4 Spectral Analysis

This chapter can be omitted without any loss of continuity. The reader with no background in Fourier or complex analysis should go straight to Chapter 5. The spectral representation of a stationary time series $\{X_t\}$ essentially decomposes $\{X_t\}$ into a sum of sinusoidal components with uncorrelated random coefficients. In conjunction with this decomposition there is a corresponding decomposition into sinusoids of the autocovariance function of $\{X_t\}$. The spectral decomposition is thus an analogue for stationary processes of the more familiar Fourier representation of deterministic functions. The analysis of stationary processes by means of their spectral representation is often referred to as the "frequency domain analysis" of time series or "spectral analysis." It is equivalent to "time domain" analysis based on the autocovariance function, but provides an alternative way of viewing the process, which for some applications may be more illuminating. For example, in the design of a structure subject to a randomly fluctuating load, it is important to be aware of the presence in the loading force of a large sinusoidal component with a particular frequency to ensure that this is not a resonant frequency of the structure. The spectral point of view is also particularly useful in the analysis of multivariate stationary processes and in the analysis of linear filters. In Section 4.1 we introduce the spectral density of a stationary process $\{X_t\}$, which specifies the frequency decomposition of the autocovariance function, and the closely related spectral representation (or frequency decomposition) of the process $\{X_t\}$ itself. Section 4.2 deals with the periodogram, a sample-based function

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from which we obtain estimators of the spectral density. In Section 4.3 we discuss time-invariant linear filters from a spectral point of view and in Section 4.4 we use the results to derive the spectral density of an arbitrary ARMA process.

4.1 Spectral Densities

Suppose that $\{X_t\}$ is a zero-mean stationary time series with autocovariance function $\gamma(\cdot)$ satisfying $\sum_{h=-\infty}^{\infty}|\gamma(h)| < \infty$. The **spectral density** of $\{X_t\}$ is the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h), \quad -\infty < \lambda < \infty, \tag{4.1.1}$$

where $e^{i\lambda} = \cos(\lambda) + i\sin(\lambda)$ and $i = \sqrt{-1}$. The summability of $|\gamma(\cdot)|$ implies that the series in (4.1.1) converges absolutely (since $\left|e^{ih\lambda}\right|^2 = \cos^2(h\lambda) + \sin^2(h\lambda) = 1$). Since cos and sin have period 2π, so also does f, and it suffices to confine attention to the values of f, on the interval $(-\pi, \pi]$.

Basic Properties of f:

(a) f is even, i.e., $f(\lambda) = f(-\lambda)$, (4.1.2)

(b) $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$, and (4.1.3)

(c) $$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda)\, d\lambda. \tag{4.1.4}$$

Proof Since $\sin(\cdot)$ is an odd function and $\cos(\cdot)$ and $\gamma(\cdot)$ are even functions, we have

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}(\cos(h\lambda) - i\sin(h\lambda))\gamma(h) \\ &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\cos(-h\lambda)\gamma(h) + 0 \\ &= f(-\lambda). \end{aligned}$$

For each positive integer N define

$$\begin{aligned} f_N(\lambda) &= \frac{1}{2\pi N} E\left(\left|\sum_{r=1}^{N} X_r e^{-ir\lambda}\right|^2\right) \\ &= \frac{1}{2\pi N} E\left(\sum_{r=1}^{N} X_r e^{-ir\lambda} \sum_{s=1}^{N} X_s e^{is\lambda}\right) \\ &= \frac{1}{2\pi N}\sum_{|h|<N}(N - |h|)e^{-ih\lambda}\gamma(h). \end{aligned}$$

Clearly, the function f_N is nonnegative for each N, and since $f_N(\lambda) \to \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) = f(\lambda)$ as $N \to \infty$, f must also be nonnegative. This proves (4.1.3). Turning to (4.1.4),

$$\begin{aligned}\int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)\,d\lambda &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{i(k-h)\lambda}\gamma(h)\,d\lambda \\ &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma(h)\int_{-\pi}^{\pi} e^{i(k-h)\lambda}\,d\lambda \\ &= \gamma(k),\end{aligned}$$

since the only nonzero summand in the second line is the one for which $h = k$ (see Problem 4.1). ■

Equation (4.1.4) expresses the autocovariances of a stationary time series with absolutely summable ACVF as the Fourier coefficients of the nonnegative even function on $(-\pi, \pi]$ defined by (4.1.1). However, even if $\sum_{h=-\infty}^{\infty}|\gamma(h)| = \infty$, there may exist a corresponding spectral density defined as follows.

Definition 4.1.1

A function f is the **spectral density** of a stationary time series $\{X_t\}$ with ACVF $\gamma(\cdot)$ if

(i) $f(\lambda) \ge 0$ for all $\lambda \in (-\pi, \pi]$, and

(ii) $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\,d\lambda$ for all integers h.

Remark 1. Spectral densities are essentially unique. That is, if f and g are two spectral densities corresponding to the autocovariance function $\gamma(\cdot)$, i.e., $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\,d\lambda = \int_{-\pi}^{\pi} e^{ih\lambda} g(\lambda)\,d\lambda$ for all integers h, then f and g have the same Fourier coefficients and hence are equal (see, for example, Brockwell and Davis (1991), Section 2.8). □

The following proposition characterizes spectral densities.

Proposition 4.1.1 *A real-valued function f defined on $(-\pi, \pi]$ is the spectral density of a real-valued stationary process if and only if*

(i) $f(\lambda) = f(-\lambda)$,
(ii) $f(\lambda) \ge 0$, *and*
(iii) $\int_{-\pi}^{\pi} f(\lambda)\,d\lambda < \infty$.

Proof If $\gamma(\cdot)$ is absolutely summable, then (i)–(iii) follow from the basic properties of f, (4.1.2)–(4.1.4). For the argument in the general case, see Brockwell and Davis (1991), Section 4.3.

Conversely, suppose f satisfies (i)–(iii). Then it is easy to check, using (i), that the function defined by

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\,d\lambda$$

is even. Moreover, if $a_r \in \mathbb{R}$, $r = 1, \ldots, n$, then

$$\begin{aligned}\sum_{r,s=1}^{n} a_r \gamma(r-s) a_s &= \int_{-\pi}^{\pi} \sum_{r,s=1}^{n} a_r a_s e^{i\lambda(r-s)} f(\lambda)\, d\lambda \\ &= \int_{-\pi}^{\pi} \left| \sum_{r=1}^{n} a_r e^{i\lambda r} \right|^2 f(\lambda)\, d\lambda \\ &\geq 0,\end{aligned}$$

so that $\gamma(\cdot)$ is also nonnegative definite and therefore, by Theorem 2.1.1, is an autocovariance function. ■

Corollary 4.1.1 *An absolutely summable function $\gamma(\cdot)$ is the autocovariance function of a stationary time series if and only if it is even and*

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \geq 0, \quad \textit{for all } \lambda \in (-\pi, \pi], \tag{4.1.5}$$

in which case $f(\cdot)$ is the spectral density of $\gamma(\cdot)$.

Proof We have already established the necessity of (4.1.5). Now suppose (4.1.5) holds. Applying Proposition 4.1.1 (the assumptions are easily checked) we conclude that f is the spectral density of some autocovariance function. But this ACVF must be $\gamma(\cdot)$, since $\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)\, d\lambda$ for all integers k. ■

Example 4.1.1 Using Corollary 4.1.1, it is a simple matter to show that the function defined by

$$\kappa(h) = \begin{cases} 1, & \text{if } h = 0, \\ \rho, & \text{if } h = \pm 1, \\ 0, & \text{otherwise,} \end{cases}$$

is the ACVF of a stationary time series if and only if $|\rho| \leq \frac{1}{2}$ (see Example 2.1.1). Since $\kappa(\cdot)$ is even and nonzero only at lags $0, \pm 1$, it follows from the corollary that κ is an ACVF if and only if the function

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = \frac{1}{2\pi} [1 + 2\rho \cos \lambda]$$

is nonnegative for *all* $\lambda \in (-\pi, \pi]$. But this occurs if and only if $|\rho| \leq \frac{1}{2}$. □

As illustrated in the previous example, Corollary 4.1.1 provides us with a powerful tool for checking whether or not an absolutely summable function on the integers is an autocovariance function. It is much simpler and much more informative than direct verification of nonnegative definiteness as required in Theorem 2.1.1.

