 - 4ac < 0$ is

$$y(t) = e^{-bt/2a}\left[c_1\cos\beta t + c_2\sin\beta t\right], \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

Remark 1. Strictly speaking, we must verify that the formula

$$\frac{d}{dt}e^{rt} = re^{rt}$$

is true even for r complex, before we can assert that $e^{r_1 t}$ and $e^{r_2 t}$ are complex-valued solutions of (1). To this end, we compute

$$\begin{aligned} \frac{d}{dt}e^{(\alpha + i\beta)t} &= \frac{d}{dt}e^{\alpha t}\left[\cos\beta t + i\sin\beta t\right] \\ &= e^{\alpha t}\left[(\alpha\cos\beta t - \beta\sin\beta t) + i(\alpha\sin\beta t + \beta\cos\beta t)\right] \end{aligned}$$

and this equals $(\alpha+i\beta)e^{(\alpha+i\beta)t}$, since

$$\begin{aligned}(\alpha+i\beta)e^{(\alpha+i\beta)t}&=(\alpha+i\beta)e^{\alpha t}[\cos\beta t+i\sin\beta t]\\&=e^{\alpha t}[(\alpha\cos\beta t-\beta\sin\beta t)+i(\alpha\sin\beta t+\beta\cos\beta t)].\end{aligned}$$

Thus, $(d/dt)e^{rt}=re^{rt}$, even for r complex.

Remark 2. At first glance, one might think that e^{r_2t} would give rise to two additional solutions of (1). This is not the case, though, since

$$\begin{aligned}e^{r_2t}&=e^{-(b/2a)t}e^{-i\beta t},\qquad \beta=\sqrt{4ac-b^2}\,/2a\\&=e^{-bt/2a}[\cos(-\beta t)+i\sin(-\beta t)]=e^{-bt/2a}[\cos\beta t-i\sin\beta t].\end{aligned}$$

Hence,

$$\mathrm{Re}\{e^{r_2t}\}=e^{-bt/2a}\cos\beta t=y_1(t)$$

and

$$\mathrm{Im}\{e^{r_2t}\}=-e^{-bt/2a}\sin\beta t=-y_2(t).$$

Example 1. Find two linearly independent real-valued solutions of the differential equation

$$4\frac{d^2y}{dt^2}+4\frac{dy}{dt}+5y=0. \tag{7}$$

Solution. The characteristic equation $4r^2+4r+5=0$ has complex roots $r_1=-\frac{1}{2}+i$ and $r_2=-\frac{1}{2}-i$. Consequently,

$$e^{r_1t}=e^{(-1/2+i)t}=e^{-t/2}\cos t+ie^{-t/2}\sin t$$

is a complex-valued solution of (7). Therefore, by Lemma 1,

$$\mathrm{Re}\{e^{r_1t}\}=e^{-t/2}\cos t\quad\text{and}\quad\mathrm{Im}\{e^{r_1t}\}=e^{-t/2}\sin t$$

are two linearly independent real-valued solutions of (7).

Example 2. Find the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+4y=0;\qquad y(0)=1,\quad y'(0)=1.$$

Solution. The characteristic equation $r^2+2r+4=0$ has complex roots $r_1=-1+\sqrt{3}\,i$ and $r_2=-1-\sqrt{3}\,i$. Hence,

$$e^{r_1t}=e^{(-1+\sqrt{3}\,i)t}=e^{-t}\cos\sqrt{3}\,t+ie^{-t}\sin\sqrt{3}\,t$$

is a complex-valued solution of $y''+2y'+4y=0$. Therefore, by Lemma 1, both

$$\mathrm{Re}\{e^{r_1t}\}=e^{-t}\cos\sqrt{3}\,t\quad\text{and}\quad\mathrm{Im}\{e^{r_1t}\}=e^{-t}\sin\sqrt{3}\,t$$

are real-valued solutions. Consequently,

$$y(t)=e^{-t}[c_1\cos\sqrt{3}\,t+c_2\sin\sqrt{3}\,t]$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$1 = y(0) = c_1$$

and

$$1 = y'(0) = -c_1 + \sqrt{3}\, c_2.$$

This implies that

$$c_1 = 1, c_2 = \frac{2}{\sqrt{3}} \quad \text{and} \quad y(t) = e^{-t}\left[\cos\sqrt{3}\, t + \frac{2}{\sqrt{3}}\sin\sqrt{3}\, t\right].$$

Exercises

Find the general solution of each of the following equations.

1. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$

2. $2\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 4y = 0$

3. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = 0$

4. $4\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + y = 0$

Solve each of the following initial-value problems.

5. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 2y = 0; \quad y(0) = 1,\ y'(0) = -2$

6. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = 0; \quad y(0) = 0,\ y'(0) = 2$

7. Assume that $b^2 - 4ac < 0$. Show that

$$y_1(t) = e^{(-b/2a)(t-t_0)}\cos\beta(t-t_0)$$

and

$$y_2(t) = e^{(-b/2a)(t-t_0)}\sin\beta(t-t_0), \qquad \beta = \frac{\sqrt{4ac-b^2}}{2a}$$

are solutions of (1), for any number t_0.

Solve each of the following initial-value problems.

8. $2\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + 3y = 0; \quad y(1) = 1,\ y'(1) = 1$

9. $3\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 4y = 0; \quad y(2) = 1,\ y'(2) = -1$

10. Verify that $W[e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t] = \beta e^{2\alpha t}$.

11. Show that $e^{i\omega t}$ is a complex-valued solution of the differential equation $y'' + \omega^2 y = 0$. Find two real-valued solutions.

12. Show that $(\cos t + i\sin t)^r = \cos rt + i\sin rt$. Use this result to obtain the double angle formulas $\sin 2t = 2\sin t\cos t$ and $\cos 2t = \cos^2 t - \sin^2 t$.

13. Show that

$$(\cos t_1 + i\sin t_1)(\cos t_2 + i\sin t_2) = \cos(t_1+t_2) + i\sin(t_1+t_2).$$

Use this result to obtain the trigonometric identities

$$\cos(t_1+t_2) = \cos t_1 \cos t_2 - \sin t_1 \sin t_2,$$
$$\sin(t_1+t_2) = \sin t_1 \cos t_2 + \cos t_1 \sin t_2.$$

14. Show that any complex number $a+ib$ can be written in the form $Ae^{i\theta}$, where $A=\sqrt{a^2+b^2}$ and $\tan\theta = b/a$.

15. Defining the two possible square roots of a complex number $Ae^{i\theta}$ as $\pm\sqrt{A}\, e^{i\theta/2}$, compute the square roots of i, $1+i$, $-i$, $\sqrt{i}$.

16. Use Problem 14 to find the three cube roots of i.

17. (a) Let $r_1 = \lambda + i\mu$ be a complex root of $r^2 + (\alpha-1)r + \beta = 0$. Show that

$$t^{\lambda+i\mu} = t^{\lambda}t^{i\mu} = t^{\lambda}e^{(\ln t)i\mu} = t^{\lambda}[\cos\mu\ln t + i\sin\mu\ln t]$$

is a complex-valued solution of Euler's equation

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0. \tag{*}$$

(b) Show that $t^{\lambda}\cos\mu\ln t$ and $t^{\lambda}\sin\mu\ln t$ are real-valued solutions of (*).

Find the general solution of each of the following equations.

18. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = 0, \quad t>0$

19. $t^2\dfrac{d^2y}{dt^2} + 2t\dfrac{dy}{dt} + 2y = 0, \quad t>0$

2.2.2 Equal roots; reduction of order

If $b^2 = 4ac$, then the characteristic equation $ar^2 + br + c = 0$ has real equal roots $r_1 = r_2 = -b/2a$. In this case, we obtain only one solution

$$y_1(t) = e^{-bt/2a}$$

of the differential equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{1}$$

Our problem is to find a second solution which is independent of y_1. One approach to this problem is to try some additional guesses. A second, and much more clever approach is to try and use our knowledge of $y_1(t)$ to help us find a second independent solution. More generally, suppose that we know one solution $y = y_1(t)$ of the second-order linear equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{2}$$

Can we use this solution to help us find a second independent solution?

The answer to this question is yes. Once we find one solution $y=y_1(t)$ of (2), we can reduce the problem of finding all solutions of (2) to that of solving a first-order linear homogeneous equation. This is accomplished by defining a new dependent variable v through the substitution

$$y(t)=y_1(t)v(t).$$

Then

$$\frac{dy}{dt}=v\frac{dy_1}{dt}+y_1\frac{dv}{dt}$$

and

$$\frac{d^2y}{dt^2}=v\frac{d^2y_1}{dt^2}+2\frac{dv}{dt}\frac{dy_1}{dt}+y_1\frac{d^2v}{dt^2}.$$

Consequently.

$$\begin{aligned}
L[y]&=v\frac{d^2y_1}{dt^2}+2\frac{dv}{dt}\frac{dy_1}{dt}+y_1\frac{d^2v}{dt^2}+p(t)\left[v\frac{dy_1}{dt}+y_1\frac{dv}{dt}\right]+q(t)vy_1\\
&=y_1\frac{d^2v}{dt^2}+\left[2\frac{dy_1}{dt}+p(t)y_1\right]\frac{dv}{dt}+\left[\frac{d^2y_1}{dt^2}+p(t)\frac{dy_1}{dt}+q(t)y_1\right]v\\
&=y_1\frac{d^2v}{dt^2}+\left[2\frac{dy_1}{dt}+p(t)y_1\right]\frac{dv}{dt},
\end{aligned}$$

since $y_1(t)$ is a solution of $L[y]=0$. Hence, $y(t)=y_1(t)v(t)$ is a solution of (2) if v satisfies the differential equation

$$y_1\frac{d^2v}{dt^2}+\left[2\frac{dy_1}{dt}+p(t)y_1\right]\frac{dv}{dt}=0. \tag{3}$$

Now, observe that Equation (3) is really a first-order linear equation for dv/dt. Its solution is

$$\begin{aligned}
\frac{dv}{dt}&=c\exp\left[-\int\left[2\frac{y_1'(t)}{y_1(t)}+p(t)\right]dt\right]\\
&=c\exp\left(-\int p(t)\,dt\right)\exp\left[-2\int\frac{y_1'(t)}{y_1(t)}dt\right]\\
&=\frac{c\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}.
\end{aligned} \tag{4}$$

Since we only need one solution $v(t)$ of (3), we set $c=1$ in (4). Integrating this equation with respect to t, and setting the constant of integration equal to zero, we obtain that $v(t)=\int u(t)\,dt$, where

$$u(t)=\frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}. \tag{5}$$

Hence,

$$y_2(t)=v(t)y_1(t)=y_1(t)\int u(t)\,dt \tag{6}$$

is a second solution of (2). This solution is independent of y_1, for if $y_2(t)$ were a constant multiple of $y_1(t)$ then $v(t)$ would be constant, and consequently, its derivative would vanish identically. However, from (4)

$$\frac{dv}{dt}=\frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)},$$

and this quantity is never zero.

Remark 1. In writing $v(t)=\int u(t)\,dt$, we set the constant of integration equal to zero. Choosing a nonzero constant of integration would only add a constant multiple of $y_1(t)$ to $y_2(t)$. Similarly, the effect of choosing a constant c other than one in Equation (4) would be to multiply $y_2(t)$ by c.

Remark 2. The method we have just presented for solving Equation (2) is known as the method of *reduction of order*, since the substitution $y(t)=y_1(t)v(t)$ reduces the problem of solving the second-order equation (2) to that of solving a first-order equation.

Application to the case of equal roots: In the case of equal roots, we found $y_1(t)=e^{-bt/2a}$ as one solution of the equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0. \tag{7}$$

We can find a second solution from Equations (5) and (6). It is important to realize though, that Equations (5) and (6) were derived under the assumption that our differential equation was written in the form

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0;$$

that is, the coefficient of y'' was one. In our equation, the coefficient of y'' is a. Hence, we must divide Equation (7) by a to obtain the equivalent equation

$$\frac{d^2y}{dt^2}+\frac{b}{a}\frac{dy}{dt}+\frac{c}{a}y=0.$$

Now, we can insert $p(t)=b/a$ into (5) to obtain that

$$u(t)=\frac{\exp\left(-\int \frac{b}{a}\,dt\right)}{\left[e^{-bt/2a}\right]^2}=\frac{e^{-bt/a}}{e^{-bt/a}}=1.$$

Hence,

$$y_2(t)=y_1(t)\int dt=ty_1(t)$$

is a second solution of (7). The functions $y_1(t)$ and $y_2(t)$ are clearly linearly independent on the interval $-\infty<t<\infty$. Therefore, the general solution of (7) in the case of equal roots is

$$y(t)=c_1e^{-bt/2a}+c_2te^{-bt/2a}=\left[c_1+c_2t\right]e^{-bt/2a}$$

Example 1. Find the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=0; \qquad y(0)=1, \quad y'(0)=3.$$

Solution. The characteristic equation $r^2+4r+4=(r+2)^2=0$ has two equal roots $r_1=r_2=-2$. Hence,

$$y(t)=c_1e^{-2t}+c_2te^{-2t}$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$1=y(0)=c_1$$

and

$$3=y'(0)=-2c_1+c_2.$$

This implies that $c_1=1$ and $c_2=5$, so that $y(t)=(1+5t)e^{-2t}$.

Example 2. Find the solution $y(t)$ of the initial-value problem

$$(1-t^2)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0; \qquad y(0)=3, \quad y'(0)=-4$$

on the interval $-1<t<1$.

Solution. Clearly, $y_1(t)=t$ is one solution of the differential equation

$$(1-t^2)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0. \tag{8}$$

We will use the method of reduction of order to find a second solution $y_2(t)$ of (8). To this end, divide both sides of (8) by $1-t^2$ to obtain the equivalent equation

$$\frac{d^2y}{dt^2}+\frac{2t}{1-t^2}\frac{dy}{dt}-\frac{2}{1-t^2}y=0.$$

Then, from (5)

$$u(t)=\frac{\exp\left(-\int \frac{2t}{1-t^2}\,dt\right)}{y_1^2(t)}=\frac{e^{\ln(1-t^2)}}{t^2}=\frac{1-t^2}{t^2},$$

and

$$y_2(t)=t\int \frac{1-t^2}{t^2}\,dt=-t\left(\frac{1}{t}+t\right)=-(1+t^2)$$

is a second solution of (8). Therefore,

$$y(t)=c_1t-c_2(1+t^2)$$

for some choice of constants c_1, c_2. (Notice that all solutions of (9) are continuous at $t=\pm 1$ even though the differential equation is not defined at these points. Thus, it does not necessarily follow that the solutions of a differential equation are discontinuous at a point where the differential equation is not defined—but this is often the case.) The constants c_1 and c_2 are determined from the initial conditions

$$3=y(0)=-c_2 \quad \text{and} \quad -4=y'(0)=c_1.$$

Hence, $y(t)=-4t+3(1+t^2)$.

Exercises

Find the general solution of each of the following equations

1. $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+9y=0$

2. $4\frac{d^2y}{dt^2}-12\frac{dy}{dt}+9y=0$

Solve each of the following initial-value problems.

3. $9\frac{d^2y}{dt^2}+6\frac{dy}{dt}+y=0$; $y(0)=1$, $y'(0)=0$

4. $4\frac{d^2y}{dt^2}-4\frac{dy}{dt}+y=0$; $y(0)=0$, $y'(0)=3$

5. Suppose $b^2=4ac$. Show that

$$y_1(t)=e^{-b(t-t_0)/2a} \quad \text{and} \quad y_2(t)=(t-t_0)e^{-b(t-t_0)/2a}$$

are solutions of (1) for every choice of t_0.

Solve the following initial-value problems.

6. $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=0$; $y(2)=1$, $y'(2)=-1$

7. $9\frac{d^2y}{dt^2}-12\frac{dy}{dt}+4y=0$; $y(\pi)=0$, $y'(\pi)=2$

8. Let a, b and c be positive numbers. Prove that every solution of the differential equation $ay''+by'+cy=0$ approaches zero as t approaches infinity.

9. Here is an alternate and very elegant way of finding a second solution $y_2(t)$ of (1).

(a) Assume that $b^2=4ac$. Show that

$$L[e^{rt}]=a(e^{rt})''+b(e^{rt})'+ce^{rt}=a(r-r_1)^2e^{rt}$$

for $r_1=-b/2a$.

(b) Show that

$$(\partial/\partial r)L[e^{rt}]=L[(\partial/\partial r)e^{rt}]=L[te^{rt}]=2a(r-r_1)e^{rt}+at(r-r_1)^2e^{rt}.$$

(c) Conclude from (a) and (b) that $L[te^{r_1t}]=0$. Hence, $y_2(t)=te^{r_1t}$ is a second solution of (1).

Use the method of reduction of order to find the general solution of the following differential equations.

10. $\dfrac{d^2y}{dt^2}-\dfrac{2(t+1)}{(t^2+2t-1)}\dfrac{dy}{dt}+\dfrac{2}{(t^2+2t-1)}y=0 \quad (y_1(t)=t+1)$

11. $\dfrac{d^2y}{dt^2}-4t\dfrac{dy}{dt}+(4t^2-2)y=0 \quad (y_1(t)=e^{t^2})$

12. $(1-t^2)\dfrac{d^2y}{dt^2}-2t\dfrac{dy}{dt}+2y=0 \quad (y_1(t)=t)$

13. $(1+t^2)\dfrac{d^2y}{dt^2}-2t\dfrac{dy}{dt}+2y=0 \quad (y_1(t)=t)$

14. $(1-t^2)\dfrac{d^2y}{dt^2}-2t\dfrac{dy}{dt}+6y=0 \quad (y_1(t)=3t^2-1)$

15. $(2t+1)\dfrac{d^2y}{dt^2}-4(t+1)\dfrac{dy}{dt}+4y=0 \quad (y_1(t)=t+1)$

16. $t^2\dfrac{d^2y}{dt^2}+t\dfrac{dy}{dt}+\left(t^2-\dfrac{1}{4}\right)y=0 \quad \left(y_1(t)=\dfrac{\sin t}{\sqrt{t}}\right)$

17. Given that the equation

$$t\frac{d^2y}{dt^2}-(1+3t)\frac{dy}{dt}+3y=0$$

has a solution of the form e^{ct}, for some constant c, find the general solution.

18. (a) Show that t^r is a solution of Euler's equation

$$t^2y''+\alpha ty'+\beta y=0, \quad t>0$$

if $r^2+(\alpha-1)r+\beta=0$.

(b) Suppose that $(\alpha-1)^2=4\beta$. Using the method of reduction of order, show that $(\ln t)t^{(1-\alpha)/2}$ is a second solution of Euler's equation.

Find the general solution of each of the following equations.

19. $t^2\dfrac{d^2y}{dt^2}+3t\dfrac{dy}{dt}+y=0$

20. $t^2\dfrac{d^2y}{dt^2}-t\dfrac{dy}{dt}+y=0$

2.3 The nonhomogeneous equation

We turn our attention now to the nonhomogeneous equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=g(t) \tag{1}$$

where the functions $p(t)$, $q(t)$ and $g(t)$ are continuous on an open interval $\alpha<t<\beta$. An important clue as to the nature of all solutions of (1) is provided by the first-order linear equation

$$\frac{dy}{dt}-2ty=-t. \tag{2}$$

The general solution of this equation is

$$y(t)=ce^{t^2}+\tfrac{1}{2}.$$

Now, observe that this solution is the sum of two terms: the first term, ce^{t^2}, is the general solution of the homogeneous equation

$$\frac{dy}{dt}-2ty=0 \tag{3}$$

while the second term, $\frac{1}{2}$, is a solution of the nonhomogeneous equation (2). In other words, every solution $y(t)$ of (2) is the sum of a particular solution, $\psi(t)=\frac{1}{2}$, with a solution ce^{t^2} of the homogeneous equation. A similar situation prevails in the case of second-order equations, as we now show.

Theorem 5. *Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the homogeneous equation*

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{4}$$

and let $\psi(t)$ be any particular solution of the nonhomogeneous equation (1). *Then, every solution $y(t)$ of* (1) *must be of the form*

$$y(t)=c_1y_1(t)+c_2y_2(t)+\psi(t)$$

for some choice of constants c_1, c_2.

The proof of Theorem 5 relies heavily on the following lemma.

Lemma 1. *The difference of any two solutions of the nonhomogeneous equation* (1) *is a solution of the homogeneous equation* (4).

PROOF. Let $\psi_1(t)$ and $\psi_2(t)$ be two solutions of (1). By the linearity of L,

$$L[\psi_1-\psi_2](t)=L[\psi_1](t)-L[\psi_2](t)=g(t)-g(t)=0.$$

Hence, $\psi_1(t)-\psi_2(t)$ is a solution of the homogeneous equation (4). □

We can now give a very simple proof of Theorem 5.

PROOF OF THEOREM 5. Let $y(t)$ be any solution of (1). By Lemma 1, the function $\phi(t)=y(t)-\psi(t)$ is a solution of the homogeneous equation (4). But every solution $\phi(t)$ of the homogeneous equation (4) is of the form $\phi(t)=c_1y_1(t)+c_2y_2(t)$, for some choice of constants c_1, c_2. Therefore,

$$y(t)=\phi(t)+\psi(t)=c_1y_1(t)+c_2y_2(t)+\psi(t). \qquad \square$$

Remark. Theorem 5 is an extremely useful theorem since it reduces the problem of finding all solutions of (1) to the much simpler problem of finding just two solutions of the homogeneous equation (4), and one solution of the nonhomogeneous equation (1).

Example 1. Find the general solution of the equation

$$\frac{d^2y}{dt^2}+y=t. \tag{5}$$

Solution. The functions $y_1(t)=\cos t$ and $y_2(t)=\sin t$ are two linearly independent solutions of the homogeneous equation $y''+y=0$. Moreover, $\psi(t)=t$ is obviously a particular solution of (5). Therefore, by Theorem 5, every solution $y(t)$ of (5) must be of the form

$$y(t)=c_1\cos t+c_2\sin t+t.$$

Example 2. Three solutions of a certain second-order nonhomogeneous linear equation are

$$\psi_1(t)=t, \qquad \psi_2(t)=t+e^t, \quad \text{and} \quad \psi_3(t)=1+t+e^t.$$

Find the general solution of this equation.
Solution. By Lemma 1, the functions

$$\psi_2(t)-\psi_1(t)=e^t \quad \text{and} \quad \psi_3(t)-\psi_2(t)=1$$

are solutions of the corresponding homogeneous equation. Moreover, these functions are obviously linearly independent. Therefore, by Theorem 5, every solution $y(t)$ of this equation must be of the form

$$y(t)=c_1e^t+c_2+t.$$

EXERCISES

1. Three solutions of a certain second-order nonhomogeneous linear equation are

$$\psi_1(t)=t^2, \psi_2(t)=t^2+e^{2t}$$

and

$$\psi_3(t)=1+t^2+2e^{2t}.$$

Find the general solution of this equation.

2. Three solutions of a certain second-order linear nonhomogeneous equation are

$$\psi_1(t) = 1 + e^{t^2}, \psi_2(t) = 1 + te^{t^2}$$

and

$$\psi_3(t) = (t+1)e^{t^2} + 1$$

Find the general solution of this equation.

3. Three solutions of a second-order linear equation $L[y] = g(t)$ are

$$\psi_1(t) = 3e^t + e^{t^2}, \psi_2(t) = 7e^t + e^{t^2}$$

and

$$\psi_3(t) = 5e^t + e^{-t^3} + e^{t^2}.$$

Find the solution of the initial-value problem

$$L[y] = g; \qquad y(0) = 1, \quad y'(0) = 2.$$

4. Let a, b and c be positive constants. Show that the difference of any two solutions of the equation

$$ay'' + by' + cy = g(t)$$

approaches zero as t approaches infinity.

5. Let $\psi(t)$ be a solution of the nonhomogeneous equation (1), and let $\phi(t)$ be a solution of the homogeneous equation (4). Show that $\psi(t) + \phi(t)$ is again a solution of (1).

2.4 The method of variation of parameters

In this section we describe a very general method for finding a particular solution $\psi(t)$ of the nonhomogeneous equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \tag{1}$$

once the solutions of the homogeneous equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \tag{2}$$

are known. The basic principle of this method is to use our knowledge of the solutions of the homogeneous equation to help us find a solution of the nonhomogeneous equation.

Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the homogeneous equation (2). We will try to find a particular solution $\psi(t)$ of the nonhomogeneous equation (1) of the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t); \tag{3}$$

that is, we will try to find functions $u_1(t)$ and $u_2(t)$ so that the linear combination $u_1(t)y_1(t) + u_2(t)y_2(t)$ is a solution of (1). At first glance, this would appear to be a dumb thing to do, since we are replacing the problem of finding one unknown function $\psi(t)$ by the seemingly harder problem of finding two unknown functions $u_1(t)$ and $u_2(t)$. However, by playing our cards right, we will be able to find $u_1(t)$ and $u_2(t)$ as the solutions of two very simple first-order equations. We accomplish this in the following manner. Observe that the differential equation (1) imposes only one condition on the two unknown functions $u_1(t)$ and $u_2(t)$. Therefore, we have a certain "freedom" in choosing $u_1(t)$ and $u_2(t)$. Our goal is to impose an additional condition on $u_1(t)$ and $u_2(t)$ which will make the expression $L[u_1y_1+u_2y_2]$ as simple as possible. Computing

$$\begin{aligned}\frac{d}{dt}\psi(t)&=\frac{d}{dt}[u_1y_1+u_2y_2]\\&=[u_1y_1'+u_2y_2']+[u_1'y_1+u_2'y_2]\end{aligned}$$

we see that $d^2\psi/dt^2$, and consequently $L[\psi]$, will contain no second-order derivatives of u_1 and u_2 if

$$y_1(t)u_1'(t)+y_2(t)u_2'(t)=0. \tag{4}$$

This suggests that we impose the condition (4) on the functions $u_1(t)$ and $u_2(t)$. In this case, then,

$$\begin{aligned}L[\psi]&=[u_1y_1'+u_2y_2']'+p(t)[u_1y_1'+u_2y_2']+q(t)[u_1y_1+u_2y_2]\\&=u_1'y_1'+u_2'y_2'+u_1[y_1''+p(t)y_1'+q(t)y_1]+u_2[y_2''+p(t)y_2'+q(t)y_2]\\&=u_1'y_1'+u_2'y_2'\end{aligned}$$

since both $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation $L[y]=0$. Consequently, $\psi(t)=u_1y_1+u_2y_2$ is a solution of the nonhomogeneous equation (1) if $u_1(t)$ and $u_2(t)$ satisfy the two equations

$$\begin{aligned}y_1(t)u_1'(t)+y_2(t)u_2'(t)&=0\\y_1'(t)u_1'(t)+y_2'(t)u_2'(t)&=g(t).\end{aligned}$$

Multiplying the first equation by $y_2'(t)$, the second equation by $y_2(t)$, and subtracting gives

$$[y_1(t)y_2'(t)-y_1'(t)y_2(t)]u_1'(t)=-g(t)y_2(t),$$

while multiplying the first equation by $y_1'(t)$, the second equation by $y_1(t)$, and subtracting gives

$$[y_1(t)y_2'(t)-y_1'(t)y_2(t)]u_2'(t)=g(t)y_1(t).$$

Hence,

$$u_1'(t)=-\frac{g(t)y_2(t)}{W[y_1,y_2](t)}\quad\text{and}\quad u_2'(t)=\frac{g(t)y_1(t)}{W[y_1,y_2](t)}. \tag{5}$$

Finally, we obtain $u_1(t)$ and $u_2(t)$ by integrating the right-hand sides of (5).

Remark. The general solution of the homogeneous equation (2) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

By letting c_1 and c_2 vary with time, we obtain a solution of the nonhomogeneous equation. Hence, this method is known as the method of variation of parameters.

Example 1.
(a) Find a particular solution $\psi(t)$ of the equation

$$\frac{d^2y}{dt^2} + y = \tan t \tag{6}$$

on the interval $-\pi/2 < t < \pi/2$.
(b) Find the solution $y(t)$ of (6) which satisfies the initial conditions $y(0) = 1, y'(0) = 1$.
Solution.
(a) The functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are two linearly independent solutions of the homogeneous equation $y'' + y = 0$ with

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2 = (\cos t)\cos t - (-\sin t)\sin t = 1.$$

Thus, from (5),

$$u_1'(t) = -\tan t \sin t \quad \text{and} \quad u_2'(t) = \tan t \cos t. \tag{7}$$

Integrating the first equation of (7) gives

$$\begin{aligned} u_1(t) &= -\int \tan t \sin t\, dt = -\int \frac{\sin^2 t}{\cos t} dt \\ &= \int \frac{\cos^2 t - 1}{\cos t} dt = \sin t - \ln|\sec t + \tan t|. \\ &= \sin t - \ln(\sec t + \tan t), \ -\frac{\pi}{2} < t < \frac{\pi}{2} \end{aligned}$$

while integrating the second equation of (7) gives

$$u_2(t) = \int \tan t \cos t\, dt = \int \sin t\, dt = -\cos t.$$

Consequently,

$$\begin{aligned} \psi(t) &= \cos t[\sin t - \ln(\sec t + \tan t)] + \sin t(-\cos t) \\ &= -\cos t \ln(\sec t + \tan t) \end{aligned}$$

is a particular solution of (6) on the interval $-\pi/2 < t < \pi/2$.
(b) By Theorem 5 of Section 2.3,

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln(\sec t + \tan t)$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$1 = y(0) = c_1 \quad \text{and} \quad 1 = y'(0) = c_2 - 1.$$

Hence, $c_1 = 1$, $c_2 = 2$ and

$$y(t) = \cos t + 2\sin t - \cos t \ln(\sec t + \tan t).$$

Remark. Equation (5) determines $u_1(t)$ and $u_2(t)$ up to two constants of integration. We usually take these constants to be zero, since the effect of choosing nonzero constants is to add a solution of the homogeneous equation to $\psi(t)$.

Exercises

Find the general solution of each of the following equations.

1. $\dfrac{d^2y}{dt^2} + y = \sec t, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

2. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = te^{2t}$

3. $2\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + y = (t^2+1)e^t$

4. $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = te^{3t} + 1$

Solve each of the following initial-value problems.

5. $3y'' + 4y' + y = (\sin t)e^{-t}$; $y(0) = 1$, $y'(0) = 0$

6. $y'' + 4y' + 4y = t^{5/2}e^{-2t}$; $y(0) = y'(0) = 0$

7. $y'' - 3y' + 2y = \sqrt{t+1}$; $y(0) = y'(0) = 0$

8. $y'' - y = f(t)$; $y(0) = y'(0) = 0$

Warning. It must be remembered, while doing Problems 3 and 5, that Equation (5) was derived under the assumption that the coefficient of y'' was one.

9. Find two linearly independent solutions of $t^2y'' - 2y = 0$ of the form $y(t) = t^r$. Using these solutions, find the general solution of $t^2y'' - 2y = t^2$.

10. One solution of the equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (*)$$

is $(1+t)^2$, and the Wronskian of any two solutions of (*) is constant. Find the general solution of

$$y'' + p(t)y' + q(t)y = 1 + t.$$

11. Find the general solution of $y'' + (1/4t^2)y = f\cos t$, $t > 0$, given that $y_1(t) = \sqrt{t}$ is a solution of the homogeneous equation.

12. Find the general solution of the equation

$$\frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\frac{dy}{dt} + \frac{2}{1+t^2}y = 1+t^2.$$

13. Show that $\sec t + \tan t$ is positive for $-\pi/2 < t < \pi/2$.

2.5 The method of judicious guessing

A serious disadvantage of the method of variation of parameters is that the integrations required are often quite difficult. In certain cases, it is usually much simpler to guess a particular solution. In this section we will establish a systematic method for guessing solutions of the equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t) \tag{1}$$

where a, b and c are constants, and $g(t)$ has one of several special forms.

Consider first the differential equation

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = a_0 + a_1t + \dots + a_nt^n. \tag{2}$$

We seek a function $\psi(t)$ such that the three functions $a\psi''$, $b\psi'$ and $c\psi$ add up to a given polynomial of degree n. The obvious choice for $\psi(t)$ is a polynomial of degree n. Thus, we set

$$\psi(t) = A_0 + A_1t + \dots + A_nt^n \tag{3}$$

and compute

$$\begin{aligned} L[\psi](t) &= a\psi''(t) + b\psi'(t) + c\psi(t) \\ &= a\left[2A_2 + \dots + n(n-1)A_nt^{n-2}\right] + b\left[A_1 + \dots + nA_nt^{n-1}\right] \\ &\quad + c\left[A_0 + A_1t + \dots + A_nt^n\right] \\ &= cA_nt^n + (cA_{n-1} + nbA_n)t^{n-1} + \dots + (cA_0 + bA_1 + 2aA_2). \end{aligned}$$

Equating coefficients of like powers of t in the equation

$$L[\psi](t) = a_0 + a_1t + \dots + a_nt^n$$

gives

$$cA_n = a_n,\ cA_{n-1} + nbA_n = a_{n-1}, \dots, cA_0 + bA_1 + 2aA_2 = a_0. \tag{4}$$

The first equation determines $A_n = a_n/c$, for $c \neq 0$, and the remaining equations then determine $A_{n-1}, \dots, A_0$ successively. Thus, Equation (1) has a particular solution $\psi(t)$ of the form (3), for $c \neq 0$.

We run into trouble when $c = 0$, since then the first equation of (4) has no solution A_n. This difficulty is to be expected though, for if $c = 0$, then $L[\psi] = a\psi'' + b\psi'$ is a polynomial of degree $n-1$, while the right hand side of (2) is a polynomial of degree n. To guarantee that $a\psi''+b\psi'$ is a polynomial of degree n, we must take ψ as a polynomial of degree $n+1$. Thus, we set

$$\psi(t)=t[A_0+A_1t+\ldots+A_nt^n]. \tag{5}$$

We have omitted the constant term in (5) since $y=$constant is a solution of the homogeneous equation $ay''+by'=0$, and thus can be subtracted from $\psi(t)$. The coefficients $A_0, A_1,\ldots,A_n$ are determined uniquely (see Exercise 19) from the equation

$$a\psi''+b\psi'=a_0+a_1t+\ldots+a_nt^n$$

if $b\neq 0$.

