5 Separation of variables and Fourier series

5.1 Two point boundary-value problems

In the applications which we will study in this chapter, we will be confronted with the following problem.

Problem: For which values of λ can we find nontrivial functions $y(x)$ which satisfy

$$\frac{d^2y}{dx^2}+\lambda y=0; \qquad ay(0)+by'(0)=0, \quad cy(l)+dy'(l)=0? \tag{1}$$

Equation (1) is called a boundary-value problem, since we prescribe information about the solution $y(x)$ and its derivative $y'(x)$ at two distinct points, $x=0$ and $x=l$. In an initial-value problem, on the other hand, we prescribe the value of y and its derivative at a single point $x=x_0$.

Our intuitive feeling, at this point, is that the boundary-value problem (1) has nontrivial solutions $y(x)$ only for certain exceptional values λ. To wit, $y(x)=0$ is certainly one solution of (1), and the existence–uniqueness theorem for second-order linear equations would seem to imply that a solution $y(x)$ of $y''+\lambda y=0$ is determined uniquely once we prescribe two additional pieces of information. Let us test our intuition on the following simple, but extremely important example.

Example 1. For which values of λ does the boundary-value problem

$$\frac{d^2y}{dx^2}+\lambda y=0; \qquad y(0)=0, \quad y(l)=0 \tag{2}$$

have nontrivial solutions?

Solution.

(i) $\lambda=0$. Every solution $y(x)$ of the differential equation $y''=0$ is of the form $y(x)=c_1x+c_2$, for some choice of constants c_1 and c_2. The condition $y(0)=0$ implies that $c_2=0$, and the condition $y(l)=0$ then implies that $c_1=0$. Thus, $y(x)=0$ is the only solution of the boundary-value problem (2), for $\lambda=0$.

(ii) $\lambda<0$: In this case, every solution $y(x)$ of $y''+\lambda y=0$ is of the form $y(x)=c_1e^{\sqrt{-\lambda}\,x}+c_2e^{-\sqrt{-\lambda}\,x}$, for some choice of constants c_1 and c_2. The boundary conditions $y(0)=y(l)=0$ imply that

$$c_1+c_2=0, \qquad e^{\sqrt{-\lambda}\,l}c_1+e^{-\sqrt{-\lambda}\,l}c_2=0. \tag{3}$$

The system of equations (3) has a nonzero solution c_1,c_2 if, and only if,

$$\det\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\,l} & e^{-\sqrt{-\lambda}\,l} \end{pmatrix}=e^{-\sqrt{-\lambda}\,l}-e^{\sqrt{-\lambda}\,l}=0.$$

This implies that $e^{\sqrt{-\lambda}\,l}=e^{-\sqrt{-\lambda}\,l}$, or $e^{2\sqrt{-\lambda}\,l}=1$. But this is impossible, since e^z is greater than one for $z>0$. Hence, $c_1=c_2=0$ and the boundary-value problem (2) has no nontrivial solutions $y(x)$ when λ is negative.

(iii) $\lambda>0$: In this case, every solution $y(x)$ of $y''+\lambda y=0$ is of the form $y(x)=c_1\cos\sqrt{\lambda}\,x+c_2\sin\sqrt{\lambda}\,x$, for some choice of constants c_1 and c_2. The condition $y(0)=0$ implies that $c_1=0$, and the condition $y(l)=0$ then implies that $c_2\sin\sqrt{\lambda}\,l=0$. This equation is satisfied, for any choice of c_2, if $\sqrt{\lambda}\,l=n\pi$, or $\lambda=n^2\pi^2/l^2$, for some positive integer n. Hence, the boundary-value problem (2) has nontrivial solutions $y(x)=c\sin n\pi x/l$ for $\lambda=n^2\pi^2/l^2$, $n=1,2,\ldots$.

Remark. Our calculations for the case $\lambda<0$ can be simplified if we write every solution $y(x)$ in the form $y=c_1\cosh\sqrt{-\lambda}\,x+c_2\sinh\sqrt{-\lambda}\,x$, where

$$\cosh\sqrt{-\lambda}\,x=\frac{e^{\sqrt{-\lambda}\,x}+e^{-\sqrt{-\lambda}\,x}}{2}$$

and

$$\sinh\sqrt{-\lambda}\,x=\frac{e^{\sqrt{-\lambda}\,x}-e^{-\sqrt{-\lambda}\,x}}{2}.$$

The condition $y(0)=0$ implies that $c_1=0$, and the condition $y(l)=0$ then implies that $c_2\sinh\sqrt{-\lambda}\,l=0$. But $\sinh z$ is positive for $z>0$. Hence, $c_2=0$, and $y(x)=0$.

Example 1 is indicative of the general boundary-value problem (1). Indeed, we have the following remarkable theorem which we state, but do not prove.

Theorem 1. *The boundary-value problem* (1) *has nontrivial solutions* $y(x)$ *only for a denumerable set of values* $\lambda_1, \lambda_2, \ldots$, *where* $\lambda_1 \leq \lambda_2 \ldots$, *and* λ_n *approaches infinity as n approaches infinity. These special values of* λ *are called eigenvalues of* (1), *and the nontrivial solutions* $y(x)$ *are called eigenfunctions of* (1). *In this terminology, the eigenvalues of* (2) *are* $\pi^2/l^2, 4\pi^2/l^2, 9\pi^2/l^2, \ldots$, *and the eigenfunctions of* (2) *are all constant multiples of* $\sin \pi x/l, \sin 2\pi x/l, \ldots$.

There is a very natural explanation of why we use the terms eigenvalue and eigenfunction in this context. Let **V** be the set of all functions $y(x)$ which have two continuous derivatives and which satisfy $ay(0)+by'(0)=0, cy(l)+dy'(l)=0$. Clearly, **V** is a vector space, of infinite dimension. Consider now the linear operator, or transformation L, defined by the equation

$$[Ly](x) = -\frac{d^2y}{dx^2}(x). \tag{4}$$

The solutions $y(x)$ of (1) are those functions y in **V** for which $Ly = \lambda y$. That is to say, the solutions $y(x)$ of (1) are exactly those functions y in **V** which are transformed by L into multiples λ of themselves.

Example 2. Find the eigenvalues and eigenfunctions of the boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0; \qquad y(0)+y'(0)=0, \quad y(1)=0. \tag{5}$$

Solution.

(i) $\lambda = 0$. Every solution $y(x)$ of $y''=0$ is of the form $y(x)=c_1x+c_2$, for some choice of constants c_1 and c_2. The conditions $y(0)+y'(0)=0$ and $y(1)=0$ both imply that $c_2=-c_1$. Hence, $y(x)=c(x-1)$, $c\neq 0$, is a nontrivial solution of (5) when $\lambda=0$; i.e., $y(x)=c(x-1)$, $c\neq 0$, is an eigenfunction of (5) with eigenvalue zero.

(ii) $\lambda<0$. In this case, every solution $y(x)$ of $y''+\lambda y=0$ is of the form $y(x)=c_1\cosh\sqrt{-\lambda}\, x + c_2\sinh\sqrt{-\lambda}\, x$, for some choice of constants c_1 and c_2. The boundary conditions $y(0)+y'(0)=0$ and $y(1)=0$ imply that

$$c_1 + \sqrt{-\lambda}\, c_2 = 0, \qquad \cosh\sqrt{-\lambda}\, c_1 + \sinh\sqrt{-\lambda}\, c_2 = 0. \tag{6}$$

(Observe that $(\cosh x)' = \sinh x$ and $(\sinh x)' = \cosh x$.) The system of equations (6) has a nontrivial solution c_1, c_2 if, and only if,

$$\det\begin{pmatrix} 1 & \sqrt{-\lambda} \\ \cosh\sqrt{-\lambda} & \sinh\sqrt{-\lambda} \end{pmatrix} = \sinh\sqrt{-\lambda} - \sqrt{-\lambda}\cosh\sqrt{-\lambda} = 0.$$

This implies that

$$\sinh\sqrt{-\lambda} = \sqrt{-\lambda}\cosh\sqrt{-\lambda}\,. \tag{7}$$

But Equation (7) has no solution $\lambda<0$. To see this, let $z=\sqrt{-\lambda}$, and consider the function $h(z)=z\cosh z-\sinh z$. This function is zero for $z=0$ and is positive for $z>0$, since its derivative

$$h'(z)=\cosh z+z\sinh z-\cosh z=z\sinh z$$

is strictly positive for $z>0$. Hence, no negative number λ can satisfy (7).

(iii) $\lambda>0$. In this case, every solution $y(x)$ of $y''+\lambda y=0$ is of the form $y(x)=c_1\cos\sqrt{\lambda}\,x+c_2\sin\sqrt{\lambda}\,x$, for some choice of constants c_1 and c_2. The boundary conditions imply that

$$c_1+\sqrt{\lambda}\,c_2=0, \qquad \cos\sqrt{\lambda}\,c_1+\sin\sqrt{\lambda}\,c_2=0. \tag{8}$$

The system of equations (8) has a nontrivial solution c_1, c_2 if, and only if,

$$\det\begin{pmatrix} 1 & \sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{pmatrix}=\sin\sqrt{\lambda}-\sqrt{\lambda}\cos\sqrt{\lambda}=0.$$

This implies that

$$\tan\sqrt{\lambda}=\sqrt{\lambda}\,. \tag{9}$$

To find those values of λ which satisfy (9), we set $\xi=\sqrt{\lambda}$ and draw the graphs of the functions $\eta=\xi$ and $\eta=\tan\xi$ in the $\xi-\eta$ plane (see Figure 1); the ξ coordinate of each point of intersection of these curves is then a root of the equation $\xi=\tan\xi$. It is clear that these curves intersect exactly once in the interval $\pi/2<\xi<3\pi/2$, and this occurs at a point $\xi_1>\pi$. Similarly, these two curves intersect exactly once in the interval $3\pi/2<\xi<5\pi/2$, and this occurs at a point $\xi_2>2\pi$. More generally, the curves $\eta=\xi$ and $\eta=\tan\xi$ intersect exactly once in the interval

$$\frac{(2n-1)\pi}{2}<\xi<\frac{(2n+1)\pi}{2}$$

and this occurs at a point $\xi_n>n\pi$.