Not all autocovariance functions have a spectral density. For example, the stationary time series

$$X_t = A \cos(\omega t) + B \sin(\omega t), \tag{4.1.6}$$

where A and B are uncorrelated random variables with mean 0 and variance 1, has ACVF $\gamma(h) = \cos(\omega h)$ (Problem 2.2), which is not expressible as $\int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda$, with f a function on $(-\pi, \pi]$. Nevertheless, $\gamma(\cdot)$ can be written as the Fourier transform of the discrete distribution function

$$F(\lambda) = \begin{cases} 0 & \text{if } \lambda < -\omega, \\ 0.5 & \text{if } -\omega \le \lambda < \omega, \\ 1.0 & \text{if } \lambda \ge \omega, \end{cases}$$

i.e.,

$$\cos(\omega h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda),$$

where the integral is as defined in Section A.1. As the following theorem states (see Brockwell and Davis (1991), p. 117), every ACVF is the Fourier transform of a (generalized) distribution function on $[-\pi, \pi]$. This representation is called the **spectral representation of the ACVF**.

Theorem 4.1.1 (Spectral Representation of the ACVF) *A function $\gamma(\cdot)$ defined on the integers is the ACVF of a stationary time series if and only if there exists a right-continuous, nondecreasing, bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$ such that*

$$\gamma(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda) \tag{4.1.7}$$

for all integers h. (For real-valued time series, F is symmetric in the sense that $\int_{(a,b]} dF(x) = \int_{[-b,-a)} dF(x)$ for all a and b such that $0 < a < b$.)

Remark 2. The function F is a **generalized distribution function** on $[-\pi, \pi]$ in the sense that $G(\lambda) = F(\lambda)/F(\pi)$ is a probability distribution function on $[-\pi, \pi]$. Note that since $F(\pi) = \gamma(0) = \text{Var}(X_1)$, the ACF of $\{X_t\}$ has spectral representation

$$\rho(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dG(\lambda).$$

The function F in (4.1.7) is called the **spectral distribution function** of $\gamma(\cdot)$. If $F(\lambda)$ can be expressed as $F(\lambda) = \int_{-\pi}^{\lambda} f(y)\, dy$ for all $\lambda \in [-\pi, \pi]$, then f is the **spectral density function** and the time series is said to have a **continuous spectrum**. If F is a discrete distribution function (i.e., if G is a discrete probability distribution function), then the time series is said to have a **discrete spectrum**. The time series (4.1.6) has a discrete spectrum. □

Example 4.1.2 Linear Combination of Sinusoids

Consider now the process obtained by adding uncorrelated processes of the type defined in (4.1.6), i.e.,

$$X_t = \sum_{j=1}^{k} (A_j \cos(\omega_j t) + B_j \sin(\omega_j t)), \quad 0 < \omega_1 < \cdots < \omega_k < \pi, \tag{4.1.8}$$

where $A_1, B_1, \ldots, A_k, B_k$ are uncorrelated random variables with $E(A_j) = E(B_j) = 0$ and $\text{Var}(A_j) = \text{Var}(B_j) = \sigma_j^2$, $j = 1, \ldots, k$. By Problem 4.5, the ACVF of this time series is $\gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(\omega_j h)$ and its spectral distribution function is $F(\lambda) = \sum_{j=1}^{k} \sigma_j^2 F_j(\lambda)$, where

$$F_j(\lambda) = \begin{cases} 0 & \text{if } \lambda < -\omega_j, \\ 0.5 & \text{if } -\omega_j \le \lambda < \omega_j, \\ 1.0 & \text{if } \lambda \ge \omega_j. \end{cases}$$

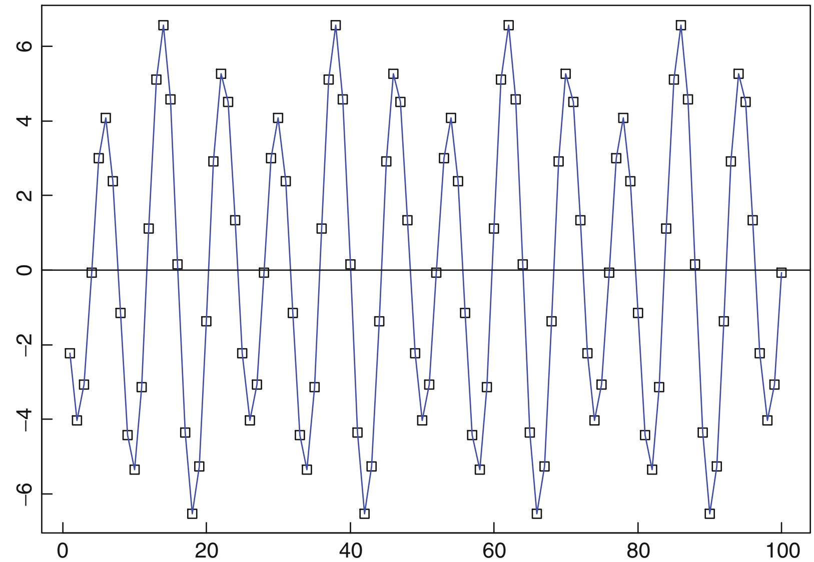

Figure 4-1
A sample path of size 100 from the time series in Example 4.1.2

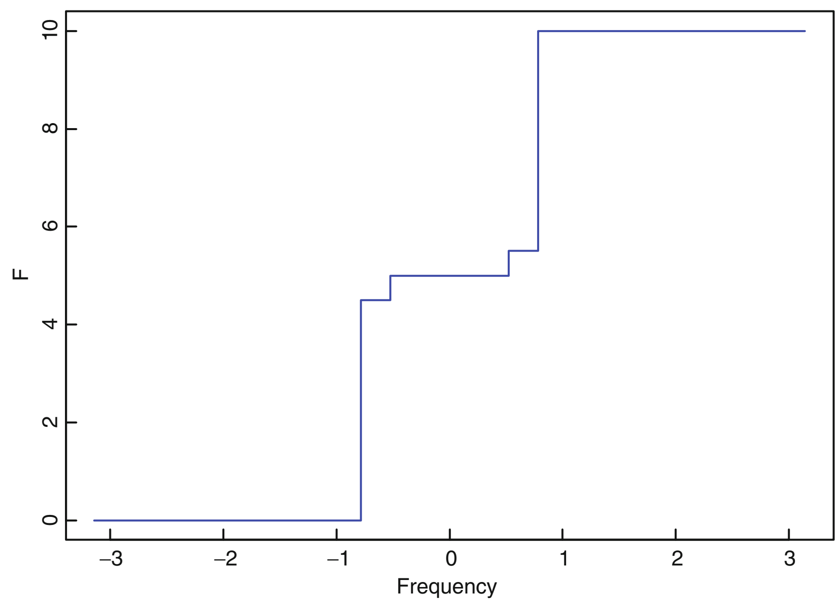

Figure 4-2
The spectral distribution function $F(\lambda)$, $-\pi \le \lambda \le \pi$, of the time series in Example 4.1.2

A sample path of this time series with $k = 2$, $\omega_1 = \pi/4$, $\omega_2 = \pi/6$, $\sigma_1^2 = 9$, and $\sigma_2^2 = 1$ is plotted in Figure 4-1. Not surprisingly, the sample path closely approximates a sinusoid with frequency $\omega_1 = \pi/4$ (and period $2\pi/\omega_1 = 8$). The general features of this sample path could have been deduced from the spectral distribution function (see Figure 4-2), which places 90% of its total mass at the frequencies $\pm\pi/4$. This means that 90% of the variance of X_t is contributed by the term $A_1 \cos(\omega_1 t) + B_1 \cos(\omega_1 t)$, which is a sinusoid with period 8.