Finally, the case $b=c=0$ is trivial to handle since the differential equation (2) can then be integrated immediately to yield a particular solution $\psi(t)$ of the form

$$\psi(t)=\frac{1}{a}\left[\frac{a_0t^2}{1\cdot 2}+\frac{a_1t^3}{2\cdot 3}+\ldots+\frac{a_nt^{n+2}}{(n+1)(n+2)}\right].$$

Summary. The differential equation (2) has a solution $\psi(t)$ of the form

$$\psi(t)=\begin{cases} A_0+A_1t+\ldots+A_nt^n, & c\neq 0\\ t(A_0+A_1t+\ldots+A_nt^n), & c=0,\ b\neq 0\\ t^2(A_0+A_1t+\ldots+A_nt^n), & c=b=0\end{cases}.$$

Example 1. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=t^2. \tag{6}$$

Solution. We set $\psi(t)=A_0+A_1t+A_2t^2$ and compute

$$\begin{aligned}L[\psi](t)&=\psi''(t)+\psi'(t)+\psi(t)\\&=2A_2+(A_1+2A_2t)+A_0+A_1t+A_2t^2\\&=(A_0+A_1+2A_2)+(A_1+2A_2)t+A_2t^2.\end{aligned}$$

Equating coefficients of like powers of t in the equation $L[\psi](t)=t^2$ gives

$$A_2=1,\qquad A_1+2A_2=0$$

and

$$A_0+A_1+2A_2=0.$$

The first equation tells us that $A_2=1$, the second equation then tells us that $A_1=-2$, and the third equation then tells us that $A_0=0$. Hence,

$$\psi(t)=-2t+t^2$$

is a particular solution of (6).

Let us now re-do this problem using the method of variation of parameters. It is easily verified that

$$y_1(t)=e^{-t/2}\cos\sqrt{3}\,t/2 \quad \text{and} \quad y_2(t)=e^{-t/2}\sin\sqrt{3}\,t/2$$

are two solutions of the homogeneous equation $L[y]=0$. Hence,

$$\psi(t)=u_1(t)e^{-t/2}\cos\sqrt{3}\,t/2+u_2(t)e^{-t/2}\sin\sqrt{3}\,t/2$$

is a particular solution of (6), where

$$u_1(t)=\int\frac{-t^2e^{-t/2}\sin\sqrt{3}\,t/2}{W[y_1,y_2](t)}dt=\frac{-2}{\sqrt{3}}\int t^2e^{t/2}\sin\sqrt{3}\,t/2\,dt$$

and

$$u_2(t)=\int\frac{t^2e^{-t/2}\cos\sqrt{3}\,t/2}{W[y_1,y_2](t)}dt=\frac{2}{\sqrt{3}}\int t^2e^{t/2}\cos\sqrt{3}\,t/2\,dt.$$

These integrations are extremely difficult to perform. Thus, the method of guessing is certainly preferrable, in this problem at least, to the method of variation of parameters.

Consider now the differential equation

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=(a_0+a_1t+\ldots+a_nt^n)e^{\alpha t}. \tag{7}$$

We would like to remove the factor $e^{\alpha t}$ from the right-hand side of (7), so as to reduce this equation to Equation (2). This is accomplished by setting $y(t)=e^{\alpha t}v(t)$. Then,

$$y'=e^{\alpha t}(v'+\alpha v) \quad \text{and} \quad y''=e^{\alpha t}(v''+2\alpha v'+\alpha^2v)$$

so that

$$L[y]=e^{\alpha t}\left[av''+(2a\alpha+b)v'+(a\alpha^2+b\alpha+c)v\right].$$

Consequently, $y(t)=e^{\alpha t}v(t)$ is a solution of (7) if, and only if,

$$a\frac{d^2v}{dt^2}+(2a\alpha+b)\frac{dv}{dt}+(a\alpha^2+b\alpha+c)v=a_0+a_1t+\ldots+a_nt^n. \tag{8}$$

In finding a particular solution $v(t)$ of (8), we must distinguish as to whether (i) $a\alpha^2+b\alpha+c\neq0$; (ii) $a\alpha^2+b\alpha+c=0$, but $2a\alpha+b\neq0$; and (iii) both $a\alpha^2+b\alpha+c$ and $2a\alpha+b=0$. The first case means that α is not a root of the characteristic equation

$$ar^2+br+c=0. \tag{9}$$

In other words, $e^{\alpha t}$ is not a solution of the homogeneous equation $L[y]=0$. The second condition means that α is a single root of the characteristic equation (9). This implies that $e^{\alpha t}$ is a solution of the homogeneous equation, but $te^{\alpha t}$ is not. Finally, the third condition means that α is a double root of the characteristic equation (9), so that both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation. Hence, Equation (7) has a particular solution $\psi(t)$ of the form (i) $\psi(t)=(A_0+\ldots+A_nt^n)e^{\alpha t}$, if $e^{\alpha t}$ is not a solution of the homogeneous equation; (ii) $\psi(t)=t(A_0+\ldots+A_nt^n)e^{\alpha t}$, if $e^{\alpha t}$ is a solution of the homogeneous equation but $te^{\alpha t}$ is not; and (iii) $\psi(t)=t^2(A_0+\ldots+A_nt^n)e^{\alpha t}$ if both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation.

Remark. There are two ways of computing a particular solution $\psi(t)$ of (7). Either we make the substitution $y=e^{\alpha t}v$ and find $v(t)$ from (8), or we guess a solution $\psi(t)$ of the form $e^{\alpha t}$ times a suitable polynomial in t. If α is a double root of the characteristic equation (9), or if $n\geqslant 2$, then it is advisable to set $y=e^{\alpha t}v$ and then find $v(t)$ from (8). Otherwise, we guess $\psi(t)$ directly.

Example 2. Find the general solution of the equation

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=(1+t+\ldots+t^{27})e^{2t}. \tag{10}$$

Solution. The characteristic equation $r^2-4r+4=0$ has equal roots $r_1=r_2=2$. Hence, $y_1(t)=e^{2t}$ and $y_2(t)=te^{2t}$ are solutions of the homogeneous equation $y''-4y'+4y=0$. To find a particular solution $\psi(t)$ of (10), we set $y=e^{2t}v$. Then, of necessity,

$$\frac{d^2v}{dt^2}=1+t+t^2+\ldots+t^{27}.$$

Integrating this equation twice, and setting the constants of integration equal to zero gives

$$v(t)=\frac{t^2}{1\cdot 2}+\frac{t^3}{2\cdot 3}+\ldots+\frac{t^{29}}{28\cdot 29}.$$

Hence, the general solution of (10) is

$$\begin{aligned} y(t)&=c_1e^{2t}+c_2te^{2t}+e^{2t}\left[\frac{t^2}{1\cdot 2}+\ldots+\frac{t^{29}}{28\cdot 29}\right]\\ &=e^{2t}\left[c_1+c_2t+\frac{t^2}{1\cdot 2}+\ldots+\frac{t^{29}}{28\cdot 29}\right]. \end{aligned}$$

It would be sheer madness (and a terrible waste of paper) to plug the expression

$$\psi(t)=t^2(A_0+A_1t+\ldots+A_{27}t^{27})e^{2t}$$

into (10) and then solve for the coefficients $A_0, A_1,\ldots,A_{27}$.

Example 3. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=(1+t)e^{3t}.$$

Solution. In this case, e^{3t} is not a solution of the homogeneous equation $y''-3y'+2y=0$. Thus, we set $\psi(t)=(A_0+A_1t)e^{3t}$. Computing

$$\begin{aligned}L[\psi](t)&=\psi''-3\psi'+2\psi\\&=e^{3t}\left[(9A_0+6A_1+9A_1t)-3(3A_0+A_1+3A_1t)+2(A_0+A_1t)\right]\\&=e^{3t}\left[(2A_0+3A_1)+2A_1t\right]\end{aligned}$$

and cancelling off the factor e^{3t} from both sides of the equation

$$L[\psi](t)=(1+t)e^{3t},$$

gives

$$2A_1t+(2A_0+3A_1)=1+t.$$

This implies that $2A_1=1$ and $2A_0+3A_1=1$. Hence, $A_1=\frac{1}{2}$, $A_0=-\frac{1}{4}$ and $\psi(t)=(-\frac{1}{4}+t/2)e^{3t}$.

Finally, we consider the differential equation

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=(a_0+a_1t+\dots+a_nt^n)\times\begin{cases}\cos\omega t\\ \sin\omega t\end{cases}. \tag{11}$$

We can reduce the problem of finding a particular solution $\psi(t)$ of (11) to the simpler problem of finding a particular solution of (7) with the aid of the following simple but extremely useful lemma.

Lemma 1. *Let $y(t)=u(t)+iv(t)$ be a complex-valued solution of the equation*

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=g(t)=g_1(t)+ig_2(t) \tag{12}$$

where a, b and c are real. This means, of course, that

$$a\left[u''(t)+iv''(t)\right]+b\left[u'(t)+iv'(t)\right]+c\left[u(t)+iv(t)\right]=g_1(t)+ig_2(t). \tag{13}$$

Then, $L[u](t)=g_1(t)$ and $L[v](t)=g_2(t)$.

PROOF. Equating real and imaginary parts in (13) gives

$$au''(t)+bu'(t)+cu(t)=g_1(t)$$

and

$$av''(t)+bv'(t)+cv(t)=g_2(t). \qquad \square$$

Now, let $\phi(t)=u(t)+iv(t)$ be a particular solution of the equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=(a_0+\dots+a_nt^n)e^{i\omega t}. \tag{14}$$

The real part of the right-hand side of (14) is $(a_0+\dots+a_nt^n)\cos\omega t$, while the imaginary part is $(a_0 + \ldots + a_n t^n)\sin\omega t$. Hence, by Lemma 1

$$u(t) = \text{Re}\{\phi(t)\}$$

is a solution of

$$ay'' + by' + cy = (a_0 + \ldots + a_n t^n)\cos\omega t$$

while

$$v(t) = \text{Im}\{\phi(t)\}$$

is a solution of

$$ay'' + by' + cy = (a_0 + \ldots + a_n t^n)\sin\omega t.$$

Example 4. Find a particular solution $\psi(t)$ of the equation

$$L[y] = \frac{d^2y}{dt^2} + 4y = \sin 2t. \tag{15}$$

Solution. We will find $\psi(t)$ as the imaginary part of a complex-valued solution $\phi(t)$ of the equation

$$L[y] = \frac{d^2y}{dt^2} + 4y = e^{2it}. \tag{16}$$

To this end, observe that the characteristic equation $r^2 + 4 = 0$ has complex roots $r = \pm 2i$. Therefore, Equation (16) has a particular solution $\phi(t)$ of the form $\phi(t) = A_0 t e^{2it}$. Computing

$$\phi'(t) = A_0(1 + 2it)e^{2it} \quad \text{and} \quad \phi''(t) = A_0(4i - 4t)e^{2it}$$

we see that

$$L[\phi](t) = \phi''(t) + 4\phi(t) = 4iA_0 e^{2it}.$$

Hence, $A_0 = 1/4i = -i/4$ and

$$\phi(t) = -\frac{it}{4}e^{2it} = -\frac{it}{4}(\cos 2t + i\sin 2t) = \frac{t}{4}\sin 2t - i\frac{t}{4}\cos 2t.$$

Therefore, $\psi(t) = \text{Im}\{\phi(t)\} = -(t/4)\cos 2t$ is a particular solution of (15).

Example 5. Find a particular solution $\psi(t)$ of the equation

$$\frac{d^2y}{dt^2} + 4y = \cos 2t. \tag{17}$$

Solution. From Example 4, $\phi(t) = (t/4)\sin 2t - i(t/4)\cos 2t$ is a complex-valued solution of (16). Therefore,

$$\psi(t) = \text{Re}\{\phi(t)\} = \frac{t}{4}\sin 2t$$

is a particular solution of (17).

Example 6. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=te^t\cos t. \tag{18}$$

Solution. Observe that $te^t\cos t$ is the real part of $te^{(1+i)t}$. Therefore, we can find $\psi(t)$ as the real part of a complex-valued solution $\phi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=te^{(1+i)t}. \tag{19}$$

To this end, observe that $1+i$ is not a root of the characteristic equation $r^2+2r+1=0$. Therefore, Equation (19) has a particular solution $\phi(t)$ of the form $\phi(t)=(A_0+A_1t)e^{(1+i)t}$. Computing $L[\phi]=\phi''+2\phi'+\phi$, and using the identity

$$(1+i)^2+2(1+i)+1=\left[(1+i)+1\right]^2=(2+i)^2$$

we see that

$$\left[(2+i)^2A_1t+(2+i)^2A_0+2(2+i)A_1\right]=t.$$

Equating coefficients of like powers of t in this equation gives

$$(2+i)^2A_1=1$$

and

$$(2+i)A_0+2A_1=0.$$

This implies that $A_1=1/(2+i)^2$ and $A_0=-2/(2+i)^3$, so that

$$\phi(t)=\left[\frac{-2}{(2+i)^3}+\frac{t}{(2+i)^2}\right]e^{(1+i)t}.$$

After a little algebra, we find that

$$\phi(t)=\frac{e^t}{125}\left\{\left[(15t-4)\cos t+(20t-22)\sin t\right]\right.$$
$$\left.+i\left[(22-20t)\cos t+(15t-4)\sin t\right]\right\}.$$

Hence,

$$\psi(t)=\mathrm{Re}\{\phi(t)\}=\frac{e^t}{125}\left[(15t-4)\cos t+(20t-22)\sin t\right].$$

Remark. The method of judicious guessing also applies to the equation

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=\sum_{j=1}^{n}p_j(t)e^{\alpha_jt} \tag{20}$$

where the $p_j(t), j=1,\ldots,n$ are polynomials in t. Let $\psi_j(t)$ be a particular solution of the equation

$$L[y]=p_j(t)e^{\alpha_j t}, \qquad j=1,\dots,n.$$

Then, $\psi(t)=\sum_{j=1}^n \psi_j(t)$ is a solution of (20) since

$$L[\psi]=L\left[\sum_{j=1}^n \psi_j\right]=\sum_{j=1}^n L[\psi_j]=\sum_{j=1}^n p_j(t)e^{\alpha_j t}.$$

Thus, to find a particular solution of the equation

$$y''+y'+y=e^t+t\sin t$$

we find particular solutions $\psi_1(t)$ and $\psi_2(t)$ of the equations

$$y''+y'+y=e^t \quad \text{and} \quad y''+y'+y=t\sin t$$

respectively, and then add these two solutions together.

Exercises

Find a particular solution of each of the following equations.

1. $y''+3y=t^3-1$

2. $y''+4y'+4y=te^{\alpha t}$

3. $y''-y=t^2e^t$

4. $y''+y'+y=1+t+t^2$

5. $y''+2y'+y=e^{-t}$

6. $y''+5y'+4y=t^2e^{7t}$

7. $y''+4y=t\sin 2t$

8. $y''-6y'+9y=(3t^7-5t^4)e^{3t}$

9. $y''-2y'+5y=2\cos^2 t$

10. $y''-2y'+5y=2(\cos^2 t)e^t$

11. $y''+y'-6y=\sin t+te^{2t}$

12. $y''+y'+4y=t^2+(2t+3)(1+\cos t)$

13. $y''-3y'+2y=e^t+e^{2t}$

14. $y''+2y'=1+t^2+e^{-2t}$

15. $y''+y=\cos t\cos 2t$

16. $y''+y=\cos t\cos 2t\cos 3t$.

17. (a) Show that $\cos^3\omega t=\frac{1}{4}\mathrm{Re}\{e^{3i\omega t}+3e^{i\omega t}\}$.
Hint: $\cos\omega t=(e^{i\omega t}+e^{-i\omega t})/2$.
(b) Find a particular solution of the equation

$$10y''+0.2y'+1000y=5+20\cos^3 10t$$

18. (a) Let $L[y]=y''-2r_1y'+r_1^2y$. Show that

$$L\left[e^{r_1t}v(t)\right]=e^{r_1t}v''(t).$$

(b) Find the general solution of the equation

$$y''-6y'+9y=t^{3/2}e^{3t}.$$

19. Let $\psi(t)=t(A_0+\dots+A_nt^n)$, and assume that $b\neq 0$. Show that the equation $a\psi''+b\psi'=a_0+\dots+a_nt^n$ determines $A_0,\dots,A_n$ uniquely.

2.6 Mechanical vibrations

Consider the case where a small object of mass m is attached to an elastic spring of length l, which is suspended from a rigid horizontal support (see Figure 1). (An elastic spring has the property that if it is stretched or compressed a distance Δl which is small compared to its natural length l, then it will exert a restoring force of magnitude $k\Delta l$. The constant k is called the spring-constant, and is a measure of the stiffness of the spring.) In addition, the mass and spring may be immersed in a medium, such as oil, which impedes the motion of an object through it. Engineers usually refer to such systems as spring-mass-dashpot systems, or as seismic instruments, since they are similar, in principle, to a seismograph which is used to detect motions of the earth's surface.

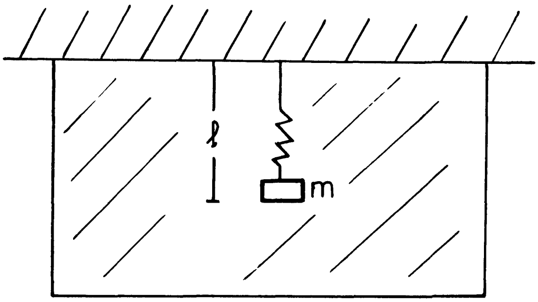

Figure 1

Spring-mass-dashpot systems have many diverse applications. For example, the shock absorbers in our automobiles are simple spring-mass-dashpot systems. Also, most heavy gun emplacements are attached to such systems so as to minimize the "recoil" effect of the gun. The usefulness of these devices will become apparent after we set up and solve the differential equation of motion of the mass m.

In calculating the motion of the mass m, it will be convenient for us to measure distances from the equilibrium position of the mass, rather than the horizontal support. The equilibrium position of the mass is that point where the mass will hang at rest if no external forces act upon it. In equilibrium, the weight mg of the mass is exactly balanced by the restoring force of the spring. Thus, in its equilibrium position, the spring has been stretched a distance Δl, where $k\Delta l = mg$. We let $y = 0$ denote this equilibrium position, and we take the downward direction as positive. Let $y(t)$ denote the position of the mass at time t. To find $y(t)$, we must compute the total force acting on the mass m. This force is the sum of four separate forces W, R, D and F.

(i) The force $W = mg$ is the weight of the mass pulling it downward. This force is positive, since the downward direction is the positive y direction.

(ii) The force R is the restoring force of the spring, and it is proportional to the elongation, or compression, $\Delta l + y$ of the spring. It always acts to restore the spring to its natural length. If $\Delta l + y > 0$, then R is negative, so that $R = -k(\Delta l + y)$, and if $\Delta l + y < 0$, then R is positive, so that $R = -k(\Delta l + y)$. In either case,

$$R = -k(\Delta l + y).$$

(iii) The force D is the damping, or drag force, which the medium exerts on the mass m. (Most media, such as oil and air, tend to resist the motion of an object through it.) This force always acts in the direction opposite the direction of motion, and is usually directly proportional to the magnitude of the velocity dy/dt. If the velocity is positive; that is, the mass is moving in the downward direction, then $D = -c\,dy/dt$, and if the velocity is negative, then $D = -c\,dy/dt$. In either case,

$$D = -c\,dy/dt.$$

(iv) The force F is the external force applied to the mass. This force is directed upward or downward, depending as to whether F is positive or negative. In general, this external force will depend explicitly on time.

From Newton's second law of motion (see Section 1.7)

$$\begin{aligned} m\frac{d^2y}{dt^2} &= W + R + D + F \\ &= mg - k(\Delta l + y) - c\frac{dy}{dt} + F(t) \\ &= -ky - c\frac{dy}{dt} + F(t), \end{aligned}$$

since $mg = k\,\Delta l$. Hence, the position $y(t)$ of the mass satisfies the second-order linear differential equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t) \tag{1}$$

where m, c and k are nonnegative constants. We adopt here the mks system of units so that F is measured in newtons, y is measured in meters, and t is measured in seconds. In this case, the units of k are N/m, the units of c are N·s/m, and the units of m are kilograms (N·s^2/m)

(a) *Free vibrations*:

We consider first the simplest case of free undamped motion. In this case, Equation (1) reduces to

$$m\frac{d^2y}{dt^2} + ky = 0 \quad \text{or} \quad \frac{d^2y}{dt^2} + \omega_0^2 y = 0 \tag{2}$$

where $\omega_0^2 = k/m$. The general solution of (2) is

$$y(t) = a\cos\omega_0 t + b\sin\omega_0 t. \tag{3}$$

In order to analyze the solution (3), it is convenient to rewrite it as a single cosine function. This is accomplished by means of the following lemma.

Lemma 1. *Any function $y(t)$ of the form* (3) *can be written in the simpler form*

$$y(t) = R\cos(\omega_0 t - \delta) \tag{4}$$

where $R = \sqrt{a^2+b^2}$ and $\delta = \tan^{-1} b/a$.

PROOF. We will verify that the two expressions (3) and (4) are equal. To this end, compute

$$R\cos(\omega_0 t - \delta) = R\cos\omega_0 t\cos\delta + R\sin\omega_0 t\sin\delta$$

and observe from Figure 2 that $R\cos\delta = a$ and $R\sin\delta = b$. Hence,

$$R\cos(\omega_0 t - \delta) = a\cos\omega_0 t + b\sin\omega_0 t. \qquad \square$$

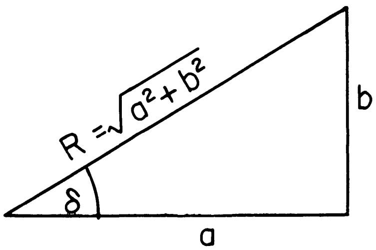

Figure 2

In Figure 3 we have graphed the function $y = R\cos(\omega_0 t - \delta)$. Notice that $y(t)$ always lies between $-R$ and $+R$, and that the motion of the mass is periodic—it repeats itself over every time interval of length $2\pi/\omega_0$. This type of motion is called simple harmonic motion; R is called the amplitude of the motion, δ the phase angle of the motion, $T_0 = 2\pi/\omega_0$ the natural period of the motion, and $\omega_0 = \sqrt{k/m}$ the natural frequency of the system.

(b) *Damped free vibrations*:

If we now include the effect of damping, then the differential equation governing the motion of the mass is

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0. \tag{5}$$

The roots of the characteristic equation $mr^2 + cr + k = 0$ are

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} \quad \text{and} \quad r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

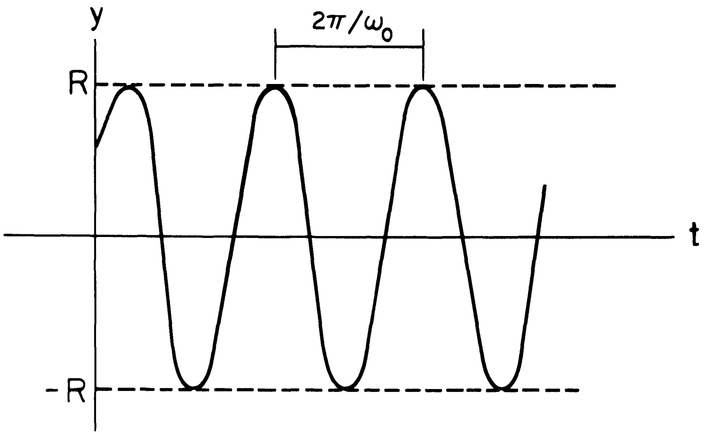

Figure 3. Graph of $y(t)=R\cos(\omega_0 t-\delta)$

Thus, there are three cases to consider, depending as to whether c^2-4km is positive, negative or zero.

(i) $c^2-4km>0$. In this case both r_1 and r_2 are negative, and every solution $y(t)$ of (5) has the form

$$y(t)=ae^{r_1t}+be^{r_2t}.$$

(ii) $c^2-4km=0$. In this case, every solution $y(t)$ of (5) is of the form

$$y(t)=(a+bt)e^{-ct/2m}.$$

(iii) $c^2-4km<0$. In this case, every solution $y(t)$ of (5) is of the form

$$y(t)=e^{-ct/2m}\left[a\cos\mu t+b\sin\mu t\right], \qquad \mu=\frac{\sqrt{4km-c^2}}{2m}.$$

The first two cases are referred to as overdamped and critically damped, respectively. They represent motions in which the originally displaced mass creeps back to its equilibrium position. Depending on the initial conditions, it may be possible to overshoot the equilibrium position once, but no more than once (see Exercises 2-3). The third case, which is referred to as an underdamped motion, occurs quite often in mechanical systems and represents a damped vibration. To see this, we use Lemma 1 to rewrite the function

$$y(t)=e^{-ct/2m}\left[a\cos\mu t+b\sin\mu t\right]$$

in the form

$$y(t)=Re^{-ct/2m}\cos(\mu t-\delta).$$

The displacement y oscillates between the curves $y=\pm Re^{-ct/2m}$, and thus represents a cosine curve with decreasing amplitude, as shown in Figure 4.

Now, observe that the motion of the mass always dies out eventually if there is damping in the system. In other words, any initial disturbance of

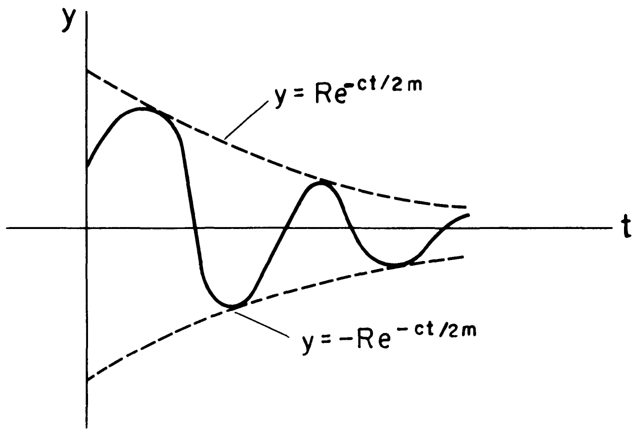

Figure 4. Graph of $\mathrm{Re}^{-ct/2m}\cos(\mu t-\delta)$

the system is dissipated by the damping present in the system. This is one reason why spring-mass-dashpot systems are so useful in mechanical systems: they can be used to damp out any undesirable disturbances. For example, the shock transmitted to an automobile by a bump in the road is dissipated by the shock absorbers in the car, and the momentum from the recoil of a gun barrel is dissipated by a spring-mass-dashpot system attached to the gun.

(c) *Damped forced vibrations*:

If we now introduce an external force $F(t)=F_0\cos\omega t$, then the differential equation governing the motion of the mass is

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=F_0\cos\omega t. \tag{6}$$

Using the method of judicious guessing, we can find a particular solution $\psi(t)$ of (6) of the form

$$\begin{aligned}\psi(t)&=\frac{F_0}{(k-m\omega^2)^2+c^2\omega^2}\left[(k-m\omega^2)\cos\omega t+c\omega\sin\omega t\right]\\&=\frac{F_0}{(k-m\omega^2)^2+c^2\omega^2}\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}\cos(\omega t-\delta)\\&=\frac{F_0\cos(\omega t-\delta)}{\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}}\end{aligned} \tag{7}$$

where $\tan\delta=c\omega/(k-m\omega^2)$. Hence, every solution $y(t)$ of (6) must be of the form

$$y(t)=\phi(t)+\psi(t)=\phi(t)+\frac{F_0\cos(\omega t-\delta)}{\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}} \tag{8}$$

where $\phi(t)$ is a solution of the homogeneous equation

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=0. \tag{9}$$

We have already seen though, that every solution $y=\phi(t)$ of (9) approaches zero as t approaches infinity. Thus, for large t, the equation $y(t)=\psi(t)$ describes very accurately the position of the mass m, regardless of its initial position and velocity. For this reason, $\psi(t)$ is called the steady state part of the solution (8), while $\phi(t)$ is called the transient part of the solution.

(d) *Forced free vibrations*:

We now remove the damping from our system and consider the case of forced free vibrations where the forcing term is periodic and has the form $F(t)=F_0\cos\omega t$. In this case, the differential equation governing the motion of the mass m is

$$\frac{d^2y}{dt^2}+\omega_0^2y=\frac{F_0}{m}\cos\omega t, \qquad \omega_0^2=k/m. \tag{10}$$

The case $\omega\neq\omega_0$ is uninteresting; every solution $y(t)$ of (10) has the form

$$y(t)=c_1\cos\omega_0t+c_2\sin\omega_0t+\frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t,$$

and thus is the sum of two periodic functions of different periods. The interesting case is when $\omega=\omega_0$; that is, when the frequency ω of the external force equals the natural frequency of the system. This case is called the *resonance* case, and the differential equation of motion for the mass m is

$$\frac{d^2y}{dt^2}+\omega_0^2y=\frac{F_0}{m}\cos\omega_0t. \tag{11}$$

We will find a particular solution $\psi(t)$ of (11) as the real part of a complex-valued solution $\phi(t)$ of the equation

$$\frac{d^2y}{dt^2}+\omega_0^2y=\frac{F_0}{m}e^{i\omega_0t}. \tag{12}$$

Since $e^{i\omega_0t}$ is a solution of the homogeneous equation $y''+\omega_0^2y=0$, we know that (12) has a particular solution $\phi(t)=Ate^{i\omega_0t}$, for some constant A. Computing

$$\phi''+\omega_0^2\phi=2i\omega_0Ae^{i\omega_0t}$$

we see that

$$A=\frac{1}{2i\omega_0}\frac{F_0}{m}=\frac{-iF_0}{2m\omega_0}.$$

Hence,

$$\begin{aligned}\phi(t)&=\frac{-iF_0t}{2m\omega_0}(\cos\omega_0t+i\sin\omega_0t)\\&=\frac{F_0t}{2m\omega_0}\sin\omega_0t-i\frac{F_0t}{2m\omega_0}\cos\omega_0t\end{aligned}$$

is a particular solution of (12), and

$$\psi(t)=\text{Re}\{\phi(t)\}=\frac{F_0t}{2m\omega_0}\sin\omega_0t$$

is a particular solution of (11). Consequently, every solution $y(t)$ of (11) is of the form

$$y(t)=c_1\cos\omega_0t+c_2\sin\omega_0t+\frac{F_0t}{2m\omega_0}\sin\omega_0t \tag{13}$$

for some choice of constants c_1,c_2.

Now, the sum of the first two terms in (13) is a periodic function of time. The third term, though, represents an oscillation with increasing amplitude, as shown in Figure 5. Thus, the forcing term $F_0\cos\omega t$, if it is in resonance with the natural frequency of the system, will always cause unbounded oscillations. Such a phenomenon was responsible for the collapse of the Tacoma Bridge, (see Section 2.6.1) and many other mechanical catastrophes.

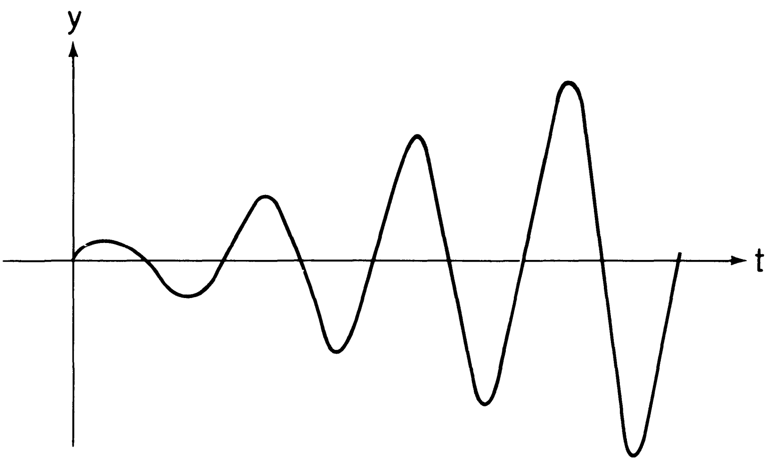

Figure 5. Graph of $f(t)=At\sin\omega_0t$

EXERCISES

1. It is found experimentally that a 1 kg mass stretches a spring 49/320 m. If the mass is pulled down an additional 1/4 m and released, find the amplitude, period and frequency of the resulting motion, neglecting air resistance (use $g = 9.8\ \text{m/s}^2$).

2. Let $y(t) = Ae^{r_1 t} + Be^{r_2 t}$, with $|A| + |B| \neq 0$.
(a) Show that $y(t)$ is zero at most once.
(b) Show that $y'(t)$ is zero at most once.

3. Let $y(t) = (A + Bt)e^{rt}$, with $|A| + |B| \neq 0$.
(a) Show that $y(t)$ is zero at most once.
(b) Show that $y'(t)$ is zero at most once.

4. A small object of mass 1 kg is attached to a spring with spring constant 2N/m. This spring–mass system is immersed in a viscous medium with damping constant 3 N·s/m. At time $t = 0$, the mass is lowered 1/2 m below its equilibrium position, and released. Show that the mass will creep back to its equilibrium position as t approaches infinity.

5. A small object of mass 1 kg is attached to a spring with spring-constant 1 N/m and is immersed in a viscous medium with damping constant 2 N·s/m. At time $t = 0$, the mass is lowered 1/4 m and given an initial velocity of 1 m/s in the upward direction. Show that the mass will overshoot its equilibrium position once, and then creep back to equilibrium.

6. A small object of mass 4 kg is attached to an elastic spring with spring-constant 64 N/m, and is acted upon by an external force $F(t) = A\cos^3\omega t$. Find all values of ω at which resonance occurs.