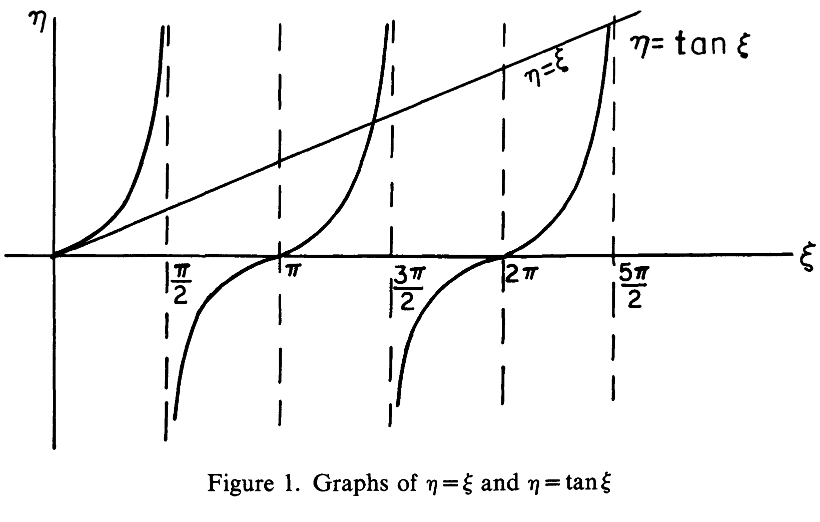


Figure 1. Graphs of $\eta=\xi$ and $\eta=\tan\xi$

Finally, the curves $\eta=\xi$ and $\eta=\tan\xi$ do not intersect in the interval $0<\xi<\pi/2$. To prove this, set $h(\xi)=\tan\xi-\xi$ and compute

$$h'(\xi)=\sec^2\xi-1=\tan^2\xi.$$

This quantity is positive for $0<\xi<\pi/2$. Consequently, the eigenvalues of (5) are $\lambda_1=\xi_1^2, \lambda_2=\xi_2^2,\dots$, and the eigenfunction of (5) are all constant multiples of the functions $-\sqrt{\lambda_1}\cos\sqrt{\lambda_1}\,x+\sin\sqrt{\lambda_1}\,x,\ -\sqrt{\lambda_2}\cos\sqrt{\lambda_2}\,x+\sin\sqrt{\lambda_2}\,x,\dots$. We cannot compute λ_n exactly. Nevertheless, we know that

$$n^2\pi^2<\lambda_n<(2n+1)^2\pi^2/4.$$

In addition, it is clear that λ_n approaches $(2n+1)^2\pi^2/4$ as n approaches infinity.

Exercises

Find the eigenvalues and eigenfunctions of each of the following boundary-value problems.

1. $y''+\lambda y=0;\quad y(0)=0,\ y'(l)=0$

2. $y''+\lambda y=0;\quad y'(0)=0,\ y'(l)=0$

3. $y''-\lambda y=0;\quad y'(0)=0,\ y'(l)=0$

4. $y''+\lambda y=0;\quad y'(0)=0,\ y(l)=0$

5. $y''+\lambda y=0;\quad y(0)=0,\ y(\pi)-y'(\pi)=0$

6. $y''+\lambda y=0;\quad y(0)-y'(0)=0,\ y(1)=0$

7. $y''+\lambda y=0;\quad y(0)-y'(0)=0,\ y(\pi)-y'(\pi)=0$

8. For which values of λ does the boundary-value problem

$$y''-2y'+(1+\lambda)y=0;\qquad y(0)=0,\ y(1)=0$$

have a nontrivial solution?

9. For which values of λ does the boundary-value problem

$$y''+\lambda y=0;\qquad y(0)=y(2\pi),\ y'(0)=y'(2\pi)$$

have a nontrivial solution?

10. Consider the boundary-value problem

$$y''+\lambda y=f(t);\qquad y(0)=0,\ y(1)=0 \tag{*}$$

(a) Show that (*) has a unique solution $y(t)$ if λ is not an eigenvalue of the homogeneous problem.
(b) Show that (*) may have no solution $y(t)$ if λ is an eigenvalue of the homogeneous problem.
(c) Let λ be an eigenvalue of the homogeneous problem. Determine conditions on f so that (*) has a solution $y(t)$. Is this solution unique?

5.2 Introduction to partial differential equations

Up to this point, the differential equations that we have studied have all been relations involving one or more functions of a single variable, and their derivatives. In this sense, these differential equations are *ordinary* differential equations. On the other hand, many important problems in applied mathematics give rise to *partial* differential equations. A partial differential equation is a relation involving one or more functions of *several* variables, and their partial derivatives. For example, the equation

$$\frac{\partial^3 u}{\partial x^3}+\left(\frac{\partial u}{\partial t}\right)^2=\frac{\partial^2 u}{\partial x^2}$$

is a partial differential equation for the function $u(x,t)$, and the equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

are a system of partial differential equations for the two functions $u(x,y)$ and $v(x,y)$. The order of a partial differential equation is the order of the highest partial derivative that appears in the equation. For example, the order of the partial differential equation

$$\frac{\partial^2 u}{\partial t^2}=2\frac{\partial^2 u}{\partial x \partial t}+u$$

is two, since the order of the highest partial derivative that appears in this equation is two.

There are three classical partial differential equations of order two which appear quite often in applications, and which dominate the theory of partial differential equations. These equations are

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \tag{2}$$

and

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{3}$$

Equation (1) is known as the heat equation, and it appears in the study of heat conduction and other diffusion processes. For example, consider a thin metal bar of length l whose surface is insulated. Let $u(x,t)$ denote the temperature in the bar at the point x at time t. This function satisfies the partial differential equation (1) for $0<x<l$. The constant α^2 is known as the thermal diffusivity of the bar, and it depends solely on the material from which the bar is made.

Equation (2) is known as the wave equation, and it appears in the study of acoustic waves, water waves and electromagnetic waves. Some form of this equation, or a generalization of it, almost invariably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. (We will gain some insight into why this is so in Section 5.7.) The wave equation also appears in the study of mechanical vibrations. Suppose, for example, that an elastic string of length l, such as a violin string or guy wire, is set in motion so that it vibrates in a vertical plane. Let $u(x,t)$ denote the vertical displacement of the string at the point x at time t (see Figure 1). If all damping effects, such as air resistance, are negligible, and if the amplitude of the motion is not too large, then $u(x,t)$ will satisfy the partial differential equation (2) on the interval $0 \leqslant x \leqslant l$. In this case, the constant c^2 is H/ρ, where H is the horizontal component of the tension in the string, and ρ is the mass per unit length of the string.

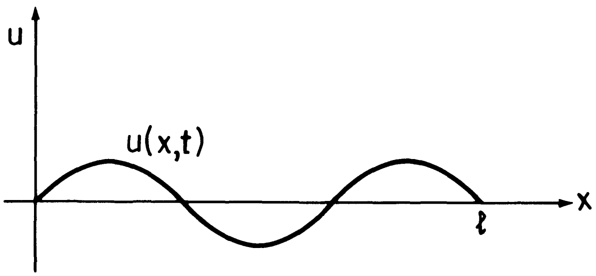


Figure 1

Equation (3) is known as Laplace's equation, and is the most famous of all partial differential equations. It arises in the study of such diverse applications as steady state heat flow, vibrating membranes, and electric and gravitational potentials. For this reason, Laplace's equation is often referred to as the potential equation.

In addition to the differential equation (1), (2), or (3), we will often impose initial and boundary conditions on the function u. These conditions will be dictated to us by the physical and biological problems themselves; they will be chosen so as to guarantee that our equation has a unique solution.

As a model case for the heat equation (1), we consider a thin metal bar of length l whose sides are insulated, and we let $u(x,t)$ denote the temperature in the bar at the point x at time t. In order to dermine the temperature in the bar at any time t we need to know (i) the initial temperature distribution in the bar, and (ii) what is happening at the ends of the bar. Are they held at constant temperatures, say 0°C, or are they insulated, so that no heat can pass through them? (This latter condition implies that $u_x(0,t) = u_x(l,t) = 0$.) Thus, a "well posed" problem for diffusion processes is the heat equation (1), together with the initial condition $u(x,0)=f(x)$, $0<x<l$, and the boundary conditions $u(0,t)=u(l,t)=0$, or $u_x(0,t)=u_x(l,t)=0$.

As a model case for the wave equation, we consider an elastic string of length l, whose ends are fixed, and which is set in motion in a vertical plane. In order to determine the position $u(x,t)$ of the string at any time t we need to know (i) the initial position of the string, and (ii) the initial velocity of the string. It is also implicit that $u(0,t)=u(l,t)=0$. Thus, a well posed problem for wave propagation is the differential equation (2) together with the initial conditions $u(x,0)=f(x)$, $u_t(x,0)=g(x)$, and the boundary conditions $u(0,t)=u(l,t)=0$.

The partial differential equation (3) does not contain the time t, so that we do not expect any "initial conditions" to be imposed here. In the problems that arise in applications, we are given u, or its normal derivative, on the boundary of a given region R, and we seek to determine $u(x,y)$ inside R. The problem of finding a solution of Laplace's equation which takes on given boundary values is known as a Dirichlet problem, while the problem of finding a solution of Laplace's equation whose normal derivative takes on given boundary values is known as a Neumann problem.

In Section 5.3 we will develop a very powerful method, known as the method of separation of variables, for solving the boundary-value problem (strictly speaking, we should say "initial boundary-value problem")

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}; \qquad u(x,0)=f(x), \quad 0<x<l; \quad u(0,t)=u(l,t)=0.$$

After developing the theory of Fourier series in Sections 5.4 and 5.5, we will show that the method of separation of variables can also be used to solve more general problems of heat conduction, and several important problems of wave propagation and potential theory.

5.3 The heat equation; separation of variables

Consider the boundary-value problem

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}; \qquad u(x,0)=f(x), \quad 0<x<l; \quad u(0,t)=u(l,t)=0. \quad (1)$$

Our goal is to find the solution $u(x,t)$ of (1). To this end, it is helpful to recall how we solved the initial-value problem

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0; \qquad y(0)=y_0, \quad y'(0)=y_0'. \quad (2)$$

First we showed that the differential equation

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \quad (3)$$

is linear; that is, any linear combination of solutions of (3) is again a solution of (3). And then, we found the solution $y(t)$ of (2) by taking an appropriate linear combination $c_1y_1(t)+c_2y_2(t)$ of two linearly independent solutions $y_1(t)$ and $y_2(t)$ of (3). Now, it is easily verified that any linear combination $c_1u_1(x,t)+\ldots+c_nu_n(x,t)$ of solutions $u_1(x,t),\ldots,u_n(x,t)$ of

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2} \tag{4}$$

is again a solution of (4). In addition, if $u_1(x,t),\ldots,u_n(x,t)$ satisfy the boundary conditions $u(0,t)=u(l,t)=0$, then the linear combination $c_1u_1+\ldots+c_nu_n$ also satisfies these boundary conditions. This suggests the following "game plan" for solving the boundary-value problem (1):

(a) Find as many solutions $u_1(x,t),u_2(x,t),\ldots$ as we can of the boundary-value problem

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}; \qquad u(0,t)=u(l,t)=0. \tag{5}$$

(b) Find the solution $u(x,t)$ of (1) by taking an appropriate linear combination of the functions $u_n(x,t)$, $n=1,2,\ldots$.