□

The remarkable feature of Example 4.1.2 is that *every* zero-mean stationary process can be expressed as a superposition of uncorrelated sinusoids with frequencies $\omega \in [0, \pi]$. In general, however, a stationary process is a superposition of infinitely many sinusoids rather than a finite number as in (4.1.8). The required generalization of (4.1.8) that allows for this is called a stochastic integral, written as

$$X_t = \int_{(-\pi,\pi]} e^{ih\lambda} dZ(\lambda), \tag{4.1.9}$$

where $\{Z(\lambda), -\pi < \lambda \le \pi\}$ is a complex-valued process with orthogonal (or uncorrelated) increments. The representation (4.1.9) of a zero-mean stationary process $\{X_t\}$ is called the **spectral representation of the process** and should be compared with the corresponding spectral representation (4.1.7) of the autocovariance function $\gamma(\cdot)$. The underlying technical aspects of stochastic integration are beyond the scope of this book; however, in the simple case of the process (4.1.8) it is not difficult to see that it can be reexpressed in the form (4.1.9) by choosing

$$dZ(\lambda) = \begin{cases} \dfrac{A_j + iB_j}{2}, & \text{if } \lambda = -\omega_j \text{ and } j \in \{1, \dots, k\}, \\ \dfrac{A_j - iB_j}{2}, & \text{if } \lambda = \omega_j \text{ and } j \in \{1, \dots, k\}, \\ 0, & \text{otherwise.} \end{cases}$$

For this example it is also clear that

$$E(dZ(\lambda)\overline{dZ(\lambda)}) = \begin{cases} \dfrac{\sigma_j^2}{2}, & \text{if } \lambda = \pm\omega_j, \\ 0, & \text{otherwise.} \end{cases}$$

In general, the connection between $dZ(\lambda)$ and the spectral distribution function of the process can be expressed symbolically as

$$E(dZ(\lambda)\overline{dZ(\lambda)}) = \begin{cases} F(\lambda) - F(\lambda-), & \text{for a discrete spectrum,} \\ f(\lambda)d\lambda, & \text{for a continuous spectrum.} \end{cases} \tag{4.1.10}$$

These relations show that a large jump in the spectral distribution function (or a large peak in the spectral density) at frequency $\pm\omega$ indicates the presence in the time series of strong sinusoidal components with frequencies at (or near) ω radians per unit time. The period of a sinusoid with frequency ω radians per unit time is $2\pi/\omega$.

Example 4.1.3 White Noise

If $\{X_t\} \sim \text{WN}(0, \sigma^2)$, then $\gamma(0) = \sigma^2$ and $\gamma(h) = 0$ for all $|h| > 0$. This process has a flat spectral density (see Problem 4.2)

$$f(\lambda) = \frac{\sigma^2}{2\pi}, \quad -\pi \le \lambda \le \pi.$$

A process with this spectral density is called **white noise**, since each frequency in the spectrum contributes equally to the variance of the process.

□

Example 4.1.4 The Spectral Density of an AR(1) Process

If $\{X_t\}$ is a causal AR(1) process satisfying the equation,

$$X_t = \phi X_{t-1} + Z_t,$$

where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, then from (4.1.1), $\{X_t\}$ has spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi(1-\phi^2)}\left(1 + \sum_{h=1}^{\infty} \phi^h \left(e^{-ih\lambda} + e^{ih\lambda}\right)\right)$$

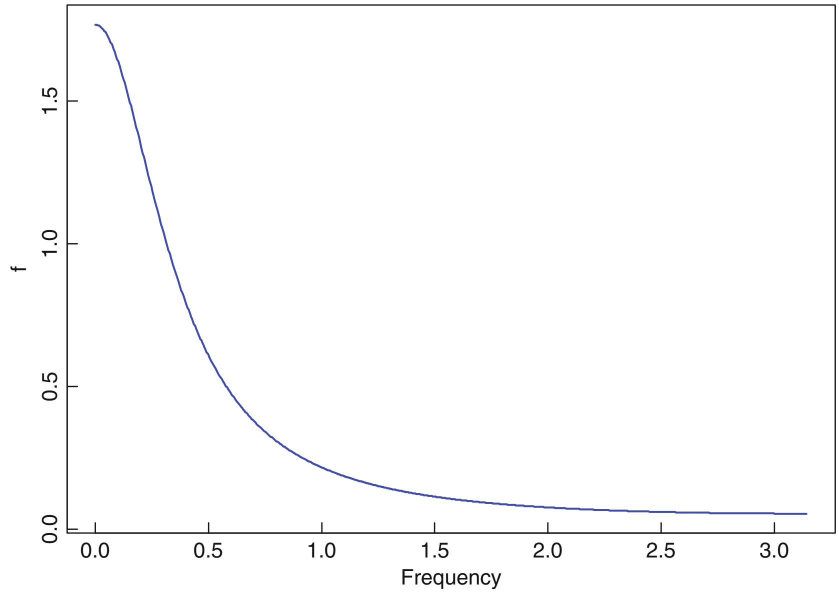

Figure 4-3
The spectral density $f(\lambda)$, $0 \le \lambda \le \pi$, of $X_t = 0.7X_{t-1} + Z_t$, where $\{Z_t\} \sim WN(0, \sigma^2)$

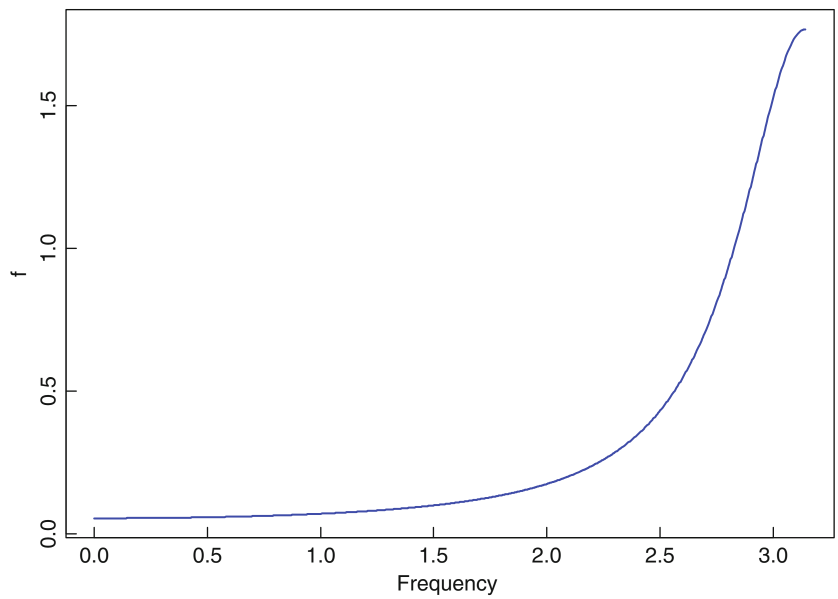

Figure 4-4
The spectral density $f(\lambda)$, $0 \le \lambda \le \pi$, of $X_t = -0.7X_{t-1} + Z_t$, where $\{Z_t\} \sim WN(0, \sigma^2)$

$$= \frac{\sigma^2}{2\pi\left(1-\phi^2\right)}\left(1 + \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}} + \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}}\right)$$
$$= \frac{\sigma^2}{2\pi}\left(1 - 2\phi\cos\lambda + \phi^2\right)^{-1}.$$

Graphs of $f(\lambda)$, $0 \le \lambda \le \pi$, are displayed in Figures 4-3 and 4-4 for $\phi = 0.7$ and $\phi = -0.7$. Observe that for $\phi = 0.7$ the density is large for low frequencies and small for high frequencies. This is not unexpected, since when $\phi = 0.7$ the process has a positive ACF with a large value at lag one (see Figure 4-5), making the series smooth with relatively few high-frequency components. On the other hand, for $\phi = -0.7$ the ACF has a large negative value at lag one (see Figure 4-6), producing a series that