7. The gun of a U.S. M60 tank is attached to a spring–mass–dashpot system with spring-constant $100\alpha^2$ and damping constant 200α, in their appropriate units. The mass of the gun is 100 kg. Assume that the displacement $y(t)$ of the gun from its rest position after being fired at time $t = 0$ satisfies the initial-value problem

$$100y'' + 200\alpha y' + 100\alpha^2 y = 0;\ y(0) = 0,\ y'(0) = 100\ \text{m/s}.$$

It is desired that one second later, the quantity $y^2 + (y')^2$ be less than .01. How large must α be to guarantee that this is so? (The spring–mass–dashpot mechanism in the M60 tanks supplied by the U.S. to Israel are critically damped, for this situation is preferable in desert warfare where one has to fire again as quickly as possible).

8. A spring–mass–dashpot system has the property that the spring constant k is 9 times its mass m, and the damping constant c is 6 times its mass. At time $t = 0$, the mass, which is hanging at rest, is acted upon by an external force $F(t) = (3\sin 3t)$ N. The spring will break if it is stretched an additional 5 m from its equilibrium position. Show that the spring will not break if $m \geq 1/5$ kg.

9. A spring–mass–dashpot system with $m = 1$, $k = 2$ and $c = 2$ (in their respective units) hangs in equilibrium. At time $t = 0$, an external force $F(t) = \pi - t$ N acts for a time interval π. Find the position of the mass at anytime $t > \pi$.

10. A 1 kg mass is attached to a spring with spring constant $k=64$ N/m. With the mass on the spring at rest in the equilibrium position at time $t=0$, an external force $F(t)=(\frac{1}{2}t)$ N is applied until time $t_1=7\pi/16$ seconds, at which time it is removed. Assuming no damping, find the frequency and amplitude of the resulting oscillation.

11. A 1 kg mass is attached to a spring with spring constant $k=4$ N/m, and hangs in equilibrium. An external force $F(t)=(1+t+\sin 2t)$ N is applied to the mass beginning at time $t=0$. If the spring is stretched a length $(1/2+\pi/4)$ m or more from its equilibrium position, then it will break. Assuming no damping present, find the time at which the spring breaks.

12. A small object of mass 1 kg is attached to a spring with spring constant $k=1$ N/m. This spring-mass system is then immersed in a viscous medium with damping constant c. An external force $F(t)=(3-\cos t)$ N is applied to the system. Determine the minimum positive value of c so that the magnitude of the steady state solution does not exceed 5 m.

13. Determine a particular solution $\psi(t)$ of $my''+cy'+ky=F_0\cos\omega t$, of the form $\psi(t)=A\cos(\omega t-\phi)$. Show that the amplitude A is a maximum when $\omega^2=\omega_0^2-\frac{1}{2}(c/m)^2$. This value of ω is called the *resonant frequency* of the system. What happens when $\omega_0^2<\frac{1}{2}(c/m)^2$?

2.6.1 The Tacoma Bridge disaster

On July 1, 1940, the Tacoma Narrows Bridge at Puget Sound in the state of Washington was completed and opened to traffic. From the day of its opening the bridge began undergoing vertical oscillations, and it soon was nicknamed "Galloping Gertie." Strange as it may seem, traffic on the bridge increased tremendously as a result of its novel behavior. People came from hundreds of miles in their cars to enjoy the curious thrill of riding over a galloping, rolling bridge. For four months, the bridge did a thriving business. As each day passed, the authorities in charge became more and more confident of the safety of the bridge—so much so, in fact, that they were planning to cancel the insurance policy on the bridge.

Starting at about 7:00 on the morning of November 7, 1940, the bridge began undulating persistently for three hours. Segments of the span were heaving periodically up and down as much as three feet. At about 10:00 a.m., something seemed to snap and the bridge began oscillating wildly. At one moment, one edge of the roadway was twenty-eight feet higher than the other; the next moment it was twenty-eight feet lower than the other edge. At 10:30 a.m. the bridge began cracking, and finally, at 11:10 a.m. the entire bridge came crashing down. Fortunately, only one car was on the bridge at the time of its failure. It belonged to a newspaper reporter who had to abandon the car and its sole remaining occupant, a pet dog, when the bridge began its violent twisting motion. The reporter reached safety, torn and bleeding, by crawling on hands and knees, desperately clutching the curb of the bridge. His dog went down with the car and the span—the only life lost in the disaster.

There were many humorous and ironic incidents associated with the collapse of the Tacoma Bridge. When the bridge began heaving violently, the authorities notified Professor F. B. Farquharson of the University of Washington. Professor Farquharson had conducted numerous tests on a simulated model of the bridge and had assured everyone of its stability. The professor was the last man on the bridge. Even when the span was tilting more than twenty-eight feet up and down, he was making scientific observations with little or no anticipation of the imminent collapse of the bridge. When the motion increased in violence, he made his way to safety by scientifically following the yellow line in the middle of the roadway. The professor was one of the most surprised men when the span crashed into the water.

One of the insurance policies covering the bridge had been written by a local travel agent who had pocketed the premium and had neglected to report the policy, in the amount of $800,000, to his company. When he later received his prison sentence, he ironically pointed out that his embezzlement would never have been discovered if the bridge had only remained up for another week, at which time the bridge officials had planned to cancel all of the policies.

A large sign near the bridge approach advertised a local bank with the slogan "as safe as the Tacoma Bridge." Immediately following the collapse of the bridge, several representatives of the bank rushed out to remove the billboard.

After the collapse of the Tacoma Bridge, the governor of the state of Washington made an emotional speech, in which he declared "We are going to build the exact same bridge, exactly as before." Upon hearing this, the noted engineer Von Karman sent a telegram to the governor stating "If you build the exact same bridge exactly as before, it will fall into the exact same river exactly as before."

The collapse of the Tacoma Bridge was due to an aerodynamical phenomenon known as *stall flutter*. This can be explained very briefly in the following manner. If there is an obstacle in a stream of air, or liquid, then a "vortex street" is formed behind the obstacle, with the vortices flowing off at a definite periodicity, which depends on the shape and dimension of the structure as well as on the velocity of the stream (see Figure 1). As a result of the vortices separating alternately from either side of the obstacle, it is acted upon by a periodic force perpendicular to the direction of the stream, and of magnitude $F_0 \cos \omega t$. The coefficient F_0 depends on the shape of the structure. The poorer the streamlining of the structure; the larger the coefficient F_0, and hence the amplitude of the force. For example, flow around an airplane wing at small angles of attack is very smooth, so that the vortex street is not well defined and the coefficient F_0 is very small. The poorly streamlined structure of a suspension bridge is another

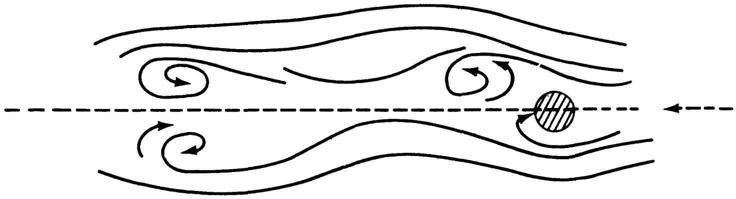

Figure 1

matter, and it is natural to expect that a force of large amplitude will be set up. Thus, a structure suspended in an air stream experiences the effect of this force and hence goes into a state of forced vibrations. The amount of danger from this type of motion depends on how close the natural frequency of the structure (remember that bridges are made of steel, a highly elastic material) is to the frequency of the driving force. If the two frequencies are the same, resonance occurs, and the oscillations will be destructive if the system does not have a sufficient amount of damping. It has now been established that oscillations of this type were responsible for the collapse of the Tacoma Bridge. In addition, resonances produced by the separation of vortices have been observed in steel factory chimneys, and in the periscopes of submarines.

The phenomenon of resonance was also responsible for the collapse of the Broughton suspension bridge near Manchester, England in 1831. This occurred when a column of soldiers marched in cadence over the bridge, thereby setting up a periodic force of rather large amplitude. The frequency of this force was equal to the natural frequency of the bridge. Thus, very large oscillations were induced, and the bridge collapsed. It is for this reason that soldiers are ordered to break cadence when crossing a bridge.

Epilog. The father of one of my students is an engineer who worked on the construction of the Bronx Whitestone Bridge in New York City. He informed me that the original plans for this bridge were very similar to those of the Tacoma Bridge. These plans were hastily redrawn following the collapse of the Tacoma Bridge.

2.6.2 Electrical networks

We now briefly study a simple series circuit, as shown in Figure 1 below. The symbol E represents a source of electromotive force. This may be a battery or a generator which produces a potential difference (or voltage), that causes a current I to flow through the circuit when the switch S is closed. The symbol R represents a resistance to the flow of current such as that produced by a lightbulb or toaster. When current flows through a coil of wire L, a magnetic field is produced which opposes any change in the current through the coil. The change in voltage produced by the coil is proportional to the rate of change of the current, and the constant of propor-

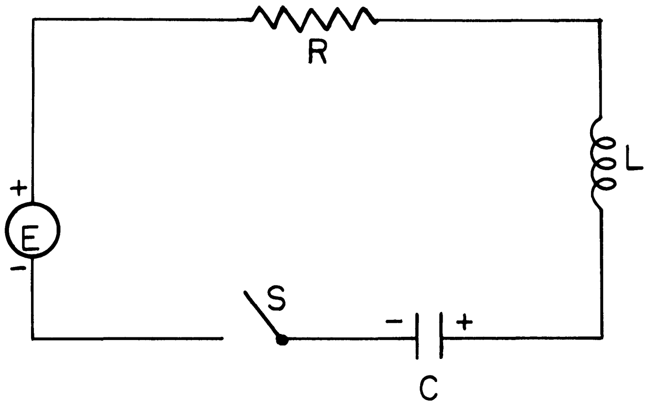

Figure 1. A simple series circuit

tionality is called the inductance L of the coil. A capacitor, or condenser, indicated by C, usually consists of two metal plates separated by a material through which very little current can flow. A capacitor has the effect of reversing the flow of current as one plate or the other becomes charged.

Let $Q(t)$ be the charge on the capacitor at time t. To derive a differential equation which is satisfied by $Q(t)$ we use the following.

Kirchoff's second law: In a closed circuit, the impressed voltage equals the sum of the voltage drops in the rest of the circuit.

Now,

(i) The voltage drop across a resistance of R ohms equals RI (Ohm's law).
(ii) The voltage drop across an inductance of L henrys equals $L(dI/dt)$.
(iii) The voltage drop across a capacitance of C farads equals Q/C.

Hence,

$$E(t) = L\frac{dI}{dt} + RI + \frac{Q}{C},$$

and since $I(t) = dQ(t)/dt$, we see that

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \tag{1}$$

Notice the resemblance of Equation (1) to the equation of a vibrating mass. Among the similarities with mechanical vibrations, electrical circuits also have the property of resonance. Unlike mechanical systems, though, resonance is put to good use in electrical systems. For example, the tuning knob of a radio is used to vary the capacitance in the tuning circuit. In this manner, the resonant frequency (see Exercise 13, Section 2.6) is changed until it agrees with the frequency of one of the incoming radio signals. The amplitude of the current produced by this signal will be much greater than that of all other signals. In this way, the tuning circuit picks out the desired station.

EXERCISES

1. Suppose that a simple series circuit has no resistance and no impressed voltage. Show that the charge Q on the capacitor is periodic in time, with frequency $\omega_0 = \sqrt{1/LC}$. The quantity $\sqrt{1/LC}$ is called the natural frequency of the circuit.

2. Suppose that a simple series circuit consisting of an inductor, a resistor and a capacitor is open, and that there is an initial charge $Q_0 = 10^{-8}$ coulombs on the capacitor. Find the charge on the capacitor and the current flowing in the circuit after the switch is closed for each of the following cases.
   (a) $L = 0.5$ henrys, $C = 10^{-5}$ farads, $R = 1000$ ohms
   (b) $L = 1$ henry, $C = 10^{-4}$ farads, $R = 200$ ohms
   (c) $L = 2$ henrys, $C = 10^{-6}$ farads, $R = 2000$ ohms

3. A simple series circuit has an inductor of 1 henry, a capacitor of 10^{-6} farads, and a resistor of 1000 ohms. The initial charge on the capacitor is zero. If a 12 volt battery is connected to the circuit, and the circuit is closed at $t = 0$, find the charge on the capacitor 1 second later, and the steady state charge.

4. A capacitor of 10^{-3} farads is in series with an electromotive force of 12 volts and an inductor of 1 henry. At $t = 0$, both Q and I are zero.
   (a) Find the natural frequency and period of the electrical oscillations.
   (b) Find the maximum charge on the capacitor, and the maximum current flowing in the circuit.

5. Show that if there is no resistance in a circuit, and the impressed voltage is of the form $E_0 \sin \omega t$, then the charge on the capacitor will become unbounded as $t \to \infty$ if $\omega = \sqrt{1/LC}$. This is the phenomenon of resonance.

6. Consider the differential equation

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = E_0 \cos \omega t. \tag{i}$$

   We find a particular solution $\psi(t)$ of (i) as the real part of a particular solution $\phi(t)$ of

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = E_0 e^{i\omega t}. \tag{ii}$$

   (a) Show that

$$i\omega\phi(t) = \frac{E_0}{R + i\left(\omega L - \dfrac{1}{\omega C}\right)} e^{i\omega t}.$$

   (b) The quantity $Z = R + i(\omega L - 1/\omega C)$ is known as the complex impedance of the circuit. The reciprocal of Z is called the admittance, and the real and imaginary parts of $1/Z$ are called the conductance and susceptance. Determine the admittance, conductance and susceptance.

7. Consider a simple series circuit with given values of L, R and C, and an impressed voltage $E_0 \sin \omega t$. For which value of ω will the steady state current be a maximum?

2.7 A model for the detection of diabetes

Diabetes mellitus is a disease of metabolism which is characterized by too much sugar in the blood and urine. In diabetes, the body is unable to burn off all its sugars, starches, and carbohydrates because of an insufficient supply of insulin. Diabetes is usually diagnosed by means of a glucose tolerance test (GTT). In this test the patient comes to the hospital after an overnight fast and is given a large dose of glucose (sugar in the form in which it usually appears in the bloodstream). During the next three to five hours several measurements are made of the concentration of glucose in the patient's blood, and these measurements are used in the diagnosis of diabetes. A very serious difficulty associated with this method of diagnosis is that there is no universally accepted criterion for interpreting the results of a glucose tolerance test. Three physicians interpreting the results of a GTT may come up with three different diagnoses. In one case recently, in Rhode Island, one physician, after reviewing the results of a GTT, came up with a diagnosis of diabetes. A second physician declared the patient to be normal. To settle the question, the results of the GTT were sent to a specialist in Boston. After examining these results, the specialist concluded that the patient was suffering from a pituitary tumor.

In the mid 1960's Drs. Rosevear and Molnar of the Mayo Clinic and Ackerman and Gatewood of the University of Minnesota discovered a fairly reliable criterion for interpreting the results of a glucose tolerance test. Their discovery arose from a very simple model they developed for the blood glucose regulatory system. Their model is based on the following simple and fairly well known facts of elementary biology.

1. Glucose plays an important role in the metabolism of any vertebrate since it is a source of energy for all tissues and organs. For each individual there is an optimal blood glucose concentration, and any excessive deviation from this optimal concentration leads to severe pathological conditions and potentially death.

2. While blood glucose levels tend to be autoregulatory, they are also influenced and controlled by a wide variety of hormones and other metabolites. Among these are the following.

(i) *Insulin*, a hormone secreted by the β cells of the pancreas. After we eat any carbohydrates, our G.I. tract sends a signal to the pancreas to secrete more insulin. In addition, the glucose in our blood directly stimulates the β cells of the pancreas to secrete insulin. It is generally believed that insulin facilitates tissue uptake of glucose by attaching itself to the impermeable membrane walls, thus allowing glucose to pass through the membranes to the center of the cells, where most of the biological and chemical activity takes place. Without sufficient insulin, the body cannot avail itself of all the energy it needs.

(ii) *Glucagon*, a hormone secreted by the α cells of the pancreas. Any excess glucose is stored in the liver in the form of glycogen. In times of need this glycogen is converted back into glucose. The hormone glucagon increases the rate of breakdown of glycogen into glucose. Evidence collected thus far clearly indicates that hypoglycemia (low blood sugar) and fasting promote the secretion of glucagon while increased blood glucose levels suppress its secretion.

(iii) *Epinephrine* (adrenalin), a hormone secreted by the adrenal medulla. Epinephrine is part of an emergency mechanism to quickly increase the concentration of glucose in the blood in times of extreme hypoglycemia. Like glucagon, epinephrine increases the rate of breakdown of glycogen into glucose. In addition, it directly inhibits glucose uptake by muscle tissue; it acts directly on the pancreas to inhibit insulin secretion; and it aids in the conversion of lactate to glucose in the liver.

(iv) *Glucocorticoids*, hormones such as cortisol which are secreted by the adrenal cortex. Glucocorticoids play an important role in the metabolism of carbohydrates.

(v) *Thyroxin*, a hormone secreted by the thyroid gland. This hormone aids the liver in forming glucose from non-carbohydrate sources such as glycerol, lactate and amino acids.

(vi) *Growth hormone* (somatotropin), a hormone secreted by the anterior pituitary gland. Not only does growth hormone affect glucose levels in a direct manner, but it also tends to "block" insulin. It is believed that growth hormone decreases the sensitivity of muscle and adipose membrane to insulin, thereby reducing the effectiveness of insulin in promoting glucose uptake.

The aim of Ackerman et al was to construct a model which would accurately describe the blood glucose regulatory system during a glucose tolerance test, and in which one or two parameters would yield criteria for distinguishing normal individuals from mild diabetics and pre-diabetics. Their model is a very simplified one, requiring only a limited number of blood samples during a GTT. It centers attention on two concentrations, that of glucose in the blood, labelled G, and that of the net hormonal concentration, labelled H. The latter is interpreted to represent the cumulative effect of all the pertinent hormones. Those hormones such as insulin which decrease blood glucose concentrations are considered to increase H, while those hormones such as cortisol which increase blood glucose concentrations are considered to decrease H. Now there are two reasons why such a simplified model can still provide an accurate description of the blood glucose regulatory system. First, studies have shown that under normal, or close to normal conditions, the interaction of one hormone, namely insulin, with blood glucose so predominates that a simple "lumped parameter model" is quite adequate. Second, evidence indicates that normoglycemia does not depend, necessarily, on the normalcy of each kinetic mechanism of the blood glucose regulatory system. Rather, it depends on the overall performance of the blood glucose regulatory system, and this system is dominated by insulin-glucose interactions.

The basic model is described analytically by the equations

$$\frac{dG}{dt} = F_1(G,H) + J(t) \tag{1}$$

$$\frac{dH}{dt} = F_2(G,H). \tag{2}$$

The dependence of F_1 and F_2 on G and H signify that changes in G and H are determined by the values of both G and H. The function $J(t)$ is the external rate at which the blood glucose concentration is being increased. Now, we assume that G and H have achieved optimal values G_0 and H_0 by the time the fasting patient has arrived at the hospital. This implies that $F_1(G_0,H_0)=0$ and $F_2(G_0,H_0)=0$. Since we are interested here in the deviations of G and H from their optimal values, we make the substitution

$$g = G - G_0, \qquad h = H - H_0.$$

Then,

$$\frac{dg}{dt} = F_1(G_0+g, H_0+h) + J(t),$$

$$\frac{dh}{dt} = F_2(G_0+g, H_0+h).$$

Now, observe that

$$F_1(G_0+g,H_0+h) = F_1(G_0,H_0) + \frac{\partial F_1(G_0,H_0)}{\partial G}g + \frac{\partial F_1(G_0,H_0)}{\partial H}h + e_1$$

and

$$F_2(G_0+g,H_0+h) = F_2(G_0,H_0) + \frac{\partial F_2(G_0,H_0)}{\partial G}g + \frac{\partial F_2(G_0,H_0)}{\partial H}h + e_2$$

where e_1 and e_2 are very small compared to g and h. Hence, assuming that G and H deviate only slightly from G_0 and H_0, and therefore neglecting the terms e_1 and e_2, we see that

$$\frac{dg}{dt} = \frac{\partial F_1(G_0,H_0)}{\partial G}g + \frac{\partial F_1(G_0,H_0)}{\partial H}h + J(t) \tag{3}$$

$$\frac{dh}{dt} = \frac{\partial F_2(G_0,H_0)}{\partial G}g + \frac{\partial F_2(G_0,H_0)}{\partial H}h. \tag{4}$$

Now, there are no means, a priori, of determining the numbers

$$\frac{\partial F_1(G_0,H_0)}{\partial G}, \quad \frac{\partial F_1(G_0,H_0)}{\partial H}, \quad \frac{\partial F_2(G_0,H_0)}{\partial G} \quad \text{and} \quad \frac{\partial F_2(G_0,H_0)}{\partial H}.$$

However, we can determine their signs. Referring to Figure 1, we see that dg/dt is negative for $g>0$ and $h=0$, since the blood glucose concentration

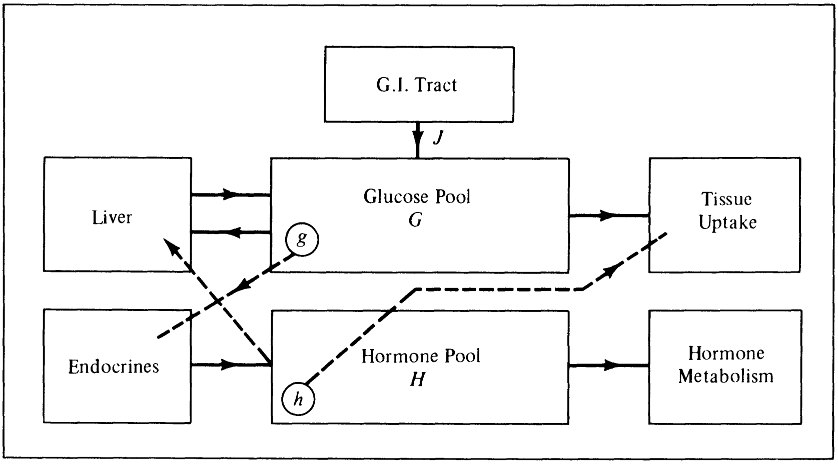

Figure 1. Simplified model of the blood glucose regulatory system

will be decreasing through tissue uptake of glucose and the storing of excess glucose in the liver in the form of glycogen. Consequently $\partial F_1(G_0,H_0)/\partial G$ must be negative. Similarly, $\partial F_1(G_0,H_0)/\partial H$ is negative since a positive value of h tends to decrease blood glucose levels by facilitating tissue uptake of glucose and by increasing the rate at which glucose is converted to glycogen. The number $\partial F_2(G_0,H_0)/\partial G$ must be positive since a positive value of g causes the endocrine glands to secrete those hormones which tend to increase H. Finally, $\partial F_2(G_0,H_0)/\partial H$ must be negative, since the concentration of hormones in the blood decreases through hormone metabolism.

Thus, we can write Equations (3) and (4) in the form

$$\frac{dg}{dt} = -m_1 g - m_2 h + J(t) \tag{5}$$

$$\frac{dh}{dt} = -m_3 h + m_4 g \tag{6}$$

where m_1, m_2, m_3, and m_4 are positive constants. Equations (5) and (6) are two first-order equations for g and h. However, since we only measure the concentration of glucose in the blood, we would like to remove the variable h. This can be accomplished as follows: Differentiating (5) with respect to t gives

$$\frac{d^2g}{dt^2} = -m_1\frac{dg}{dt} - m_2\frac{dh}{dt} + \frac{dJ}{dt}.$$

Substituting for dh/dt from (6) we obtain that

$$\frac{d^2g}{dt^2} = -m_1\frac{dg}{dt} + m_2m_3h - m_2m_4g + \frac{dJ}{dt}. \tag{7}$$

Next, observe from (5) that $m_2h = (-dg/dt) - m_1g + J(t)$. Consequently,

$g(t)$ satisfies the second-order linear differential equation

$$\frac{d^2g}{dt^2}+(m_1+m_3)\frac{dg}{dt}+(m_1m_3+m_2m_4)g=m_3J+\frac{dJ}{dt}.$$

We rewrite this equation in the form

$$\frac{d^2g}{dt^2}+2\alpha\frac{dg}{dt}+\omega_0^2g=S(t) \tag{8}$$

where $\alpha=(m_1+m_3)/2$, $\omega_0^2=m_1m_3+m_2m_4$, and $S(t)=m_3J+dJ/dt$.

Notice that the right-hand side of (8) is identically zero except for the very short time interval in which the glucose load is being ingested. We will learn to deal with such functions in Section 2.12. For our purposes here, let $t=0$ be the time at which the glucose load has been completely ingested. Then, for $t\geqslant 0$, $g(t)$ satisfies the second-order linear homogeneous equation

$$\frac{d^2g}{dt^2}+2\alpha\frac{dg}{dt}+\omega_0^2g=0. \tag{9}$$

This equation has positive coefficients. Hence, by the analysis in Section 2.6, (see also Exercise 8, Section 2.2.2) $g(t)$ approaches zero as t approaches infinity. Thus our model certainly conforms to reality in predicting that the blood glucose concentration tends to return eventually to its optimal concentration.

The solutions $g(t)$ of (9) are of three different types, depending as to whether $\alpha^2-\omega_0^2$ is positive, negative, or zero. These three types, of course, correspond to the overdamped, critically damped and underdamped cases discussed in Section 2.6. We will assume that $\alpha^2-\omega_0^2$ is negative; the other two cases are treated in a similar manner. If $\alpha^2-\omega_0^2<0$, then the characteristic equation of Equation (9) has complex roots. It is easily verified in this case (see Exercise 1) that every solution $g(t)$ of (9) is of the form

$$g(t)=Ae^{-\alpha t}\cos(\omega t-\delta),\qquad \omega^2=\omega_0^2-\alpha^2. \tag{10}$$

Consequently,

$$G(t)=G_0+Ae^{-\alpha t}\cos(\omega t-\delta). \tag{11}$$

Now there are five unknowns G_0, A, α, ω_0, and δ in (11). One way of determining them is as follows. The patient's blood glucose concentration before the glucose load is ingested is G_0. Hence, we can determine G_0 by measuring the patient's blood glucose concentration immediately upon his arrival at the hospital. Next, if we take four additional measurements G_1, G_2, G_3, and G_4 of the patient's blood glucose concentration at times t_1, t_2, t_3, and t_4, then we can determine A, α, ω_0, and δ from the four equations

$$G_j=G_0+Ae^{-\alpha t_j}\cos(\omega t_j-\delta),\qquad j=1,2,3,4.$$

A second, and better method of determining G_0, A, α, ω_0, and δ is to take n measurements $G_1,G_2,\ldots,G_n$ of the patient's blood glucose concentration at times $t_1, t_2, \ldots, t_n$. Typically n is 6 or 7. We then find optimal values for G_0, A, α, ω_0, and δ such that the least square error

$$E = \sum_{j=1}^{n} \left[G_j - G_0 - Ae^{-\alpha t_j}\cos(\omega t_j - \delta) \right]^2$$

is minimized. The problem of minimizing E can be solved on a digital computer, and Ackerman et al (see reference at end of section) provide a complete Fortran program for determining optimal values for G_0, A, α, ω_0, and δ. This method is preferrable to the first method since Equation (11) is only an approximate formula for $G(t)$. Consequently, it is possible to find values G_0, A, α, ω_0, and δ so that Equation (11) is satisfied exactly at four points t_1, t_2, t_3, and t_4 but yields a poor fit to the data at other times. The second method usually offers a better fit to the data on the entire time interval since it involves more measurements.

In numerous experiments, Ackerman et al observed that a slight error in measuring G could produce a very large error in the value of α. Hence, any criterion for diagnosing diabetes that involves the parameter α is unreliable. However, the parameter ω_0, the natural frequency of the system, was relatively insensitive to experimental errors in measuring G. Thus, we may regard a value of ω_0 as the basic descriptor of the response to a glucose tolerance test. For discussion purposes, it is more convenient to use the corresponding natural period $T_0 = 2\pi/\omega_0$. The remarkable fact is that data from a variety of sources indicated that *a value of less than four hours for T_0 indicated normalcy, while appreciably more than four hours implied mild diabetes.*

Remark 1. The usual period between meals in our culture is about 4 hours. This suggests the interesting possibility that sociological factors may also play a role in the blood glucose regulatory system.

Remark 2. We wish to emphasize that the model described above can only be used to diagnose mild diabetes or pre-diabetes, since we have assumed throughout that the deviation g of G from its optimal value G_0 is small. Very large deviations of G from G_0 usually indicate severe diabetes or diabetes insipidus, which is a disorder of the posterior lobe of the pituitary gland.

A serious shortcoming of this simplified model is that it sometimes yields a poor fit to the data in the time period three to five hours after ingestion of the glucose load. This indicates, of course, that variables such as epinephrine and glucagon play an important role in this time period. Thus these variables should be included as separate variables in our model, rather than being lumped together with insulin. In fact, evidence indicates that levels of epinephrine may rise dramatically during the recovery phase of the GTT response, when glucose levels have been lowered below fasting

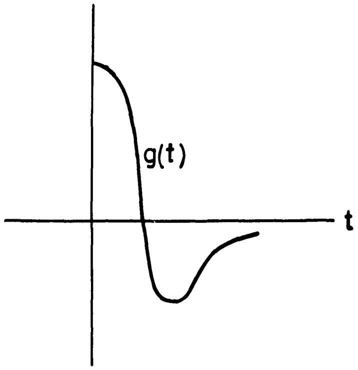

Figure 2. Graph of $g(t)$ if $\alpha^2-\omega_0^2>0$

levels. This can also be seen directly from Equation (9). If $\alpha^2-\omega_0^2>0$, then $g(t)$ may have the form described in Figure 2. Note that $g(t)$ drops very rapidly from a fairly high value to a negative one. It is quite conceivable, therefore, that the body will interpret this as an extreme emergency and thereby secrete a large amount of epinephrine.

Medical researchers have long recognized the need of including epinephrine as a separate variable in any model of the blood glucose regulatory system. However, they were stymied by the fact that there was no reliable method of measuring the concentration of epinephrine in the blood. Thus, they had to assume, for all practical purposes, the the level of epinephrine remained constant during the course of a glucose tolerance test. This author has just been informed that researchers at Rhode Island Hospital have devised an accurate method of measuring the concentration of epinephrine in the blood. Thus we will be able to develop and test more accurate models of the blood glucose regulatory system. Hopefully, this will lead to more reliable criteria for the diagnosis of diabetes.

Reference

E. Ackerman, L. Gatewood, J. Rosevear, and G. Molnar, Blood glucose regulation and diabetes, Chapter 4 in *Concepts and Models of Biomathematics*, F. Heinmets, ed., Marcel Dekker, 1969, 131–156.

Exercises

1. Derive Equation (10).

2. A patient arrives at the hospital after an overnight fast with a blood glucose concentration of 70 mg glucose/100 ml blood (mg glucose/100 ml blood = milligrams of glucose per 100 milliliters of blood). His blood glucose concentration 1 hour, 2 hours, and 3 hours after fully absorbing a large amount of glucose is 95, 65, and 75 mg glucose/100 ml blood, respectively. Show that this patient is normal. *Hint*: In the underdamped case, the time interval between two successive zeros of $G-G_0$ exceeds one half the natural period.

According to a famous diabetologist, the blood glucose concentration of a nondiabetic who has just absorbed a large amount of glucose will be at or below the fasting level in 2 hours or less. Exercises 3 and 4 compare the diagnoses of this diabetologist with those of Ackerman et al.

3. The deviation $g(t)$ of a patient's blood glucose concentration from its optimal concentration satisfies the differential equation $(d^2g/dt^2)+2\alpha(dg/dt)+\alpha^2 g=0$ immediately after he fully absorbs a large amount of glucose. The time t is measured in minutes, so that the units of α are reciprocal minutes. Show that this patient is normal according to Ackerman et al, if $\alpha>\pi/120$ (min), and that this patient is normal according to the famous diabetologist if

$$g'(0) < -\left(\tfrac{1}{120}+\alpha\right)g(0).$$

4. A patient's blood glucose concentration $G(t)$ satisfies the initial-value problem

$$\frac{d^2G}{dt^2}+\frac{1}{20\,(\text{min})}\frac{dG}{dt}+\frac{1}{2500\,(\text{min})^2}G$$

$$=\frac{1}{2500\,(\text{min})^2}75 \text{ mg glucose}/100 \text{ ml blood};$$

$$G(0)=150 \text{ mg glucose}/100 \text{ ml blood},$$

$$G'(0)=-\alpha G(0)/(\text{min}); \qquad \alpha \geqslant \frac{1}{200}\,\frac{1-4e^{18/5}}{1-e^{18/5}}$$

immediately after he fully absorbs a large amount of glucose. This patient's optimal blood glucose concentration is 75 mg glucose/100 ml blood. Show that this patient is a diabetic according to Ackerman et al, but is normal according to the famous diabetologist.

2.8 Series solutions

We return now to the general homogeneous linear second-order equation

$$L[y]=P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0 \tag{1}$$

with $P(t)$ unequal to zero in the interval $\alpha<t<\beta$. It was shown in Section 2.1 that every solution $y(t)$ of (1) can be written in the form $y(t)=c_1y_1(t)+c_2y_2(t)$, where $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of (1). Thus, the problem of finding all solutions of (1) is reduced to the simpler problem of finding just two solutions. In Section 2.2 we handled the special case where P, Q, and R are constants. The next simplest case is when $P(t)$, $Q(t)$, and $R(t)$ are polynomials in t. In this case, the form of the differential equation suggests that we guess a polynomial solution $y(t)$ of (1). If $y(t)$ is a polynomial in t, then the three functions $P(t)y''(t)$, $Q(t)y'(t)$, and $R(t)y(t)$ are again polynomials in t. Thus, in principle, we can determine a polynomial solution $y(t)$ of (1) by setting the sums of the coefficients of like powers of t in the expression $L[y](t)$ equal to zero. We illustrate this method with the following example.