(a) Since we don't know, as yet, how to solve any partial differential equations, we must reduce the problem of solving (5) to that of solving one or more ordinary differential equations. This is accomplished by setting $u(x,t)=X(x)T(t)$ (hence the name separation of variables). Computing

$$\frac{\partial u}{\partial t}=XT' \quad\text{and}\quad \frac{\partial^2 u}{\partial x^2}=X''T$$

we see that $u(x,t)=X(x)T(t)$ is a solution of the equation $u_t=\alpha^2u_{xx}$ ($u_t=\partial u/\partial t$ and $u_{xx}=\partial^2u/\partial x^2$) if

$$XT'=\alpha^2X''T. \tag{6}$$

Dividing both sides of (6) by α^2XT gives

$$\frac{X''}{X}=\frac{T'}{\alpha^2T}. \tag{7}$$

Now, observe that the left-hand side of (7) is a function of x alone, while the right-hand side of (7) is a function of t alone. This implies that

$$\frac{X''}{X}=-\lambda, \quad\text{and}\quad \frac{T'}{\alpha^2T}=-\lambda \tag{8}$$

for some constant λ. (The only way that a function of x can equal a function of t is if both are constant. To convince yourself of this, let $f(x)=g(t)$ and fix t_0. Then, $f(x)=g(t_0)$ for all x, so that $f(x)=\text{constant}=c_1$, and this immediately implies that $g(t)$ also equals c_1.) In addition, the boundary conditions

$$0=u(0,t)=X(0)T(t),$$

and

$$0=u(l,t)=X(l)T(t)$$

imply that $X(0)=0$ and $X(l)=0$ (otherwise, u must be identically zero). Thus, $u(x,t)=X(x)T(t)$ is a solution of (5) if

$$X''+\lambda X=0; \qquad X(0)=0, \quad X(l)=0 \tag{9}$$

and

$$T'+\lambda\alpha^2 T=0. \tag{10}$$

At this point, the constant λ is arbitrary. However, we know from Example 1 of Section 5.1 that the boundary-value problem (9) has a nontrivial solution $X(x)$ only if $\lambda=\lambda_n=n^2\pi^2/l^2$, $n=1,2,\ldots$; and in this case,

$$X(x)=X_n(x)=\sin\frac{n\pi x}{l}.$$

Equation (10), in turn, implies that

$$T(t)=T_n(t)=e^{-\alpha^2 n^2\pi^2 t/l^2}.$$

(Actually, we should multiply both $X_n(x)$ and $T_n(t)$ by constants; however, we omit these constants here since we will soon be taking linear combinations of the functions $X_n(x)T_n(t)$.) Hence,

$$u_n(x,t)=\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2}$$

is a nontrivial solution of (5) for every positive integer n.

(b) Suppose that $f(x)$ is a finite linear combination of the functions $\sin n\pi x/l$; that is,

$$f(x)=\sum_{n=1}^{N} c_n\sin\frac{n\pi x}{l}.$$

Then,

$$u(x,t)=\sum_{n=1}^{N} c_n\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2}$$

is the desired solution of (1), since it is a linear combination of solutions of (5), and it satisfies the initial condition

$$u(x,0)=\sum_{n=1}^{N} c_n\sin\frac{n\pi x}{l}=f(x), \qquad 0<x<l.$$

Unfortunately, though, most functions $f(x)$ cannot be expanded as a finite linear combination of the functions $\sin n\pi x/l$, $n=1,2,\ldots$, on the interval $0<x<l$. This leads us to ask the following question.

Question: Can an arbitrary function $f(x)$ be written as an *infinite* linear combination of the functions $\sin n\pi x/l$, $n=1,2,\ldots$, on the interval $0<x<$

l? In other words, given an arbitrary function f, can we find constants $c_1, c_2, \ldots$, such that

$$f(x) = c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + \ldots = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}; \qquad 0 < x < l?$$

Remarkably, the answer to this question is yes, as we show in Section 5.5.

Example 1. At time $t=0$, the temperature $u(x,0)$ in a thin copper rod ($\alpha^2 = 1.14$) of length one is $2\sin 3\pi x + 5\sin 8\pi x$, $0 \leqslant x \leqslant 1$. The ends of the rod are packed in ice, so as to maintain them at 0°C. Find the temperature $u(x,t)$ in the rod at any time $t > 0$.

Solution. The temperature $u(x,t)$ satisfies the boundary-value problem

$$\frac{\partial u}{\partial t} = 1.14\frac{\partial^2 u}{\partial x^2}; \qquad \begin{cases} u(x,0) = 2\sin 3\pi x + 5\sin 8\pi x, & 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

and this implies that

$$u(x,t) = 2\sin 3\pi x\, e^{-9(1.14)\pi^2 t} + 5\sin 8\pi x\, e^{-64(1.14)\pi^2 t}.$$

Exercises

Find a solution $u(x,t)$ of the following problems.

1. $\dfrac{\partial u}{\partial t} = 1.71\dfrac{\partial^2 u}{\partial x^2}$; $\begin{cases} u(x,0) = \sin \pi x/2 + 3\sin 5\pi x/2, & 0 < x < 2 \\ u(0,t) = u(2,t) = 0 \end{cases}$

2. $\dfrac{\partial u}{\partial t} = 1.14\dfrac{\partial^2 u}{\partial x^2}$; $\begin{cases} u(x,0) = \sin \pi x/2 - 3\sin 2\pi x, & 0 < x < 2 \\ u(0,t) = u(2,t) = 0 \end{cases}$

3. Use the method of separation of variables to solve the boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u; \qquad \begin{cases} u(x,0) = 3\sin 2\pi x - 7\sin 4\pi x, & 0 < x < 10 \\ u(0,t) = u(10,t) = 0 \end{cases}$$

Use the method of separation of variables to solve each of the following boundary-value problems.

4. $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial y}$; $u(0,y) = e^y + e^{-2y}$

5. $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial y}$; $u(t,0) = e^{-3t} + e^{2t}$

6. $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial y} + u$; $u(0,y) = 2e^{-y} - e^{2y}$

7. $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial y} - u$; $u(t,0) = e^{-5t} + 2e^{-7t} - 14e^{13t}$

8. Determine whether the method of separation of variables can be used to replace each of the following partial differential equations by pairs of ordinary differential equations. If so, find the equations.

(a) $tu_{tt}+u_x=0$ (b) $tu_{xx}+xu_t=0$

(c) $u_{xx}+(x-y)u_{yy}=0$ (d) $u_{xx}+2u_{xt}+u_t=0$

9. The heat equation in two space dimensions is

$$u_t=\alpha^2(u_{xx}+u_{yy}). \tag{*}$$

(a) Assuming that $u(x,y,t)=X(x)Y(y)T(t)$, find ordinary differential equations satisfied by X, Y, and T.

(b) Find solutions $u(x,y,t)$ of (*) which satisfy the boundary conditions $u(0,y,t)=0$, $u(a,y,t)=0$, $u(x,0,t)=0$, and $u(x,b,t)=0$.

10. The heat equation in two space dimensions may be expressed in terms of polar coordinates as

$$u_t=\alpha^2\left[u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}\right].$$

Assuming that $u(r,\theta,t)=R(r)\Theta(\theta)T(t)$, find ordinary differential equations satisfied by R, Θ, and T.

5.4 Fourier series

On December 21, 1807, an engineer named Joseph Fourier announced to the prestigious French Academy of Sciences that an arbitrary function $f(x)$ could be expanded in an infinite series of sines and cosines. Specifically, let $f(x)$ be defined on the interval $-l\leqslant x\leqslant l$, and compute the numbers

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx, \qquad n=0,1,2,\ldots \tag{1}$$

and

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx, \qquad n=1,2,\ldots \tag{2}$$

Then, the infinite series

$$\frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+b_1\sin\frac{\pi x}{l}+\ldots=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right] \tag{3}$$

converges to $f(x)$. Fourier's announcement caused a loud furor in the Academy. Many of its prominent members, including the famous mathematician Lagrange, thought this result to be pure nonsense, since at that time it could not be placed on a rigorous foundation. However, mathematicians have now developed the theory of "Fourier series" to such an extent that whole volumes have been written on it. (Just recently, in fact, they have succeeded in establishing exceedingly sharp conditions for the Fourier series (3) to converge. This result ranks as one of the great mathematical theorems of the twentieth century.) The following theorem, while not the most general theorem possible, covers most of the situations that arise in applications.

Theorem 2. *Let f and f' be piecewise continuous on the interval $-l \leqslant x \leqslant l$. (This means that f and f' have only a finite number of discontinuities on this interval, and both f and f' have right- and left-hand limits at each point of discontinuity.) Compute the numbers a_n and b_n from* (1) *and* (2) *and form the infinite series* (3). *This series, which is called the Fourier series for f on the interval $-l \leqslant x \leqslant l$, converges to $f(x)$ if f is continuous at x, and to $\frac{1}{2}[f(x+0)+f(x-0)]$* if f is discontinuous at x. At $x = \pm l$, the Fourier series* (3) *converges to $\frac{1}{2}[f(l)+f(-l)]$, where $f(\pm l)$ is the limit of $f(x)$ as x approaches $\pm l$.*

Remark. The quantity $\frac{1}{2}[f(x+0)+f(x-0)]$ is the average of the right- and left-hand limits of f at the point x. If we define $f(x)$ to be the average of the right- and left-hand limits of f at any point of discontinuity x, then the Fourier series (3) converges to $f(x)$ for all points x in the interval $-l < x < l$.

Example 1. Let f be the function which is 0 for $-1 \leqslant x < 0$ and 1 for $0 \leqslant x \leqslant 1$. Compute the Fourier series for f on the interval $-1 \leqslant x \leqslant 1$.
Solution. In this problem, $l = 1$. Hence, from (1) and (2),

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 dx = 1,$$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_0^1 \cos n\pi x\,dx = 0, \qquad n \geqslant 1$$

and

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_0^1 \sin n\pi x\,dx$$

$$= \frac{1}{n\pi}(1-\cos n\pi) = \frac{1-(-1)^n}{n\pi}, \qquad n \geqslant 1.$$

Notice that $b_n = 0$ for n even, and $b_n = 2/n\pi$ for n odd. Hence, the Fourier series for f on the interval $-1 \leqslant x \leqslant 1$ is

$$\frac{1}{2} + \frac{2\sin\pi x}{\pi} + \frac{2\sin 3\pi x}{3\pi} + \frac{2\sin 5\pi x}{5\pi} + \dots.$$

By Theorem 2, this series converges to 0 if $-1 < x < 0$, and to 1 if $0 < x < 1$. At $x = -1$, 0, and $+1$, this series reduces to the single number $\frac{1}{2}$, which is the value predicted for it by Theorem 2.

*The quantity $f(x+0)$ denotes the limit from the right of f at the point x. Similarly, $f(x-0)$ denotes the limit of f from the left.