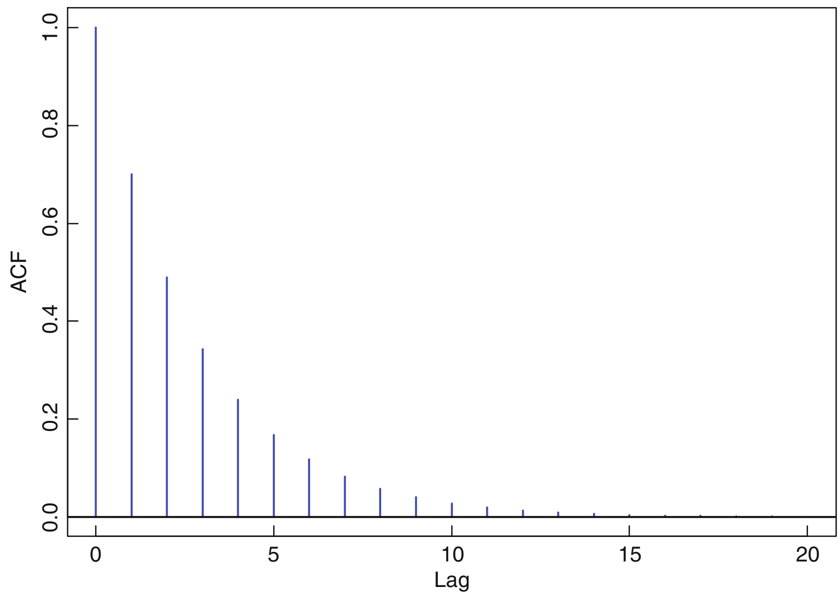

Figure 4-5
The ACF of the AR(1) process $X_t = 0.7X_{t-1} + Z_t$

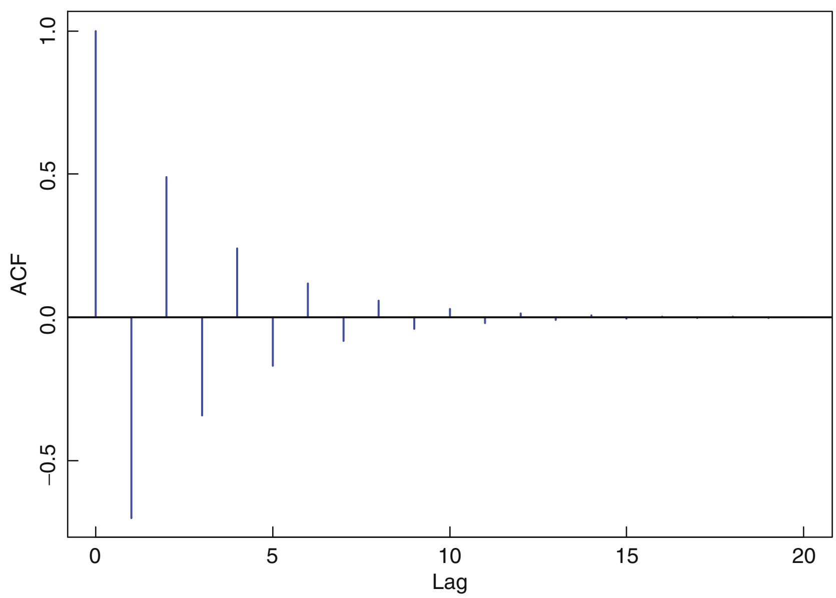

Figure 4-6
The ACF of the AR(1) process $X_t = -0.7X_{t-1} + Z_t$

fluctuates rapidly about its mean value. In this case the series has a large contribution from high-frequency components as reflected by the size of the spectral density near frequency π.

□

Example 4.1.5 Spectral Density of an MA(1) Process

If

$$X_t = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim \mathrm{WN}\left(0, \sigma^2\right)$, then from (4.1.1),

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + \theta^2 + \theta\left(e^{-i\lambda} + e^{i\lambda}\right)\right) = \frac{\sigma^2}{2\pi}\left(1 + 2\theta\cos\lambda + \theta^2\right).$$

This function is shown in Figures 4-7 and 4-8 for the values $\theta = 0.9$ and $\theta = -0.9$. Interpretations of the graphs analogous to those in Example 4.1.4 can again be made.

□

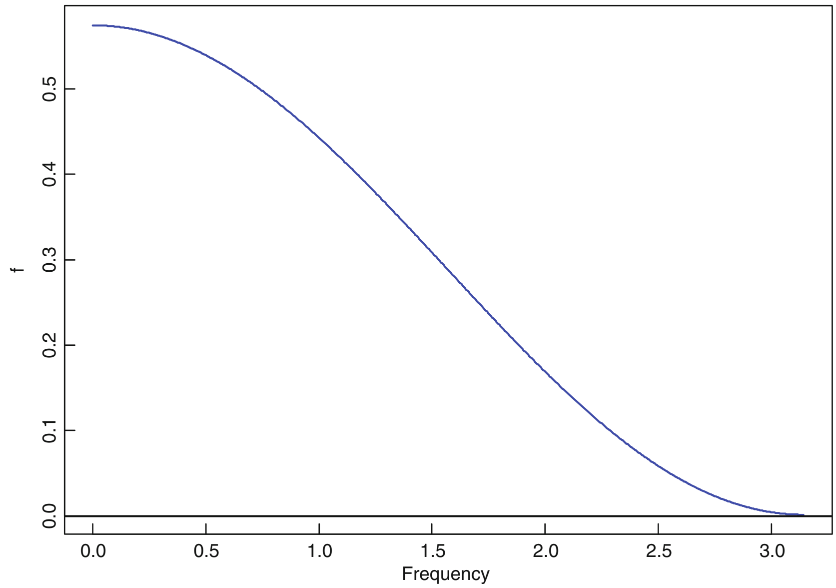

Figure 4-7
The spectral density $f(\lambda)$, $0 \le \lambda \le \pi$, of $X_t = Z_t + 0.9Z_{t-1}$ where $\{Z_t\} \sim WN(0, \sigma^2)$

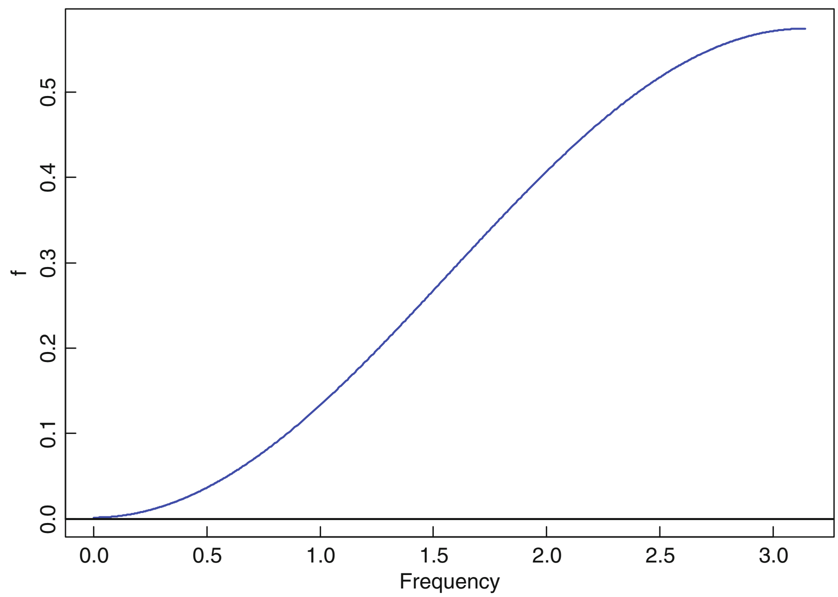

Figure 4-8
The spectral density $f(\lambda)$, $0 \le \lambda \le \pi$, of $X_t = Z_t - 0.9Z_{t-1}$ where $\{Z_t\} \sim WN(0, \sigma^2)$

4.2 The Periodogram

If $\{X_t\}$ is a stationary time series $\{X_t\}$ with ACVF $\gamma(\cdot)$ and spectral density $f(\cdot)$, then just as the sample ACVF $\hat{\gamma}(\cdot)$ of the observations $\{x_1, \ldots, x_n\}$ can be regarded as a sample analogue of $\gamma(\cdot)$, so also can the periodogram $I_n(\cdot)$ of the observations be regarded as a sample analogue of $2\pi f(\cdot)$.

To introduce the periodogram, we consider the vector of complex numbers

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n,$$

where $\mathbb{C}^n$ denotes the set of all column vectors with complex-valued components. Now let $\omega_k = 2\pi k/n$, where k is any integer between $-(n-1)/2$ and $n/2$ (inclusive), i.e.,

$$\omega_k = \frac{2\pi k}{n}, \quad k = -\left[\frac{n-1}{2}\right], \dots, \left[\frac{n}{2}\right], \tag{4.2.1}$$

where $[y]$ denotes the largest integer less than or equal to y. We shall refer to the set F_n of these values as the **Fourier frequencies** associated with sample size n, noting that F_n is a subset of the interval $(-\pi, \pi]$. Correspondingly, we introduce the n vectors

$$\mathbf{e}_k = \frac{1}{\sqrt{n}} \begin{bmatrix} e^{i\omega_k} \\ e^{2i\omega_k} \\ \vdots \\ e^{ni\omega_k} \end{bmatrix}, \qquad k = -\left[\frac{n-1}{2}\right], \dots, \left[\frac{n}{2}\right]. \tag{4.2.2}$$

Now $\mathbf{e}_1, \dots, \mathbf{e}_n$ are **orthonormal** in the sense that

$$\mathbf{e}_j{}^*\mathbf{e}_k = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \tag{4.2.3}$$

where $\mathbf{e}_j{}^*$ denotes the row vector whose kth component is the complex conjugate of the kth component of $\mathbf{e}_j$ (see Problem 4.3). This implies that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{C}^n$, so that any $\mathbf{x} \in \mathbb{C}^n$ can be expressed as the sum of n components,

$$\mathbf{x} = \sum_{k=-[(n-1)/2]}^{[n/2]} a_k \mathbf{e}_k. \tag{4.2.4}$$

The coefficients a_k are easily found by multiplying (4.2.4) on the left by $\mathbf{e}_k{}^*$ and using (4.2.3). Thus,

$$a_k = \mathbf{e}_k{}^*\mathbf{x} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-it\omega_k}. \tag{4.2.5}$$

The sequence $\{a_k\}$ is called the **discrete Fourier transform** of the sequence $\{x_1, \dots, x_n\}$.