Example 1. Find two linearly independent solutions of the equation

$$L[y]=\frac{d^2y}{dt^2}-2t\frac{dy}{dt}-2y=0. \tag{2}$$

Solution. We will try to find 2 polynomial solutions of (2). Now, it is not obvious, a priori, what the degree of any polynomial solution of (2) should be. Nor is it evident that we will be able to get away with a polynomial of finite degree. Therefore, we set

$$y(t)=a_0+a_1t+a_2t^2+\ldots=\sum_{n=0}^{\infty}a_nt^n.$$

Computing

$$\frac{dy}{dt}=a_1+2a_2t+3a_3t^2+\ldots=\sum_{n=0}^{\infty}na_nt^{n-1}$$

and

$$\frac{d^2y}{dt^2}=2a_2+6a_3t+\ldots=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2},$$

we see that $y(t)$ is a solution of (2) if

$$\begin{aligned}L[y](t)&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}-2t\sum_{n=0}^{\infty}na_nt^{n-1}-2\sum_{n=0}^{\infty}a_nt^n\\&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}-2\sum_{n=0}^{\infty}na_nt^n-2\sum_{n=0}^{\infty}a_nt^n=0.\end{aligned} \tag{3}$$

Our next step is to rewrite the first summation in (3) so that the exponent of the general term is n, instead of $n-2$. This is accomplished by increasing every n underneath the summation sign by 2, and decreasing the lower limit by 2, that is,

$$\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}=\sum_{n=-2}^{\infty}(n+2)(n+1)a_{n+2}t^n.$$

(If you don't believe this, you can verify it by writing out the first few terms in both summations. If you still don't believe this and want a formal proof, set $m=n-2$. When n is zero, m is -2 and when n is infinity, m is infinity. Therefore

$$\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}=\sum_{m=-2}^{\infty}(m+2)(m+1)a_{m+2}t^m,$$

and since m is a dummy variable, we may replace it by n.) Moreover, observe that the contribution to this sum from $n=-2$ and $n=-1$ is zero since the factor $(n+2)(n+1)$ vanishes in both these instances. Hence,

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n$$

and we can rewrite (3) in the form

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n - 2\sum_{n=0}^{\infty} na_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0. \tag{4}$$

Setting the sum of the coefficients of like powers of t in (4) equal to zero gives

$$(n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0$$

so that

$$a_{n+2} = \frac{2(n+1)a_n}{(n+2)(n+1)} = \frac{2a_n}{n+2}. \tag{5}$$

Equation (5) is a recurrence formula for the coefficients $a_0, a_1, a_2, a_3, \ldots$. The coefficient a_n determines the coefficient a_{n+2}. Thus, a_0 determines a_2 through the relation $a_2 = 2a_0/2 = a_0$; a_2, in turn, determines a_4 through the relation $a_4 = 2a_2/(2+2) = a_0/2$; and so on. Similarly, a_1 determines a_3 through the relation $a_3 = 2a_1/(2+1) = 2a_1/3$; a_3, in turn, determines a_5 through the relation $a_5 = 2a_3/(3+2) = 4a_1/3\cdot 5$; and so on. Consequently, all the coefficients are determined uniquely once a_0 and a_1 are prescribed. The values of a_0 and a_1 are completely arbitrary. This is to be expected, though, for if

$$y(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$

then the values of y and y' at $t=0$ are a_0 and a_1 respectively. Thus, the coefficients a_0 and a_1 must be arbitrary until specific initial conditions are imposed on y.

To find two solutions of (2), we choose two different sets of values of a_0 and a_1. The simplest possible choices are (i) $a_0 = 1, a_1 = 0$; (ii) $a_0 = 0$, $a_1 = 1$.

(i) $$a_0 = 1, \qquad a_1 = 0.$$

In this case, all the odd coefficients $a_1, a_3, a_5, \ldots$ are zero since $a_3 = 2a_1/3 = 0$, $a_5 = 2a_3/5 = 0$, and so on. The even coefficients are determined from the relations

$$a_2 = a_0 = 1, \qquad a_4 = \frac{2a_2}{4} = \frac{1}{2}, \qquad a_6 = \frac{2a_4}{6} = \frac{1}{2\cdot 3},$$

and so on. Proceeding inductively, we find that

$$a_{2n} = \frac{1}{2\cdot 3\cdots n} = \frac{1}{n!}.$$

Hence,

$$y_1(t) = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \ldots = e^{t^2}$$

is one solution of (2).

(ii) $$a_0=0, \qquad a_1=1.$$

In this case, all the even coefficients are zero, and the odd coefficients are determined from the relations

$$a_3=\frac{2a_1}{3}=\frac{2}{3}, \qquad a_5=\frac{2a_3}{5}=\frac{2}{5}\frac{2}{3}, \qquad a_7=\frac{2a_5}{7}=\frac{2}{7}\frac{2}{5}\frac{2}{3},$$

and so on. Proceeding inductively, we find that

$$a_{2n+1}=\frac{2^n}{3\cdot 5\cdots(2n+1)}.$$

Thus,

$$y_2(t)=t+\frac{2t^3}{3}+\frac{2^2t^5}{3\cdot 5}+\ldots=\sum_{n=0}^{\infty}\frac{2^n t^{2n+1}}{3\cdot 5\cdots(2n+1)}$$

is a second solution of (2).

Now, observe that $y_1(t)$ and $y_2(t)$ are polynomials of infinite degree, even though the coefficients $P(t)=1$, $Q(t)=-2t$, and $R(t)=-2$ are polynomials of finite degree. Such polynomials are called power series. Before proceeding further, we will briefly review some of the important properties of power series.

1. An infinite series
$$y(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots=\sum_{n=0}^{\infty}a_n(t-t_0)^n \tag{6}$$
is called a power series about $t=t_0$.
2. All power series have an interval of convergence. This means that there exists a nonnegative number ρ such that the infinite series (6) converges for $|t-t_0|<\rho$, and diverges for $|t-t_0|>\rho$. The number ρ is called the radius of convergence of the power series.
3. The power series (6) can be differentiated and integrated term by term, and the resultant series have the same interval of convergence.
4. The simplest method (if it works) for determining the interval of convergence of the power series (6) is the Cauchy ratio test. Suppose that the absolute value of a_{n+1}/a_n approaches a limit λ as n approaches infinity. Then, the power series (6) converges for $|t-t_0|<1/\lambda$, and diverges for $|t-t_0|>1/\lambda$.
5. The product of two power series $\sum_{n=0}^{\infty}a_n(t-t_0)^n$ and $\sum_{n=0}^{\infty}b_n(t-t_0)^n$ is again a power series of the form $\sum_{n=0}^{\infty}c_n(t-t_0)^n$, with $c_n=a_0b_n+a_1b_{n-1}+\ldots+a_nb_0$. The quotient
$$\frac{a_0+a_1t+a_2t^2+\ldots}{b_0+b_1t+b_2t^2+\ldots}$$
of two power series is again a power series, provided that $b_0\neq 0$.

6. Many of the functions $f(t)$ that arise in applications can be expanded in power series; that is, we can find coefficients $a_0, a_1, a_2, \ldots$ so that

$$f(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots=\sum_{n=0}^{\infty} a_n(t-t_0)^n. \tag{7}$$

   Such functions are said to be *analytic* at $t=t_0$, and the series (7) is called the Taylor series of f about $t=t_0$. It can easily be shown that if f admits such an expansion, then, of necessity, $a_n=f^{(n)}(t_0)/n!$, where $f^{(n)}(t)=d^nf(t)/dt^n$.
7. The interval of convergence of the Taylor series of a function $f(t)$, about t_0, can be determined directly through the Cauchy ratio test and other similar methods, or indirectly, through the following theorem of complex analysis.

Theorem 6. *Let the variable t assume complex values, and let z_0 be the point closest to t_0 at which f or one of its derivatives fails to exist. Compute the distance ρ, in the complex plane, between t_0 and z_0. Then, the Taylor series of f about t_0 converges for $|t-t_0|<\rho$, and diverges for $|t-t_0|>\rho$.*

As an illustration of Theorem 6, consider the function $f(t)=1/(1+t^2)$. The Taylor series of f about $t=0$ is

$$\frac{1}{1+t^2}=1-t^2+t^4-t^6+\ldots,$$

and this series has radius of convergence one. Although the function $(1+t^2)^{-1}$ is perfectly well behaved for t real, it goes to infinity when $t=\pm i$, and the distance of each of these points from the origin is one.

A second application of Theorem 6 is that the radius of convergence of the Taylor series about $t=0$ of the quotient of two polynomials $a(t)$ and $b(t)$, is the magnitude of the smallest zero of $b(t)$.

At this point we make the important observation that it really wasn't necessary to assume that the functions $P(t)$, $Q(t)$, and $R(t)$ in (1) are polynomials. The method used to solve Example 1 should also be applicable to the more general differential equation

$$L[y]=P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0$$

where $P(t)$, $Q(t)$, and $R(t)$ are power series about t_0. (Of course, we would expect the algebra to be much more cumbersome in this case.) If

$$P(t)=p_0+p_1(t-t_0)+\ldots,\qquad Q(t)=q_0+q_1(t-t_0)+\ldots,$$
$$R(t)=r_0+r_1(t-t_0)+\ldots$$

and $y(t)=a_0+a_1(t-t_0)+\ldots$, then $L[y](t)$ will be the sum of three power series about $t=t_0$. Consequently, we should be able to find a recurrence formula for the coefficients a_n by setting the sum of the coefficients of like powers of t in the expression $L[y](t)$ equal to zero. This is the content of the following theorem, which we quote without proof.

Theorem 7. *Let the functions $Q(t)/P(t)$ and $R(t)/P(t)$ have convergent Taylor series expansions about $t=t_0$, for $|t-t_0|<\rho$. Then, every solution $y(t)$ of the differential equation*

$$P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0 \tag{8}$$

is analytic at $t=t_0$, and the radius of convergence of its Taylor series expansion about $t=t_0$ is at least ρ. The coefficients $a_2,a_3,\ldots$, in the Taylor series expansion

$$y(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots \tag{9}$$

are determined by plugging the series (9) *into the differential equation* (8) *and setting the sum of the coefficients of like powers of t in this expression equal to zero.*

Remark. The interval of convergence of the Taylor series expansion of any solution $y(t)$ of (8) is determined, usually, by the interval of convergence of the power series $Q(t)/P(t)$ and $R(t)/P(t)$, rather than by the interval of convergence of the power series $P(t)$, $Q(t)$, and $R(t)$. This is because the differential equation (8) must be put in the standard form

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0$$

whenever we examine questions of existence and uniqueness.

Example 2.
(a) Find two linearly independent solutions of

$$L[y]=\frac{d^2y}{dt^2}+\frac{3t}{1+t^2}\frac{dy}{dt}+\frac{1}{1+t^2}y=0. \tag{10}$$

(b) Find the solution $y(t)$ of (10) which satisfies the initial conditions $y(0)=2$, $y'(0)=3$.

Solution.
(a) The *wrong* way to do this problem is to expand the functions $3t/(1+t^2)$ and $1/(1+t^2)$ in power series about $t=0$. The right way to do this problem is to multiply both sides of (10) by $1+t^2$ to obtain the equivalent equation

$$L[y]=(1+t^2)\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+y=0.$$

We do the problem this way because the algebra is much less cumbersome when the coefficients of the differential equation (8) are polynomials than when they are power series. Setting $y(t)=\sum_{n=0}^{\infty}a_nt^n$, we compute

$$\begin{aligned}L[y](t)&=(1+t^2)\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+3t\sum_{n=0}^{\infty}na_nt^{n-1}+\sum_{n=0}^{\infty}a_nt^n\\&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}\left[n(n-1)+3n+1\right]a_nt^n\\&=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)^2a_nt^n.\end{aligned}$$

Setting the sum of the coefficients of like powers of t equal to zero gives $(n+2)(n+1)a_{n+2}+(n+1)^2a_n=0$. Hence,

$$a_{n+2}=-\frac{(n+1)^2a_n}{(n+2)(n+1)}=-\frac{(n+1)a_n}{n+2}. \tag{11}$$

Equation (11) is a recurrence formula for the coefficients $a_2,a_3,\dots$ in terms of a_0 and a_1. To find two linearly independent solutions of (10), we choose the two simplest cases (i) $a_0=1$, $a_1=0$; and (ii) $a_0=0$, $a_1=1$.

(i) $$a_0=1,\qquad a_1=0.$$

In this case, all the odd coefficients are zero since $a_3=-2a_1/3=0$, $a_5=-4a_3/5=0$, and so on. The even coefficients are determined from the relations

$$a_2=-\frac{a_0}{2}=-\frac{1}{2},\qquad a_4=-\frac{3a_2}{4}=\frac{1\cdot 3}{2\cdot 4},\qquad a_6=-\frac{5a_4}{6}=-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$$

and so on. Proceeding inductively, we find that

$$a_{2n}=(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}=(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^nn!}.$$

Thus,

$$y_1(t)=1-\frac{t^2}{2}+\frac{1\cdot 3}{2\cdot 4}t^4+\dots=\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^nn!}t^{2n} \tag{12}$$

is one solution of (10). The ratio of the $(n+1)$st term to the nth term of $y_1(t)$ is

$$-\frac{1\cdot 3\cdots(2n-1)(2n+1)t^{2n+2}}{2^{n+1}(n+1)!}\times\frac{2^nn!}{1\cdot 3\cdots(2n-1)t^{2n}}=\frac{-(2n+1)t^2}{2(n+1)},$$

and the absolute value of this quantity approaches t^2 as n approaches infinity. Hence, by the Cauchy ratio test, the infinite series (12) converges for $|t|<1$, and diverges for $|t|>1$.

(ii) $$a_0=0,\qquad a_1=1.$$

In this case, all the even coefficients are zero, and the odd coefficients are determined from the relations

$$a_3 = -\frac{2a_1}{3} = -\frac{2}{3}, \qquad a_5 = -\frac{4a_3}{5} = \frac{2\cdot 4}{3\cdot 5}, \qquad a_7 = -\frac{6a_5}{7} = -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7},$$

and so on. Proceeding inductively, we find that

$$a_{2n+1} = (-1)^n \frac{2\cdot 4\cdots 2n}{3\cdot 5\cdots(2n+1)} = \frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)}.$$

Thus,

$$y_2(t) = t - \frac{2}{3}t^3 + \frac{2\cdot 4}{3\cdot 5}t^5 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)} t^{2n+1} \tag{13}$$

is a second solution of (10), and it is easily verified that this solution, too, converges for $|t|<1$, and diverges for $|t|>1$. This, of course, is not very surprising, since the Taylor series expansions about $t=0$ of the functions $3t/(1+t^2)$ and $1/(1+t^2)$ only converge for $|t|<1$.

(b) The solution $y_1(t)$ satisfies the initial conditions $y(0)=1, y'(0)=0$, while $y_2(t)$ satisfies the initial conditions $y(0)=0, y'(0)=1$. Hence $y(t)=2y_1(t)+3y_2(t)$.

Example 3. Solve the initial-value problem

$$L[y] = \frac{d^2y}{dt^2} + t^2\frac{dy}{dt} + 2ty = 0; \qquad y(0)=1, \quad y'(0)=0.$$

Solution. Setting $y(t) = \sum_{n=0}^{\infty} a_n t^n$, we compute

$$\begin{aligned} L[y](t) &= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + t^2 \sum_{n=0}^{\infty} n a_n t^{n-1} + 2t \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} n a_n t^{n+1} + 2\sum_{n=0}^{\infty} a_n t^{n+1} \\ &= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} (n+2)a_n t^{n+1}. \end{aligned}$$

Our next step is to rewrite the first summation so that the exponent of the general term is $n+1$ instead of $n-2$. This is accomplished by increasing every n underneath the summation sign by 3, and decreasing the lower limit by 3; that is,

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} &= \sum_{n=-3}^{\infty} (n+3)(n+2)a_{n+3} t^{n+1} \\ &= \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3} t^{n+1}. \end{aligned}$$

Therefore,

$$L[y](t)=\sum_{n=-1}^{\infty}(n+3)(n+2)a_{n+3}t^{n+1}+\sum_{n=0}^{\infty}(n+2)a_nt^{n+1}$$
$$=2a_2+\sum_{n=0}^{\infty}(n+3)(n+2)a_{n+3}t^{n+1}+\sum_{n=0}^{\infty}(n+2)a_nt^{n+1}.$$

Setting the sums of the coefficients of like powers of t equal to zero gives

$$2a_2=0,\quad \text{and}\quad (n+3)(n+2)a_{n+3}+(n+2)a_n=0;\qquad n=0,1,2,\ldots$$

Consequently,

$$a_2=0,\quad \text{and}\quad a_{n+3}=-\frac{a_n}{n+3};\qquad n\geqslant 0. \tag{14}$$

The recurrence formula (14) determines a_3 in terms of a_0, a_4 in terms of a_1, a_5 in terms of a_2, and so on. Since $a_2=0$, we see that $a_5, a_8, a_{11},\ldots$ are all zero, regardless of the values of a_0 and a_1. To satify the initial conditions, we set $a_0=1$ and $a_1=0$. Then, from (14), $a_4, a_7, a_{10},\ldots$ are all zero, while

$$a_3=-\frac{a_0}{3}=-\frac{1}{3},\qquad a_6=-\frac{a_3}{6}=\frac{1}{3\cdot 6},\qquad a_9=-\frac{a_6}{9}=-\frac{1}{3\cdot 6\cdot 9}$$

and so on. Proceeding inductively, we find that

$$a_{3n}=\frac{(-1)^n}{3\cdot 6\cdots 3n}=\frac{(-1)^n}{3^n 1\cdot 2\cdots n}=\frac{(-1)^n}{3^n n!}.$$

Hence,

$$y(t)=1-\frac{t^3}{3}+\frac{t^6}{3\cdot 6}-\frac{t^9}{3\cdot 6\cdot 9}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n t^{3n}}{3^n n!}.$$

By Theorem 7, this series converges for all t, since the power series t^2 and $2t$ obviously converge for all t. (We could also verify this directly using the Cauchy ratio test.)

Example 4. Solve the initial-value problem

$$L[y]=(t^2-2t)\frac{d^2y}{dt^2}+5(t-1)\frac{dy}{dt}+3y=0;\qquad y(1)=7,\ \ y'(1)=3. \tag{15}$$

Solution. Since the initial conditions are given at $t=1$, we will express the coefficients of the differential equation (15) as polynomials in $(t-1)$, and then we will find $y(t)$ as a power series centered about $t=1$. To this end, observe that

$$t^2-2t=t(t-2)=\left[(t-1)+1\right]\left[(t-1)-1\right]=(t-1)^2-1.$$

Hence, the differential equation (15) can be written in the form

$$L[y]=\left[(t-1)^2-1\right]\frac{d^2y}{dt^2}+5(t-1)\frac{dy}{dt}+3y=0.$$

Setting $y(t)=\sum_{n=0}^{\infty}a_n(t-1)^n$, we compute

$$\begin{aligned}L[y](t)&=\left[(t-1)^2-1\right]\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^{n-2}\\&\quad+5(t-1)\sum_{n=0}^{\infty}na_n(t-1)^{n-1}+3\sum_{n=0}^{\infty}a_n(t-1)^n\\&=-\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^{n-2}\\&\quad+\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^n+\sum_{n=0}^{\infty}(5n+3)a_n(t-1)^n\\&=-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(t-1)^n+\sum_{n=0}^{\infty}(n^2+4n+3)a_n(t-1)^n.\end{aligned}$$

Setting the sums of the coefficients of like powers of t equal to zero gives $-(n+2)(n+1)a_{n+2}+(n^2+4n+3)a_n=0$, so that

$$a_{n+2}=\frac{n^2+4n+3}{(n+2)(n+1)}a_n=\frac{n+3}{n+2}a_n,\qquad n\geqslant 0. \tag{16}$$

To satisfy the initial conditions, we set $a_0=7$ and $a_1=3$. Then, from (16),

$$\begin{aligned}a_2&=\frac{3}{2}a_0=\frac{3}{2}\cdot 7, & a_4&=\frac{5}{4}a_2=\frac{5\cdot 3}{4\cdot 2}\cdot 7, & a_6&=\frac{7}{6}a_4=\frac{7\cdot 5\cdot 3}{6\cdot 4\cdot 2}\cdot 7,\ldots\\ a_3&=\frac{4}{3}a_1=\frac{4}{3}\cdot 3, & a_5&=\frac{6}{5}a_3=\frac{6\cdot 4}{5\cdot 3}\cdot 3, & a_7&=\frac{8}{7}a_5=\frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3}\cdot 3,\ldots\end{aligned}$$

and so on. Proceeding inductively, we find that

$$a_{2n}=\frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)}\cdot 7\quad\text{and}\quad a_{2n+1}=\frac{4\cdot 6\cdots(2n+2)}{3\cdot 5\cdots(2n+1)}\cdot 3\qquad(\text{for } n\geqslant 1).$$

Hence,

$$\begin{aligned}y(t)&=7+3(t-1)+\frac{3}{2}\cdot 7(t-1)^2+\frac{4}{3}\cdot 3(t-1)^3+\ldots\\&=7+7\sum_{n=1}^{\infty}\frac{3\cdot 5\cdots(2n+1)(t-1)^{2n}}{2^n n!}+3(t-1)+3\sum_{n=1}^{\infty}\frac{2^n(n+1)!(t-1)^{2n+1}}{3\cdot 5\cdots(2n+1)}.\end{aligned}$$

Example 5. Solve the initial-value problem

$$L[y]=(1-t)\frac{d^2y}{dt^2}+\frac{dy}{dt}+(1-t)y=0;\qquad y(0)=1,\ \ y'(0)=1.$$

Solution. Setting $y(t)=\sum_{n=0}^{\infty}a_nt^n$, we compute

$$
\begin{aligned}
L[y](t) &= (1-t)\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} \\
&\quad + \sum_{n=0}^{\infty} n a_n t^{n-1} + (1-t)\sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n t^{n-1} \\
&\quad + \sum_{n=0}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} n(n-2)a_n t^{n-1} \\
&\quad + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} (n+1)(n-1)a_{n+1} t^n \\
&\quad + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=1}^{\infty} a_{n-1} t^n \\
&= 2a_2 + a_1 + a_0 \\
&\quad + \sum_{n=1}^{\infty} \left\{(n+2)(n+1)a_{n+2} - (n+1)(n-1)a_{n+1} + a_n - a_{n-1}\right\} t^n.
\end{aligned}
$$

Setting the coefficients of each power of t equal to zero gives

$$
a_2 = -\frac{a_1 + a_0}{2} \quad \text{and} \quad a_{n+2} = \frac{(n+1)(n-1)a_{n+1} - a_n + a_{n-1}}{(n+2)(n+1)}, \qquad n \geq 1. \tag{17}
$$

To satisfy the initial conditions, we set $a_0 = 1$ and $a_1 = 1$. Then, from (17),

$$
a_2 = -1, \qquad a_3 = \frac{-a_1 + a_0}{6} = 0, \qquad a_4 = \frac{3a_3 - a_2 + a_1}{12} = \frac{1}{6},
$$

$$
a_5 = \frac{8a_4 - a_3 + a_2}{20} = \frac{1}{60}, \qquad a_6 = \frac{15a_5 - a_4 + a_3}{30} = \frac{1}{360}
$$

and so on. Unfortunately, though, we cannot discern a general pattern for the coefficients a_n as we did in the previous examples. (This is because the coefficient a_{n+2} depends on the values of a_{n+1}, a_n, and a_{n-1}, while in our previous examples, the coefficient a_{n+2} depended on only one of its predecessors.) This is not a serious problem, though, for we can find the coefficients a_n quite easily with the aid of a digital computer. Sample Pascal and Fortran programs to compute the coefficients $a_2, \ldots, a_n$ in terms of a_0 and a_1, and to evaluate the "approximate" solution

$$
y(t) \cong a_0 + a_1 t + \ldots + a_n t^n
$$

at any point t are given below. These programs have variable values for a_0 and a_1, so they can also be used to solve the more general initial-value problem

$$(1-t)\frac{d^2y}{dt^2}+\frac{dy}{dt}+(1-t)y=0; \qquad y(0)=a_0, \quad y'(0)=a_1.$$

Pascal Program

```
Program Series (input, output);

var
  A: array[0..199] of real;
  T, sum: real;
  k, N: integer;

begin
  readln(A[0], A[1], T, N);
  page;
  A[2] := -0.5*(A[1]+A[0]);
  sum := A[0]+A[1]*T+A[2]*T*T;
  for k := 1 to N-2 do
    begin
      A[k+2] := ((k+1)*(k-1)*A[k+1]-A[k]+A[k-1])
                /((k+1)*(k+2));
      sum := sum+A[k+2]*exp((k+2)*ln(T));
    end;
  writeln('For N = ' ,N:3, ' and T = ' ,T:6:4);
  writeln('the sum is: ', sum:11:9);
end.
```

Fortran Program

```
      DIMENSION A(200)
      READ (5, 10) A0, A(1), T, N
10    FORMAT (3F15.8, I5)
      A(2)= -0.5*(A(1)+A0)
      A(3)=(A0-A(1))/2.*3.
      SUM=A0+A(1)*T+A(2)*T**2+A(3)*T**3
      NA=N-2
      D0 20 K=2,NA
      A(K+2)=(A(K-1)-A(K)+(K+1.)*(K-1.)*
     1 A(K+1))/(K+1.)*(K+2.)
      SUM=SUM+A(K+2)*T**(K+2)
20    CONTINUE
      WRITE (6,30) N,T,SUM
30    FORMAT (1H1, 'FOR N=', I3, ', AND T=',F10.4/1H, 'THE
     1 SUM IS', F20.9)
      CALL EXIT
      END
```

See also C Program 14 in Appendix C for a sample C program.
Setting $A[0]=1$, $A[1]=1$, ($A(1)=1$ for the Fortran program), $T=0.5$, and $N=20$ in these programs gives

$$y\left(\tfrac{1}{2}\right) \cong a_0 + a_1\left(\tfrac{1}{2}\right) + \ldots + a_{20}\left(\tfrac{1}{2}\right)^{20} = 1.26104174.$$

This result is correct to eight significant decimal places, since any larger value of N yields the same result.

Exercises

Find the general solution of each of the following equations.

1. $y'' + ty' + y = 0$

2. $y'' - ty = 0$

3. $(2+t^2)y'' - ty' - 3y = 0$

4. $y'' - t^3 y = 0$

Solve each of the following initial-value problems.

5. $t(2-t)y'' - 6(t-1)y' - 4y = 0$; $y(1)=1$, $y'(1)=0$

6. $y'' + t^2 y = 0$; $y(0)=2$, $y'(0)=-1$

7. $y'' - t^3 y = 0$; $y(0)=0$, $y'(0)=-2$

8. $y'' + (t^2+2t+1)y' - (4+4t)y = 0$; $y(-1)=0$, $y'(-1)=1$

9. The equation $y'' - 2ty' + \lambda y = 0$, λ constant, is known as the Hermite differential equation, and it appears in many areas of mathematics and physics.
(a) Find 2 linearly independent solutions of the Hermite equation.
(b) Show that the Hermite equation has a polynomial solution of degree n if $\lambda = 2n$. This polynomial, when properly normalized; that is, when multiplied by a suitable constant, is known as the Hermite polynomial $H_n(t)$.

10. The equation $(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0$, α constant, is known as the Legendre differential equation, and it appears in many areas of mathematics and physics.
(a) Find 2 linearly independent solutions of the Legendre equation.
(b) Show that the Legendre differential equation has a polynomial solution of degree n if $\alpha = n$.
(c) The Legendre polynomial $P_n(t)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ which satisfies the condition $P_n(1) = 1$. Find $P_0(t)$, $P_1(t)$, $P_2(t)$, and $P_3(t)$.

11. The equation $(1-t^2)y'' - ty' + \alpha^2 y = 0$, α constant, is known as the Tchebycheff differential equation, and it appears in many areas of mathematics and physics.
(a) Find 2 linearly independent solutions of the Tchebycheff equation.
(b) Show that the Tchebycheff equation has a polynomial solution of degree n if $\alpha = n$. These polynomials, when properly normalized, are called the Tchebycheff polynomials.

12. (a) Find 2 linearly independent solutions of

$$y'' + t^3 y' + 3t^2 y = 0.$$

(b) Find the first 5 terms in the Taylor series expansion about $t = 0$ of the solution $y(t)$ of the initial-value problem

$$y'' + t^3 y' + 3t^2 y = e^t; \qquad y(0) = 0, \quad y'(0) = 0.$$

In each of Problems 13–17, (a) Find the first 5 terms in the Taylor series expansion $\sum_{n=0}^{\infty} a_n t^n$ of the solution $y(t)$ of the given initial-value problem. (b) Write a computer program to find the first $N+1$ coefficients $a_0, a_1, \ldots, a_N$, and to evaluate the polynomial $a_0 + a_1 t + \ldots + a_N t^N$. Then, obtain an approximation of $y(\frac{1}{2})$ by evaluating $\sum_{n=0}^{20} a_n (\frac{1}{2})^n$.

13. $(1-t)y'' + ty' + y = 0; \quad y(0) = 1, \ y'(0) = 0$

14. $y'' + y' + ty = 0; \quad y(0) = -1, \ y'(0) = 2$

15. $y'' + ty' + e^t y = 0; \quad y(0) = 1, \ y'(0) = 0$

16. $y'' + y' + e^t y = 0; \quad y(0) = 0, \ y'(0) = -1$

17. $y'' + y' + e^{-t} y = 0; \quad y(0) = 3, \ y'(0) = 5$

2.8.1 Singular points, Euler equations

The differential equation

$$L[y] = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \tag{1}$$

is said to be singular at $t = t_0$ if $P(t_0) = 0$. Solutions of (1) frequently become very large, or oscillate very rapidly, in a neighborhood of the singular point t_0. Thus, solutions of (1) may not even be continuous, let along analytic at t_0, and the method of power series solution will fail to work, in general.

Our goal is to find a class of singular equations which we can solve for t near t_0. To this end we will first study a very simple equation, known as Euler's equation, which is singular, but easily solvable. We will then use the Euler equation to motivate a more general class of singular equations which are also solvable in the vicinity of the singular point.

Definition. The differential equation

$$L[y] = t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0. \tag{2}$$

where α and β are constants is known as Euler's equation.

We will assume at first, for simplicity, that $t>0$. Observe that t^2y'' and ty' are both multiples of t^r if $y=t^r$. This suggests that we try $y=t^r$ as a solution of (2). Computing

$$\frac{d}{dt}t^r = rt^{r-1} \quad \text{and} \quad \frac{d^2}{dt^2}t^r = r(r-1)t^{r-2}$$

we see that

$$\begin{aligned} L[t^r] &= r(r-1)t^r + \alpha r t^r + \beta t^r \\ &= [r(r-1)+\alpha r+\beta]t^r \\ &= F(r)t^r \end{aligned} \tag{3}$$

where

$$\begin{aligned} F(r) &= r(r-1)+\alpha r+\beta \\ &= r^2+(\alpha-1)r+\beta. \end{aligned} \tag{4}$$

Hence, $y=t^r$ is a solution of (2) if, and only if, r is a solution of the quadratic equation

$$r^2+(\alpha-1)r+\beta=0. \tag{5}$$

The solutions r_1, r_2 of (5) are

$$r_1 = -\tfrac{1}{2}\left[(\alpha-1)+\sqrt{(\alpha-1)^2-4\beta}\,\right]$$

$$r_2 = -\tfrac{1}{2}\left[(\alpha-1)-\sqrt{(\alpha-1)^2-4\beta}\,\right].$$

Just as in the case of constant coefficients, we must examine separately the cases where $(\alpha-1)^2-4\beta$ is positive, negative, and zero.

Case 1. $(\alpha-1)^2-4\beta>0$. In this case Equation (5) has two real, unequal roots, and thus (2) has two solutions of the form $y_1(t)=t^{r_1}$, $y_2(t)=t^{r_2}$. Clearly, t^{r_1} and t^{r_2} are independent if $r_1\neq r_2$. Thus the general solution of (2) is (for $t>0$)

$$y(t)=c_1t^{r_1}+c_2t^{r_2}.$$

Example 1. Find the general solution of

$$L[y]=t^2\frac{d^2y}{dt^2}+4t\frac{dy}{dt}+2y=0, \qquad t>0. \tag{6}$$

Solution. Substituting $y=t^r$ in (6) gives

$$\begin{aligned} L[t^r] &= r(r-1)t^r+4rt^r+2t^r \\ &= [r(r-1)+4r+2]t^r \\ &= (r^2+3r+2)t^r \\ &= (r+1)(r+2)t^r \end{aligned}$$

Hence $r_1 = -1$, $r_2 = -2$ and

$$y(t) = c_1 t^{-1} + c_2 t^{-2} = \frac{c_1}{t} + \frac{c_2}{t^2}$$

is the general solution of (6).