Example 2. Let f be the function which is 1 for $-2 \leqslant x < 0$ and x for $0 \leqslant x \leqslant 2$. Compute the Fourier series for f on the interval $-2 \leqslant x \leqslant 2$.
Solution. In this problem $l = 2$. Hence from (1) and (2),

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0} dx + \frac{1}{2}\int_{0}^{2} x\,dx = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{(n\pi)^2}(\cos n\pi - 1), \qquad n \geqslant 1$$

and

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{0}\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx$$

$$= -\frac{1}{n\pi}(1 + \cos n\pi), \qquad n \geqslant 1.$$

Notice that $a_n = 0$ if n is even; $a_n = -4/n^2\pi^2$ if n is odd; $b_n = 0$ if n is odd; and $b_n = -2/n\pi$ if n is even. Hence, the Fourier series for f on the interval $-2 \leqslant x \leqslant 2$ is

$$1 - \frac{4}{\pi^2}\cos\frac{\pi x}{2} - \frac{1}{\pi}\sin\pi x - \frac{4}{9\pi^2}\cos\frac{3\pi x}{2} - \frac{1}{2\pi}\sin 2\pi x + \dots$$

$$= 1 - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x/2}{(2n+1)^2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}. \tag{4}$$

By Theorem 2, this series converges to 1 if $-2 < x < 0$; to x, if $0 < x < 2$; to $\frac{1}{2}$ if $x = 0$; and to $\frac{3}{2}$ if $x \pm 2$. Now, at $x = 0$, the Fourier series (4) is

$$1 - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right].$$

Thus, we deduce the remarkable identity

$$\frac{1}{2} = 1 - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right]$$

or

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

The Fourier coefficients a_n and b_n defined by (1) and (2) can be derived in a simple manner. Indeed, if a piecewise continuous function f can be expanded in a series of sines and cosines on the interval $-l \leqslant x \leqslant l$, then, of necessity, this series must be the Fourier series (3). We prove this in the following manner. Suppose that f is piecewise continuous, and that

$$f(x)=\frac{c_0}{2}+\sum_{k=1}^{\infty}\left[c_k\cos\frac{k\pi x}{l}+d_k\sin\frac{k\pi x}{l}\right] \tag{5}$$

for some numbers c_k and d_k. Equation (5) is assumed to hold at all but a finite number of points in the interval $-l\leqslant x\leqslant l$. Integrating both sides of (5) between $-l$ and l gives $c_0l=\int_{-l}^{l}f(x)dx$, since

$$\int_{-l}^{l}\cos\frac{k\pi x}{l}dx=\int_{-l}^{l}\sin\frac{k\pi x}{l}dx=0; \qquad k=1,2,\ldots*$$

Similarly, multiplying both sides of (5) by $\cos n\pi x/l$ and integrating between $-l$ and l gives

$$lc_n=\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx$$

while multiplying both sides of (5) by $\sin n\pi x/l$ and integrating between $-l$ and l gives

$$ld_n=\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx.$$

This follows immediately from the relations (see Exercise 19)

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}\cos\frac{k\pi x}{l}dx=\begin{cases}0, & k\neq n\\ l, & k=n\end{cases} \tag{6}$$

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}\sin\frac{k\pi x}{l}dx=0 \tag{7}$$

and

$$\int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{k\pi x}{l}dx=\begin{cases}0, & k\neq n\\ l, & k=n\end{cases}. \tag{8}$$

Hence, the coefficients c_n and d_n must equal the Fourier coefficients a_n and b_n. In particular, therefore, a function f can be expanded in one, and only one, Fourier series on the interval $-l\leqslant x\leqslant l$.

Example 3. Find the Fourier series for the function $f(x)=\cos^2x$ on the interval $-\pi\leqslant x\leqslant\pi$.
Solution. By the preceding remark, the function $f(x)=\cos^2x$ has a unique Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{\pi}+b_n\sin\frac{n\pi x}{\pi}\right]=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos nx+b_n\sin nx\right]$$

*It can be shown that it is permissible to integrate the series (5) term by term.

on the interval $-\pi \leqslant x \leqslant \pi$. But we already know that

$$\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}.$$

Hence, the Fourier series for $\cos^2 x$ on the interval $-\pi \leqslant x \leqslant \pi$ must be $\frac{1}{2} + \frac{1}{2}\cos 2x$.

The functions $\cos n\pi x/l$ and $\sin n\pi x/l$, $n=1,2,\ldots$ all have the interesting property that they are periodic with period $2l$; that is, they repeat themselves over every interval of length $2l$. This follows trivially from the identities

$$\cos\frac{n\pi}{l}(x+2l) = \cos\left(\frac{n\pi x}{l} + 2n\pi\right) = \cos\frac{n\pi x}{l}$$

and

$$\sin\frac{n\pi}{l}(x+2l) = \sin\left(\frac{n\pi x}{l} + 2n\pi\right) = \sin\frac{n\pi x}{l}.$$

Hence, the Fourier series (3) converges for all x to a periodic function $F(x)$. This function is called the periodic extension of $f(x)$. It is defined by the equations

$$\begin{cases} F(x) = f(x), & -l < x < l \\ F(x) = \frac{1}{2}\left[f(l) + f(-l)\right], & x = \pm l \\ F(x+2l) = F(x). & \end{cases}$$

For example, the periodic extension of the function $f(x) = x$ is described in Figure 1, and the periodic extension of the function $f(x) = |x|$ is the sawtoothed function described in Figure 2.

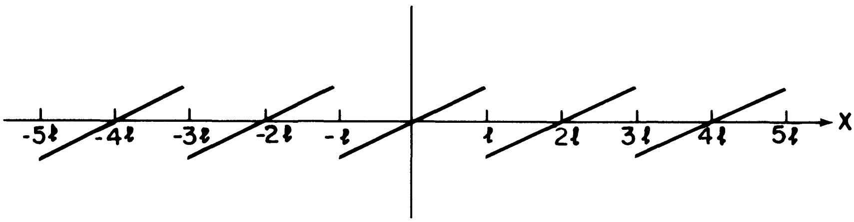


Figure 1. Periodic extension of $f(x) = x$

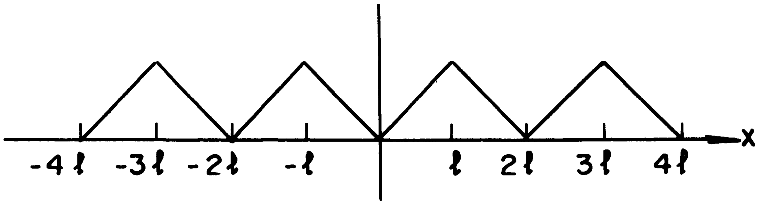


Figure 2. Periodic extension of $f(x) = |x|$

EXERCISES

In each of Problems 1–13, find the Fourier series for the given function f on the prescribed interval.

1. $f(x)=\begin{cases} -1, & -1\leqslant x<0 \\ 1, & 0\leqslant x\leqslant 1 \end{cases};\quad |x|\leqslant 1$

2. $f(x)=\begin{cases} x, & -2\leqslant x<0 \\ 0, & 0\leqslant x\leqslant 2 \end{cases};\quad |x|\leqslant 2$

3. $f(x)=x;\quad -1\leqslant x\leqslant 1$

4. $f(x)=\begin{cases} -x, & -1\leqslant x<0 \\ x, & 0\leqslant x\leqslant 1 \end{cases};\quad |x|\leqslant 1$

5. $f(x)=\begin{cases} 1, & -2\leqslant x<0 \\ 0, & 0\leqslant x<1 \\ 1, & 1\leqslant x\leqslant 2 \end{cases};\quad |x|\leqslant 2$

6. $f(x)=\begin{cases} 0, & -2\leqslant x<1 \\ 3, & 1\leqslant x\leqslant 2 \end{cases};\quad |x|\leqslant 2$

7. $f(x)=\begin{cases} 0, & -l\leqslant x<0 \\ e^x, & 0\leqslant x\leqslant l \end{cases};\quad |x|\leqslant l$

8. $f(x)=\begin{cases} e^x, & -l\leqslant x<0 \\ 0, & 0\leqslant x\leqslant l \end{cases};\quad |x|\leqslant l$

9. $f(x)=\begin{cases} e^{-x}, & -l\leqslant x<0 \\ e^x, & 0\leqslant x\leqslant l \end{cases};\quad -l\leqslant x\leqslant l$

10. $f(x)=e^x;\quad |x|\leqslant l$

11. $f(x)=e^{-x};\quad |x|\leqslant l$

12. $f(x)=\sin^2 x;\quad |x|\leqslant \pi$

13. $f(x)=\sin^3 x;\quad |x|\leqslant \pi$

14. Let $f(x)=(\pi\cos ax)/2a\sin a\pi$, a not an integer.
(a) Find the Fourier series for f on the interval $-\pi\leqslant x\leqslant \pi$.
(b) Show that this series converges at $x=\pi$ to the value $(\pi/2a)\cot\pi a$.
(c) Use this result to sum the series

$$\frac{1}{1^2-a^2}+\frac{1}{2^2-a^2}+\frac{1}{3^2-a^2}+\cdots.$$

15. Suppose that f and f' are piecewise continuous on the interval $-l\leqslant x\leqslant l$. Show that the Fourier coefficients a_n and b_n approach zero as n approaches infinity.

16. Let

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right].$$

Show that

$$\frac{1}{l}\int_{-l}^{l}f^2(x)\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

This relation is known as Parseval's identity. *Hint*: Square the Fourier series for f and integrate term by term.

17. (a) Find the Fourier series for the function $f(x)=x^2$ on the interval $-\pi \leqslant x \leqslant \pi$.

(b) Use Parseval's identity to show that

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}.$$

18. If the Dirac delta function $\delta(x)$ had a Fourier series on the interval $-l \leqslant x \leqslant l$, what would it be?

19. Derive Equations (6)–(8). *Hint*: Use the trigonometric identities

$$\sin A \cos B=\tfrac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\sin A \sin B=\tfrac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos A \cos B=\tfrac{1}{2}[\cos(A+B)+\cos(A-B)].$$

5.5 Even and odd functions

There are certain special cases when the Fourier series of a function f reduces to a pure cosine or a pure sine series. These special cases occur when f is even or odd.

Definition. A function f is said to be *even* if $f(-x)=f(x)$.

Example 1. The function $f(x)=x^2$ is even since

$$f(-x)=(-x)^2=x^2=f(x).$$

Example 2. The function $f(x)=\cos n\pi x/l$ is even since

$$f(-x)=\cos\frac{-n\pi x}{l}=\cos\frac{n\pi x}{l}=f(x).$$

Definition. A function f is said to be *odd* if $f(-x)=-f(x)$.

Example 3. The function $f(x)=x$ is odd since

$$f(-x)=-x=-f(x).$$

Example 4. The function $f(x)=\sin n\pi x/l$ is odd since

$$f(-x)=\sin\frac{-n\pi x}{l}=-\sin\frac{n\pi x}{l}=-f(x).$$

Even and odd functions satisfy the following elementary properties.