Remark 1. The tth component of (4.2.4) can be written as

$$x_t = \sum_{k=-[(n-1)/2]}^{[n/2]} a_k[\cos(\omega_k t) + i \sin(\omega_k t)], \quad t = 1, \dots, n, \tag{4.2.6}$$

showing that (4.2.4) is just a way of representing x_t as a linear combination of sine waves with frequencies $\omega_k \in F_n$. □

Definition 4.2.1

The **periodogram** of $\{x_1, \dots, x_n\}$ is the function

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^{n} x_t e^{-it\lambda} \right|^2. \tag{4.2.7}$$

Remark 2. If λ is one of the Fourier frequencies ω_k, then $I_n(\omega_k) = |a_k|^2$, and so from (4.2.4) and (4.2.3) we find at once that the squared length of $\mathbf{x}$ is

$$\sum_{t=1}^{n} |x_t|^2 = \mathbf{x}^*\mathbf{x} = \sum_{k=-[(n-1)/2]}^{[n/2]} |a_k|^2 = \sum_{k=-[(n-1)/2]}^{[n/2]} I_n(\omega_k).$$

The value of the periodogram at frequency ω_k is thus the contribution to this sum of squares from the "frequency ω_k" term $a_k\mathbf{e}_k$ in (4.2.4). □

The next proposition shows that $I_n(\lambda)$ can be regarded as a sample analogue of $2\pi f(\lambda)$. Recall that if $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then

$$2\pi f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma(h)e^{-ih\lambda}, \quad \lambda \in (-\pi, \pi]. \tag{4.2.8}$$

Proposition 4.2.1 *If $x_1, \ldots, x_n$ are any real numbers and ω_k is any of the nonzero Fourier frequencies $2\pi k/n$ in $(-\pi, \pi]$, then*

$$I_n(\omega_k) = \sum_{|h|<n} \hat{\gamma}(h)e^{-ih\omega_k}, \tag{4.2.9}$$

where $\hat{\gamma}(h)$ is the sample ACVF of $x_1, \ldots, x_n$.

Proof Since $\sum_{t=1}^{n} e^{-it\omega_k} = 0$ if $\omega_k \neq 0$, we can subtract the sample mean $\bar{x}$ from x_t in the defining equation (4.2.7) of $I_n(\omega_k)$. Hence,

$$\begin{aligned} I_n(\omega_k) &= n^{-1}\sum_{s=1}^{n}\sum_{t=1}^{n}(x_s - \bar{x})(x_t - \bar{x})e^{-i(s-t)\omega_k} \\ &= \sum_{|h|<n} \hat{\gamma}(h)e^{-ih\omega_k}. \end{aligned}$$

■

In view of the similarity between (4.2.8) and (4.2.9), a natural estimate of the spectral density $f(\lambda)$ is $I_n(\lambda)/(2\pi)$. For a very large class of stationary time series $\{X_t\}$ with strictly positive spectral density, it can be shown that for any fixed frequencies $\lambda_1, \ldots, \lambda_m$ such that $0 < \lambda_1 < \cdots < \lambda_m < \pi$, the joint distribution function $F_n(x_1, \ldots, x_m)$ of the periodogram values $(I_n(\lambda_1), \ldots, I_n(\lambda_m))$ converges, as $n \to \infty$, to $F(x_1, \ldots, x_m)$, where

$$F(x_1, \ldots, x_m) = \begin{cases} \prod_{i=1}^{m}\left(1 - \exp\left\{\dfrac{-x_i}{2\pi f(\lambda_i)}\right\}\right), & \text{if } x_1, \ldots, x_m > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.10}$$

Thus for large n the periodogram ordinates $(I_n(\lambda_1), \ldots, I_n(\lambda_m))$ are approximately distributed as independent exponential random variables with means $2\pi f(\lambda_1), \ldots, 2\pi f(\lambda_m)$, respectively. In particular, for each fixed $\lambda \in (0, \pi)$ and $\epsilon > 0$,

$$P[|I_n(\lambda) - 2\pi f(\lambda)| > \epsilon] \to p > 0, \text{ as } n \to \infty,$$

so the probability of an estimation error larger than ϵ cannot be made arbitrarily small by choosing a sufficiently large sample size n. Thus, $I_n(\lambda)$ is not a *consistent* estimator of $2\pi f(\lambda)$.

Since for large n the periodogram ordinates at fixed frequencies are approximately independent with variances changing only slightly over small frequency intervals, we might hope to construct a consistent estimator of $f(\lambda)$ by averaging the periodogram estimates in a small frequency interval containing λ, provided that we can choose the interval in such a way that its width decreases to zero while at the same time the number of Fourier frequencies in the interval increases to ∞ as $n \to \infty$. This can indeed be done, since the number of Fourier frequencies in any *fixed* frequency interval increases approximately linearly with n. Consider, for example, the estimator

$$\tilde{f}(\lambda) = \frac{1}{2\pi} \sum_{|j| \le m} (2m+1)^{-1} I_n(g(n, \lambda) + 2\pi j/n), \tag{4.2.11}$$

where $m = \sqrt{n}$ and $g(n, \lambda)$ is the multiple of $2\pi/n$ closest to λ. The number of periodogram ordinates being averaged is approximately $2\sqrt{n}$, and the width of the frequency interval over which the average is taken is approximately $4\pi/\sqrt{n}$. It can be shown (see Brockwell and Davis (1991), Section 11.4) that this estimator is consistent for the spectral density f. The argument in fact establishes the consistency of a whole class of estimators defined as follows.

Definition 4.2.2 A **discrete spectral average estimator** of the spectral density $f(\lambda)$ has the form

$$\hat{f}(\lambda) = \frac{1}{2\pi} \sum_{|j| \le m_n} W_n(j) I_n(g(n, \lambda) + 2\pi j/n), \tag{4.2.12}$$

where the **bandwidths** m_n satisfy

$$m_n \to \infty \text{ and } m_n/n \to 0 \text{ as } n \to \infty, \tag{4.2.13}$$

and the **weight functions** $W_n(\cdot)$ satisfy

$$W_n(j) = W_n(-j),\ W_n(j) \ge 0 \text{ for all } j, \tag{4.2.14}$$

$$\sum_{|j| \le m_n} W_n(j) = 1, \tag{4.2.15}$$

and

$$\sum_{|j| \le m_n} W_n^2(j) \to 0 \text{ as } n \to \infty. \tag{4.2.16}$$

Remark 3. The conditions imposed on the sequences $\{m_n\}$ and $\{W_n(\cdot)\}$ ensure consistency of $\hat{f}(\lambda)$ for $f(\lambda)$ for a very large class of stationary processes (see Brockwell and Davis (1991), Theorem 10.4.1) including all the ARMA processes considered in this book. The conditions (4.2.13) simply mean that the number of terms in the weighted average (4.2.12) goes to ∞ as $n \to \infty$ while at the same time the width of the *frequency* interval over which the average is taken goes to zero. The conditions on $\{W_n(\cdot)\}$ ensure that the mean and variance of $\hat{f}(\lambda)$ converge as $n \to \infty$ to $f(\lambda)$ and 0, respectively. Under the conditions of Brockwell and Davis (1991), Theorem 10.4.1, it can be shown, in fact, that

$$\lim_{n \to \infty} E\hat{f}(\lambda) = f(\lambda)$$

and

$$\lim_{n \to \infty} \left(\sum_{|j| \le m_n} W_n^2(j) \right)^{-1} \operatorname{Cov}(\hat{f}(\lambda), \hat{f}(\nu)) = \begin{cases} 2f^2(\lambda) & \text{if } \lambda = \nu = 0 \text{ or } \pi, \\ f^2(\lambda) & \text{if } 0 < \lambda = \nu < \pi, \\ 0 & \text{if } \lambda \ne \nu. \end{cases}$$