Case 2. $(\alpha - 1)^2 - 4\beta = 0$. In this case

$$r_1 = r_2 = \frac{1-\alpha}{2}$$

and we have only one solution $y = t^{r_1}$ of (2). A second solution (see Exercise 11) can be found by the method of reduction of order. However, we would like to present here an alternate method of obtaining y_2 which will generalize very nicely in Section 2.8.3. Observe that $F(r) = (r - r_1)^2$ in the case of equal roots. Hence

$$L[t^r] = (r - r_1)^2 t^r. \tag{7}$$

Taking partial derivatives of both sides of (7) with respect to r gives

$$\frac{\partial}{\partial r} L[t^r] = L\left[\frac{\partial}{\partial r} t^r\right] = \frac{\partial}{\partial r}\left[(r - r_1)^2 t^r\right].$$

Since $\partial(t^r)/\partial r = t^r \ln t$, we see that

$$L[t^r \ln t] = (r - r_1)^2 t^r \ln t + 2(r - r_1)t^r. \tag{8}$$

The right hand side of (8) vanishes when $r = r_1$. Hence,

$$L[t^{r_1} \ln t] = 0$$

which implies that $y_2(t) = t^{r_1} \ln t$ is a second solution of (2). Since t^{r_1} and $t^{r_1} \ln t$ are obviously linearly independent, the general solution of (2) in the case of equal roots is

$$y(t) = (c_1 + c_2 \ln t)t^r, \qquad t > 0.$$

Example 2. Find the general solution of

$$L[y] = t^2 \frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0, \qquad t > 0. \tag{9}$$

Solution. Substituting $y = t^r$ in (9) gives

$$\begin{aligned} L[t^r] &= r(r-1)t^r - 5rt^r + 9t^r \\ &= [r(r-1) - 5r + 9]t^r \\ &= (r^2 - 6r + 9)t^r \\ &= (r-3)^2 t^r. \end{aligned}$$

The equation $(r-3)^2=0$ has $r=3$ as a double root. Hence,

$$y_1(t)=t^3,\ y_2(t)=t^3\ln t$$

and the general solution of (9) is

$$y(t)=(c_1+c_2\ln t)t^3, \qquad t>0.$$

Case 3. $(\alpha-1)^2-4\beta<0$. In this case.

$$r_1=\lambda+i\mu \quad \text{and} \quad r_2=\lambda-i\mu$$

with

$$\lambda=\frac{1-\alpha}{2},\ \mu=\frac{\left[4\beta-(\alpha-1)^2\right]^{1/2}}{2} \tag{10}$$

are complex roots. Hence,

$$\begin{aligned}\phi(t)&=t^{\lambda+i\mu}=t^{\lambda}t^{i\mu}\\&=t^{\lambda}\left(e^{\ln t}\right)^{i\mu}=t^{\lambda}e^{i\mu\ln t}\\&=t^{\lambda}\left[\cos(\mu\ln t)+i\sin(\mu\ln t)\right]\end{aligned}$$

is a complex-valued solution of (2). But then (see Section 2.2.1)

$$y_1(t)=\text{Re}\{\phi(t)\}=t^{\lambda}\cos(\mu\ln t)$$

and

$$y_2(t)=\text{Im}\{\phi(t)\}=t^{\lambda}\sin(\mu\ln t)$$

are two real-valued independent solutions of (2). Hence, the general solution of (2), in the case of complex roots, is

$$y(t)=t^{\lambda}\left[c_1\cos(\mu\ln t)+c_2\sin(\mu\ln t)\right]$$

with λ and μ given by (10).

Example 3. Find the general solution of the equation

$$L[y]=t^2y''-5ty'+25y=0, \qquad t>0. \tag{11}$$

Solution. Substituting $y=t^r$ in (11) gives

$$\begin{aligned}L[t^r]&=r(r-1)t^r-5rt^r+25t^r\\&=[r(r-1)-5r+25]t^r\\&=[r^2-6r+25]t^r\end{aligned}$$

The roots of the equation $r^2-6r+25=0$ are

$$\frac{6\pm\sqrt{36-100}}{2}=3\pm4i$$

so that

$$\begin{aligned}\phi(t) &= t^{3+4i} = t^3 t^{4i}\\ &= t^3 e^{(\ln t)4i} = t^3 e^{i(4\ln t)}\\ &= t^3[\cos(4\ln t) + i\sin(4\ln t)]\end{aligned}$$

is a complex-valued solution of (11). Consequently,

$$y_1(t) = \text{Re}\{\phi(t)\} = t^3\cos(4\ln t)$$

and

$$y_2(t) = \text{Im}\{\phi(t)\} = t^3\sin(4\ln t)$$

are two independent solutions of (11), and the general solution is

$$y(t) = t^3[c_1\cos(4\ln t) + c_2\sin(4\ln t)], \qquad t > 0.$$

Let us now return to the case $t < 0$. One difficulty is that t^r may not be defined if t is negative. For example, $(-1)^{1/2}$ equals i, which is imaginary. A second difficulty is that $\ln t$ is not defined for negative t. We overcome both of these difficulties with the following clever change of variable. Set

$$t = -x, \qquad x > 0,$$

and let $y = u(x)$, $x > 0$. Observe, from the chain rule, that

$$\frac{dy}{dt} = \frac{du}{dx}\frac{dx}{dt} = -\frac{du}{dx}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{du}{dx}\right) = \frac{d}{dx}\left(-\frac{du}{dx}\right)\frac{dx}{dt} = \frac{d^2u}{dx^2}.$$

Thus, we can rewrite (2) in the form

$$(-x)^2\frac{d^2u}{dx^2} + \alpha(-x)\left(-\frac{du}{dx}\right) + \beta u = 0$$

or

$$x^2\frac{d^2u}{dx^2} + \alpha x\frac{du}{dx} + \beta u = 0, \qquad x > 0 \tag{12}$$

But Equation (12) is exactly the same as (2) with t replaced by x and y replaced by u. Hence, Equation (12) has solutions of the form

$$u(x) = \begin{cases} c_1 x^{r_1} + c_2 x^{r_2} \\ (c_1 + c_2 \ln x)x^{r_1} \\ [c_1\cos(\mu\ln x) + c_2\sin(\mu\ln x)]x^{\lambda} \end{cases} \tag{13}$$

depending on whether $(\alpha-1)^2-4\beta$ is positive, zero, or negative. Observe now that

$$x=-t=|t|$$

for negative t. Thus, for negative t, the solutions of (2) have one of the forms

$$\begin{cases} c_1|t|^{r_1}+c_2|t|^{r_2} \\ [c_1+c_2\ln|t|]|t|^{r_1} \\ [c_1\cos(\mu\ln|t|)+c_2\sin(\mu\ln|t|)]|t|^{\lambda} \end{cases}$$

Remark. The equation

$$(t-t_0)^2\frac{d^2y}{dt^2}+\alpha(t-t_0)\frac{dy}{dt}+\beta y=0 \tag{14}$$

is also an Euler equation, with a singularity at $t=t_0$ instead of $t=0$. In this case we look for solutions of the form $(t-t_0)^r$. Alternately, we can reduce (14) to (2) by the change of variable $x=t-t_0$.

EXERCISES

In Problems 1–8, find the general solution of the given equation.

1. $t^2y''+5ty'-5y=0$

2. $2t^2y''+3ty'-y=0$

3. $(t-1)^2y''-2(t-1)y'+2y=0$

4. $t^2y''+3ty'+y=0$

5. $t^2y''-ty'+y=0$

6. $(t-2)^2y''+5(t-2)y'+4y=0$

7. $t^2y''+ty'+y=0$

8. $t^2y''+3ty'+2y=0$

9. Solve the initial-value problem

$$t^2y''-ty'-2y=0; \quad y(1)=0, \quad y'(1)=1$$

on the interval $0<t<\infty$.

10. Solve the initial-value problem

$$t^2y''-3ty'+4y=0; \quad y(1)=1, \quad y'(1)=0$$

on the interval $0<t<\infty$.

11. Use the method of reduction of order to show that $y_2(t)=t^{r_1}\ln t$ in the case of equal roots.

2.8.2 Regular singular points, the method of Frobenius

Our goal now is to find a class of singular differential equations which is more general than the Euler equation

$$t^2\frac{d^2y}{dt^2}+\alpha t\frac{dy}{dt}+\beta y=0 \tag{1}$$

but which is also solvable by analytical techniques. To this end we rewrite (1) in the form

$$\frac{d^2y}{dt^2}+\frac{\alpha}{t}\frac{dy}{dt}+\frac{\beta}{t^2}y=0. \tag{2}$$

A very natural generalization of (2) is the equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{3}$$

where $p(t)$ and $q(t)$ can be expanded in series of the form

$$\begin{aligned}p(t)&=\frac{p_0}{t}+p_1+p_2t+p_3t^2+\cdots\\ q(t)&=\frac{q_0}{t^2}+\frac{q_1}{t}+q_2+q_3t+q_4t^2+\cdots\end{aligned} \tag{4}$$

Definition. The equation (3) is said to have a *regular singular point* at $t=0$ if $p(t)$ and $q(t)$ have series expansions of the form (4). Equivalently, $t=0$ is a regular singular point of (3) if the functions $tp(t)$ and $t^2q(t)$ are analytic at $t=0$. Equation (3) is said to have a regular singular point at $t=t_0$ if the functions $(t-t_0)p(t)$ and $(t-t_0)^2q(t)$ are analytic at $t=t_0$. A singular point of (3) which is not regular is called *irregular*.

Example 1. Classify the singular points of Bessel's equation of order ν

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-\nu^2)y=0, \tag{5}$$

where ν is a constant.

Solution. Here $P(t)=t^2$ vanishes at $t=0$. Hence, $t=0$ is the only singular point of (5). Dividing both sides of (5) by t^2 gives

$$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}+\left(1-\frac{\nu^2}{t^2}\right)y=0.$$

Observe that

$$tp(t)=1 \quad\text{and}\quad t^2q(t)=t^2-\nu^2$$

are both analytic at $t=0$. Hence Bessel's equation of order ν has a regular singular point at $t=0$.

Example 2. Classify the singular points of the Legendre equation

$$(1-t^2)\frac{d^2y}{dt^2}-2t\frac{dy}{dt}+\alpha(\alpha+1)y=0 \tag{6}$$

where α is a constant.

Solution. Since $1-t^2$ vanishes when $t=1$ and -1, we see that (6) is singular at $t=\pm 1$. Dividing both sides of (6) by $1-t^2$ gives

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\alpha\frac{(\alpha+1)}{1-t^2}y=0.$$

Observe that

$$(t-1)p(t)=-(t-1)\frac{2t}{1-t^2}=\frac{2t}{1+t}$$

and

$$(t-1)^2q(t)=\alpha(\alpha+1)\frac{(t-1)^2}{1-t^2}=\alpha(\alpha+1)\frac{1-t}{1+t}$$

are analytic at $t=1$. Similarly, both $(t+1)p(t)$ and $(t+1)^2q(t)$ are analytic at $t=-1$. Hence, $t=1$ and $t=-1$ are regular singular points of (6).

Example 3. Show that $t=0$ is an irregular singular point of the equation

$$t^2\frac{d^2y}{dt^2}+3\frac{dy}{dt}+ty=0. \tag{7}$$

Solution. Dividing through by t^2 gives

$$\frac{d^2y}{dt^2}+\frac{3}{t^2}\frac{dy}{dt}+\frac{1}{t}y=0.$$

In this case, the function

$$tp(t)=t\left(\frac{3}{t^2}\right)=\frac{3}{t}$$

is not analytic at $t=0$. Hence $t=0$ is an irregular singular point of (7).

We return now to the equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{8}$$

where $t=0$ is a regular singular point. For simplicity, we will restrict ourselves to the interval $t>0$. Multiplying (8) through by t^2 gives the equivalent equation

$$L[y]=t^2\frac{d^2y}{dt^2}+t(tp(t))\frac{dy}{dt}+t^2q(t)y=0. \tag{9}$$

We can view Equation (9) as being obtained from (1) by adding higher powers of t to the coefficients α and β. This suggests that we might be able to obtain solutions of (9) by adding terms of the form t^{r+1}, t^{r+2},... to the solutions t^r of (1). Specifically, we will try to obtain solutions of (9) of the form

$$y(t)=\sum_{n=0}^{\infty}a_nt^{n+r}=t^r\sum_{n=0}^{\infty}a_nt^n.$$

Example 4. Find two linearly independent solutions of the equation

$$L[y]=2t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty=0, \qquad 0<t<\infty. \tag{10}$$

Solution. Let

$$y(t)=\sum_{n=0}^{\infty} a_n t^{n+r}, \qquad a_0\neq 0.$$

Computing

$$y'(t)=\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}$$

and

$$y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2}$$

we see that

$$\begin{aligned}
L[y]&=t^r\left[2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n-1}\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}(n+r)a_nt^{n-1}+\sum_{n=0}^{\infty}a_nt^{n+1}\right]\\
&=t^r\left[2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n-1}\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}(n+r)a_nt^{n-1}+\sum_{n=2}^{\infty}a_{n-2}t^{n-1}\right]\\
&=\left[2r(r-1)a_0+ra_0\right]t^{r-1}+\left[2(1+r)ra_1+(1+r)a_1\right]t^r\\
&\qquad+\sum_{n=2}^{\infty}\left[2(n+r)(n+r-1)a_n+(n+r)a_n+a_{n-2}\right]t^{n+r-1}
\end{aligned}$$

Setting the coefficients of each power of t equal to zero gives

(i) $2r(r-1)a_0+ra_0=r(2r-1)a_0=0,$
(ii) $2(r+1)ra_1+(r+1)a_1=(r+1)(2r+1)a_1=0,$
and
(iii) $2(n+r)(n+r-1)a_n+(n+r)a_n=(n+r)[2(n+r)-1]a_n=-a_{n-2},$
$n\geqslant 2.$

The first equation determines r; it implies that $r=0$ or $r=\frac{1}{2}$. The second equation then forces a_1 to be zero, and the third equation determines a_n for $n\geqslant 2$.

(i) $r=0$. In this case, the recurrence formula (iii) is

$$a_n=\frac{-a_{n-2}}{n(2n-1)},\qquad n\geqslant 2.$$

Since $a_1=0$, we see that all of the odd coefficients are zero. The even coefficients are determined from the relations

$$a_2=\frac{-a_0}{2\cdot 3},\quad a_4=\frac{-a_2}{4\cdot 7}=\frac{a_0}{2\cdot 4\cdot 3\cdot 7},\quad a_6=\frac{-a_4}{6\cdot 11}=\frac{-a_0}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11}$$

and so on. Setting $a_0=1$, we see that

$$y_1(t)=1-\frac{t^2}{2\cdot 3}+\frac{t^4}{2\cdot 4\cdot 3\cdot 7}+\cdots=1+\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{2^n n!3\cdot 7\cdots(4n-1)}$$

is one solution of (10). It is easily verified, using the Cauchy ratio test, that this series converges for all t.

(ii) $r=\frac{1}{2}$. In this case, the recurrence formula (iii) is

$$a_n=\frac{-a_{n-2}}{(n+\frac{1}{2})[2(n+\frac{1}{2})-1]}=\frac{-a_{n-2}}{n(2n+1)},\quad n\geqslant 2.$$

Again, all of the odd coefficients are zero. The even coefficients are determined from the relations

$$a_2=\frac{-a_0}{2\cdot 5},\quad a_4=\frac{-a_2}{4\cdot 9}=\frac{a_0}{2\cdot 4\cdot 5\cdot 9},\quad a_6=\frac{-a_4}{6\cdot 13}=\frac{-a_0}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}$$

and so on. Setting $a_0=1$, we see that

$$y_2(t)=t^{1/2}\left[1-\frac{t^2}{2\cdot 5}+\frac{t^4}{2\cdot 4\cdot 5\cdot 9}+\cdots\right]$$
$$=t^{1/2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{2^n n!5\cdot 9\cdots(4n+1)}\right]$$

is a second solution of (10) on the interval $0<t<\infty$.

Remark. Multiplying both sides of (10) by t gives

$$2t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+t^2y=0.$$

This equation can be viewed as a generalization of the Euler equation

$$2t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}=0. \tag{11}$$

Equation (11) has solutions of the form t^r, where

$$2r(r-1)+r=0.$$

This equation is equivalent to Equation (i) which determined r for the solutions of (10).

Let us now see whether our technique, which is known as the method of Frobenius, works in general for Equation (9). (We will assume throughout this section that $t>0$.) By assumption, this equation can be written in the form

$$L[y]=t^2\frac{d^2y}{dt^2}+t[p_0+p_1t+p_2t^2+\cdots]\frac{dy}{dt}+[q_0+q_1t+q_2t^2+\cdots]y=0.$$

Set

$$y(t)=\sum_{n=0}^{\infty}a_nt^{n+r}, \text{ with } a_0\neq 0.$$

Computing

$$y'(t)=\sum_{n=0}^{\infty}(n+r)a_nt^{n+r-1}$$

and

$$y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-2}$$

we see that

$$L[y]=t^r\left\{\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^n+\left(\sum_{m=0}^{\infty}p_mt^m\right)\left[\sum_{n=0}^{\infty}(n+r)a_nt^n\right]\right.$$
$$\left.+\left(\sum_{m=0}^{\infty}q_mt^m\right)\left(\sum_{n=0}^{\infty}a_nt^n\right)\right\}.$$

Multiplying through and collecting terms gives

$$\begin{aligned}L[y]=&[r(r-1)+p_0r+q_0]a_0t^r\\&+\{[(1+r)r+p_0(1+r)+q_0]a_1+(rp_1+q_1)a_0\}t^{r+1}\\&\vdots\\&+\Big\{[(n+r)(n+r-1)+p_0(n+r)+q_0]a_n\\&+\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]a_k\Big\}t^{n+r}\\&+\cdots.\end{aligned}$$

This expression can be simplified if we set

$$F(r)=r(r-1)+p_0r+q_0. \tag{12}$$

Then,

$$L[y] = a_0F(r)t^r + [a_1F(1+r) + (rp_1+q_1)a_0]t^{1+r} + \cdots$$
$$+ a_nF(n+r)t^{n+r} + \left\{ \sum_{k=0}^{n-1} [(k+r)p_{n-k} + q_{n-k}]a_k \right\} t^{n+r}$$
$$+ \cdots.$$

Setting the coefficient of each power of t equal to zero gives

$$F(r) = r(r-1) + p_0r + q_0 = 0 \tag{13}$$

and

$$F(n+r)a_n = -\sum_{k=0}^{n-1} [(k+r)p_{n-k} + q_{n-k}]a_k, \quad n \geq 1. \tag{14}$$

Equation (13) is called the *indicial* equation of (9). It is a quadratic equation in r, and its roots determine the two possible values r_1 and r_2 of r for which there may be solutions of (9) of the form

$$\sum_{n=0}^{\infty} a_n t^{n+r}.$$

Note that the indicial equation (13) is exactly the equation we would obtain in looking for solutions t^r of the Euler equation

$$t^2\frac{d^2y}{dt^2} + p_0t\frac{dy}{dt} + q_0y = 0.$$

Equation (14) shows that, in general, a_n depends on r and all the preceding coefficients $a_0, a_1, \ldots, a_{n-1}$. We can solve it recursively for a_n provided that $F(1+r), F(2+r), \ldots, F(n+r)$ are not zero. Observe though that if $F(n+r) = 0$ for some positive integer n, then $n+r$ is a root of the indicial equation (13). Consequently, if (13) has two real roots r_1, r_2 with $r_1 > r_2$ and $r_1 - r_2$ not an integer, then Equation (9) has two solutions of the form

$$y_1(t) = t^{r_1}\sum_{n=0}^{\infty} a_n(r_1)t^n, \; y_2(t) = t^{r_2}\sum_{n=0}^{\infty} a_n(r_2)t^n,$$

and these solutions can be shown to converge wherever $tp(t)$ and $t^2q(t)$ both converge.

Remark. We have introduced the notation $a_n(r_1)$ and $a_n(r_2)$ to emphasize that a_n is determined after we choose $r = r_1$ or r_2.

Example 5. Find the general solution of the equation

$$L[y] = 4t\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 3y = 0. \tag{15}$$

Solution. Equation (15) has a regular singular point at $t=0$ since

$$tp(t)=\tfrac{3}{4} \quad \text{and} \quad t^2q(t)=\tfrac{3}{4}t$$

are both analytic at $t=0$. Set

$$y(t)=\sum_{n=0}^{\infty} a_n t^{n+r}, \quad a_0 \neq 0.$$

Computing

$$y'(t)=\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}$$

and

$$y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2}$$

we see that

$$\begin{aligned}L[y]&=4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-1}\\&\quad+3\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}+3\sum_{n=0}^{\infty}a_n t^{n+r}\\&=\sum_{n=0}^{\infty}\left[4(n+r)(n+r-1)+3(n+r)\right]a_n t^{n+r-1}+\sum_{n=1}^{\infty}3a_{n-1}t^{n+r-1}.\end{aligned}$$

Setting the sum of coefficients of like powers of t equal to zero gives

$$4r(r-1)+3r=4r^2-r=r(4r-1)=0 \tag{16}$$

and

$$\left[4(n+r)(n+r-1)+3(n+r)\right]a_n \equiv (n+r)\left[4(n+r)-1\right]a_n=-3a_{n-1}, \quad n \geqslant 1. \tag{17}$$

Equation (16) is the indicial equation, and it implies that $r=0$ or $r=\frac{1}{4}$. Since these roots do not differ by an integer, we can find two solutions of (15) of the form

$$\sum_{n=0}^{\infty} a_n t^{n+r}$$

with a_n determined from (17).

$r=0$. In this case the recurrence relation (17) reduces to

$$a_n=-3\frac{a_{n-1}}{4n(n-1)+3n}=\frac{-3a_{n-1}}{n(4n-1)}.$$

Setting $a_0 = 1$ gives

$$a_1 = -1, \quad a_2 = \frac{-3a_1}{2\cdot 7} = 3\frac{1}{2\cdot 7},$$
$$a_3 = \frac{-3a_2}{3\cdot 11} = -3^2\frac{1}{2\cdot 3\cdot 7\cdot 11},$$
$$a_4 = \frac{-3a_3}{4\cdot 15} = 3^3\frac{1}{2\cdot 3\cdot 4\cdot 7\cdot 11\cdot 15},$$

and, in general,

$$a_n = \frac{(-1)^n 3^{n-1}}{n!7\cdot 11\cdot 15\cdots(4n-1)}.$$

Hence,

$$y_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n-1}}{n!7\cdot 11\cdot 15\cdots(4n-1)} t^n \tag{18}$$

is one solution of (15). It is easily seen, using the Cauchy ratio test, that $y_1(t)$ converges for all t. Hence $y_1(t)$ is an analytic solution of (15).

$r = \frac{1}{4}$. In this case the recurrence relation (17) reduces to

$$a_n = \frac{-3a_{n-1}}{(n+\frac{1}{4})[4(n-\frac{3}{4})+3]} = \frac{-3a_{n-1}}{n(4n+1)}, \quad n \geq 1.$$

Setting $a_0 = 1$ gives

$$a_1 = \frac{-3}{5}, \quad a_2 = \frac{3^2}{2\cdot 5\cdot 9}, \quad a_3 = \frac{-3^3}{2\cdot 3\cdot 5\cdot 9\cdot 13},$$
$$a_4 = \frac{3^4}{2\cdot 3\cdot 4\cdot 5\cdot 9\cdot 13\cdot 17}, \ldots .$$

Proceeding inductively, we see that

$$a_n = \frac{(-1)^n 3^n}{n!5\cdot 9\cdot 13\cdots(4n+1)}.$$

Hence,

$$y_2(t) = t^{1/4} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!5\cdot 9\cdot 13\cdots(4n+1)} t^n$$

is a second solution of (15). It can easily be shown, using the Cauchy ratio test, that this solution converges for all positive t. Note, however, that $y_2(t)$ is not differentiable at $t = 0$.

The method of Frobenius hits a snag in two separate instances. The first instance occurs when the indicial equation (13) has equal roots $r_1 = r_2$. In this case we can only find one solution of (9) of the form

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty} a_n t^n.$$

In the next section we will prove that (9) has a second solution $y_2(t)$ of the form

$$y_2(t)=y_1(t)\ln t+t^{r_1}\sum_{n=0}^{\infty} b_n t^n$$

and show how to compute the coefficients b_n. The computation of the b_n is usually a very formidable problem. We wish to point out here, though, that in many physical applications the solution $y_2(t)$ is rejected on the grounds that it is singular. Thus, it often suffices to find $y_1(t)$ alone. It is also possible to find a second solution $y_2(t)$ by the method of reduction of order, but this too is usually very cumbersome.

The second snag in the method of Frobenius occurs when the roots r_1, r_2 of the indicial equation differ by a positive integer. Suppose that $r_1=r_2+N$, where N is a positive integer. In this case, we can find one solution of the form

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty} a_n t^n.$$

However, it may not be possible to find a second solution $y_2(t)$ of the form

$$y_2(t)=t^{r_2}\sum_{n=0}^{\infty} b_n t^n.$$

This is because $F(r_2+n)=0$ when $n=N$. Thus, the left hand side of (14) becomes

$$0\cdot a_N=-\sum_{k=0}^{N-1}\left[(k+r_2)p_{N-k}+q_{N-k}\right]a_k \tag{19}$$

when $n=N$. This equation cannot be satisfied for any choice of a_N, if

$$\sum_{k=0}^{N-1}\left[(k+r_2)p_{N-k}+q_{N-k}\right]a_k\neq 0.$$

In this case (see Section 2.8.3), Equation (9) has a second solution of the form

$$y_2(t)=y_1(t)\ln t+t^{r_2}\sum_{n=0}^{\infty} b_n t^n$$

where again, the computation of the b_n is a formidable problem.

On the other hand, if the sum on the right hand side of (19) vanishes, then a_N is arbitrary, and we can obtain a second solution of the form

$$y_2(t) = t^{r_2} \sum_{n=0}^{\infty} b_n t^n.$$

We illustrate this situation with the following example.

Example 6. Find two solutions of Bessel's equation of order $\frac{1}{2}$,

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - 1/4)y = 0, \quad 0 < t < \infty. \tag{20}$$

Solution. This equation has a regular singular point at $t = 0$ since

$$tp(t) = 1 \quad \text{and} \quad t^2 q(t) = t^2 - \tfrac{1}{4}$$

are both analytic at $t = 0$. Set

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}, \quad a_0 \neq 0.$$

Computing

$$y'(t) = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1}$$

and

$$y''(t) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}$$

we see that

$$\begin{aligned} L[y] &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n t^{n+r} \\ &\quad + \sum_{n=0}^{\infty} a_n t^{n+r+2} - \tfrac{1}{4} \sum_{n=0}^{\infty} a_n t^{n+r} \\ &= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + (n+r) - \tfrac{1}{4}\right] a_n t^{n+r} + \sum_{n=2}^{\infty} a_{n-2} t^{n+r}. \end{aligned}$$

Setting the sum of coefficients of like powers of t equal to zero gives

$$F(r)a_0 = \left[r(r-1) + r - \tfrac{1}{4}\right]a_0 = \left(r^2 - \tfrac{1}{4}\right)a_0 = 0 \tag{i}$$

$$F(1+r)a_1 = \left[(1+r)r + (1+r) - \tfrac{1}{4}\right]a_1 = \left[(1+r)^2 - \tfrac{1}{4}\right]a_1 = 0 \tag{ii}$$

and

$$F(n+r)a_n = \left[(n+r)^2 - \tfrac{1}{4}\right]a_n = -a_{n-2}, \quad n \geq 2 \tag{iii}$$

Equation (i) is the indicial equation, and it implies that $r_1 = \frac{1}{2}$, $r_2 = -\frac{1}{2}$.
$r_1 = \frac{1}{2}$: Set $a_0 = 1$. Equation (ii) forces a_1 to be zero, and the recurrence

relation (iii) implies that

$$a_n = \frac{-a_{n-2}}{F(n+\frac{1}{2})} = \frac{-a_{n-2}}{n(n+1)}, \quad n \geq 2.$$

This, in turn, implies that all the odd coefficients $a_3, a_5, \ldots,$ are zero, and the even coefficients are given by

$$a_2 = \frac{-a_0}{2\cdot 3} = \frac{-1}{2\cdot 3} = -\frac{1}{3!}$$

$$a_4 = \frac{-a_2}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4\cdot 5} = \frac{1}{5!}$$

$$a_6 = \frac{-a_4}{6\cdot 7} = \frac{-1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = -\frac{1}{7!}$$

and so on. Proceeding inductively, we see that

$$a_{2n} = \frac{(-1)^n}{(2n)!(2n+1)}$$

Hence

$$y_1(t) = t^{1/2}\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots\right)$$

is one solution of (20). This solution can be rewritten in the form

$$y_1(t) = \frac{t^{1/2}}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)$$
$$= \frac{1}{\sqrt{t}} \sin t.$$

$r_2 = -\frac{1}{2}$: Set $a_0 = 1$. Since $1 + r_2 = \frac{1}{2}$ is also a root of the indicial equation, we could, conceivably, run into trouble when trying to solve for a_1. However, Equation (ii) is automatically satisfied, regardless of the value of a_1. We will set $a_1 = 0$. (A nonzero value of a_1 will just reproduce a multiple of $y_1(t)$). The recurrence relation (iii) becomes

$$a_n = \frac{-a_{n-2}}{(n-\frac{1}{2})^2 - \frac{1}{4}} = \frac{-a_{n-2}}{n^2 - n} = \frac{-a_{n-2}}{n(n-1)}, \quad n \geq 2.$$

All the odd coefficients are again zero, and the even coefficients are

$$a_2 = \frac{-a_0}{2\cdot 1} = -\frac{1}{2!}$$

$$a_4 = \frac{-a_2}{4\cdot 3} = \frac{1}{4!}$$

$$a_6 = \frac{-a_4}{6\cdot 5} = -\frac{1}{6!}$$

and so on. Proceeding inductively, we see that

$$a_{2n}=\frac{(-1)^n}{(2n)!}.$$

Hence,

$$\begin{aligned}y_2(t)&=t^{-1/2}\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots\right)\\&=\frac{1}{\sqrt{t}}\cos t\end{aligned}$$

is a second solution of (20).

Remark 1. If r is a complex root of the indicial equation, then

$$y(t)=t^r\sum_{n=0}^{\infty}a_nt^n$$

is a complex-valued solution of (9). It is easily verified in this case that both the real and imaginary parts of $y(t)$ are real-valued solutions of (9).

Remark 2. We must set

$$y(t)=|t|^r\sum_{n=0}^{\infty}a_nt^n$$

if we want to solve (9) on an interval where t is negative. The proof is exactly analogous to the proof for the Euler equation in Section 2.8.1, and is left as an exercise for the reader.

We summarize the results of this section in the following theorem.

Theorem 8. *Consider the differential equation* (9) *where* $t=0$ *is a regular singular point. Then, the functions* $tp(t)$ *and* $t^2q(t)$ *are analytic at* $t=0$ *with power series expansions*

$$tp(t)=p_0+p_1t+p_2t^2+\cdots,\ t^2q(t)=q_0+q_1t+q_2t^2+\cdots$$

which converge for $|t|<\rho$. *Let* r_1 *and* r_2 *be the two roots of the indicial equation*

$$r(r-1)+p_0r+q_0=0$$

with $r_1\geq r_2$ *if they are real. Then, Equation* (9) *has two linearly independent solutions* $y_1(t)$ *and* $y_2(t)$ *on the interval* $0<t<\rho$ *of the following form*:

(a) *If* r_1-r_2 *is not a positive integer, then*

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty}a_nt^n,\quad y_2(t)=t^{r_2}\sum_{n=0}^{\infty}b_nt^n.$$

(b) *If* $r_1 = r_2$, *then*

$$y_1(t) = t^{r_1}\sum_{n=0}^{\infty} a_n t^n, \quad y_2(t) = y_1(t)\ln t + t^{r_1}\sum_{n=0}^{\infty} b_n t^n.$$

(c) *If* $r_1 - r_2 = N$, *a positive integer, then*

$$y_1(t) = t^{r_1}\sum_{n=0}^{\infty} a_n t^n, \quad y_2(t) = ay_1(t)\ln t + t^{r_2}\sum_{n=0}^{\infty} b_n t^n$$

where the constant a may turn out to be zero.

Exercises

In each of Problems 1–6, determine whether the specified value of t is a regular singular point of the given differential equation.

1. $t(t-2)^2y'' + ty' + y = 0;\ t = 0$

2. $t(t-2)^2y'' + ty' + y = 0;\ t = 2$

3. $(\sin t)y'' + (\cos t)y' + \frac{1}{t}y = 0;\ t = 0$

4. $(e^t - 1)y'' + e^ty' + y = 0;\ t = 0$

5. $(1-t^2)y'' + \frac{1}{\sin(t+1)}y' + y = 0;\ t = -1$

6. $t^3y'' + (\sin t^2)y' + ty = 0;\ t = 0$

Find the general solution of each of the following equations.

7. $2t^2y'' + 3ty' - (1+t)y = 0$

8. $2ty'' + (1-2t)y' - y = 0$

9. $2ty'' + (1+t)y' - 2y = 0$

10. $2t^2y'' - ty' + (1+t)y = 0$

11. $4ty'' + 3y' - 3y = 0$

12. $2t^2y'' + (t^2 - t)y' + y = 0$

In each of Problems 13–18, find two independent solutions of the given equation. In each problem, the roots of the indicial equation differ by a positive integer, but two solutions exist of the form $t^r\sum_{n=0}^{\infty} a_n t^n$.

13. $t^2y'' - ty' - (t^2 + \frac{5}{4})y = 0$

14. $t^2y'' + (t - t^2)y' - y = 0$

15. $ty'' - (t^2 + 2)y' + ty = 0$

16. $t^2y'' + (3t - t^2)y' - ty = 0$

17. $t^2y'' + t(t+1)y' - y = 0$

18. $ty'' - (4+t)y' + 2y = 0$

19. Consider the equation

$$t^2y'' + (t^2 - 3t)y' + 3y = 0 \qquad (*)$$

(a) Show that $r = 1$ and $r = 3$ are the two roots of the indicial equation of $(*)$.
(b) Find a power series solution of $(*)$ of the form

$$y_1(t) = t^3\sum_{n=0}^{\infty} a_n t^n, \quad a_0 = 1.$$

(c) Show that $y_1(t)=t^3e^{-t}$.
(d) Show that $(*)$ has no solution of the form

$$t\sum_{n=0}^{\infty} b_n t^n.$$

(e) Find a second solution of $(*)$ using the method of reduction of order. Leave your answer in integral form.

20. Consider the equation

$$t^2y''+ty'-(1+t)y=0.$$

(a) Show that $r=-1$ and $r=1$ are the two roots of the indicial equation.
(b) Find one solution of the form

$$y_1(t)=t\sum_{n=0}^{\infty} a_n t^n.$$

(c) Find a second solution using the method of reduction of order.

21. Consider the equation

$$ty''+ty'+2y=0.$$

(a) Show that $r=0$ and $r=1$ are the two roots of the indicial equation.
(b) Find one solution of the form

$$y_1(t)=t\sum_{n=0}^{\infty} a_n t^n.$$

(c) Find a second solution using the method of reduction of order.

22. Consider the equation

$$ty''+(1-t^2)y'+4ty=0.$$

(a) Show that $r=0$ is a double root of the indicial equation.
(b) Find one solution of the form $y_1(t)=\Sigma_{n=0}^{\infty} a_n t^n$.
(c) Find a second solution using the method of reduction of order.