1. The product of two even functions is even.
2. The product of two odd functions is even.
3. The product of an odd function with an even function is odd.

The proofs of these assertions are trivial and follow immediately from the definitions. For example, let f and g be odd and let $h(x)=f(x)g(x)$. This function h is even since

$$h(-x)=f(-x)g(-x)=[-f(x)][-g(x)]=f(x)g(x)=h(x).$$

In addition to the multiplicative properties 1–3, even and odd functions satisfy the following integral properties.

4. The integral of an odd function f over a symmetric interval $[-l,l]$ is zero; that is, $\int_{-l}^{l} f(x)\,dx=0$ if f is odd.
5. The integral of an even function f over the interval $[-l,l]$ is twice the integral of f over the interval $[0,l]$; that is,

$$\int_{-l}^{l} f(x)\,dx=2\int_{0}^{l} f(x)\,dx$$

if f is even.

Proof of property 4. If f is odd, then the area under the curve of f between $-l$ and 0 is the negative of the area under the curve of f between 0 and l. Hence, $\int_{-l}^{l} f(x)\,dx=0$ if f is odd. □

Proof of property 5. If f is even, then the area under the curve of f between $-l$ and 0 equals the area under the curve of f between 0 and l. Hence,

$$\int_{-l}^{l} f(x)\,dx=\int_{-l}^{0} f(x)\,dx+\int_{0}^{l} f(x)\,dx=2\int_{0}^{l} f(x)\,dx$$

if f is even. □

Concerning even and odd functions, we have the following important lemma.

Lemma 1.

(a) *The Fourier series for an even function is a pure cosine series; that is, it contains no terms of the form* $\sin n\pi x/l$.

(b) *The Fourier series for an odd function is a pure sine series; that is, it contains no terms of the form* $\cos n\pi x/l$.

Proof. (a) If f is even, then the function $f(x)\sin n\pi x/l$ is odd. Thus, by Property 4, the coefficients

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx, \qquad n=1,2,3,\ldots$$

in the Fourier series for f are all zero.

(b) If f is odd, then the function $f(x)\cos n\pi x/l$ is also odd. Consequently, by Property 4, the coefficients

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad n=0,1,2,\ldots$$

in the Fourier series for f are all zero. □

We are now in a position to prove the following extremely important extension of Theorem 2. This theorem will enable us to solve the heat conduction problem of Section 5.3 and many other boundary-value problems that arise in applications.

Theorem 3. *Let f and f' be piecewise continuous on the interval $0 \leqslant x \leqslant l$. Then, on this interval, $f(x)$ can be expanded in either a pure cosine series*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l},$$

or a pure sine series

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

In the former case, the coefficients a_n are given by the formula

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \qquad n=0,1,2,\ldots \tag{1}$$

while in the latter case, the coefficients b_n are given by the formula

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \tag{2}$$

PROOF. Consider first the function

$$F(x)=\begin{cases} f(x), & 0\leqslant x\leqslant l \\ f(-x), & -l\leqslant x<0 \end{cases}.$$

The graph of $F(x)$ is described in Figure 1, and it is easily seen that F is even. (For this reason, F is called the even extension of f.) Hence, by

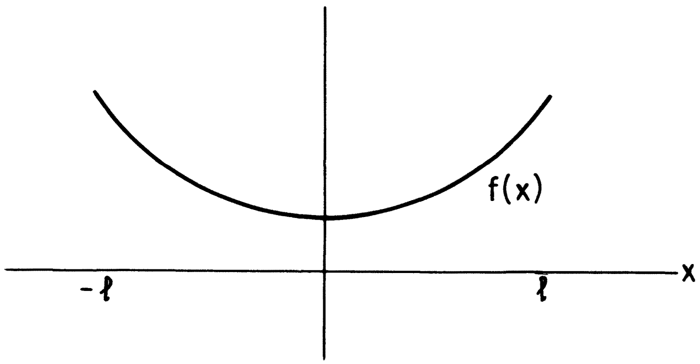


Figure 1. Graph of $F(x)$

Lemma 1, the Fourier series for F on the interval $-l \leqslant x \leqslant l$ is

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}; \qquad a_n=\frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}\,dx. \tag{3}$$

Now, observe that the function $F(x)\cos n\pi x/l$ is even. Thus, by Property 5

$$a_n=\frac{2}{l}\int_0^l F(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

Finally, since $F(x)=f(x)$, $0\leqslant x\leqslant l$, we conclude from (3) that

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}, \qquad 0\leqslant x\leqslant l.$$

Observe too, that the series (3) converges to $f(x)$ for $x=0$ and $x=l$.

To show that $f(x)$ can also be expanded in a pure sine series, we consider the function

$$G(x)=\begin{cases} f(x), & 0<x<l \\ -f(-x), & -l<x<0 \\ 0, & x=0,\pm l. \end{cases}$$

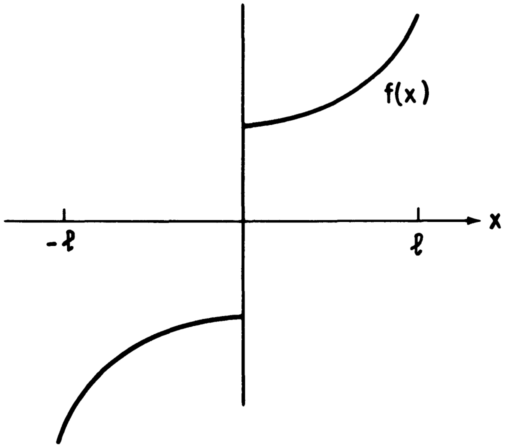


Figure 2. Graph of $G(x)$

The graph of $G(x)$ is described in Figure 2, and it is easily seen that G is odd. (For this reason, G is called the odd extension of f.) Hence, by Lemma 1, the Fourier series for G on the interval $-l\leqslant x\leqslant l$ is

$$G(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}; \qquad b_n=\frac{1}{l}\int_{-l}^{l} G(x)\sin\frac{n\pi x}{l}\,dx. \tag{4}$$

Now, observe that the function $G(x)\sin n\pi x/l$ is even. Thus, by Property 5,

$$b_n=\frac{2}{l}\int_0^l G(x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Finally, since $G(x)=f(x), 0<x<l$, we conclude from (4) that

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l},\qquad 0<x<l.$$

Observe too, that the series (4) is zero for $x=0$ and $x=l$. □

Example 5. Expand the function $f(x)=1$ in a pure sine series on the interval $0<x<\pi$.

Solution. By Theorem 3, $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$, where

$$b_n=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx=\frac{2}{n\pi}(1-\cos n\pi)=\begin{cases}0, & n \text{ even}\\ \dfrac{4}{n\pi}, & n \text{ odd}.\end{cases}$$

Hence,

$$1=\frac{4}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\dots\right],\qquad 0<x<\pi.$$

Example 6. Expand the function $f(x)=e^x$ in a pure cosine series on the interval $0\leqslant x\leqslant 1$.

Solution. By Theorem 3, $f(x)=a_0/2+\sum_{n=1}^{\infty}a_n\cos n\pi x$, where

$$a_0=2\int_0^1 e^x\,dx=2(e-1)$$

and

$$\begin{aligned}a_n&=2\int_0^1 e^x\cos n\pi x\,dx=2\,\mathrm{Re}\int_0^1 e^x e^{in\pi x}\,dx\\&=2\,\mathrm{Re}\int_0^1 e^{(1+in\pi)x}\,dx=2\,\mathrm{Re}\left\{\frac{e^{1+in\pi}-1}{1+in\pi}\right\}\\&=2\,\mathrm{Re}\left\{\frac{(e\cos n\pi-1)(1-in\pi)}{1+n^2\pi^2}\right\}=\frac{2(e\cos n\pi-1)}{1+n^2\pi^2}.\end{aligned}$$

Hence,

$$e^x=e-1+2\sum_{n=1}^{\infty}\frac{(e\cos n\pi-1)}{1+n^2\pi^2}\cos n\pi x,\qquad 0\leqslant x\leqslant 1.$$

EXERCISES

Expand each of the following functions in a Fourier cosine series on the prescribed interval.

1. $f(x)=e^{-x};\quad 0\leqslant x\leqslant 1$

2. $f(x)=\begin{cases}0, & 0\leqslant x\leqslant 1\\ 1, & 1<x\leqslant 2\end{cases};\quad 0\leqslant x\leqslant 2$

3. $f(x)=\begin{cases}x, & 0\leqslant x<a\\ a, & a\leqslant x\leqslant 2a\end{cases};\quad 0\leqslant x\leqslant 2a$

4. $f(x)=\cos^2 x;\quad 0\leqslant x\leqslant \pi$

5. $f(x)=\begin{cases}x, & 0\leqslant x\leqslant l/2\\ l-x, & l/2\leqslant x\leqslant l\end{cases};\quad 0\leqslant x\leqslant l$

Expand each of the following functions in a Fourier sine series on the prescribed interval.

6. $f(x)=e^{-x};\quad 0<x<1$

7. $f(x)=\begin{cases}0, & 0<x\leqslant 1\\ 1, & 1<x<2\end{cases};\quad 0<x<2$

8. $f(x)=\begin{cases}x, & 0<x<a\\ a, & a\leqslant x<2a\end{cases};\quad 0<x<2a$

9. $f(x)=2\sin x\cos x;\quad 0<x<\pi$

10. $f(x)=\begin{cases}x, & 0<x<l/2\\ l-x, & l/2\leqslant x<l\end{cases};\quad 0<x<l$

11. (a) Expand the function $f(x)=\sin x$ in a Fourier cosine series on the interval $0\leqslant x\leqslant\pi$.
(b) Expand the function $f(x)=\cos x$ in a Fourier sine series on the interval $0<x<\pi$.
(c) Can you expand the function $f(x)=\sin x$ in a Fourier cosine series on the interval $-\pi\leqslant x\leqslant\pi$? Explain.

5.6 Return to the heat equation

We return now to the boundary-value problem

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2};\qquad \begin{cases}u(x,0)=f(x)\\ u(0,t)=u(l,t)=0\end{cases}. \tag{1}$$

We showed in Section 5.3 that the function

$$u(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{l}e^{-\alpha^2n^2\pi^2t/l^2}$$

is (formally) a solution of the boundary-value problem

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2};\qquad u(0,t)=u(l,t)=0 \tag{2}$$

for any choice of constants $c_1,c_2,\ldots$. This led us to ask whether we can find constants $c_1, c_2, \ldots$ such that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = f(x), \qquad 0<x<l. \tag{3}$$

As we showed in Section 5.5, the answer to this question is yes; if we choose

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx,$$

then the Fourier series $\sum_{n=1}^{\infty} c_n \sin n\pi x/l$ converges to $f(x)$ if f is continuous at the point x. Thus,

$$u(x,t) = \frac{2}{l}\sum_{n=1}^{\infty}\left[\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right]\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2} \tag{4}$$

is the desired solution of (1).