□

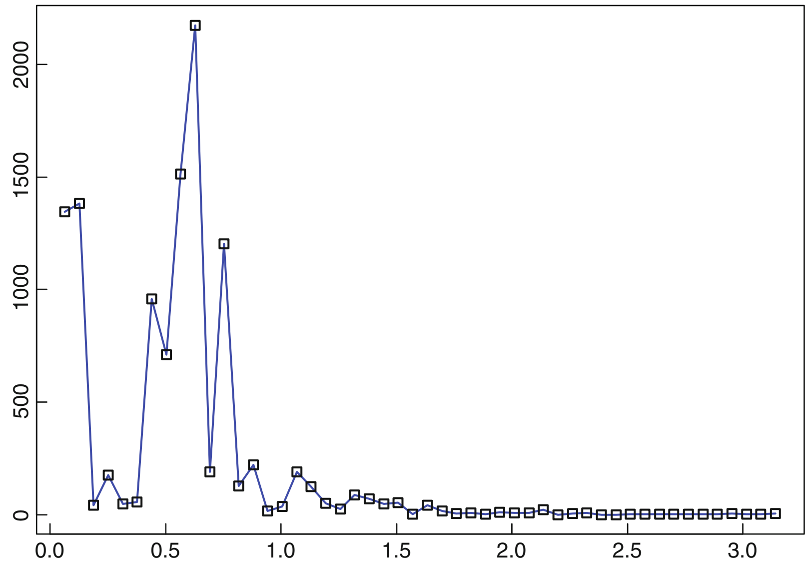

Figure 4-9 The spectral density estimate, $I_{100}(\lambda)/(2\pi)$, $0 < \lambda \le \pi$, of the sunspot numbers, 1770–1869

Example 4.2.1 For the simple moving average estimator with $m_n = \sqrt{n}$ and $W_n(j) = (2m_n + 1)^{-1}$, $|j| \le m_n$, Remark 3 gives

$$\left(2\sqrt{n}+1\right) \text{Var}\left(\hat{f}(\lambda)\right) \to \begin{cases} 2f^2(\lambda) & \text{if } \lambda = 0 \text{ or } \pi, \\ f^2(\lambda) & \text{if } 0 < \lambda < \pi. \end{cases}$$

□

In practice, when the sample size n is a fixed finite number, the choice of m and $\{W(\cdot)\}$ involves a compromise between achieving small bias and small variance for the estimator $\hat{f}(\lambda)$. A weight function that assigns roughly equal weights to a broad band of frequencies will produce an estimate of $f(\lambda)$ that, although smooth, may have a large bias, since the estimate of $f(\lambda)$ depends on the values of I_n at frequencies distant from λ. On the other hand, a weight function that assigns most of its weight to a narrow frequency band centered at zero will give an estimator with relatively small bias, but with a larger variance. In practice it is advisable to experiment with a range of weight functions and to select the one that appears to strike a satisfactory balance between bias and variance.

The option `Spectrum>Smoothed Periodogram` in the program ITSM allows the user to apply up to 50 successive discrete spectral average filters with weights $W(j) = 1/(2m+1)$, $j = -m, -m+1, \ldots, m$, to the periodogram. The value of m for each filter can be specified arbitrarily, and the weights of the filter corresponding to the combined effect (the **convolution** of the component filters) is displayed by the program. The program computes the corresponding discrete spectral average estimators $\hat{f}(\lambda)$, $0 \le \lambda \le \pi$.

Example 4.2.2 The Sunspot Numbers, 1770–1869

Figure 4-9 displays a plot of $(2\pi)^{-1}$ times the periodogram of the annual sunspot numbers (obtained by opening the project SUNSPOTS.TSM in ITSM and selecting `Spectrum>Periodogram`). Figure 4-10 shows the result of applying the discrete spectral weights $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$ (corresponding to $m = 1$, $W(j) = 1/(2m+1)$, $|j| \le m$). It is obtained from ITSM by selecting `Spectrum>Smoothed Periodogram`, entering 1 for the number of Daniell filters, 1 for the order m, and clicking on `Apply`. As expected, with such a small value of m, not much smoothing of the periodogram occurs. If we change the number of Daniell filters to 2 and set the order of the first

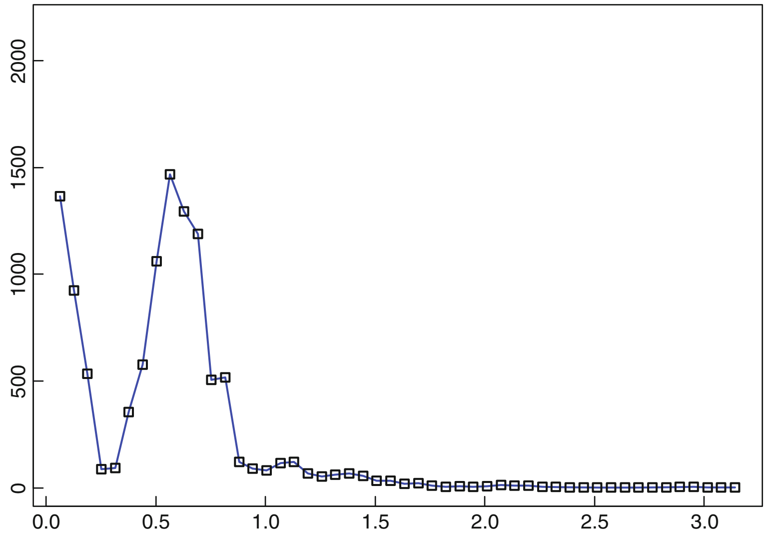

Figure 4-10 The spectral density estimate, $\hat{f}(\lambda)$, $0 < \lambda \leq \pi$, of the sunspot numbers, 1770–1869, with weights $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$

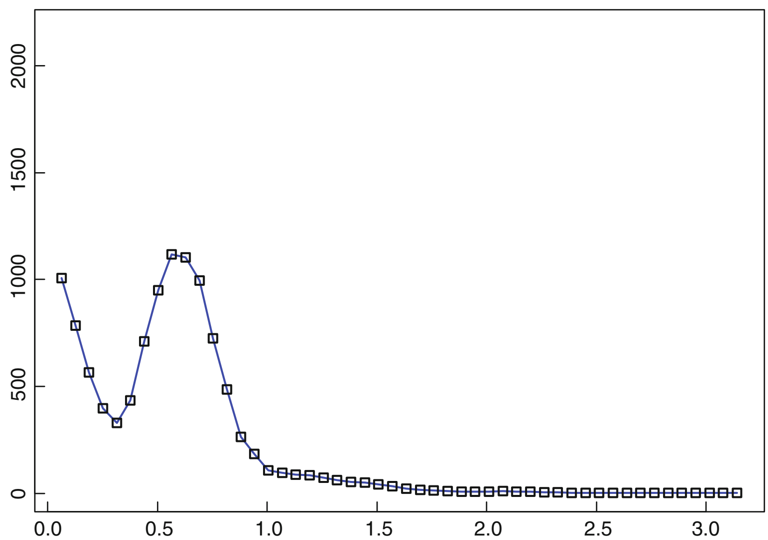

Figure 4-11 The spectral density estimate, $\hat{f}(\lambda)$, $0 < \lambda \leq \pi$, of the sunspot numbers, 1770–1869, with weights $\left\{\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{3}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{15}\right\}$

filter to 1 and the order of the second filter to 2, we obtain a combined filter with a more dispersed set of weights, $W(0) = W(1) = \frac{3}{15}$, $W(2) = \frac{2}{15}$, $W(3) = \frac{1}{15}$. Clicking on `Apply` will then give the smoother spectral estimate shown in Figure 4-11. When you are satisfied with the smoothed estimate click `OK`, and the dialog box will close. All three spectral density estimates show a well-defined peak at the frequency $\omega_{10} = 2\pi/10$ radians per year, in keeping with the suggestion from the graph of the data itself that the sunspot series contains an approximate cycle with period around 10 or 11 years.