23. Consider the Bessel equation of order zero

$$t^2y''+ty'+t^2y=0.$$

(a) Show that $r=0$ is a double root of the indicial equation.
(b) Find one solution of the form

$$y_1(t)=1-\frac{t^2}{2^2}+\frac{t^4}{2^2\cdot 4^2}-\frac{t^6}{2^2\cdot 4^2\cdot 6^2}+\cdots.$$

This solution is known as $J_0(t)$.
(c) Find a second solution using the method of reduction of order.

24. Consider the Bessel equation of order ν

$$t^2y''+ty'+(t^2-\nu^2)y=0$$

where ν is real and positive.
(a) Find a power series solution

$$J_\nu(t)=\frac{t^\nu}{2^\nu\nu!}\sum_{n=0}^{\infty}a_nt^n,\quad a_0=1.$$

This function $J_\nu(t)$ is called the Bessel function of order ν.
(b) Find a second solution if 2ν is not an integer.

25. The differential equation

$$ty''+(1-t)y'+\lambda y=0,\quad \lambda \text{ constant},$$

is called the Laguerre differential equation.
(a) Show that the indicial equation is $r^2=0$.
(b) Find a solution $y(t)$ of the Laguerre equation of the form $\sum_{n=0}^{\infty}a_nt^n$.
(c) Show that this solution reduces to a polynomial if $\lambda=n$.

26. The differential equation

$$t(1-t)y''+[\gamma-(1+\alpha+\beta)t]y'-\alpha\beta y=0$$

where α, β, and γ are constants, is known as the hypergeometric equation.
(a) Show that $t=0$ is a regular singular point and that the roots of the indicial equation are 0 and $1-\gamma$.
(b) Show that $t=1$ is also a regular singular point, and that the roots of the indicial equation are now 0 and $\gamma-\alpha-\beta$.
(c) Assume that γ is not an integer. Find two solutions $y_1(t)$ and $y_2(t)$ of the hypergeometric equation of the form

$$y_1(t)=\sum_{n=0}^{\infty}a_nt^n,\quad y_2(t)=t^{1-\gamma}\sum_{n=0}^{\infty}b_nt^n.$$

27. (a) Show that the equation

$$2(\sin t)y''+(1-t)y'-2y=0$$

has two solutions of the form

$$y_1(t)=\sum_{n=0}^{\infty}a_nt^n,\quad y_2(t)=t^{1/2}\sum_{n=0}^{\infty}b_nt^n.$$

(b) Find the first 5 terms in these series expansions assuming that $a_0=b_0=1$.

28. Let $y(t)=u(t)+iv(t)$ be a complex-valued solution of (3) with $p(t)$ and $q(t)$ real. Show that both $u(t)$ and $v(t)$ are real-valued solutions of (3).

29. (a) Show that the indicial equation of

$$t^2y''+ty'+(1+t)y=0 \qquad (*)$$

has complex roots $r=\pm i$.

(b) Show that (*) has 2 linearly independent solutions $y(t)$ of the form

$$y(t) = \sin(\ln t) \sum_{n=0}^{\infty} a_n t^n + \cos(\ln t) \sum_{n=0}^{\infty} b_n t^n.$$

2.8.3 Equal roots, and roots differing by an integer

Equal roots.
We run into trouble if the indicial equation has equal roots $r_1 = r_2$ because then the differential equation

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \tag{1}$$

has only one solution of the form

$$y(t) = t^r \sum_{n=0}^{\infty} a_n t^n. \tag{2}$$

The method of finding a second solution is very similar to the method used in finding a second solution of Euler's equation, in the case of equal roots. Let us rewrite (2) in the form

$$y(t) = y(t,r) = t^r \sum_{n=0}^{\infty} a_n(r) t^n$$

to emphasize that the solution $y(t)$ depends on our choice of r. Then (see Section 2.8.2)

$$L[y](t,r) = a_0 F(r) t^r$$
$$+ \sum_{n=1}^{\infty} \left\{ a_n(r) F(n+r) + \sum_{k=0}^{n-1} \left[(k+r) p_{n-k} + q_{n-k} \right] a_k \right\} t^{n+r}.$$

We now think of r as a continuous variable and determine a_n as a function of r by requiring that the coefficient of t^{n+r} be zero for $n \geq 1$. Thus

$$a_n(r) = \frac{-\sum_{k=0}^{n-1} \left[(k+r) p_{n-k} + q_{n-k} \right] a_k}{F(n+r)}$$

With this choice of $a_n(r)$, we see that

$$L[y](t,r) = a_0 F(r) t^r. \tag{3}$$

In the case of equal roots, $F(r) = (r - r_1)^2$, so that (3) can be written in the form

$$L[y](t,r) = a_0 (r - r_1)^2 t^r.$$

Since $L[y](t, r_1) = 0$, we obtain one solution

$$y_1(t) = t^{r_1} \left[a_0 + \sum_{n=1}^{\infty} a_n(r_1) t^n \right].$$

Observe now, that

$$\begin{aligned}\frac{\partial}{\partial r}L[y](t,r)&=L\left[\frac{\partial y}{\partial r}\right](t,r)\\&=\frac{\partial}{\partial r}a_0(r-r_1)^2t^r\\&=2a_0(r-r_1)t^r+a_0(r-r_1)^2(\ln t)t^r\end{aligned}$$

also vanishes when $r=r_1$. Thus

$$\begin{aligned}y_2(t)&=\frac{\partial}{\partial r}y_1(t,r)|_{r=r_1}\\&=\frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty}a_n(r)t^{n+r}\right]_{r=r_1}\\&=\sum_{n=0}^{\infty}[a_n(r_1)t^{n+r_1}]\ln t+\sum_{n=0}^{\infty}a_n'(r_1)t^{n+r_1}\\&=y_1(t)\ln t+\sum_{n=0}^{\infty}a_n'(r_1)t^{n+r_1}\end{aligned}$$

is a second solution of (1).

Example 1. Find two solutions of Bessel's equation of order zero

$$L[y]=t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+t^2y=0,\quad t>0. \tag{4}$$

Solution. Set

$$y(t)=\sum_{n=0}^{\infty}a_nt^{n+r}.$$

Computing

$$y'(t)=\sum_{n=0}^{\infty}(n+r)a_nt^{n+r-1}$$

and

$$y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-2}$$

we see that

$$\begin{aligned}L[y]&=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nt^{n+r}+\sum_{n=0}^{\infty}a_nt^{n+r+2}\\&=\sum_{n=0}^{\infty}(n+r)^2a_nt^{n+r}+\sum_{n=2}^{\infty}a_{n-2}t^{n+r}.\end{aligned}$$

Setting the sums of like powers of t equal to zero gives

(i) $r^2a_0 = F(r)a_0 = 0$
(ii) $(1+r)^2a_1 = F(1+r)a_1 = 0$
and
(iii) $(n+r)^2a_n = F(n+r)a_n = -a_{n-2},\ n \geq 2.$

Equation (i) is the indicial equation, and it has equal roots $r_1 = r_2 = 0$. Equation (ii) forces a_1 to be zero, and the recurrence relation (iii) says that

$$a_n = \frac{-a_{n-2}}{(n+r)^2}.$$

Clearly, $a_3 = a_5 = a_7 = \cdots = 0$. The even coefficients are given by

$$a_2(r) = \frac{-a_0}{(2+r)^2} = \frac{-1}{(2+r)^2}$$

$$a_4(r) = \frac{-a_2}{(4+r)^2} = \frac{1}{(2+r)^2(4+r)^2}$$

and so on. Proceeding inductively, we see that

$$a_{2n}(r) = \frac{(-1)^n}{(2+r)^2(4+r)^2\cdots(2n+r)^2}.$$

To determine $y_1(t)$, we set $r = 0$. Then

$$a_2(0) = \frac{-1}{2^2}$$

$$a_4(0) = \frac{1}{2^2 \cdot 4^2} = \frac{1}{2^4}\frac{1}{(2!)^2}$$

$$a_6(0) = \frac{-1}{2^2 \cdot 4^2 \cdot 6^2} = \frac{-1}{2^6(3!)^2}$$

and in general

$$a_{2n}(0) = \frac{(-1)^n}{2^2 \cdot 4^2 \cdots (2n)^2} = \frac{(-1)^n}{2^{2n}(n!)^2}.$$

Hence,

$$y_1(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^4 \cdot (2!)^2} - \frac{t^6}{2^6(3!)^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$$

is one solution of (4). This solution is often referred to as the Bessel function of the first kind of order zero, and is denoted by $J_0(t)$.

To obtain a second solution of (4) we set

$$y_2(t)=y_1(t)\ln t+\sum_{n=0}^{\infty}a'_{2n}(0)t^{2n}.$$

To compute $a'_{2n}(0)$, observe that

$$\begin{aligned}\frac{a'_{2n}(r)}{a_{2n}(r)}&=\frac{d}{dr}\ln|a_{2n}(r)|=\frac{d}{dr}\ln(2+r)^{-2}\cdots(2n+r)^{-2}\\&=-2\frac{d}{dr}[\ln(2+r)+\ln(4+r)+\cdots+\ln(2n+r)]\\&=-2\left(\frac{1}{2+r}+\frac{1}{4+r}+\cdots+\frac{1}{2n+r}\right).\end{aligned}$$

Hence,

$$\begin{aligned}a'_{2n}(0)&=-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)a_{2n}(0)\\&=-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)a_{2n}(0).\end{aligned}$$

Setting

$$H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\tag{5}$$

we see that

$$a'_{2n}(0)=\frac{-H_n(-1)^n}{2^{2n}(n!)^2}=\frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}$$

and thus

$$y_2(t)=y_1(t)\ln t+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}t^{2n}$$

is a second solution of (4) with H_n given by (5).

Roots differing by a positive integer. Suppose that r_2 and $r_1=r_2+N$, N a positive integer, are the roots of the indicial equation. Then we can certainly find one solution of (1) of the form

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty}a_n(r_1)t^n.$$

As we mentioned previously, it may not be possible to find a second solution of the form

$$t^{r_2}\sum_{n=0}^{\infty}b_nt^n.$$

In this case, Equation (1) will have a second solution of the form

$$y_2(t) = \frac{\partial}{\partial r} y(t,r)\bigg|_{r=r_2}$$

$$= a y_1(t)\ln t + \sum_{n=0}^{\infty} a_n'(r_2) t^{n+r_2}$$

where a is a constant, and

$$y(t,r) = t^r \sum_{n=0}^{\infty} a_n(r) t^n$$

with

$$a_0 = a_0(r) = r - r_2.$$

The proof of this result can be found in more advanced books on differential equations. In Exercise 5, we develop a simple proof, using the method of reduction of order, to show why a logarithm term will be present.

Remark. It is usually very difficult, and quite cumbersome, to obtain the second solution $y_2(t)$ when a logarithm term is present. Beginning and intermediate students are not expected, usually, to perform such calculations. We have included several exercises for the more industrious students. In these problems, and in similar problems which occur in applications, it is often more than sufficient to find just the first few terms in the series expansion of $y_2(t)$, and this can usually be accomplished using the method of reduction of order.

Exercises

In Problems 1 and 2, show that the roots of the indicial equation are equal, and find two independent solutions of the given equation.

1. $ty'' + y' - 4y = 0$

2. $t^2y'' - t(1+t)y' + y = 0$

3. (a) Show that $r=-1$ and $r=1$ are the roots of the indicial equation for Bessel's equation of order one

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-1)y = 0.$$

(b) Find a solution:

$$J_1(t) = \tfrac{1}{2}t \sum_{n=0}^{\infty} a_n t^n,\ a_0 = 1.$$

$J_1(t)$ is called the Bessel function of order one.

(c) Find a second solution:

$$y_2(t) = -J_1(t)\ln t + \frac{1}{t}\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^n (H_n + H_{n-1})}{2^{2n} n!(n-1)!} t^{2n}\right].$$

4. Consider the equation

$$ty''+3y'-3y=0, \qquad t>0.$$

(a) Show that $r=0$ and $r=-2$ are the roots of the indicial equation.
(b) Find a solution

$$y_1(t)=\sum_{n=0}^{\infty} a_n t^n.$$

(c) Find a second solution

$$y_2(t)=y_1(t)\ln t+\frac{1}{t^2}-\frac{1}{t}+\frac{1}{4}+\frac{11}{36}t+\frac{31}{576}t^2+\cdots.$$

5. This exercise gives an alternate proof of some of the results of this section, using the method of reduction of order.
(a) Let $t=0$ be a regular singular point of the equation

$$t^2y''+tp(t)y'+q(t)y=0 \tag{i}$$

Show that the substitution $y=t^r z$ reduces (i) to the equation

$$t^2z''+[2r+p(t)]tz'+[r(r-1)+rp(t)+q(t)]z=0. \tag{ii}$$

(b) Let r be a root of the indicial equation. Show that (ii) has an analytic solution $z_1(t)=\sum_{n=0}^{\infty}a_nt^n$.
(c) Set $z_2(t)=z_1(t)v(t)$. Show that

$$v(t)=\int u(t)\,dt, \qquad \text{where } u(t)=\frac{e^{-\int[2r+p(t)]/t\,dt}}{z_1^2(t)}.$$

(d) Suppose that $r=r_0$ is a double root of the indicial equation. Show that $2r_0+p_0=1$, and conclude therefore that

$$u(t)=\frac{u_0}{t}+u_1+u_2t+\cdots.$$

(e) Use the result in (d) to show that $y_2(t)$ has an $\ln t$ term in the case of equal roots.
(f) Suppose the roots of the indicial equation are r_0 and r_0-N, N a positive integer. Show that $2r_0+p_0=1+N$, and conclude therefore, that

$$u(t)=\frac{1}{t^{1+N}}\hat{u}(t)$$

where $\hat{u}(t)$ is analytic at $t=0$.
(g) Use the result in (f) to show that $y_2(t)$ has an $\ln t$ term if the coefficient of t^N in the expansion of $\hat{u}(t)$ is nonzero. Show, in addition, that if this coefficient is zero, then

$$v(t)=\frac{v_{-N}}{t^N}+\cdots+\frac{v_{-1}}{t}+v_1t+v_2t^2+\cdots$$

and $y_2(t)$ has no $\ln t$ term.

2.9 The method of Laplace transforms

In this section we describe a very different and extremely clever way of solving the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t); \qquad y(0)=y_0, \quad y'(0)=y_0' \tag{1}$$

where a, b and c are constants. This method, which is known as the method of Laplace transforms, is especially useful in two cases which arise quite often in applications. The first case is when $f(t)$ is a discontinuous function of time. The second case is when $f(t)$ is zero except for a very short time interval in which it is very large.

To put the method of Laplace transforms into proper perspective, we consider the following hypothetical situation. Suppose that we want to multiply the numbers 3.163 and 16.38 together, but that we have forgotten completely how to multiply. We only remember how to add. Being good mathematicians, we ask ourselves the following question.

Question: Is it possible to reduce the problem of multiplying the two numbers 3.163 and 16.38 together to the simpler problem of adding two numbers together?

The answer to this question, of course, is yes, and is obtained as follows. First, we consult our logarithm tables and find that $\ln 3.163 = 1.15152094$, and $\ln 16.38 = 2.79606108$. Then, we add these two numbers together to yield 3.94758202. Finally, we consult our anti-logarithm tables and find that $3.94758202 = \ln 51.80994$. Hence, we conclude that $3.163 \times 16.38 = 51.80994$.

The key point in this analysis is that the operation of multiplication is replaced by the simpler operation of addition when we work with the logarithms of numbers, rather than with the numbers themselves. We represent this schematically in Table 1. In the method to be discussed below, the unknown function $y(t)$ will be replaced by a new function $Y(s)$, known as the Laplace transform of $y(t)$. This association will have the property that $y'(t)$ will be replaced by $sY(s)-y(0)$. Thus, the operation of differentiation with respect to t will be replaced, essentially, by the operation of multiplication with respect to s. In this manner, we will replace the initial-value problem (1) by an algebraic equation which can be solved explicitly for $Y(s)$. Once we know $Y(s)$, we can consult our "anti-Laplace transform" tables and recover $y(t)$.

Table 1

a	$\longrightarrow$	$\ln a$
b	$\longrightarrow$	$\ln b$
$a \cdot b$	$\longrightarrow$	$\ln a + \ln b$

We begin with the definition of the Laplace transform.

Definition. Let $f(t)$ be defined for $0 \leqslant t < \infty$. The Laplace transform of $f(t)$, which is denoted by $F(s)$, or $\mathcal{L}\{f(t)\}$, is given by the formula

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \tag{2}$$

where

$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\,dt.$$

Example 1. Compute the Laplace transform of the function $f(t) = 1$.
Solution. From (2),

$$\mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st}\,dt = \lim_{A\to\infty} \frac{1-e^{-sA}}{s}$$

$$= \begin{cases} \dfrac{1}{s}, & s>0 \\ \infty, & s \leqslant 0 \end{cases}.$$

Example 2. Compute the Laplace transform of the function e^{at}.
Solution. From (2),

$$\mathcal{L}\{e^{at}\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{at}\,dt = \lim_{A\to\infty} \frac{e^{(a-s)A}-1}{a-s}$$

$$= \begin{cases} \dfrac{1}{s-a}, & s>a \\ \infty, & s \leqslant a \end{cases}.$$

Example 3. Compute the Laplace transform of the functions $\cos\omega t$ and $\sin\omega t$.
Solution. From (2),

$$\mathcal{L}\{\cos\omega t\} = \int_0^\infty e^{-st} \cos\omega t\,dt \quad \text{and} \quad \mathcal{L}\{\sin\omega t\} = \int_0^\infty e^{-st} \sin\omega t\,dt.$$

Now, observe that

$$\mathcal{L}\{\cos\omega t\} + i\mathcal{L}\{\sin\omega t\} = \int_0^\infty e^{-st} e^{i\omega t}\,dt = \lim_{A\to\infty} \int_0^A e^{(i\omega - s)t}\,dt$$

$$= \lim_{A\to\infty} \frac{e^{(i\omega - s)A}-1}{i\omega - s}$$

$$= \begin{cases} \dfrac{1}{s-i\omega} = \dfrac{s+i\omega}{s^2+\omega^2}, & s>0 \\ \text{undefined}, & s \leqslant 0 \end{cases}.$$

Equating real and imaginary parts in this equation gives

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2} \quad \text{and} \quad \mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}, \qquad s>0.$$

Equation (2) associates with every function $f(t)$ a new function, which we call $F(s)$. As the notation $\mathcal{L}\{f(t)\}$ suggests, the Laplace transform is an operator acting on functions. It is also a linear operator, since

$$\begin{aligned}\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\} &= \int_0^\infty e^{-st}[c_1f_1(t)+c_2f_2(t)]\,dt \\ &= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt \\ &= c_1\mathcal{L}\{f_1(t)\}+c_2\mathcal{L}\{f_2(t)\}.\end{aligned}$$

It is to be noted, though, that whereas $f(t)$ is defined for $0 \leq t < \infty$, its Laplace transform is usually defined in a different interval. For example, the Laplace transform of e^{2t} is only defined for $2 < s < \infty$, and the Laplace transform of e^{8t} is only defined for $8 < s < \infty$. This is because the integral (2) will only exist, in general, if s is sufficiently large.

One very serious difficulty with the definition (2) is that this integral may fail to exist for every value of s. This is the case, for example, if $f(t) = e^{t^2}$ (see Exercise 13). To guarantee that the Laplace transform of $f(t)$ exists at least in some interval $s > s_0$, we impose the following conditions on $f(t)$.

(i) The function $f(t)$ is piecewise continuous. This means that $f(t)$ has at most a finite number of discontinuities on any interval $0 \leq t \leq A$, and both the limit from the right and the limit from the left of f exist at every point of discontinuity. In other words, $f(t)$ has only a finite number of "jump discontinuities" in any finite interval. The graph of a typical piecewise continuous function $f(t)$ is described in Figure 1.

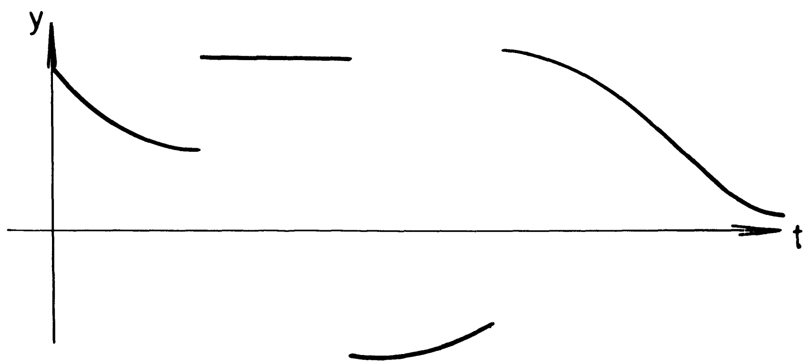

Figure 1. Graph of a typical piecewise continuous function

(ii) The function $f(t)$ is of exponential order, that is, there exist constants M and c such that

$$|f(t)| \leqslant Me^{ct}, \qquad 0 \leqslant t < \infty.$$

Lemma 1. *If $f(t)$ is piecewise continuous and of exponential order, then its Laplace transform exists for all s sufficiently large. Specifically, if $f(t)$ is piecewise continuous, and $|f(t)| \leqslant Me^{ct}$, then $F(s)$ exists for $s > c$.*

We prove Lemma 1 with the aid of the following lemma from integral calculus, which we quote without proof.

Lemma 2. *Let $g(t)$ be piecewise continuous. Then, the improper integral $\int_0^\infty g(t)\,dt$ exists if $\int_0^\infty |g(t)|\,dt$ exists. To prove that this latter integral exists, it suffices to show that there exists a constant K such that*

$$\int_0^A |g(t)|\,dt \leqslant K$$

for all A.

Remark. Notice the similarity of Lemma 2 with the theorem of infinite series (see Appendix B) which states that the infinite series $\sum a_n$ converges if $\sum |a_n|$ converges, and that $\sum |a_n|$ converges if there exists a constant K such that $|a_1| + \ldots + |a_n| \leqslant K$ for all n.

We are now in a position to prove Lemma 1.

PROOF OF LEMMA 1. Since $f(t)$ is piecewise continuous, the integral $\int_0^A e^{-st} f(t)\,dt$ exists for all A. To prove that this integral has a limit for all s sufficiently large, observe that

$$\int_0^A |e^{-st} f(t)|\,dt \leqslant M \int_0^A e^{-st} e^{ct}\,dt$$

$$= \frac{M}{c-s}\left[e^{(c-s)A} - 1\right] \leqslant \frac{M}{s-c}$$

for $s > c$. Consequently, by Lemma 2, the Laplace transform of $f(t)$ exists for $s > c$. Thus, from here on, we tacitly assume that $|f(t)| \leqslant Me^{ct}$, and $s > c$. □

The real usefulness of the Laplace transform in solving differential equations lies in the fact that the Laplace transform of $f'(t)$ is very closely related to the Laplace transform of $f(t)$. This is the content of the following important lemma.

Lemma 3. *Let* $F(s)=\mathcal{L}\{f(t)\}$. *Then*

$$\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0)=sF(s)-f(0).$$

PROOF. The proof of Lemma 3 is very elementary; we just write down the formula for the Laplace transform of $f'(t)$ and integrate by parts. To wit,

$$\begin{aligned}\mathcal{L}\{f'(t)\}&=\lim_{A\to\infty}\int_0^A e^{-st}f'(t)\,dt\\&=\lim_{A\to\infty}e^{-st}f(t)\Big|_0^A+\lim_{A\to\infty}s\int_0^A e^{-st}f(t)\,dt\\&=-f(0)+s\lim_{A\to\infty}\int_0^A e^{-st}f(t),dt\\&=-f(0)+sF(s).\end{aligned}$$ □

Our next step is to relate the Laplace transform of $f''(t)$ to the Laplace transform of $f(t)$. This is the content of Lemma 4.

Lemma 4. *Let* $F(s)=\mathcal{L}\{f(t)\}$. *Then,*

$$\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0).$$

PROOF. Using Lemma 3 twice, we see that

$$\begin{aligned}\mathcal{L}\{f''(t)\}&=s\mathcal{L}\{f'(t)\}-f'(0)\\&=s[sF(s)-f(0)]-f'(0)\\&=s^2F(s)-sf(0)-f'(0).\end{aligned}$$ □

We have now developed all the machinery necessary to reduce the problem of solving the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t);\qquad y(0)=y_0,\quad y'(0)=y_0' \tag{3}$$

to that of solving an algebraic equation. Let $Y(s)$ and $F(s)$ be the Laplace transforms of $y(t)$ and $f(t)$ respectively. Taking Laplace transforms of both sides of the differential equation gives

$$\mathcal{L}\{ay''(t)+by'(t)+cy(t)\}=F(s).$$

By the linearity of the Laplace transform operator,

$$\mathcal{L}\{ay''(t)+by'(t)+cy(t)\}=a\mathcal{L}\{y''(t)\}+b\mathcal{L}\{y'(t)\}+c\mathcal{L}\{y(t)\},$$

and from Lemmas 3 and 4

$$\mathcal{L}\{y'(t)\}=sY(s)-y_0, \qquad \mathcal{L}\{y''(t)\}=s^2Y(s)-sy_0-y_0'.$$

Hence,

$$a\left[s^2Y(s)-sy_0-y_0'\right]+b\left[sY(s)-y_0\right]+cY(s)=F(s)$$

and this *algebraic equation* implies that

$$Y(s)=\frac{(as+b)y_0}{as^2+bs+c}+\frac{ay_0'}{as^2+bs+c}+\frac{F(s)}{as^2+bs+c}. \tag{4}$$

Equation (4) tells us the Laplace transform of the solution $y(t)$ of (3). To find $y(t)$, we must consult our anti, or inverse, Laplace transform tables. Now, just as $Y(s)$ is expressed explicitly in terms of $y(t)$; that is, $Y(s) = \int_0^\infty e^{-st}y(t)\,dt$, we can write down an explicit formula for $y(t)$. However, this formula, which is written symbolically as $y(t)=\mathcal{L}^{-1}\{Y(s)\}$, involves an integration with respect to a complex variable, and this is beyond the scope of this book. Therefore, instead of using this formula, we will derive several elegant properties of the Laplace transform operator in the next section. These properties will enable us to invert many Laplace transforms by inspection; that is, by recognizing "which functions they are the Laplace transform of".

Example 4. Solve the initial-value problem

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{3t}; \qquad y(0)=1, \quad y'(0)=0.$$

Solution. Let $Y(s)=\mathcal{L}\{y(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$s^2Y(s)-s-3\left[sY(s)-1\right]+2Y(s)=\frac{1}{s-3}$$

and this implies that

$$\begin{aligned}Y(s)&=\frac{1}{(s-3)(s^2-3s+2)}+\frac{s-3}{s^2-3s+2}\\&=\frac{1}{(s-1)(s-2)(s-3)}+\frac{s-3}{(s-1)(s-2)}.\end{aligned} \tag{5}$$

To find $y(t)$, we expand each term on the right-hand side of (5) in partial fractions. Thus, we write

$$\frac{1}{(s-1)(s-2)(s-3)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}.$$

This implies that

$$A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)=1. \tag{6}$$

Setting $s=1$ in (6) gives $A=\frac{1}{2}$; setting $s=2$ gives $B=-1$; and setting $s=3$ gives $C=\frac{1}{2}$. Hence,

$$\frac{1}{(s-1)(s-2)(s-3)}=\frac{1}{2}\frac{1}{s-1}-\frac{1}{s-2}+\frac{1}{2}\frac{1}{s-3}.$$

Similarly, we write

$$\frac{s-3}{(s-1)(s-2)}=\frac{D}{s-1}+\frac{E}{s-2}$$

and this implies that

$$D(s-2)+E(s-1)=s-3. \tag{7}$$

Setting $s=1$ in (7) gives $D=2$, while setting $s=2$ gives $E=-1$. Hence,

$$\begin{aligned} Y(s)&=\frac{1}{2}\frac{1}{s-1}-\frac{1}{s-2}+\frac{1}{2}\frac{1}{s-3}+\frac{2}{s-1}-\frac{1}{s-2}\\ &=\frac{5}{2}\frac{1}{s-1}-\frac{2}{s-2}+\frac{1}{2}\frac{1}{s-3}. \end{aligned}$$

Now, we recognize the first term as being the Laplace transform of $\frac{5}{2}e^{t}$. Similarly, we recognize the second and third terms as being the Laplace transforms of $-2e^{2t}$ and $\frac{1}{2}e^{3t}$, respectively. Therefore,

$$Y(s)=\mathcal{L}\left\{\tfrac{5}{2}e^{t}-2e^{2t}+\tfrac{1}{2}e^{3t}\right\}$$

so that

$$y(t)=\tfrac{5}{2}e^{t}-2e^{2t}+\tfrac{1}{2}e^{3t}.$$

Remark. We have cheated a little bit in this problem because there are actually infinitely many functions whose Laplace transform is a given function. For example, the Laplace transform of the function

$$z(t)=\begin{cases} \frac{5}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}, & t\neq 1,2,\text{ and }3\\ 0, & t=1,2,3\end{cases}$$

is also $Y(s)$, since $z(t)$ differs from $y(t)$ at only three points.* However, there is only one *continuous* function $y(t)$ whose Laplace transform is a given function $Y(s)$, and it is in this sense that we write $y(t)=\mathcal{L}^{-1}\{Y(s)\}$.

We wish to emphasize that Example 4 is just by way of illustrating the method of Laplace transforms for solving initial-value problems. The best way of solving this particular initial-value problem is by the method of

*If $f(t)=g(t)$ except at a finite number of points, then $\int_a^b f(t)\,dt=\int_a^b g(t)\,dt$.

judicious guessing. However, even though it is longer to solve this particular initial-value problem by the method of Laplace transforms, there is still something "nice and satisfying" about this method. If we had done this problem by the method of judicious guessing, we would have first computed a particular solution $\psi(t)=\frac{1}{2}e^{3t}$. Then, we would have found two independent solutions e^t and e^{2t} of the homogeneous equation, and we would have written

$$y(t)=c_1e^t+c_2e^{2t}+\tfrac{1}{2}e^{3t}$$

as the general solution of the differential equation. Finally, we would have computed $c_1=\frac{5}{2}$ and $c_2=-2$ from the initial conditions. What is unsatisfying about this method is that we first had to find *all* the solutions of the differential equation before we could find the specific solution $y(t)$ which we were interested in. The method of Laplace transforms, on the other hand, enables us to find $y(t)$ directly, without first finding all solutions of the differential equation.

Exercises

Determine the Laplace transform of each of the following functions.

1. t

2. t^n

3. $e^{at}\cos bt$

4. $e^{at}\sin bt$

5. $\cos^2 at$

6. $\sin^2 at$

7. $\sin at\cos bt$

8. $t^2\sin t$

9. Given that $\int_0^\infty e^{-x^2}dx=\sqrt{\pi}\,/2$, find $\mathcal{L}\{t^{-1/2}\}$. *Hint*: Make the change of variable $u=\sqrt{t}$ in (2).

Show that each of the following functions are of exponential order.

10. t^n

11. $\sin at$

12. $e^{\sqrt{t}}$

13. Show that e^{t^2} does not possess a Laplace transform. *Hint*: Show that $e^{t^2-st}>e^t$ for $t>s+1$.

14. Suppose that $f(t)$ is of exponential order. Show that $F(s)=\mathcal{L}\{f(t)\}$ approaches 0 as $s\to\infty$.

Solve each of the following initial-value problems.

15. $y''-5y'+4y=e^{2t};\quad y(0)=1,\ y'(0)=-1$

16. $2y''+y'-y=e^{3t};\quad y(0)=2,\ y'(0)=0$

Find the Laplace transform of the solution of each of the following initial-value problems.

17. $y''+2y'+y=e^{-t};\quad y(0)=1,\ y'(0)=3$

18. $y''+y=t^2\sin t;\quad y(0)=y'(0)=0$

19. $y''+3y'+7y=\cos t;\quad y(0)=0, y'(0)=2$

20. $y''+y'+y=t^3;\quad y(0)=2, y'(0)=0$

21. Prove that all solutions $y(t)$ of $ay''+by'+cy=f(t)$ are of exponential order if $f(t)$ is of exponential order. *Hint*: Show that all solutions of the homogeneous equation are of exponential order. Obtain a particular solution using the method of variation of parameters, and show that it, too, is of exponential order.

22. Let $F(s)=\mathcal{L}\{f(t)\}$. Prove that

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\}=s^nF(s)-s^{n-1}f(0)-\dots-\frac{df^{(n-1)}(0)}{dt^{n-1}}.$$

Hint: Try induction.

23. Solve the initial-value problem

$$y'''-6y''+11y'-6y=e^{4t};\qquad y(0)=y'(0)=y''(0)=0$$

24. Solve the initial-value problem

$$y''-3y'+2y=e^{-t};\qquad y(t_0)=1,\quad y'(t_0)=0$$

by the method of Laplace transforms. *Hint*: Let $\phi(t)=y(t+t_0)$.

2.10 Some useful properties of Laplace transforms

In this section we derive several important properties of Laplace transforms. Using these properties, we will be able to compute the Laplace transform of most functions without performing tedious integrations, and to invert many Laplace transforms by inspection.

Property 1. If $\mathcal{L}\{f(t)\}=F(s)$, then

$$\mathcal{L}\{-tf(t)\}=\frac{d}{ds}F(s).$$

PROOF. By definition, $F(s)=\int_0^\infty e^{-st}f(t)\,dt$. Differentiating both sides of this equation with respect to s gives

$$\begin{aligned}\frac{d}{ds}F(s)&=\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt\\&=\int_0^\infty\frac{\partial}{\partial s}(e^{-st})f(t)\,dt=\int_0^\infty -te^{-st}f(t)\,dt\\&=\mathcal{L}\{-tf(t)\}.\end{aligned}$$

□

Property 1 states that the Laplace transform of the function $-tf(t)$ is the derivative of the Laplace transform of $f(t)$. Thus, if we know the Laplace transform $F(s)$ of $f(t)$, then, we don't have to perform a tedious integration to find the Laplace transform of $tf(t)$; we need only differentiate $F(s)$ and multiply by -1.