Remark. Strictly speaking, the solution (4) cannot be regarded as the solution of (1) until we rigorously justify all the limiting processes involved. Specifically, we must verify that the function $u(x,t)$ defined by (4) actually has partial derivatives with respect to x and t, and that $u(x,t)$ satisfies the heat equation $u_t = \alpha^2 u_{xx}$. (It is not true, necessarily, that an infinite sum of solutions of a linear differential equation is again a solution. Indeed, an infinite sum of solutions of a given differential equation need not even be differentiable.) However, in the case of (4) it is possible to show (see Exercise 3) that $u(x,t)$ has partial derivatives with respect to x and t of all orders, and that $u(x,t)$ satisfies the boundary-value problem (1). The argument rests heavily upon the fact that the infinite series (4) converges very rapidly, due to the presence of the factor $e^{-\alpha^2 n^2\pi^2 t/l^2}$. Indeed, the function $u(x,t)$, for fixed $t>0$, is even analytic for $0<x<l$. Thus, heat conduction is a diffusive process which instantly smooths out any discontinuities that may be present in the initial temperature distribution in the rod. Finally, we observe that $\lim_{t\to\infty} u(x,t) = 0$, for all x, regardless of the initial temperature in the rod. This is in accord with our physical intuition that the heat distribution in the rod should ultimately approach a "steady state"; that is, a state in which the temperature does not change with time.

Example 1. A thin aluminum bar ($\alpha^2 = 0.86\ \text{cm}^2/\text{s}$) 10 cm long is heated to a uniform temperature of 100°C. At time $t=0$, the ends of the bar are plunged into an ice bath at 0°C, and thereafter they are maintained at this temperature. No heat is allowed to escape through the lateral surface of the bar. Find an expression for the temperature at any point in the bar at any later time t.

Solution. Let $u(x,t)$ denote the temperature in the bar at the point x at time t. This function satisfies the boundary-value problem

$$\frac{\partial u}{\partial t}=0.86\frac{\partial^2 u}{\partial x^2};\qquad \begin{cases} u(x,0)=100, & 0<x<10 \\ u(0,t)=u(10,t)=0 \end{cases} \tag{5}$$

The solution of (5) is

$$u(x,t)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{10} e^{-0.86n^2\pi^2 t/100}$$

where

$$c_n=\frac{1}{5}\int_0^{10} 100\sin\frac{n\pi x}{10}dx=\frac{200}{n\pi}(1-\cos n\pi).$$

Notice that $c_n=0$ if n is even, and $c_n=400/n\pi$ if n is odd. Hence,

$$u(x,t)=\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\frac{\pi x}{10}}{(2n+1)}e^{-0.86(2n+1)^2\pi^2 t/100}.$$

There are several other problems of heat conduction which can be solved by the method of separation of variables. Example 2 below treats the case where the ends of the bar are also insulated, and Exercise 4 treats the case where the ends of the bar are kept at constant, but nonzero temperatures T_1 and T_2.

Example 2. Consider a thin metal rod of length l and thermal diffusivity α^2, whose sides and ends are insulated so that there is no passage of heat through them. Let the initial temperature distribution in the rod be $f(x)$. Find the temperature distribution in the rod at any later time t.

Solution. Let $u(x,t)$ denote the temperature in the rod at the point x at time t. This function satisfies the boundary-value problem

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2};\qquad \begin{cases} u(x,0)=f(x), & 0<x<l \\ u_x(0,t)=u_x(l,t)=0 \end{cases} \tag{6}$$

We solve this problem in two steps. First, we will find infinitely many solutions $u_n(x,t)=X_n(x)T_n(t)$ of the boundary-value problem

$$u_t=\alpha^2 u_{xx};\qquad u_x(0,t)=u_x(l,t)=0, \tag{7}$$

and then we will find constants $c_0, c_1, c_2, \ldots$ such that

$$u(x,t)=\sum_{n=0}^{\infty} c_n u_n(x,t)$$

satisfies the initial condition $u(x,0)=f(x)$.

Step 1: Let $u(x,t)=X(x)T(t)$. Computing

$$\frac{\partial u}{\partial t}=XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}=X''T$$

we see that $u(x,t)$ is a solution of $u_t=\alpha^2 u_{xx}$ if

$$XT'=\alpha^2 X''T, \quad \text{or} \quad \frac{X''}{X}=\frac{T'}{\alpha^2 T}. \tag{8}$$

As we showed in Section 5.3, Equation (8) implies that

$$X''+\lambda X=0 \quad \text{and} \quad T'+\lambda\alpha^2 T=0$$

for some constant λ. In addition, the boundary conditions

$$0=u_x(0,t)=X'(0)T(t) \quad \text{and} \quad 0=u_x(l,t)=X'(l)T(t)$$

imply that $X'(0)=0$ and $X'(l)=0$. Hence $u(x,t)=X(x)T(t)$ is a solution of (7) if

$$X''+\lambda X=0; \qquad X'(0)=0, \quad X'(l)=0 \tag{9}$$

and

$$T'+\lambda\alpha^2 T=0. \tag{10}$$

At this point, the constant λ is arbitrary. However, the boundary-value problem (9) has a nontrivial solution $X(x)$ (see Exercise 1, Section 5.1) only if $\lambda=n^2\pi^2/l^2$, $n=0,1,2,\ldots$, and in this case

$$X(x)=X_n(x)=\cos\frac{n\pi x}{l}.$$

Equation (10), in turn, implies that $T(t)=e^{-\alpha^2 n^2\pi^2 t/l^2}$. Hence,

$$u_n(x,t)=\cos\frac{n\pi x}{l}\, e^{-\alpha^2 n^2\pi^2 t/l^2}$$

is a solution of (7) for every nonnegative integer n.

Step 2: Observe that the linear combination

$$u(x,t)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n\cos\frac{n\pi x}{l}\, e^{-\alpha^2 n^2\pi^2 t/l^2}$$

is a solution (formally) of (7) for every choice of constants $c_0,c_1,c_2,\ldots$. Its initial value is

$$u(x,0)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n\cos\frac{n\pi x}{l}.$$

Thus, in order to satisfy the initial condition $u(x,0)=f(x)$ we must choose constants $c_0,c_1,c_2,\ldots$ such that

$$f(x)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n\cos\frac{n\pi x}{l}, \qquad 0\leqslant x\leqslant l.$$

In other words, we must expand f in a Fourier cosine series on the interval

$0 \leqslant x \leqslant l$. This is precisely the situation in Theorem 3 of Section 5.5, and we conclude, therefore, that

$$c_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

Hence,

$$u(x,t) = \frac{1}{l}\int_0^l f(x)\,dx + \frac{2}{l}\sum_{n=1}^{\infty}\left[\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx\right]\cos\frac{n\pi x}{l}\,e^{-\alpha^2 n^2\pi^2 t/l^2} \quad (11)$$

is the desired solution of (6).

Remark. Observe from (11) that the temperature in the rod ultimately approaches the steady state temperature

$$\frac{1}{l}\int_0^l f(x)\,dx.$$

This steady state temperature can be interpreted as the "average" of the initial temperature distribution in the rod.

Exercises

1. The ends $x=0$ and $x=10$ of a thin aluminum bar ($\alpha^2=0.86$) are kept at 0° C, while the surface of the bar is insulated. Find an expression for the temperature $u(x,t)$ in the bar if initially
(a) $u(x,0)=70, \quad 0<x<10$
(b) $u(x,0)=70\cos x, \quad 0<x<10$
(c) $u(x,0)=\begin{cases} 10x, & 0<x<5 \\ 10(10-x), & 5\leqslant x<10 \end{cases}$
(d) $u(x,0)=\begin{cases} 0, & 0<x<3 \\ 65, & 3\leqslant x<10 \end{cases}$

2. The ends and sides of a thin copper bar ($\alpha^2=1.14$) of length 2 are insulated so that no heat can pass through them. Find the temperature $u(x,t)$ in the bar if initially
(a) $u(x,0)=65\cos^2\pi x, \quad 0\leqslant x\leqslant 2$
(b) $u(x,0)=70\sin x, \quad 0\leqslant x\leqslant 2$
(c) $u(x,0)=\begin{cases} 60x, & 0\leqslant x<1 \\ 60(2-x), & 1\leqslant x\leqslant 2 \end{cases}$
(d) $u(x,0)=\begin{cases} 0, & 0\leqslant x<1 \\ 75, & 1\leqslant x\leqslant 2 \end{cases}$

3. Verify that the function $u(x,t)$ defined by (4) satisfies the heat equation. *Hint*: Use the Cauchy ratio test to show that the infinite series (4) can be differentiated term by term with respect to x and t.

4. A steady state solution $u(x,t)$ of the heat equation $u_t=\alpha^2 u_{xx}$ is a solution $u(x,t)$ which does not change with time.
(a) Show that all steady state solutions of the heat equation are linear functions of x; i.e., $u(x)=Ax+B$.

(b) Find a steady state solution of the boundary-value problem

$$u_t=\alpha^2u_{xx};\qquad u(0,t)=T_1,\quad u(l,t)=T_2.$$

(c) Solve the heat conduction problem

$$u_t=\alpha^2u_{xx};\qquad \begin{cases} u(x,0)=75, & 0<x<1\\ u(0,t)=20, & u(1,t)=60\end{cases}$$

Hint: Let $u(x,t)=v(x)+w(x,t)$ where $v(x)$ is the steady state solution of the boundary-value problem $u_t=\alpha^2u_{xx}$; $u(0,t)=20$, $u(1,t)=60$.

5. (a) The ends of a copper rod ($\alpha^2=1.14$) 10 cm long are maintained at 0°C, while the center of the rod is maintained at 100°C by an external heat source. Show that the temperature in the rod will ultimately approach a steady state distribution regardless of the initial temperature in the rod. *Hint*: Split this problem into two boundary-value problems.
(b) Assume that the temperature in the rod is at its steady state distribution. At time $t=0$, the external heat source is removed from the center of the rod, and placed at the left end of the rod. Find the temperature in the rod at any later time t.

6. Solve the boundary-value problem

$$u_t=u_{xx}+u;\qquad \begin{cases} u(x,0)=\cos x, & 0<x<1\\ u(0,t)=0, & u(1,t)=0.\end{cases}$$

5.7 The wave equation

We consider now the boundary-value problem

$$\frac{\partial^2u}{\partial t^2}=c^2\frac{\partial^2u}{\partial x^2};\qquad \begin{cases} u(x,0)=f(x), & u_t(x,0)=g(x)\\ u(0,t)=u(l,t)=0\end{cases}\tag{1}$$

which characterizes the propagation of waves in various media, and the mechanical vibrations of an elastic string. This problem, too, can be solved by the method of separation of variables. Specifically, we will (a) find solutions $u_n(x,t)=X_n(x)T_n(t)$ of the boundary-value problem

$$\frac{\partial^2u}{\partial t^2}=c^2\frac{\partial^2u}{\partial x^2};\qquad u(0,t)=u(l,t)=0\tag{2}$$

and (b) find the solution $u(x,t)$ of (1) by taking a suitable linear combination of the functions $u_n(x,t)$.