□

4.3 Time-Invariant Linear Filters

In Section 1.5 we saw the utility of time-invariant linear filters for smoothing the data, estimating the trend, eliminating the seasonal and/or trend components of the data, etc. A linear process is the output of a time-invariant linear filter (TLF) applied to a white noise input series. More generally, we say that the process $\{Y_t\}$ is the output of a linear filter $C = \{c_{t,k}, t, k = 0 \pm 1, \ldots\}$ applied to an input process $\{X_t\}$ if

$$Y_t = \sum_{k=-\infty}^{\infty} c_{t,k} X_k, \quad t = 0, \pm 1, \ldots. \tag{4.3.1}$$

The filter is said to be **time-invariant** if the weights $c_{t,t-k}$ are independent of t, i.e., if

$$c_{t,t-k} = \psi_k.$$

In this case,

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$$

and

$$Y_{t-s} = \sum_{k=-\infty}^{\infty} \psi_k X_{t-s-k},$$

so that the time-shifted process $\{Y_{t-s}, t = 0, \pm 1, \ldots\}$ is obtained from $\{X_{t-s}, t = 0, \pm 1, \ldots\}$ by application of the same linear filter $\Psi = \{\psi_j, j = 0, \pm 1, \ldots\}$. The TLF ψ is said to be **causal** if

$$\psi_j = 0 \text{ for } j < 0,$$

since then Y_t is expressible in terms only of $X_s,\ s \le t$.

Example 4.3.1 The filter defined by

$$Y_t = aX_{-t}, \quad t = 0, \pm 1, \ldots,$$

is linear but not time-invariant, since $c_{t,\,t-k} = 0$ except when $2t = k$. Thus, $c_{t,\,t-k}$ depends on the value of t.

□

Example 4.3.2 The Simple Moving Average

The filter

$$Y_t = (2q+1)^{-1} \sum_{|j| \le q} X_{t-j}$$

is a TLF with $\psi_j = (2q+1)^{-1}, j = -q, \ldots, q$, and $\psi_j = 0$ otherwise.

□

Spectral methods are particularly valuable in describing the behavior of time-invariant linear filters as well as in designing filters for particular purposes such as the suppression of high-frequency components. The following proposition shows how the spectral density of the output of a TLF is related to the spectral density of the input—a fundamental result in the study of time-invariant linear filters.

Proposition 4.3.1 *Let $\{X_t\}$ be a stationary time series with mean zero and spectral density $f_X(\lambda)$. Suppose that $\Psi = \{\psi_j, j = 0, \pm 1, \ldots\}$ is an absolutely summable TLF (i.e., $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$). Then the time series*

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$$

is stationary with mean zero and spectral density

$$f_Y(\lambda) = \left|\Psi\left(e^{-i\lambda}\right)\right|^2 f_X(\lambda) = \Psi\left(e^{-i\lambda}\right)\Psi\left(e^{i\lambda}\right) f_X(\lambda),$$

where $\Psi(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda}$. *(The function* $\Psi\left(e^{-i\cdot}\right)$ *is called the* **transfer function** *of the filter, and the squared modulus* $\left|\Psi\left(e^{-i\cdot}\right)\right|^2$ *is referred to as the* **power transfer function** *of the filter.)*

Proof Applying Proposition 2.2.1, we see that $\{Y_t\}$ is stationary with mean 0 and ACVF

$$\gamma_Y(h) = \sum_{j,k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h+k-j). \tag{4.3.2}$$

Since $\{X_t\}$ has spectral density $f_X(\lambda)$, we have

$$\gamma_X(h+k-j) = \int_{-\pi}^{\pi} e^{i(h-j+k)\lambda} f_X(\lambda)\, d\lambda, \tag{4.3.3}$$

which, when substituted into (4.3.2), gives

$$\begin{aligned}
\gamma_Y(h) &= \sum_{j,\,k=-\infty}^{\infty} \psi_j \psi_k \int_{-\pi}^{\pi} e^{i(h-j+k)\lambda}\, f_X(\lambda)\, d\lambda \\
&= \int_{-\pi}^{\pi} \left(\sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda} \right) \left(\sum_{k=-\infty}^{\infty} \psi_k e^{ik\lambda} \right) e^{ih\lambda}\, f_X(\lambda)\, d\lambda \\
&= \int_{-\pi}^{\pi} e^{ih\lambda} \left| \sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda} \right|^2 f_X(\lambda)\, d\lambda.
\end{aligned}$$

The last expression immediately identifies the spectral density function of $\{Y_t\}$ as

$$f_Y(\lambda) = \left|\psi\left(e^{-i\lambda}\right)\right|^2 f_X(\lambda) = \psi\left(e^{-i\lambda}\right)\psi\left(e^{i\lambda}\right) f_X(\lambda). \qquad \blacksquare$$

Remark 4. Proposition 4.3.1 allows us to analyze the net effect of applying one or more filters in succession. For example, if the input process $\{X_t\}$ with spectral density f_X is operated on sequentially by two absolutely summable TLFs Ψ_1 and Ψ_2, then the net effect is the same as that of a TLF with transfer function $\psi_1\left(e^{-i\lambda}\right)\Psi_2\left(e^{-i\lambda}\right)$ and the spectral density of the output process

$$W_t = \psi_1(B)\psi_2(B)\, X_t$$

is $\left|\psi_1\left(e^{-i\lambda}\right)\psi_2\left(e^{-i\lambda}\right)\right|^2 f_X(\lambda)$. (See also Remark 2 of Section 2.2.) □

As we saw in Section 1.5, differencing at lag s is one method for removing a seasonal component with period s from a time series. The transfer function for this filter is $1 - e^{-is\lambda}$, which is zero for all frequencies that are integer multiples of $2\pi/s$ radians per unit time. Consequently, this filter has the desired effect of removing all components with period s.

The simple moving-average filter in Example 4.3.2 has transfer function

$$\psi\left(e^{-i\lambda}\right) = D_q(\lambda),$$

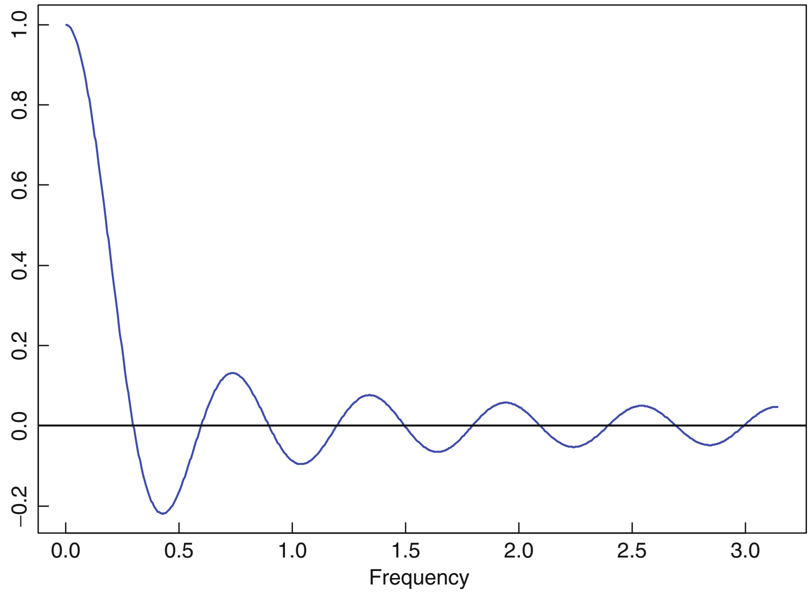

Figure 4-12
The transfer function $D_{10}(\lambda)$ for the simple moving-average filter

where $D_q(\lambda)$ is the Dirichlet kernel

$$D_q(\lambda) = (2q+1)^{-1} \sum_{|j|\le q} e^{-ij\lambda} = \begin{cases} \dfrac{\sin[(q+0.5)\lambda]}{(2q+1)\sin(\lambda/2)}, & \text{if } \lambda \neq 0, \\ 1, & \text{if } \lambda = 0. \end{cases}$$