Example 1. Compute the Laplace transform of te^t.
Solution. The Laplace transform of e^t is $1/(s-1)$. Hence, by Property 1, the Laplace transform of te^t is

$$\mathcal{L}\{te^t\} = -\frac{d}{ds}\frac{1}{s-1} = \frac{1}{(s-1)^2}.$$

Example 2. Compute the Laplace transform of t^{13}.
Solution. Using Property 1 thirteen times gives

$$\mathcal{L}\{t^{13}\} = (-1)^{13}\frac{d^{13}}{ds^{13}}\mathcal{L}\{1\} = (-1)^{13}\frac{d^{13}}{ds^{13}}\frac{1}{s} = \frac{(13)!}{s^{14}}.$$

The main usefulness of Property 1 is in inverting Laplace transforms, as the following examples illustrate.

Example 3. What function has Laplace transform $-1/(s-2)^2$?
Solution. Observe that

$$-\frac{1}{(s-2)^2} = \frac{d}{ds}\frac{1}{s-2} \quad \text{and} \quad \frac{1}{s-2} = \mathcal{L}\{e^{2t}\}.$$

Hence, by Property 1,

$$\mathcal{L}^{-1}\left\{-\frac{1}{(s-2)^2}\right\} = -te^{2t}.$$

Example 4. What function has Laplace transform $-4s/(s^2+4)^2$?
Solution. Observe that

$$-\frac{4s}{(s^2+4)^2} = \frac{d}{ds}\frac{2}{s^2+4} \quad \text{and} \quad \frac{2}{s^2+4} = \mathcal{L}\{\sin 2t\}.$$

Hence, by Property 1,

$$\mathcal{L}^{-1}\left\{-\frac{4s}{(s^2+4)^2}\right\} = -t\sin 2t.$$

Example 5. What function has Laplace transform $1/(s-4)^3$?

Solution. We recognize that

$$\frac{1}{(s-4)^3}=\frac{d^2}{ds^2}\frac{1}{2}\frac{1}{s-4}.$$

Hence, using Property 1 twice, we see that

$$\frac{1}{(s-4)^3}=\mathcal{L}\left\{\frac{1}{2}t^2e^{4t}\right\}.$$

Property 2. If $F(s)=\mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{e^{at}f(t)\}=F(s-a).$$

PROOF. By definition,

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\}&=\int_0^\infty e^{-st}e^{at}f(t)\,dt=\int_0^\infty e^{(a-s)t}f(t)\,dt\\&=\int_0^\infty e^{-(s-a)t}f(t)\,dt\equiv F(s-a).\end{aligned}$$ □

Property 2 states that the Laplace transform of $e^{at}f(t)$ evaluated at the point s equals the Laplace transform of $f(t)$ evaluated at the point $(s-a)$. Thus, if we know the Laplace transform $F(s)$ of $f(t)$, then we don't have to perform an integration to find the Laplace transform of $e^{at}f(t)$; we need only replace every s in $F(s)$ by $s-a$.

Example 6. Compute the Laplace transform of $e^{3t}\sin t$.
Solution. The Laplace transform of $\sin t$ is $1/(s^2+1)$. Therefore, to compute the Laplace transform of $e^{3t}\sin t$, we need only replace every s by $s-3$; that is,

$$\mathcal{L}\{e^{3t}\sin t\}=\frac{1}{(s-3)^2+1}.$$

The real usefulness of Property 2 is in inverting Laplace transforms, as the following examples illustrate.

Example 7. What function $g(t)$ has Laplace transform

$$G(s)=\frac{s-7}{25+(s-7)^2}?$$

Solution. Observe that

$$F(s)=\frac{s}{s^2+5^2}=\mathcal{L}\{\cos 5t\}$$

and that $G(s)$ is obtained from $F(s)$ by replacing every s by $s-7$. Hence, by Property 2,

$$\frac{s-7}{(s-7)^2+25}=\mathcal{L}\{e^{7t}\cos 5t\}.$$

Example 8. What function has Laplace transform $1/(s^2-4s+9)$?
Solution. One way of solving this problem is to expand $1/(s^2-4s+9)$ in partial fractions. A much better way is to complete the square of s^2-4s+9. Thus, we write

$$\frac{1}{s^2-4s+9}=\frac{1}{s^2-4s+4+(9-4)}=\frac{1}{(s-2)^2+5}.$$

Now,

$$\frac{1}{s^2+5}=\mathcal{L}\left\{\frac{1}{\sqrt{5}}\sin\sqrt{5}\,t\right\}.$$

Hence, by Property 2,

$$\frac{1}{s^2-4s+9}=\frac{1}{(s-2)^2+5}=\mathcal{L}\left\{\frac{1}{\sqrt{5}}e^{2t}\sin\sqrt{5}\,t\right\}.$$

Example 9. What function has Laplace transform $s/(s^2-4s+9)$?
Solution. Observe that

$$\frac{s}{s^2-4s+9}=\frac{s-2}{(s-2)^2+5}+\frac{2}{(s-2)^2+5}.$$

The function $s/(s^2+5)$ is the Laplace transform of $\cos\sqrt{5}\,t$. Therefore, by Property 2,

$$\frac{s-2}{(s-2)^2+5}=\mathcal{L}\{e^{2t}\cos\sqrt{5}\,t\},$$

and

$$\frac{s}{s^2-4s+9}=\mathcal{L}\left\{e^{2t}\cos\sqrt{5}\,t+\frac{2}{\sqrt{5}}e^{2t}\sin\sqrt{5}\,t\right\}.$$

In the previous section we showed that the Laplace transform is a linear operator; that is

$$\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\}=c_1\mathcal{L}\{f_1(t)\}+c_2\mathcal{L}\{f_2(t)\}.$$

Thus, if we know the Laplace transforms $F_1(s)$ and $F_2(s)$, of $f_1(t)$ and $f_2(t)$, then we don't have to perform any integrations to find the Laplace transform of a linear combination of $f_1(t)$ and $f_2(t)$; we need only take the same linear combination of $F_1(s)$ and $F_2(s)$. For example, two functions which appear quite often in the study of differential equations are the hyperbolic cosine and hyperbolic sine functions. These functions are defined by the equations

$$\cosh at=\frac{e^{at}+e^{-at}}{2},\qquad \sinh at=\frac{e^{at}-e^{-at}}{2}.$$

Therefore, by the linearity of the Laplace transform,

$$\mathcal{L}\{\cosh at\}=\frac{1}{2}\mathcal{L}\{e^{at}\}+\frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$=\frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]=\frac{s}{s^2-a^2}$$

and

$$\mathcal{L}\{\sinh at\}=\frac{1}{2}\mathcal{L}\{e^{at}\}-\frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$=\frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]=\frac{a}{s^2-a^2}.$$

Exercises

Use Properties 1 and 2 to find the Laplace transform of each of the following functions.

1. t^n **2.** t^ne^{at} **3.** $t\sin at$ **4.** $t^2\cos at$

5. $t^{5/2}$ (see Exercise 9, Section 2.9)

6. Let $F(s)=\mathcal{L}\{f(t)\}$, and suppose that $f(t)/t$ has a limit as t approaches zero. Prove that

$$\mathcal{L}\{f(t)/t\}=\int_s^\infty F(u)\,du. \tag{*}$$

(The assumption that $f(t)/t$ has a limit as $t\to 0$ guarantees that the integral on the right-hand side of (*) exists.)

7. Use Equation (*) of Problem 6 to find the Laplace transform of each of the following functions:

(a) $\dfrac{\sin t}{t}$ (b) $\dfrac{\cos at-1}{t}$ (c) $\dfrac{e^{at}-e^{bt}}{t}$

Find the inverse Laplace transform of each of the following functions. In several of these problems, it will be helpful to write the functions

$$p_1(s)=\frac{\alpha_1s^3+\beta_1s^2+\gamma_1s+\delta_1}{(as^2+bs+c)(ds^2+es+f)} \quad\text{and}\quad p_2(s)=\frac{\alpha_1s^2+\beta_1s+\gamma}{(as+b)(cs^2+ds+e)}$$

in the simpler form

$$p_1(s)=\frac{As+B}{as^2+bs+c}+\frac{Cs+D}{ds^2+es+f} \quad\text{and}\quad p_2(s)=\frac{A}{as+b}+\frac{Cs+D}{cs^2+ds+e}.$$

8. $\dfrac{s}{(s+a)^2+b^2}$ **9.** $\dfrac{s^2-5}{s^3+4s^2+3s}$

10. $\dfrac{1}{s(s^2+4)}$ **11.** $\dfrac{s}{s^2-3s-12}$

12. $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$ **13.** $\dfrac{3s}{(s+1)^4}$

14. $\dfrac{1}{s(s+4)^2}$

15. $\dfrac{s}{(s+1)^2(s^2+1)}$

16. $\dfrac{1}{(s^2+1)^2}$

17. Let $F(s)=\mathcal{L}\{f(t)\}$. Show that

$$f(t)=-\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

Thus, if we know how to invert $F'(s)$, then we can also invert $F(s)$.

18. Use the result of Problem 17 to invert each of the following Laplace transforms

(a) $\ln\left(\dfrac{s+a}{s-a}\right)$ (b) $\arctan\dfrac{a}{s}$ (c) $\ln\left(1-\dfrac{a^2}{s^2}\right)$

Solve each of the following initial-value problems by the method of Laplace transforms.

19. $y''+y=\sin t;\quad y(0)=1, y'(0)=2$

20. $y''+y=t\sin t;\quad y(0)=1, y'(0)=2$

21. $y''-2y'+y=te^t;\quad y(0)=0, y'(0)=0$

22. $y''-2y'+7y=\sin t;\quad y(0)=0, y'(0)=0$

23. $y''+y'+y=1+e^{-t};\quad y(0)=3, y'(0)=-5$

24. $y''+y=\begin{cases} 2, & 0\leqslant t\leqslant 3 \\ 3t-7, & 3<t<\infty \end{cases};\quad y(0)=0, y'(0)=0$

2.11 Differential equations with discontinuous right-hand sides

In many applications, the right-hand side of the differential equation $ay''+by'+cy=f(t)$ has a jump discontinuity at one or more points. For example, a particle may be moving under the influence of a force $f_1(t)$, and suddenly, at time t_1, an additional force $f_2(t)$ is applied to the particle. Such equations are often quite tedious and cumbersome to solve, using the methods developed in Sections 2.4 and 2.5. In this section we show how to handle such problems by the method of Laplace transforms. We begin by computing the Laplace transform of several simple discontinuous functions.

The simplest example of a function with a single jump discontinuity is the function

$$H_c(t)=\begin{cases} 0, & 0\leqslant t<c \\ 1, & t\geqslant c \end{cases}.$$

This function, whose graph is given in Figure 1, is often called the unit step

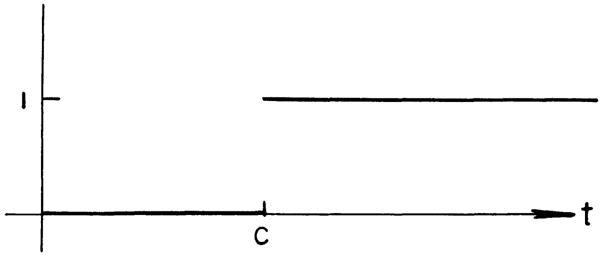

Figure 1. Graph of $H_c(t)$

function, or the Heaviside function. Its Laplace transform is

$$\begin{aligned}\mathcal{L}\{H_c(t)\} &= \int_0^\infty e^{-st}H_c(t)\,dt = \int_c^\infty e^{-st}\,dt \\ &= \lim_{A\to\infty}\int_c^A e^{-st}\,dt = \lim_{A\to\infty}\frac{e^{-cs}-e^{-sA}}{s} \\ &= \frac{e^{-cs}}{s}, \qquad s>0.\end{aligned}$$

Next, let f be any function defined on the interval $0\leqslant t<\infty$, and let g be the function obtained from f by moving the graph of f over c units to the right, as shown in Figure 2. More precisely, $g(t)=0$ for $0\leqslant t<c$, and $g(t)=f(t-c)$ for $t\geqslant c$. For example, if $c=2$ then the value of g at $t=7$ is the value of f at $t=5$. A convenient analytical expression for $g(t)$ is

$$g(t)=H_c(t)f(t-c).$$

The factor $H_c(t)$ makes g zero for $0\leqslant t<c$, and replacing the argument t of f by $t-c$ moves f over c units to the right. Since $g(t)$ is obtained in a simple manner from $f(t)$, we would expect that its Laplace transform can also be obtained in a simple manner from the Laplace transform of $f(t)$. This is indeed the case, as we now show.

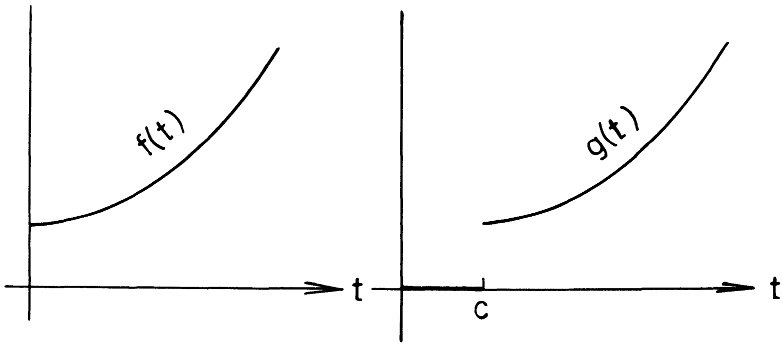

Figure 2

Property 3. Let $F(s)=\mathcal{L}\{f(t)\}$. Then,

$$\mathcal{L}\{H_c(t)f(t-c)\}=e^{-cs}F(s).$$

PROOF. By definition,

$$\mathcal{L}\{H_c(t)f(t-c)\}=\int_0^\infty e^{-st}H_c(t)f(t-c)\,dt$$
$$=\int_c^\infty e^{-st}f(t-c)\,dt.$$

This integral suggests the substitution

$$\xi=t-c.$$

Then,

$$\int_c^\infty e^{-st}f(t-c)\,dt=\int_0^\infty e^{-s(\xi+c)}f(\xi)\,d\xi$$
$$=e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)\,d\xi$$
$$=e^{-cs}F(s).$$

Hence, $\mathcal{L}\{H_c(t)f(t-c)\}=e^{-cs}\mathcal{L}\{f(t)\}$. □

Example 1. What function has Laplace transform e^{-s}/s^2?
Solution. We know that $1/s^2$ is the Laplace transform of the function t. Hence, by Property 3

$$\frac{e^{-s}}{s^2}=\mathcal{L}\{H_1(t)(t-1)\}.$$

The graph of $H_1(t)(t-1)$ is given in Figure 3.

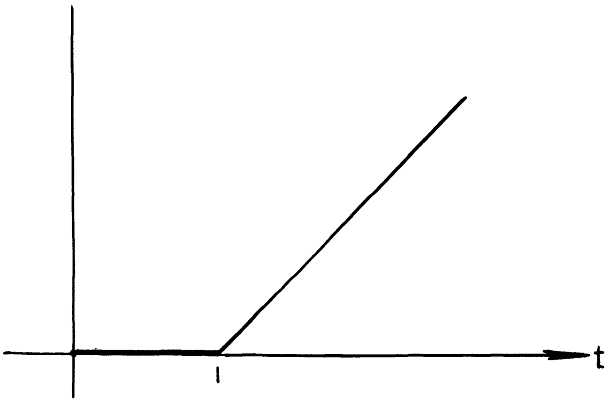

Figure 3. Graph of $H_1(t)\,(t-1)$

Example 2. What function has Laplace transform $e^{-3s}/(s^2-2s-3)$?
Solution. Observe that

$$\frac{1}{s^2-2s-3}=\frac{1}{s^2-2s+1-4}=\frac{1}{(s-1)^2-2^2}.$$

Since $1/(s^2-2^2)=\mathcal{L}\{\frac{1}{2}\sinh 2t\}$, we conclude from Property 2 that

$$\frac{1}{(s-1)^2-2^2}=\mathcal{L}\left\{\frac{1}{2}e^t\sinh 2t\right\}.$$

Consequently, from Property 3,

$$\frac{e^{-3s}}{s^2-2s-3}=\mathcal{L}\left\{\frac{1}{2}H_3(t)e^{t-3}\sinh 2(t-3)\right\}.$$

Example 3. Let $f(t)$ be the function which is t for $0\leqslant t<1$, and 0 for $t\geqslant 1$. Find the Laplace transform of f without performing any integrations.
Solution. Observe that $f(t)$ can be written in the form

$$f(t)=t\left[H_0(t)-H_1(t)\right]=t-tH_1(t).$$

Hence, from Property 1,

$$\begin{aligned}\mathcal{L}\{f(t)\}&=\mathcal{L}\{t\}-\mathcal{L}\{tH_1(t)\}\\&=\frac{1}{s^2}+\frac{d}{ds}\frac{e^{-s}}{s}=\frac{1}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}.\end{aligned}$$

Example 4. Solve the initial-value problem

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=f(t)=\begin{cases}1, & 0\leqslant t<1; \quad 0, \ 1\leqslant t<2;\\ 1, & 2\leqslant t<3; \quad 0, \ 3\leqslant t<4;\\ 1, & 4\leqslant t<5; \quad 0, \ 5\leqslant t<\infty.\end{cases} \qquad y(0)=0,\ y'(0)=0$$

Solution. Let $Y(s)=\mathcal{L}\{y(t)\}$ and $F(s)=\mathcal{L}\{f(t)\}$. Taking Laplace transforms of both sides of the differential equation gives $(s^2-3s+2)Y(s)=F(s)$, so that

$$Y(s)=\frac{F(s)}{s^2-3s+2}=\frac{F(s)}{(s-1)(s-2)}.$$

One way of computing $F(s)$ is to write $f(t)$ in the form

$$f(t)=\left[H_0(t)-H_1(t)\right]+\left[H_2(t)-H_3(t)\right]+\left[H_4(t)-H_5(t)\right].$$

Hence, by the linearity of the Laplace transform

$$F(s)=\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\frac{e^{-4s}}{s}-\frac{e^{-5s}}{s}.$$

A second way of computing $F(s)$ is to evaluate the integral

$$\int_0^\infty e^{-st}f(t)\,dt=\int_0^1 e^{-st}\,dt+\int_2^3 e^{-st}\,dt+\int_4^5 e^{-st}\,dt$$
$$=\frac{1-e^{-s}}{s}+\frac{e^{-2s}-e^{-3s}}{s}+\frac{e^{-4s}-e^{-5s}}{s}.$$

Consequently,

$$Y(s)=\frac{1-e^{-s}+e^{-2s}-e^{-3s}+e^{-4s}-e^{-5s}}{s(s-1)(s-2)}.$$

Our next step is to expand $1/s(s-1)(s-2)$ in partial fractions; i.e., we write

$$\frac{1}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}.$$

This implies that

$$A(s-1)(s-2)+Bs(s-2)+Cs(s-1)=1. \qquad (1)$$

Setting $s=0$ in (1) gives $A=\frac{1}{2}$; setting $s=1$ gives $B=-1$; and setting $s=2$ gives $C=\frac{1}{2}$. Thus,

$$\frac{1}{s(s-1)(s-2)}=\frac{1}{2}\frac{1}{s}-\frac{1}{s-1}+\frac{1}{2}\frac{1}{s-2}$$
$$=\mathcal{L}\left\{\frac{1}{2}-e^{t}+\frac{1}{2}e^{2t}\right\}.$$

Consequently, from Property 3,

$$\begin{aligned}y(t)=&\left[\tfrac{1}{2}-e^{t}+\tfrac{1}{2}e^{2t}\right]-H_1(t)\left[\tfrac{1}{2}-e^{(t-1)}+\tfrac{1}{2}e^{2(t-1)}\right]\\&+H_2(t)\left[\tfrac{1}{2}-e^{(t-2)}+\tfrac{1}{2}e^{2(t-2)}\right]-H_3(t)\left[\tfrac{1}{2}-e^{(t-3)}+\tfrac{1}{2}e^{2(t-3)}\right]\\&+H_4(t)\left[\tfrac{1}{2}-e^{(t-4)}+\tfrac{1}{2}e^{2(t-4)}\right]-H_5(t)\left[\tfrac{1}{2}-e^{(t-5)}+\tfrac{1}{2}e^{2(t-5)}\right].\end{aligned}$$

Remark. It is easily verified that the function

$$\tfrac{1}{2}-e^{(t-n)}+\tfrac{1}{2}e^{2(t-n)}$$

and its derivative are both zero at $t=n$. Hence, both $y(t)$ and $y'(t)$ are continuous functions of time, even though $f(t)$ is discontinuous at $t=1, 2, 3, 4$, and 5. More generally, both the solution $y(t)$ of the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t); \qquad y(t_0)=y_0,\quad y'(t_0)=y_0'$$

and its derivative $y'(t)$ are always continuous functions of time, if $f(t)$ is piecewise continuous. We will indicate the proof of this result in Section 2.12.

Exercises

Find the solution of each of the following initial-value problems.

1. $y''+2y'+y=2(t-3)H_3(t)$; $y(0)=2, y'(0)=1$

2. $y''+y'+y=H_\pi(t)-H_{2\pi}(t)$; $y(0)=1, y'(0)=0$

3. $y''+4y=\begin{cases} 1, & 0\leqslant t<4 \\ 0, & t>4 \end{cases}$; $y(0)=3, y'(0)=-2$

4. $y''+y=\begin{cases} \sin t, & 0\leqslant t<\pi \\ \cos t, & \pi\leqslant t<\infty \end{cases}$; $y(0)=1, y'(0)=0$

5. $y''+y=\begin{cases} \cos t, & 0\leqslant t<\pi/2 \\ 0, & \pi/2\leqslant t<\infty \end{cases}$; $y(0)=3, y'(0)=-1$

6. $y''+2y'+y=\begin{cases} \sin 2t, & 0\leqslant t<\pi/2 \\ 0, & \pi/2\leqslant t<\infty \end{cases}$; $y(0)=1, y'(0)=0$

7. $y''+y'+7y=\begin{cases} t, & 0\leqslant t<2 \\ 0, & 2\leqslant t<\infty \end{cases}$; $y(0)=0, y'(0)=0$

8. $y''+y=\begin{cases} t^2, & 0\leqslant t<1 \\ 0, & 1\leqslant t<\infty \end{cases}$; $y(0)=0, y'(0)=0$

9. $y''-2y'+y=\begin{cases} 0, & 0\leqslant t<1 \\ t, & 1\leqslant t<2 \\ 0, & 2\leqslant t<\infty \end{cases}$; $y(0)=0, y'(0)=1$

10. Find the Laplace transform of $|\sin t|$. *Hint*: Observe that

$$|\sin t|=\sin t+2\sum_{n=1}^{\infty} H_{n\pi}(t)\sin(t-n\pi).$$

11. Solve the initial-value problem of Example 4 by the method of judicious guessing. *Hint*: Find the general solution of the differential equation in each of the intervals $0<t<1$, $1<t<2$, $2<t<3$, $3<t<4$, $4<t<5$, $5<t<\infty$, and choose the arbitrary constants so that $y(t)$ and $y'(t)$ are continuous at the points $t=1$, 2, 3, 4, and 5.

2.12 The Dirac delta function

In many physical and biological applications we are often confronted with an initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t); \qquad y(0)=y_0, \quad y'(0)=y_0' \tag{1}$$

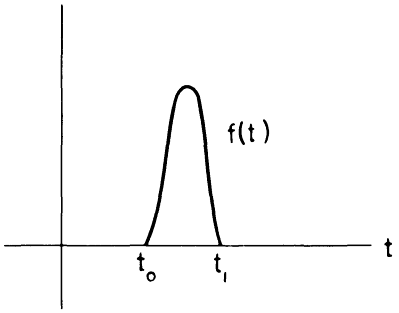

Figure 1. The graph of a typical impulsive function $f(t)$

where we do not know $f(t)$ explicitly. Such problems usually arise when we are dealing with phenomena of an impulsive nature. In these situations, the only information we have about $f(t)$ is that it is identically zero except for a very short time interval $t_0 \leqslant t \leqslant t_1$, and that its integral over this time interval is a given number $I_0 \neq 0$. If I_0 is not very small, then $f(t)$ will be quite large in the interval $t_0 \leqslant t \leqslant t_1$. Such functions are called impulsive functions, and the graph of a typical $f(t)$ is given in Figure 1.

In the early 1930's the Nobel Prize winning physicist P. A. M. Dirac developed a very controversial method for dealing with impulsive functions. His method is based on the following argument. Let t_1 get closer and closer to t_0. Then the function $f(t)/I_0$ approaches the function which is 0 for $t \neq t_0$, and ∞ for $t = t_0$, and whose integral over any interval containing t_0 is 1. We will denote this function, which is known as the Dirac delta function, by $\delta(t-t_0)$. Of course, $\delta(t-t_0)$ is not an ordinary function. However, says Dirac, let us formally operate with $\delta(t-t_0)$ as if it really were an ordinary function. Then, if we set $f(t) = I_0\delta(t-t_0)$ in (1) and impose the condition

$$\int_a^b g(t)\delta(t-t_0)\,dt = \begin{cases} g(t_0) & \text{if } a \leqslant t_0 \leqslant b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

for any continuous function $g(t)$, we will always obtain the correct solution $y(t)$.

Remark. Equation (2) is certainly a very reasonable condition to impose on $\delta(t-t_0)$. To see this, suppose that $f(t)$ is an impulsive function which is positive for $t_0 < t < t_1$, zero otherwise, and whose integral over the interval $[t_0, t_1]$ is 1. For any continuous function $g(t)$,

$$\left[\min_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t) \leqslant g(t)f(t) \leqslant \left[\max_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t).$$

Consequently,

$$\int_{t_0}^{t_1} \left[\min_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t)\,dt \leqslant \int_{t_0}^{t_1} g(t)f(t)\,dt \leqslant \int_{t_0}^{t_1} \left[\max_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t)\,dt,$$

or

$$\min_{t_0 \leqslant t \leqslant t_1} g(t) \leqslant \int_{t_0}^{t_1} g(t) f(t)\, dt \leqslant \max_{t_0 \leqslant t \leqslant t_1} g(t).$$

Thus, as $t_1 \to t_0$, $\int_{t_0}^{t_1} g(t) f(t)\, dt \to g(t_0)$.

Now, most mathematicians, of course, usually ridiculed this method. "How can you make believe that $\delta(t-t_0)$ is an ordinary function if it is obviously not," they asked. However, they never laughed too loud since Dirac and his followers always obtained the right answer. In the late 1940's, in one of the great success stories of mathematics, the French mathematician Laurent Schwartz succeeded in placing the delta function on a firm mathematical foundation. He accomplished this by enlarging the class of all functions so as to include the delta function. In this section we will first present a physical justification of the method of Dirac. Then we will illustrate how to solve the initial-value problem (1) by the method of Laplace transforms. Finally, we will indicate very briefly the "germ" of Laurent Schwartz's brilliant idea.

Physical justification of the method of Dirac. Newton's second law of motion is usually written in the form

$$\frac{d}{dt} mv(t) = f(t) \tag{3}$$

where m is the mass of the particle, v is its velocity, and $f(t)$ is the total force acting on the particle. The quantity mv is called the momentum of the particle. Integrating Equation (3) between t_0 and t_1 gives

$$mv(t_1) - mv(t_0) = \int_{t_0}^{t_1} f(t)\, dt.$$

This equation says that the change in momentum of the particle from time t_0 to time t_1 equals $\int_{t_0}^{t_1} f(t)\, dt$. Thus, the physically important quantity is the integral of the force, which is known as the impulse imparted by the force, rather than the force itself. Now, we may assume that $a > 0$ in Equation (1), for otherwise we can multiply both sides of the equation by -1 to obtain $a > 0$. In this case (see Section 2.6) we can view $y(t)$, for $t \leqslant t_0$, as the position at time t of a particle of mass a moving under the influence of the force $-b(dy/dt) - cy$. At time t_0 a force $f(t)$ is applied to the particle, and this force acts over an extremely short time interval $t_0 \leqslant t \leqslant t_1$. Since the time interval is extremely small, we may assume that the position of the particle does not change while the force $f(t)$ acts. Thus the sum result of the impulsive force $f(t)$ is that the velocity of the particle jumps by an amount I_0/a at time t_0. In other words, $y(t)$ satisfies the initial-value problem

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0; \qquad y(0) = y_0, \quad y'(0) = y_0'$$

for $0 \leqslant t < t_0$, and

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0; \qquad y(t_0)=z_0, \quad y'(t_0)=z_0'+\frac{I_0}{a} \tag{4}$$

for $t \geqslant t_0$, where z_0 and z_0' are the position and velocity of the particle just before the impulsive force acts. It is clear, therefore, that any method which correctly takes into account the momentum I_0 transferred to the particle at time t_0 by the impulsive force $f(t)$ must yield the correct answer. It is also clear that we always keep track of the momentum I_0 transferred to the particle by $f(t)$ if we replace $f(t)$ by $I_0\delta(t-t_0)$ and obey Equation (2). Hence the method of Dirac will always yield the correct answer.

Remark. We can now understand why any solution $y(t)$ of the differential equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t), \qquad f(t) \text{ a piecewise continuous function,}$$

is a continuous function of time even though $f(t)$ is discontinuous. To wit, since the integral of a piecewise continuous function is continuous, we see that $y'(t)$, must vary continuously with time. Consequently, $y(t)$ must also vary continuously with time.

Solution of Equation (1) *by the method of Laplace transforms.* In order to solve the initial-value problem (1) by the method of Laplace transforms, we need only know the Laplace transform of $\delta(t-t_0)$. This is obtained directly from the definition of the Laplace transform and Equation (2), for

$$\mathcal{L}\{\delta(t-t_0)\} \equiv \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0} \qquad (\text{for } t_0 \geqslant 0).$$

Example 1. Find the solution of the initial-value problem

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=3\delta(t-1)+\delta(t-2); \qquad y(0)=1, \quad y'(0)=1.$$

Solution. Let $Y(s)=\mathcal{L}\{y(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$s^2Y-s-1-4(sY-1)+4Y=3e^{-s}+e^{-2s}$$

or

$$(s^2-4s+4)Y(s)=s-3+3e^{-s}+e^{-2s}.$$

Consequently,

$$Y(s)=\frac{s-3}{(s-2)^2}+\frac{3e^{-s}}{(s-2)^2}+\frac{e^{-2s}}{(s-2)^2}.$$

Now, $1/(s-2)^2 = \mathfrak{L}\{te^{2t}\}$. Hence,

$$\frac{3e^{-s}}{(s-2)^2} + \frac{e^{-2s}}{(s-2)^2} = \mathfrak{L}\{3H_1(t)(t-1)e^{2(t-1)} + H_2(t)(t-2)e^{2(t-2)}\}.$$

To invert the first term of $Y(s)$, observe that

$$\frac{s-3}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2} = \mathfrak{L}\{e^{2t}\} - \mathfrak{L}\{te^{2t}\}.$$

Thus, $y(t) = (1-t)e^{2t} + 3H_1(t)(t-1)e^{2(t-1)} + H_2(t)(t-2)e^{2(t-2)}$.

It is instructive to do this problem the long way, that is, to find $y(t)$ separately in each of the intervals $0 \leqslant t < 1$, $1 \leqslant t < 2$ and $2 \leqslant t < \infty$. For $0 \leqslant t < 1$, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0; \qquad y(0) = 1, \quad y'(0) = 1.$$

The characteristic equation of this differential equation is $r^2 - 4r + 4 = 0$, whose roots are $r_1 = r_2 = 2$. Hence, any solution $y(t)$ must be of the form $y(t) = (a_1 + a_2 t)e^{2t}$. The constants a_1 and a_2 are determined from the initial conditions

$$1 = y(0) = a_1 \quad \text{and} \quad 1 = y'(0) = 2a_1 + a_2.$$

Hence, $a_1 = 1$, $a_2 = -1$ and $y(t) = (1-t)e^{2t}$ for $0 \leqslant t < 1$. Now $y(1) = 0$ and $y'(1) = -e^2$. At time $t = 1$ the derivative of $y(t)$ is suddenly increased by 3. Consequently, for $1 \leqslant t < 2$, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0; \qquad y(1) = 0, \quad y'(1) = 3 - e^2.$$

Since the initial conditions are given at $t = 1$, we write this solution in the form $y(t) = [b_1 + b_2(t-1)]e^{2(t-1)}$ (see Exercise 1). The constants b_1 and b_2 are determined from the initial conditions

$$0 = y(1) = b_1 \quad \text{and} \quad 3 - e^2 = y'(1) = 2b_1 + b_2.$$

Thus, $b_1 = 0$, $b_2 = 3 - e^2$ and $y(t) = (3 - e^2)(t-1)e^{2(t-1)}$, $1 \leqslant t < 2$. Now, $y(2) = (3 - e^2)e^2$ and $y'(2) = 3(3 - e^2)e^2$. At time $t = 2$ the derivative of $y(t)$ is suddenly increased by 1. Consequently, for $2 \leqslant t < \infty$, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0; \qquad y(2) = e^2(3 - e^2), \quad y'(2) = 1 + 3e^2(3 - e^2).$$

Hence $y(t) = [c_1 + c_2(t-2)]e^{2(t-2)}$. The constants c_1 and c_2 are determined from the equations

$$e^2(3 - e^2) = c_1 \quad \text{and} \quad 1 + 3e^2(3 - e^2) = 2c_1 + c_2.$$

Thus,

$$c_1 = e^2(3-e^2), \qquad c_2 = 1+3e^2(3-e^2)-2e^2(3-e^2) = 1+e^2(3-e^2)$$

and $y(t) = [e^2(3-e^2)+(1+e^2(3-e^2))(t-2)]e^{2(t-2)}$, $t \geq 2$. The reader should verify that this expression agrees with the expression obtained for $y(t)$ by the method of Laplace transforms.

Example 2. A particle of mass 1 is attached to a spring dashpot mechanism. The stiffness constant of the spring is 1 N/ft and the drag force exerted by the dashpot mechanism on the particle is twice its velocity. At time $t=0$, when the particle is at rest, an external force e^{-t} is applied to the system. At time $t=1$, an additional force $f(t)$ of very short duration is applied to the particle. This force imparts an impulse of 3 N·s to the particle. Find the position of the particle at any time t greater than 1.