(a) Let $u(x,t)=X(x)T(t)$. Computing

$$\frac{\partial^2u}{\partial t^2}=XT''\quad\text{and}\quad\frac{\partial^2u}{\partial x^2}=X''T$$

we see that $u(x,t)=X(x)T(t)$ is a solution of the wave equation $u_{tt}=c^2u_{xx}$

if $XT'' = c^2X''T$, or

$$\frac{T''}{c^2T} = \frac{X''}{X}. \tag{3}$$

Next, we observe that the left-hand side of (3) is a function of t alone, while the right-hand side is a function of x alone. This implies that

$$\frac{T''}{c^2T} = -\lambda = \frac{X''}{X}$$

for some constant λ. In addition, the boundary conditions

$$0 = u(0,t) = X(0)T(t), \quad \text{and} \quad 0 = u(l,t) = X(l)T(t)$$

imply that $X(0)=0$ and $X(l)=0$. Hence $u(x,t)=X(x)T(t)$ is a solution of (2) if

$$X'' + \lambda X = 0; \qquad X(0) = X(l) = 0 \tag{4}$$

and

$$T'' + \lambda c^2 T = 0. \tag{5}$$

At this point, the constant λ is arbitrary. However, the boundary-value problem (4) has a nontrivial solution $X(x)$ only if $\lambda = \lambda_n = n^2\pi^2/l^2$, and in this case,

$$X(x) = X_n(x) = \sin\frac{n\pi x}{l}.$$

Equation (5), in turn, implies that

$$T(t) = T_n(t) = a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}.$$

Hence,

$$u_n(x,t) = \sin\frac{n\pi x}{l}\left[a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right]$$

is a nontrivial solution of (2) for every positive integer n, and every pair of constants a_n, b_n.

(b) The linear combination

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right]$$

formally satisfies the boundary-value problem (2) and the initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \quad \text{and} \quad u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin\frac{n\pi x}{l}.$$

Thus, to satisfy the initial conditions $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$, we must choose the constants a_n and b_n such that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin\frac{n\pi x}{l}$$

on the interval $0<x<l$. In other words, we must expand the functions $f(x)$ and $g(x)$ in Fourier sine series on the interval $0<x<l$. This is precisely the situation in Theorem 3 of Section 5.5, and we conclude, therefore, that

$$a_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \quad \text{and} \quad b_n=\frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx.$$

For simplicity, we now restrict ourselves to the case where $g(x)$ is zero; that is, the string is released with zero initial velocity. In this case the displacement $u(x,t)$ of the string at any time $t>0$ is given by the formula

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}; \qquad a_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx. \tag{6}$$

There is a physical significance to the various terms in (6). Each term represents a particular mode in which the string vibrates. The first term ($n=1$) represents the first mode of vibration in which the string oscillates about its equilibrium position with frequency

$$\omega_1=\frac{1}{2\pi}\frac{\pi c}{l}=\frac{c}{2l} \text{ cycles per second.}$$

This lowest frequency is called the fundamental frequency, or first harmonic of the string. Similarly, the nth mode has a frequency

$$\omega_n=\frac{1}{2\pi}\frac{n\pi c}{l}=n\omega_1 \text{ cycles per second}$$

which is called the nth harmonic of the string.

In the case of the vibrating string, all the harmonic frequencies are integer multiples of the fundamental frequency ω_1. Thus, we have music in this case. Of course, if the tension in the string is not large enough, then the sound produced will be of such very low frequency that it is not in the audible range. As we increase the tension in the string, we increase the frequency, and the result is a musical note that can be heard by the human ear.

Justification of solution. We cannot prove directly, as in the case of the heat equation, that the function $u(x,t)$ defined in (6) is a solution of the wave equation. Indeed, we cannot even prove directly that the infinite series (6) has a partial derivative with respect to t and x. For example, on formally computing u_t, we obtain that

$$u_t=-\sum_{n=1}^{\infty}\frac{n\pi c}{l}a_n\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

and due to the presence of the factor n, this series may not converge. However, there is an alternate way to establish the validity of the solution (6). At the same time, we will gain additional insight into the structure of

$u(x,t)$. Observe first that

$$\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}=\frac{1}{2}\left[\sin\frac{n\pi}{l}(x-ct)+\sin\frac{n\pi}{l}(x+ct)\right].$$

Next, let F be the odd periodic extension of f on the interval $-l<x<l$; that is,

$$F(x)=\begin{cases} f(x), & 0<x<l \\ -f(-x), & -l<x<0 \end{cases} \quad \text{and} \quad F(x+2l)=F(x).$$

It is easily verified (see Exercise 6) that the Fourier series for F is

$$F(x)=\sum_{n=1}^{\infty} c_n\sin\frac{n\pi x}{l}, \qquad c_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Therefore, we can write $u(x,t)$ in the form

$$u(x,t)=\tfrac{1}{2}\left[F(x-ct)+F(x+ct)\right] \tag{7}$$

and it is now a trivial matter to show that $u(x,t)$ satisfies the wave equation if $f(x)$ has two continuous derivatives.

Equation (7) has the following interpretation. If we plot the graph of the function $y=F(x-ct)$ for any fixed t, we see that it is the same as the graph of $y=F(x)$, except that it is displaced a distance ct in the positive x direction, as shown in Figures 1a and 1b. Thus, $F(x-ct)$ is a *wave* which travels with velocity c in the positive x direction. Similarly, $F(x+ct)$ is a wave which travels with velocity c in the negative x direction. The number c represents the velocity with which a disturbance propagates along the string. If a disturbance occurs at the point x_0, then it will be felt at the point x after a time $t=(x-x_0)/c$ has elapsed. Thus, the wave equation, or some form of it, characterizes the propagation of waves in a medium where disturbances (or signals) travel with a finite, rather than infinite, velocity.

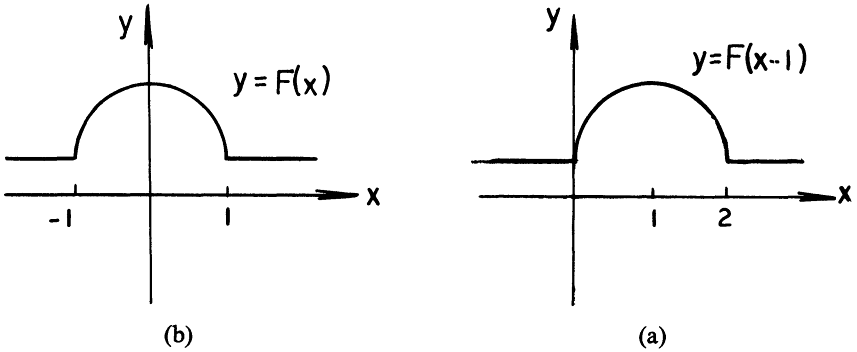


Figure 1

Exercises

Solve each of the following boundary-value problems.

1. $u_{tt}=c^2u_{xx}$; $\begin{cases} u(x,0)=\cos x-1, & u_t(x,0)=0, \quad 0\leqslant x\leqslant 2\pi \\ u(0,t)=0, & u(2\pi,t)=0 \end{cases}$

2. $u_{tt}=c^2u_{xx}$; $\begin{cases} u(x,0)=0, & u_t(x,0)=1, \quad 0\leqslant x\leqslant 1 \\ u(0,t)=0, & u(1,t)=0 \end{cases}$

3. $\begin{matrix} u_{tt}=c^2u_{xx} \\ u(0,t)=u(3,t)=0 \end{matrix}$; $u(x,0)=\begin{cases} x, & 0\leqslant x\leqslant 1, \quad u_t(x,0)=0 \\ 1, & 1\leqslant x\leqslant 2 \\ 3-x, & 2\leqslant x\leqslant 3 \end{cases}$

4. $u_{tt}=c^2u_{xx}$; $\begin{cases} u(x,0)=x\cos \pi x/2, & u_t(x,0)=0, \quad 0\leqslant x\leqslant 1 \\ u(0,t)=0, & u(1,t)=0 \end{cases}$

5. A string of length 10 ft is raised at the middle to a distance of 1 ft, and then released. Describe the motion of the string, assuming that $c^2=1$.

6. Let F be the odd periodic extension of f on the interval $-l<x<l$. Show that the Fourier series

$$\frac{2}{l}\sum_{n=1}^{\infty}\left[\int_0^l f(x)\sin\frac{n\pi x}{l}dx\right]\sin\frac{n\pi x}{l}$$

converges to $F(x)$ if F is continuous at x.

7. Show that the transformation $\xi=x-ct$, $\eta=x+ct$ reduces the wave equation to the equation $u_{\xi\eta}=0$. Conclude, therefore, that every solution $u(x,t)$ of the wave equation is of the form $u(x,t)=F(x-ct)+G(x+ct)$ for some functions F and G.

8. Show that the solution of the boundary-value problem

$$u_{tt}=c^2u_{xx}; \qquad \begin{cases} u(x,0)=f(x), & u_t(x,0)=g(x), \quad -l<x<l \\ u(0,t)=u(l,t)=0 \end{cases}$$

is

$$u(x,t)=\frac{1}{2}[F(x-ct)+F(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds$$

where F is the odd periodic extension of f.

9. The wave equation in two dimensions is $u_{tt}=c^2(u_{xx}+u_{yy})$. Find solutions of this equation by the method of separation of variables.

10. Solve the boundary-value problem

$$u_{tt}=c^2u_{xx}+u; \qquad \begin{cases} u(x,0)=f(x), & u_t(x,0)=0, \quad 0<x<l \\ u(0,t)=0, & u(l,t)=0 \end{cases}.$$

5.8 Laplace's equation

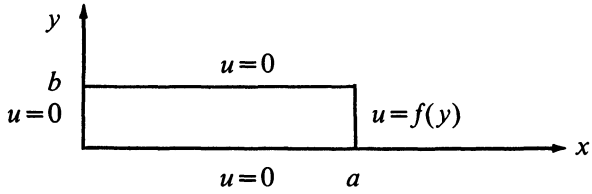
We consider now Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{1}$$

As we mentioned in Section 5.2, two important boundary-value problems that arise in connection with (1) are the Dirichlet problem and the Neumann problem. In a Dirichlet problem we seek a function $u(x,y)$ which satisfies Laplace's equation inside a region R, and which assumes prescribed values on the boundary of R. In a Neumann problem, we seek a function $u(x,y)$ which satisfies Laplace's equation inside a region R, and whose derivative in the direction normal to the boundary of R takes on prescribed values. Both of these problems can be solved by the method of separation of variables if R is a rectangle.