Solution. Let $y(t)$ be the distance of the particle from its equilibrium position. Then, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t} + 3\delta(t-1); \qquad y(0)=0, \quad y'(0)=0.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$(s^2+2s+1)Y(s) = \frac{1}{s+1} + 3e^{-s}, \quad \text{or} \quad Y(s) = \frac{1}{(s+1)^3} + \frac{3e^{-s}}{(s+1)^2}.$$

Since

$$\frac{1}{(s+1)^3} = \mathcal{L}\left\{\frac{t^2e^{-t}}{2}\right\} \quad \text{and} \quad \frac{3e^{-s}}{(s+1)^2} = 3\mathcal{L}\left\{H_1(t)(t-1)e^{-(t-1)}\right\}$$

we see that

$$y(t) = \frac{t^2e^{-t}}{2} + 3H_1(t)(t-1)e^{-(t-1)}.$$

Consequently, $y(t) = \frac{1}{2}t^2e^{-t} + 3(t-1)e^{-(t-1)}$ for $t > 1$.

We conclude this section with a very brief description of Laurent Schwartz's method for placing the delta function on a rigorous mathematical foundation. The main step in his method is to rethink our notion of "function." In Calculus, we are taught to recognize a function by its value at each time t. A much more subtle (and much more difficult) way of recognizing a function is by what it does to other functions. More precisely, let f be a piecewise continuous function defined for $-\infty < t < \infty$. To each function ϕ which is infinitely often differentiable and which vanishes for $|t|$

sufficiently large, we assign a number $K[\phi]$ according to the formula

$$K[\phi]=\int_{-\infty}^{\infty}\phi(t)f(t)\,dt. \tag{5}$$

As the notation suggests, K is an operator acting on functions. However, it differs from the operators introduced previously in that it associates a number, rather than a function, with ϕ. For this reason, we say that $K[\phi]$ is a functional, rather than a function. Now, observe that the association $\phi\to K[\phi]$ is a linear association, since

$$\begin{aligned}K[c_1\phi_1+c_2\phi_2]&=\int_{-\infty}^{\infty}(c_1\phi_1+c_2\phi_2)(t)f(t)\,dt\\&=c_1\int_{-\infty}^{\infty}\phi_1(t)f(t)\,dt+c_2\int_{-\infty}^{\infty}\phi_2(t)f(t)\,dt\\&=c_1K[\phi_1]+c_2K[\phi_2].\end{aligned}$$

Hence every piecewise continuous function defines, through (5), a linear functional on the space of all infinitely often differentiable functions which vanish for $|t|$ sufficiently large.

Now consider the functional $K[\phi]$ defined by the relation $K[\phi]=\phi(t_0)$. K is a linear functional since

$$K[c_1\phi_1+c_2\phi_2]=c_1\phi_1(t_0)+c_2\phi_2(t_0)=c_1K[\phi_1]+c_2K[\phi_2].$$

To mimic (5), we write K symbolically in the form

$$K[\phi]=\int_{-\infty}^{\infty}\phi(t)\delta(t-t_0)\,dt. \tag{6}$$

In this sense, $\delta(t-t_0)$ is a "generalized function." It is important to realize though, that we cannot speak of the value of $\delta(t-t_0)$ at any time t. The only meaningful quantity is the expression $\int_{-\infty}^{\infty}\phi(t)\delta(t-t_0)\,dt$, and we must always assign the value $\phi(t_0)$ to this expression.

Admittedly, it is very difficult to think of a function in terms of the linear functional (5) that it induces. The advantage to this way of thinking, though, is that it is now possible to assign a derivative to every piecewise continuous function and to every "generalized function." To wit, suppose that $f(t)$ is a differentiable function. Then $f'(t)$ induces the linear functional

$$K'[\phi]=\int_{-\infty}^{\infty}\phi(t)f'(t)\,dt. \tag{7}$$

Integrating by parts and using the fact that $\phi(t)$ vanishes for $|t|$ sufficiently large, we see that

$$K'[\phi]=\int_{-\infty}^{\infty}[-\phi'(t)]f(t)\,dt=K[-\phi']. \tag{8}$$

Now, notice that the formula $K'[\phi]=K[-\phi']$ makes sense even if $f(t)$ is not differentiable. This motivates the following definition.

Definition. To every linear functional $K[\phi]$ we assign the new linear functional $K'[\phi]$ by the formula $K'[\phi]=K[-\phi']$. The linear functional $K'[\phi]$ is called the derivative of $K[\phi]$ since if $K[\phi]$ is induced by a differentiable function $f(t)$ then $K'[\phi]$ is induced by $f'(t)$.

Finally, we observe from (8) that the derivative of the delta function $\delta(t-t_0)$ is the linear functional which assigns to each function ϕ the number $-\phi'(t_0)$, for if $K[\phi]=\phi(t_0)$ then $K'[\phi]=K[-\phi']=-\phi'(t_0)$. Thus,

$$\int_{-\infty}^{\infty}\phi(t)\delta'(t-t_0)\,dt=-\phi'(t_0)$$

for all differentiable functions $\phi(t)$.

Exercises

1. Let a be a fixed constant. Show that every solution of the differential equation $(d^2y/dt^2)+2\alpha(dy/dt)+\alpha^2y=0$ can be written in the form
$$y(t)=[c_1+c_2(t-a)]e^{-\alpha(t-a)}.$$

2. Solve the initial-value problem $(d^2y/dt^2)+4(dy/dt)+5y=f(t)$; $y(0)=1, y'(0)=0$, where $f(t)$ is an impulsive force which acts on the extremely short time interval $1\leqslant t\leqslant 1+\tau$, and $\int_1^{1+\tau}f(t)\,dt=2$.

3. (a) Solve the initial-value problem $(d^2y/dt^2)-3(dy/dt)+2y=f(t)$; $y(0)=1$, $y'(0)=0$, where $f(t)$ is an impulsive function which acts on the extremely short time interval $2\leqslant t\leqslant 2+\tau$, and $\int_2^{2+\tau}f(t)\,dt=-1$.
(b) Solve the initial-value problem $(d^2y/dt^2)-3(dy/dt)+2y=0$; $y(0)=1$, $y'(0)=0$, on the interval $0\leqslant t\leqslant 2$. Compute $z_0=y(2)$ and $z_0'=y'(2)$. Then solve the initial-value problem
$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=0;\qquad y(2)=z_0,\ \ y'(2)=z_0'-1,\qquad 2\leqslant t<\infty.$$
Compare this solution with the solution of part (a).

4. A particle of mass 1 is attached to a spring dashpot mechanism. The stiffness constant of the spring is 3 N/m and the drag force exerted on the particle by the dashpot mechanism is 4 times its velocity. At time $t=0$, the particle is stretched $\frac{1}{4}$ m from its equilibrium position. At time $t=3$ seconds, an impulsive force of very short duration is applied to the system. This force imparts an impulse of 2 N·s to the particle. Find the displacement of the particle from its equilibrium position.

In Exercises 5–7 solve the given initial-value problem.

5. $\dfrac{d^2y}{dt^2}+y=\sin t+\delta(t-\pi);\quad y(0)=0, y'(0)=0$

6. $\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+y=2\delta(t-1)-\delta(t-2);\quad y(0)=1, y'(0)=0$

7. $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+y=e^{-t}+3\delta(t-3);\quad y(0)=0, y'(0)=3$

8. (a) Solve the initial-value problem

$$\frac{d^2y}{dt^2}+y=\sum_{j=0}^{\infty}\delta(t-j\pi),\qquad y(0)=y'(0)=0,$$

and show that

$$y(t)=\begin{cases}\sin t, & n \text{ even}\\ 0, & n \text{ odd}\end{cases}$$

in the interval $n\pi<t<(n+1)\pi$.

(b) Solve the initial-value problem

$$\frac{d^2y}{dt^2}+y=\sum_{j=0}^{\infty}\delta(t-2j\pi),\qquad y(0)=y'(0)=0,$$

and show that $y(t)=(n+1)\sin t$ in the interval $2n\pi<t<2(n+1)\pi$.

This example indicates why soldiers are instructed to break cadence when marching across a bridge. To wit, if the soldiers are in step with the natural frequency of the steel in the bridge, then a resonance situation of the type (b) may be set up.

9. Let $f(t)$ be the function which is $\frac{1}{2}$ for $t>t_0$, 0 for $t=t_0$, and $-\frac{1}{2}$ for $t<t_0$. Let $K[\phi]$ be the linear functional

$$K[\phi]=\int_{-\infty}^{\infty}\phi(t)f(t)\,dt.$$

Show that $K'[\phi]\equiv K[-\phi']=\phi(t_0)$. Thus, $\delta(t-t_0)$ may be viewed as the derivative of $f(t)$.

2.13 The convolution integral

Consider the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t);\qquad y(0)=y_0,\quad y'(0)=y_0'. \tag{1}$$

Let $Y(s)=\mathcal{L}\{y(t)\}$ and $F(s)=\mathcal{L}\{f(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$a\left[s^2Y(s)-sy_0-y_0'\right]+b\left[sY(s)-y_0\right]+cY(s)=F(s)$$

and this implies that

$$Y(s)=\frac{as+b}{as^2+bs+c}y_0+\frac{a}{as^2+bs+c}y_0'+\frac{F(s)}{as^2+bs+c}.$$

Now, let

$$y_1(t)=\mathcal{L}^{-1}\left\{\frac{as+b}{as^2+bs+c}\right\}$$

and

$$y_2(t)=\mathcal{L}^{-1}\left\{\frac{a}{as^2+bs+c}\right\}.$$

Setting $f(t)=0$, $y_0=1$ and $y_0'=0$, we see that $y_1(t)$ is the solution of the homogeneous equation which satisfies the initial conditions $y_1(0)=1$, $y_1'(0)=0$. Similarly, by setting $f(t)=0$, $y_0=0$ and $y_0'=1$, we see that $y_2(t)$ is the solution of the homogeneous equation which satisfies the initial conditions $y_2(0)=0$, $y_2'(0)=1$. This implies that

$$\psi(t)=\mathcal{L}^{-1}\left\{\frac{F(s)}{as^2+bs+c}\right\}$$

is the particular solution of the nonhomogeneous equation which satisfies the initial conditions $\psi(0)=0$, $\psi'(0)=0$. Thus, the problem of finding a particular solution $\psi(t)$ of the nonhomogeneous equation is now reduced to the problem of finding the inverse Laplace transform of the function $F(s)/(as^2+bs+c)$. If we look carefully at this function, we see that it is the product of two Laplace transforms; that is

$$\frac{F(s)}{as^2+bs+c}=\mathcal{L}\{f(t)\}\times\mathcal{L}\left\{\frac{y_2(t)}{a}\right\}.$$

It is natural to ask whether there is any simple relationship between $\psi(t)$ and the functions $f(t)$ and $y_2(t)/a$. It would be nice, of course, if $\psi(t)$ were the product of $f(t)$ with $y_2(t)/a$, but this is obviously false. However, there is an extremely interesting way of combining two functions f and g together to form a new function $f*g$, which resembles multiplication, and for which

$$\mathcal{L}\{(f*g)(t)\}=\mathcal{L}\{f(t)\}\times\mathcal{L}\{g(t)\}.$$

This combination of f and g appears quite often in applications, and is known as the *convolution* of f with g.

Definition. The *convolution* $(f*g)(t)$ of f with g is defined by the equation

$$(f*g)(t)=\int_0^t f(t-u)\,g(u)\,du. \tag{2}$$

For example, if $f(t)=\sin 2t$ and $g(t)=e^{t^2}$, then

$$(f*g)(t)=\int_0^t \sin 2(t-u)e^{u^2}\,du.$$

The convolution operator $*$ clearly bears some resemblance to the multiplication operator since we multiply the value of f at the point $t-u$ by the value of g at the point u, and then integrate this product with respect to u. Therefore, it should not be too surprising to us that the convolution operator satisfies the following properties.

Property 1. The convolution operator obeys the commutative law of multiplication; that is, $(f*g)(t)=(g*f)(t)$.

PROOF. By definition,

$$(f*g)(t)=\int_0^t f(t-u)\,g(u)\,du.$$

Let us make the substitution $t-u=s$ in this integral. Then,

$$\begin{aligned}(f*g)(t)&=-\int_t^0 f(s)\,g(t-s)\,ds\\&=\int_0^t g(t-s)f(s)\,ds\equiv(g*f)(t).\end{aligned}$$ □

Property 2. The convolution operator satisfies the distributive law of multiplication; that is,

$$f*(g+h)=f*g+f*h.$$

PROOF. See Exercise 19. □

Property 3. The convolution operator satisfies the associative law of multiplication; that is, $(f*g)*h=f*(g*h)$.

PROOF. See Exercise 20. □

Property 4. The convolution of any function f with the zero function is zero.

PROOF. Obvious. □

On the other hand, the convolution operator differs from the multiplication operator in that $f*1\neq f$ and $f*f\neq f^2$. Indeed, the convolution of a function f with itself may even be negative.

Example 1. Compute the convolution of $f(t)=t^2$ with $g(t)=1$.

Solution. From Property 1,

$$(f*g)(t)=(g*f)(t)=\int_0^t 1\cdot u^2\,du=\frac{t^3}{3}.$$

Example 2. Compute the convolution of $f(t)=\cos t$ with itself, and show that it is not always positive.
Solution. By definition,

$$\begin{aligned}(f*f)(t)&=\int_0^t \cos(t-u)\cos u\,du\\ &=\int_0^t(\cos t\cos^2 u+\sin t\sin u\cos u)\,du\\ &=\cos t\int_0^t \frac{1+\cos 2u}{2}\,du+\sin t\int_0^t \sin u\cos u\,du\\ &=\cos t\left[\frac{t}{2}+\frac{\sin 2t}{4}\right]+\frac{\sin^3 t}{2}\\ &=\frac{t\cos t+\sin t\cos^2 t+\sin^3 t}{2}\\ &=\frac{t\cos t+\sin t(\cos^2 t+\sin^2 t)}{2}\\ &=\frac{t\cos t+\sin t}{2}.\end{aligned}$$

This function, clearly, is negative for

$$(2n+1)\pi\leqslant t\leqslant(2n+1)\pi+\tfrac{1}{2}\pi,\qquad n=0,1,2,\ldots.$$

We now show that the Laplace transform of $f*g$ is the product of the Laplace transform of f with the Laplace transform of g.

Theorem 9. $\mathcal{L}\{(f*g)(t)\}=\mathcal{L}\{f(t)\}\times\mathcal{L}\{g(t)\}$.

PROOF. By definition,

$$\mathcal{L}\{(f*g)(t)\}=\int_0^\infty e^{-st}\left[\int_0^t f(t-u)g(u)\,du\right]dt.$$

This iterated integral equals the double integral

$$\iint_R e^{-st}f(t-u)g(u)\,du\,dt$$

where R is the triangular region described in Figure 1. Integrating first

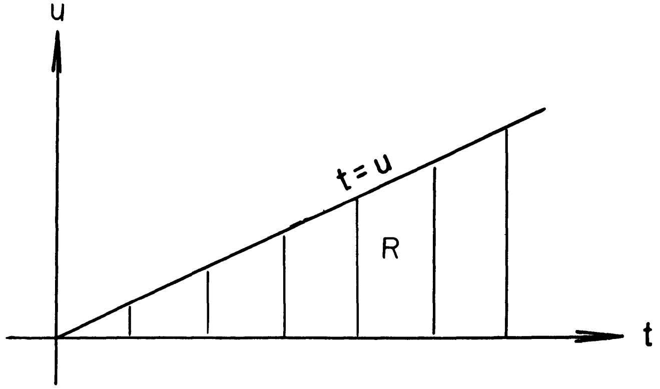

Figure 1

with respect to t, instead of u, gives

$$\mathcal{L}\{(f*g)(t)\}=\int_0^\infty g(u)\left[\int_u^\infty e^{-st}f(t-u)\,dt\right]du.$$

Setting $t-u=\xi$, we see that

$$\int_u^\infty e^{-st}f(t-u)\,dt=\int_0^\infty e^{-s(u+\xi)}f(\xi)\,d\xi.$$

Hence,

$$\begin{aligned}\mathcal{L}\{(f*g)(t)\}&=\int_0^\infty g(u)\left[\int_0^\infty e^{-su}e^{-s\xi}f(\xi)\,d\xi\right]du\\&=\left[\int_0^\infty g(u)e^{-su}\,du\right]\left[\int_0^\infty e^{-s\xi}f(\xi)\,d\xi\right]\\&\equiv\mathcal{L}\{f(t)\}\times\mathcal{L}\{g(t)\}.\end{aligned}$$ □

Example 3. Find the inverse Laplace transform of the function

$$\frac{a}{s^2(s^2+a^2)}.$$

Solution. Observe that

$$\frac{1}{s^2}=\mathcal{L}\{t\}\quad\text{and}\quad\frac{a}{s^2+a^2}=\mathcal{L}\{\sin at\}.$$

Hence, by Theorem 9

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\}=\int_0^t(t-u)\sin au\,du$$

$$=\frac{at-\sin at}{a^2}.$$

Example 4. Find the inverse Laplace transform of the function

$$\frac{1}{s(s^2+2s+2)}.$$

Solution. Observe that

$$\frac{1}{s}=\mathcal{L}\{1\} \quad \text{and} \quad \frac{1}{s^2+2s+2}=\frac{1}{(s+1)^2+1}=\mathcal{L}\{e^{-t}\sin t\}.$$

Hence, by Theorem 9,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}=\int_0^t e^{-u}\sin u\,du$$

$$=\frac{1}{2}\left[1-e^{-t}(\cos t+\sin t)\right].$$

Remark. Let $y_2(t)$ be the solution of the homogeneous equation $ay''+by'+cy=0$ which satisfies the initial conditions $y_2(0)=0, y_2'(0)=1$. Then,

$$\psi(t)=f(t)*\frac{y_2(t)}{a} \tag{3}$$

is the particular solution of the nonhomogeneous equation $ay''+by'+cy=f(t)$ which satisfies the initial conditions $\psi(0)=\psi'(0)=0$. Equation (3) is often much simpler to use than the variation of parameters formula derived in Section 2.4.

Exercises

Compute the convolution of each of the following pairs of functions.

1. $e^{at}, e^{bt}, a\neq b$ **2.** e^{at}, e^{at}

3. $\cos at, \cos bt$ **4.** $\sin at, \sin bt, a\neq b$

5. $\sin at, \sin at$ **6.** $t, \sin t$

Use Theorem 9 to invert each of the following Laplace transforms.

7. $\frac{1}{s^2(s^2+1)}$ **8.** $\frac{s}{(s+1)(s^2+4)}$ **9.** $\frac{s}{(s^2+1)^2}$

10. $\frac{1}{s(s^2+1)}$ **11.** $\frac{1}{s^2(s+1)^2}$ **12.** $\frac{1}{(s^2+1)^2}$

Use Theorem 9 to find the solution $y(t)$ of each of the following integro-differential equations.

13. $y(t)=4t-3\int_0^t y(u)\sin(t-u)\,du$

14. $y(t)=4t-3\int_0^t y(t-u)\sin u\,du$

15. $y'(t)=\sin t+\int_0^t y(t-u)\cos u\,du,\ y(0)=0$

16. $y(t)=4t^2-\int_0^t y(u)e^{-(t-u)}du$

17. $y'(t)+2y+\int_0^t y(u)du=\sin t,\ y(0)=1$

18. $y(t)=t-e^t\int_0^t y(u)e^{-u}du$

19. Prove that $f*(g+h)=f*g+f*h$.

20. Prove that $(f*g)*h=f*(g*h)$.

2.14 The method of elimination for systems

The theory of second-order linear differential equations can also ɒe used to find the solutions of two simultaneous first-order equations of the form

$$\begin{aligned} x'&=\frac{dx}{dt}=a(t)x+b(t)y+f(t)\\ y'&=\frac{dy}{dt}=c(t)x+d(t)y+g(t). \end{aligned} \tag{1}$$

The key idea is to eliminate one of the variables, say y, and then find x as the solution of a second-order linear differential equation. This technique is known as the *method of elimination*, and we illustrate it with the following two examples.

Example 1. Find all solutions of the simultaneous equations

$$\begin{aligned} x'&=2x+y+t\\ y'&=x+3y+1. \end{aligned} \tag{2}$$

Solution. First, we solve for

$$y=x'-2x-t \tag{3}$$

from the first equation of (2). Differentiating this equation gives

$$y'=x''-2x'-1=x+3y+1.$$

Then, substituting for y from (3) gives

$$x''-2x'-1=x+3(x'-2x-t)+1$$

so that

$$x''-5x'+5x=2-3t. \tag{4}$$

Equation (4) is a second-order linear equation and its solution is

$$x(t)=e^{5t/2}\left[c_1e^{\sqrt{5}\,t/2}+c_2e^{-\sqrt{5}\,t/2}\right]-\frac{(1+3t)}{5}$$

for some constants c_1 and c_2. Finally, plugging this expression into (3) gives

$$y(t)=e^{5t/2}\left[\frac{1+\sqrt{5}}{2}c_1e^{\sqrt{5}\,t/2}+\frac{1-\sqrt{5}}{2}c_2e^{-\sqrt{5}\,t/2}\right]+\frac{t-1}{5}.$$

Example 2. Find the solution of the initial-value problem

$$\begin{aligned} x'&=3x-y, \qquad x(0)=3\\ y'&=x+y, \qquad y(0)=0. \end{aligned} \tag{5}$$

Solution. From the first equation of (5),

$$y=3x-x'. \tag{6}$$

Differentiating this equation gives

$$y'=3x'-x''=x+y.$$

Then, substituting for y from (6) gives

$$3x'-x''=x+3x-x'$$

so that

$$x''-4x'+4x=0.$$

This implies that

$$x(t)=(c_1+c_2t)e^{2t}$$

for some constants c_1, c_2, and plugging this expression into (6) gives

$$y(t)=(c_1-c_2+c_2t)e^{2t}.$$

The constants c_1 and c_2 are determined from the initial conditions

$$\begin{aligned} x(0)&=3=c_1\\ y(0)&=0=c_1-c_2. \end{aligned}$$

Hence $c_1=3$, $c_2=3$ and

$$x(t)=3(1+t)e^{2t}, y(t)=3te^{2t}$$

is the solution of (5).

Remark. The simultaneous equations (1) are usually referred to as a first-order *system* of equations. Systems of equations are treated fully in Chapters 3 and 4.

Exercises

Find all solutions of each of the following systems of equations.

1. $x'=6x-3y$
$y'=2x+y$

2. $x'=-2x+y+t$
$y'=-4x+3y-1$

3. $x' = -3x + 2y$
   $y' = -x - y$

4. $x' = x + y + e^t$
   $y' = x - y - e^t$

Find the solution of each of the following initial-value problems.

5. $x' = x + y, \quad x(0) = 2$
   $y' = 4x + y, \quad y(0) = 3$

6. $x' = x - 3y, \quad x(0) = 0$
   $y' = -2x + 2y, \quad y(0) = 5$

7. $x' = x - y, \quad x(0) = 1$
   $y' = 5x - 3y, \quad y(0) = 2$

8. $x' = 3x - 2y, \quad x(0) = 1$
   $y' = 4x - y, \quad y(0) = 5$

9. $x' = 4x + 5y + 4e^t \cos t, \quad x(0) = 0$
   $y' = -2x - 2y, \quad y(0) = 0$

10. $x' = 3x - 4y + e^t, \quad x(0) = 1$
    $y' = x - y + e^t, \quad y(0) = 1$

11. $x' = 2x - 5y + \sin t, \quad x(0) = 0$
    $y' = x - 2y + \tan t, \quad y(0) = 0$

12. $x' = y + f_1(t), \quad x(0) = 0$
    $y' = -x + f_2(t), \quad y(0) = 0$

2.15 Higher-order equations

In this section we briefly discuss higher-order linear differential equations.

Definition. The equation

$$L[y] = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0(t)y = 0, \quad a_n(t) \neq 0 \qquad (1)$$

is called the general nth order homogeneous linear equation. The differential equation (1) together with the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)} \qquad (1')$$

is called an initial-value problem. The theory for Equation (1) is completely analogous to the theory for the second-order linear homogeneous equation which we studied in Sections 2.1 and 2.2. Therefore, we will state the relevant theorems without proof. Complete proofs can be obtained by generalizing the methods used in Sections 2.1 and 2.2, or by using the methods to be developed in Chapter 3.

Theorem 10. *Let $y_1(t), \ldots, y_n(t)$ be n independent solutions of (1); that is, no solution $y_j(t)$ is a linear combination of the other solutions. Then, every solution $y(t)$ of (1) is of the form*

$$y(t) = c_1 y_1(t) + \ldots + c_n y_n(t) \qquad (2)$$

for some choice of constants $c_1, \ldots, c_n$. For this reason, we say that (2) is the general solution of (1).

To find n independent solutions of (1) when the coefficients $a_0, a_1, \ldots, a_n$ do not depend on t, we compute

$$L[e^{rt}] = (a_n r^n + a_{n-1} r^{n-1} + \ldots + a_0)e^{rt}. \qquad (3)$$

This implies that e^{rt} is a solution of (1) if, and only if, r is a root of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_0 = 0. \tag{4}$$

Thus, if Equation (4) has n distinct roots $r_1, \ldots, r_n$, then the general solution of (1) is $y(t) = c_1 e^{r_1 t} + \ldots + c_n e^{r_n t}$. If $r_j = \alpha_j + i\beta_j$ is a complex root of (4), then

$$u(t) = \mathrm{Re}\{e^{r_j t}\} = e^{\alpha_j t}\cos\beta_j t$$

and

$$v(t) = \mathrm{Im}\{e^{r_j t}\} = e^{\alpha_j t}\sin\beta_j t$$

are two real-valued solutions of (1). Finally, if r_1 is a root of multiplicity k; that is, if

$$a_n r^n + \ldots + a_0 = (r - r_1)^k q(r)$$

where $q(r_1) \neq 0$, then $e^{r_1 t}, te^{r_1 t}, \ldots, t^{k-1}e^{r_1 t}$ are k independent solutions of (1). We prove this last assertion in the following manner. Observe from (3) that

$$L[e^{rt}] = (r - r_1)^k q(r) e^{rt}$$

if r_1 is a root of multiplicity k. Therefore,

$$\begin{aligned} L[t^j e^{r_1 t}] &= L\left[\frac{\partial^j}{\partial r^j} e^{rt}\right]\Bigg|_{r=r_1} \\ &= \frac{\partial^j}{\partial r^j} L[e^{rt}]\Bigg|_{r=r_1} \\ &= \frac{\partial^j}{\partial r^j}(r - r_1)^k q(r) e^{rt}\Bigg|_{r=r_1} \\ &= 0, \text{ for } 1 \leqslant j < k. \end{aligned}$$

Example 1. Find the general solution of the equation

$$\frac{d^4 y}{dt^4} + y = 0. \tag{5}$$

Solution. The characteristic equation of (5) is $r^4 + 1 = 0$. We find the roots of this equation by noting that

$$-1 = e^{i\pi} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i}.$$

Hence,

$$r_1 = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i),$$

$$r_2 = e^{3\pi i/4} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}(1-i),$$

$$r_3 = e^{5\pi i/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}(1+i),$$

and

$$r_4 = e^{7\pi i/4} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}}(1-i)$$

are 4 roots of the equation $r^4 + 1 = 0$. The roots r_3 and r_4 are the complex conjugates of r_2 and r_1, respectively. Thus,

$$e^{r_1 t} = e^{t/\sqrt{2}}\left[\cos\frac{t}{\sqrt{2}} + i\sin\frac{t}{\sqrt{2}}\right]$$

and

$$e^{r_2 t} = e^{-t/\sqrt{2}}\left[\cos\frac{t}{\sqrt{2}} + i\sin\frac{t}{\sqrt{2}}\right]$$

are 2 complex-valued solutions of (5), and this implies that

$$y_1(t) = e^{t/\sqrt{2}}\cos\frac{t}{\sqrt{2}}, \qquad y_2(t) = e^{t/\sqrt{2}}\sin\frac{t}{\sqrt{2}},$$
$$y_3(t) = e^{-t/\sqrt{2}}\cos\frac{t}{\sqrt{2}}, \quad \text{and} \quad y_4(t) = e^{-t/\sqrt{2}}\sin\frac{t}{\sqrt{2}}$$

are 4 real-valued solutions of (5). These solutions are clearly independent. Hence, the general solution of (5) is

$$y(t) = e^{t/\sqrt{2}}\left[a_1\cos\frac{t}{\sqrt{2}} + b_1\sin\frac{t}{\sqrt{2}}\right] + e^{-t/\sqrt{2}}\left[a_2\cos\frac{t}{\sqrt{2}} + b_2\sin\frac{t}{\sqrt{2}}\right].$$

Example 2. Find the general solution of the equation

$$\frac{d^4y}{dt^4} - 3\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} - \frac{dy}{dt} = 0. \tag{6}$$

Solution. The characteristic equation of (6) is

$$0 = r^4 - 3r^3 + 3r^2 - r = r(r^3 - 3r^2 + 3r - 1)$$
$$= r(r-1)^3.$$

Its roots are $r_1 = 0$ and $r_2 = 1$, with $r_2 = 1$ a root of multiplicity three. Hence, the general solution of (6) is

$$y(t) = c_1 + (c_2 + c_3 t + c_4 t^2)e^t.$$

The theory for the nonhomogeneous equation

$$L[y] = a_n(t)\frac{d^ny}{dt^n} + \dots + a_0(t)y = f(t), \quad a_n(t) \neq 0 \tag{7}$$

is also completely analogous to the theory for the second-order nonhomogeneous equation. The following results are the analogs of Lemma 1 and Theorem 5 of Section 2.3.

Lemma 1. *The difference of any two solutions of the nonhomogeneous equation* (7) *is a solution of the homogeneous equation* (1).

Theorem 11. *Let $\psi(t)$ be a particular solution of the nonhomogeneous equation* (7), *and let $y_1(t),\dots,y_n(t)$ be n independent solutions of the homogeneous equation* (1). *Then, every solution $y(t)$ of* (7) *is of the form*

$$y(t)=\psi(t)+c_1y_1(t)+\dots+c_ny_n(t)$$

for some choice of constants $c_1, c_2,\dots,c_n$.

The method of judicious guessing also applies to the nth-order equation

$$a_n\frac{d^ny}{dt^n}+\dots+a_0y=\left[b_0+b_1t+\dots+b_kt^k\right]e^{\alpha t}. \tag{8}$$

It is easily verified that Equation (8) has a particular solution $\psi(t)$ of the form

$$\psi(t)=\left[c_0+c_1t+\dots+c_kt^k\right]e^{\alpha t}$$

if $e^{\alpha t}$ is not a solution of the homogeneous equation, and

$$\psi(t)=t^j\left[c_0+c_1t+\dots+c_kt^k\right]e^{\alpha t}$$

if $t^{j-1}e^{\alpha t}$ is a solution of the homogeneous equation, but $t^je^{\alpha t}$ is not.

Example 3. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+3\frac{dy}{dt}+y=e^t. \tag{9}$$

Solution. The characteristic equation

$$r^3+3r^2+3r+1=(r+1)^3$$

has $r=-1$ as a triple root. Hence, e^t is not a solution of the homogeneous equation, and Equation (9) has a particular solution $\psi(t)$ of the form

$$\psi(t)=Ae^t.$$

Computing $L[\psi](t)=8Ae^t$, we see that $A=\frac{1}{8}$. Consequently, $\psi(t)=\frac{1}{8}e^t$ is a particular solution of (9).

There is also a variation of parameters formula for the nonhomogeneous equation (7). Let $v(t)$ be the solution of the homogeneous equation

(1) which satisfies the initial conditions $v(t_0)=0$, $v'(t_0)=0,\dots,v^{(n-2)}(t_0)=0$, $v^{(n-1)}(t_0)=1$. Then,

$$\psi(t)=\int_{t_0}^{t}\frac{v(t-s)}{a_n(s)}f(s)\,ds$$

is a particular solution of the nonhomogeneous equation (7). We will prove this assertion in Section 3.12. (This can also be proven using the method of Laplace transforms; see Section 2.13.)

Exercises

Find the general solution of each of the following equations.

1. $y'''-2y''-y'+2y=0$

2. $y'''-6y''+5y'+12y=0$

3. $y^{(iv)}-5y'''+6y''+4y'-8y=0$

4. $y'''-y''+y'-y=0$

Solve each of the following initial-value problems.

5. $y^{(iv)}+4y'''+14y''-20y'+25y=0;\quad y(0)=y'(0)=y''(0)=0,\ y'''(0)=0$

6. $y^{(iv)}-y=0;\quad y(0)=1,\ y'(0)=y''(0)=0,\ y'''(0)=-1$

7. $y^{(v)}-2y^{(iv)}+y'''=0;\quad y(0)=y'(0)=y''(0)=y'''(0)=0,\ y^{(iv)}(0)=-1$

8. Given that $y_1(t)=e^t\cos t$ is a solution of

$$y^{(iv)}-2y'''+y''+2y'-2y=0, \tag{*}$$

find the general solution of (*). *Hint*: Use this information to find the roots of the characteristic equation of (*).

Find a particular solution of each of the following equations.

9. $y'''+y'=\tan t$

10. $y^{(iv)}-y=g(t)$

11. $y^{(iv)}+y=g(t)$

12. $y'''+y'=2t^2+4\sin t$

13. $y'''-4y'=t+\cos t+2e^{-2t}$

14. $y^{(iv)}-y=t+\sin t$

15. $y^{(iv)}+2y''+y=t^2\sin t$

16. $y^{(vi)}+y''=t^2$

17. $y'''+y''+y'+y=t+e^{-t}$

18. $y^{(iv)}+4y'''+6y''+4y'+y=t^3e^{-t}$

Hint for (18): Make the substitution $y=e^{-t}v$ and solve for v. Otherwise, it will take an awfully long time to do this problem.