Example 1. Find a function $u(x,y)$ which satisfies Laplace's equation in the rectangle $0<x<a$, $0<y<b$ and which also satisfies the boundary conditions

$$\begin{aligned} u(x,0)&=0, & u(x,b)&=0\\ u(0,y)&=0, & u(a,y)&=f(y)\end{aligned}$$



(2)

Solution. We solve this problem in two steps. First, we will find functions $u_n(x,y)=X_n(x)Y_n(y)$ which satisfy the boundary-value problem

$$u_{xx}+u_{yy}=0; \qquad u(x,0)=0, \quad u(x,b)=0, \quad u(0,y)=0. \tag{3}$$

Then, we will find constants c_n such that the linear combination

$$u(x,y)=\sum_{n=1}^{\infty} c_n u_n(x,y)$$

satisfies the boundary condition $u(a,y)=f(y)$.

Step 1: Let $u(x,y)=X(x)Y(y)$. Computing $u_{xx}=X''Y$ and $u_{yy}=XY''$, we see that $u(x,y)=X(x)Y(y)$ is a solution of Laplace's equation if $X''Y+XY''=0$, or

$$\frac{Y''}{Y}=-\frac{X''}{X}. \tag{4}$$

Next, we observe that the left-hand side of (4) is a function of y alone, while the right-hand side is a function of x alone. This implies that

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda.$$

for some constant λ. In addition, the boundary conditions

$$0 = u(x,0) = X(x)Y(0), \qquad 0 = u(x,b) = X(x)Y(b),$$
$$0 = u(0,y) = X(0)Y(y)$$

imply that $Y(0)=0$, $Y(b)=0$, and $X(0)=0$. Hence $u(x,y)=XY$ is a solution of (3) if

$$Y'' + \lambda Y = 0; \qquad Y(0)=0, \quad Y(b)=0 \tag{5}$$

and

$$X'' - \lambda X = 0, \qquad X(0)=0. \tag{6}$$

At this point the constant λ is arbitrary. However, the boundary-value problem (5) has a nontrivial solution $Y(y)$ only if $\lambda = \lambda_n = n^2\pi^2/b^2$, and in this case,

$$Y(y) = Y_n(y) = \sin n\pi y/b.$$

Equation (6), in turn, implies that $X_n(x)$ is proportional to $\sinh n\pi x/b$. (The differential equation $X'' - (n^2\pi^2/b^2)X = 0$ implies that $X(x) = c_1 \cosh n\pi x/b + c_2 \sinh n\pi x/b$ for some choice of constants c_1, c_2, and the initial condition $X(0)=0$ forces c_1 to be zero.) We conclude, therefore, that

$$u_n(x,y) = \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

is a solution of (3) for every positive integer n.

Step 2: The function

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

is a solution (formally) of (3) for every choice of constants $c_1, c_2, \ldots$. Its value at $x=a$ is

$$u(a,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b}.$$

Therefore, we must choose the constants c_n such that

$$f(y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b}, \qquad 0<y<b.$$

In other words, we must expand f in a Fourier sine series on the interval $0<y<b$. This is precisely the situation described in Theorem 3 of Section

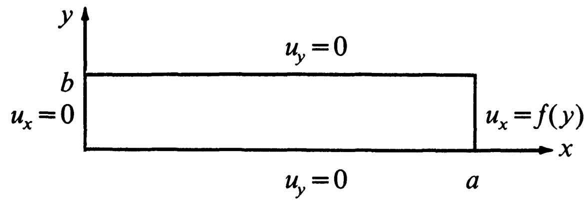
5.5, and we conclude, therefore, that

$$c_n = \frac{2}{b\sinh\frac{n\pi a}{b}}\int_0^b f(y)\sin\frac{n\pi y}{b}dy, \qquad n=1,2,\ldots.$$

Remark. The method of separation of variables can always be used to solve the Dirichlet problem for a rectangle R if u is zero on three sides of R. We can solve an arbitrary Dirichlet problem for a rectangle R by splitting it up into four problems where u is zero on three sides of R (see Exercises 1–4).

Example 2. Find a function $u(x,y)$ which satisfies Laplace's equation in the rectangle $0<x<a$, $0<y<b$, and which also satisfies the boundary conditions

$$\begin{aligned} &u_y(x,0)=0, \quad u_y(x,b)=0 \\ &u_x(0,y)=0, \quad u_x(a,y)=f(y) \end{aligned} \tag{7}$$



Solution. We attempt to solve this problem in two steps. First, we will find functions $u_n(x,y)=X_n(x)Y_n(y)$ which satisfy the boundary-value problem

$$u_{xx}+u_{yy}=0; \qquad u_y(x,0)=0, \quad u_y(x,b)=0, \quad \text{and} \quad u_x(0,y)=0. \tag{8}$$

Then, we will try and find constants c_n such that the linear combination $u(x,y)=\sum_{n=0}^{\infty}c_n u_n(x,y)$ satisfies the boundary condition $u_x(a,y)=f(y)$.

Step 1: Set $u(x,y)=X(x)Y(y)$. Then, as in Example 1,

$$\frac{Y''}{Y}=-\frac{X''}{X}=-\lambda$$

for some constant λ. The boundary conditions

$$\begin{aligned} &0=u_y(x,0)=X(x)Y'(0), \quad 0=u_y(x,b)=X(x)Y'(b), \\ &0=u_x(0,y)=X'(0)Y(y) \end{aligned}$$

imply that

$$Y'(0)=0, \quad Y'(b)=0 \quad \text{and} \quad X'(0)=0.$$

Hence, $u(x,y)=X(x)Y(y)$ is a solution of (8) if

$$Y''+\lambda Y=0; \qquad Y'(0)=0, \quad Y'(b)=0 \tag{9}$$

and

$$X''-\lambda X=0; \qquad X'(0)=0. \tag{10}$$

At this point, the constant λ is arbitrary. However, the boundary-value problem (9) has a nontrivial solution $Y(y)$ only if $\lambda=\lambda_n=n^2\pi^2/b^2, n=0,1,2,\ldots$, and in this case

$$Y(y)=Y_n(y)=\cos n\pi y/b.$$

Equation (10), in turn, implies that $X(x)$ is proportional to $\cosh n\pi x/b$. (The differential equation $X''-n^2\pi^2X/b^2=0$ implies that $X(x)=c_1\cosh n\pi x/b+c_2\sinh n\pi x/b$ for some choice of constants c_1,c_2, and the boundary condition $X'(0)=0$ forces c_2 to be zero.) We conclude, therefore, that

$$u_n(x,y)=\cosh\frac{n\pi x}{b}\cos\frac{n\pi y}{b}$$

is a solution of (8) for every nonnegative integer n.
Step 2: The function

$$u(x,y)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n\cosh\frac{n\pi x}{b}\cos\frac{n\pi y}{b}$$

is a solution (formally) of (8) for every choice of constants $c_0,c_1,c_2,\ldots$. The value of u_x at $x=a$ is

$$u_x(a,y)=\sum_{n=1}^{\infty}\frac{n\pi}{b}c_n\sinh\frac{n\pi a}{b}\cos\frac{n\pi y}{b}.$$

Therefore, we must choose the constants $c_1,c_2,\ldots,$ such that

$$f(y)=\sum_{n=1}^{\infty}\frac{n\pi}{b}c_n\sinh\frac{n\pi a}{b}\cos\frac{n\pi y}{b}, \qquad 0<y<b. \tag{11}$$

Now, Theorem 3 of Section 5.5 states that we can expand $f(y)$ in the cosine series

$$f(y)=\frac{1}{b}\int_0^b f(y)\,dy+\frac{2}{b}\sum_{n=1}^{\infty}\left[\int_0^b f(y)\cos\frac{n\pi y}{b}dy\right]\cos\frac{n\pi y}{b} \tag{12}$$

on the interval $0\leqslant y\leqslant b$. However, we cannot equate coefficients in (11) and (12) since the series (11) has no constant term. Therefore, the condition

$$\int_0^b f(y)\,dy=0$$

is necessary for this Neumann problem to have a solution. If this is the case, then

$$c_n = \frac{2}{n\pi \sinh\frac{n\pi a}{b}} \int_0^b f(y)\cos\frac{n\pi y}{b}\,dy, \qquad n \geqslant 1.$$

Finally, note that c_0 remains arbitrary, and thus the solution $u(x,y)$ is only determined up to an additive constant. This is a property of all Neumann problems.

Exercises

Solve each of the following Dirichlet problems.

1. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(x,0)=0, & u(x,b)=0 \\ u(a,y)=0, & u(0,y)=f(y) \end{matrix}$

2. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(0,y)=0, & u(a,y)=0 \\ u(x,0)=0, & u(x,b)=f(x) \end{matrix}$

Remark. You can do this problem the long way, by separation of variables, or you can try something smart, like interchanging x with y and using the result of Example 1 in the text.

3. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(0,y)=0, & u(a,y)=0 \\ u(x,b)=0, & u(x,0)=f(x) \end{matrix}$

4. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(x,0)=f(x), & u(x,b)=g(x) \\ u(0,y)=h(y), & u(a,y)=k(y) \end{matrix}$

Hint: Write $u(x,y)$ as the sum of 4 functions, each of which is zero on three sides of the rectangle.

5. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(x,0)=0, & u(x,b)=1 \\ u(0,y)=0, & u(a,y)=1 \end{matrix}$

6. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(x,b)=0, & u(x,0)=1 \\ u(0,y)=0, & u(a,y)=1 \end{matrix}$

7. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(x,0)=1, & u(x,b)=1 \\ u(0,y)=0, & u(a,y)=1 \end{matrix}$

8. $\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<a, \quad 0<y<b \end{matrix}; \quad \begin{matrix} u(x,0)=1, & u(x,b)=1 \\ u(0,y)=1, & u(a,y)=1 \end{matrix}$

Remark. *Think!*

9. Solve the boundary-value problem

$$\begin{matrix} u_{xx}+u_{yy}=u \\ 0<x<1, \quad 0<y<1 \end{matrix}; \quad \begin{matrix} u(x,0)=0, & u(x,1)=0 \\ u(0,y)=0, & u(1,y)=y \end{matrix}$$

10. (a) For which functions $f(y)$ can we find a solution $u(x,y)$ of the Neumann problem

$$\begin{matrix} u_{xx}+u_{yy}=0 \\ 0<x<1, \quad 0<y<1 \end{matrix}; \quad \begin{matrix} u_x(1,y)=0, & u_x(0,y)=f(y) \\ u_y(x,0)=0, & u_y(x,1)=0 \end{matrix}$$

(b) Solve this problem if $f(y)=\sin 2\pi y$.

11. Laplace's equation in three dimensions is

$$u_{xx}+u_{yy}+u_{zz}=0.$$

Assuming that $u=X(x)Y(y)Z(z)$, find 3 ordinary differential equations satisfied by X, Y, and Z.