A graph of D_q is given in Figure 4-12. Notice that $|D_q(\lambda)|$ is near 1 in a neighborhood of 0 and tapers off to 0 for large frequencies. This is an example of a low-pass filter. The ideal low-pass filter would have a transfer function of the form

$$\psi(e^{-i\lambda}) = \begin{cases} 1, & \text{if } |\lambda| \le \omega_c, \\ 0, & \text{if } |\lambda| > \omega_c, \end{cases}$$

where ω_c is a predetermined cutoff value. To determine the corresponding linear filter, we expand $\Psi\left(e^{-i\lambda}\right)$ as a Fourier series,

$$\psi\left(e^{-i\lambda}\right) = \sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda}, \tag{4.3.4}$$

with coefficients

$$\psi_j = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{ij\lambda}\, d\lambda = \begin{cases} \dfrac{\omega_c}{\pi}, & \text{if } j = 0, \\ \dfrac{\sin(j\omega_c)}{j\pi}, & \text{if } |j| > 0. \end{cases}$$

We can approximate the ideal low-pass filter by truncating the series in (4.3.4) at some large value q, which may depend on the length of the observed input series. In Figure 4-13 the transfer function of the ideal low-pass filter with $w_c = \pi/4$ is plotted with the approximations $\Psi^{(q)}\left(e^{-i\lambda}\right) = \sum_{j=-q}^{q} \psi_j e^{-ij\lambda}$ for $q=2$ and $q=10$. As can be seen in the figure, the approximations do not mirror Ψ very well near the cutoff value ω_c and behave like damped sinusoids for frequencies greater than ω_c. The poor approximation in the neighborhood of ω_c is typical of Fourier series approximations to functions with discontinuities, an effect known as the **Gibbs phenomenon**. Convergence factors may

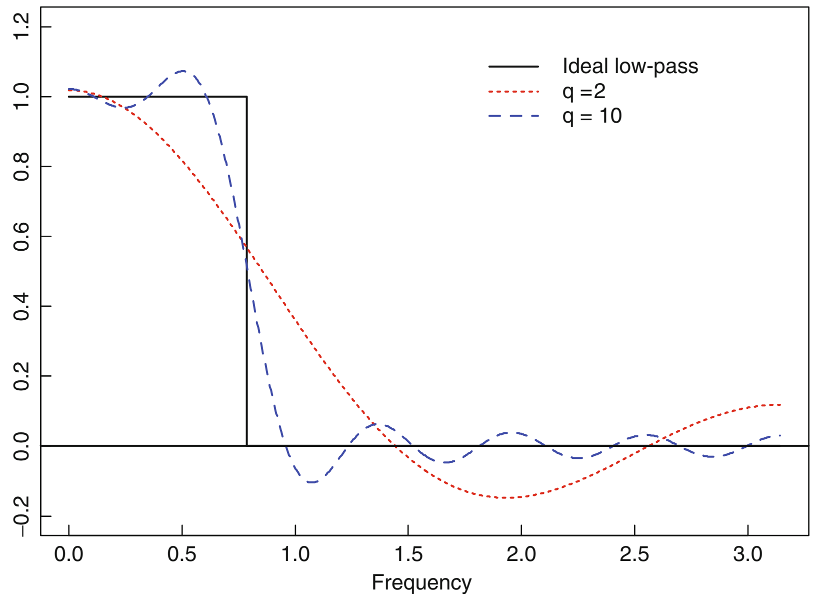

Figure 4-13 The transfer function for the ideal low-pass filter and truncated Fourier approximations $\Psi^{(q)}$ for $q = 2, 10$

be employed to help mitigate the overshoot problem at ω_c and to improve the overall approximation of $\Psi^{(q)}\left(e^{-i\cdot}\right)$ to $\Psi\left(e^{-i\cdot}\right)$ (see Bloomfield 2000).

4.4 The Spectral Density of an ARMA Process

In Section 4.1 the spectral density was computed for an MA(1) and for an AR(1) process. As an application of Proposition 4.3.1, we can now easily derive the spectral density of an arbitrary ARMA(p, q) process.

Spectral Density of an ARMA(*p,q*) Process: If $\{X_t\}$ is a causal ARMA(p, q) process satisfying $\phi(B)X_t = \theta(B)Z_t$, then

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left|\theta\left(e^{-i\lambda}\right)\right|^2}{\left|\phi\left(e^{-i\lambda}\right)\right|^2}, \quad -\pi \le \lambda \le \pi. \tag{4.4.1}$$

Because the spectral density of an ARMA process is a ratio of trigonometric polynomials, it is often called a **rational spectral density**.

Proof From (3.1.3), $\{X_t\}$ is obtained from $\{Z_t\}$ by application of the TLF with transfer function

$$\psi\left(e^{-i\lambda}\right) = \frac{\theta\left(e^{-i\lambda}\right)}{\phi\left(e^{-i\lambda}\right)}.$$

Since $\{Z_t\}$ has spectral density $f_Z(\lambda) = \sigma^2/(2\pi)$, the result now follows from Proposition 4.3.1. ■

For any specified values of the parameters $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ and σ^2, the `Spectrum>Model` option of ITSM can be used to plot the model spectral density.

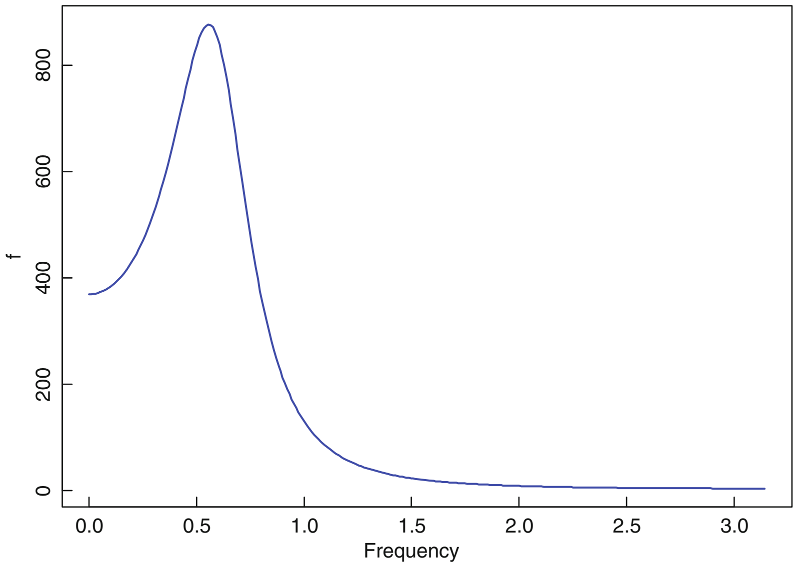

Figure 4-14 The spectral density $f_X(\lambda)$, $0 \le \lambda \le \pi$ of the AR(2) model (3.2.20) fitted to the mean-corrected sunspot series

Example 4.4.1 The Spectral Density of an AR(2) Process

For an AR(2) process (4.4.1) becomes

$$f_X(\lambda) = \frac{\sigma^2}{2\pi\left(1-\phi_1 e^{-i\lambda}-\phi_2 e^{-2i\lambda}\right)\left(1-\phi_1 e^{i\lambda}-\phi_2 e^{2i\lambda}\right)}$$
$$= \frac{\sigma^2}{2\pi\left(1+\phi_1^2+2\phi_2+\phi_2^2+2(\phi_1\phi_2-\phi_1)\cos\lambda-4\phi_2\cos^2\lambda\right)}.$$

Figure 4-14 shows the spectral density, found from the `Spectrum>Model` option of ITSM, for the model (3.2.20) fitted to the mean-corrected sunspot series. Notice the well-defined peak in the model spectral density. The frequency at which this peak occurs can be found by differentiating the denominator of the spectral density with respect to $\cos\lambda$ and setting the derivative equal to zero. This gives

$$\cos\lambda = \frac{\phi_1\phi_2-\phi_1}{4\phi_2} = 0.849.$$

The corresponding frequency is $\lambda = 0.556$ radians per year, or equivalently $c = \lambda/(2\pi) = 0.0885$ cycles per year, and the corresponding period is therefore $1/0.0885 = 11.3$ years. The model thus reflects the approximate cyclic behavior of the data already pointed out in Example 4.2.2. The model spectral density in Figure 4-14 should be compared with the rescaled periodogram of the data and the nonparametric spectral density estimates of Figures 4-9, 4-10, and 4-11.

□

Example 4.4.2 The ARMA(1,1) Process

In this case the expression (4.4.1) becomes

$$f_X(\lambda) = \frac{\sigma^2(1+\theta e^{i\lambda})(1+\theta e^{-i\lambda})}{2\pi(1-\phi e^{i\lambda})(1-\phi e^{-i\lambda})}$$
$$= \frac{\sigma^2(1+\theta^2+2\theta\cos\lambda)}{2\pi(1+\phi^2-2\phi\cos\lambda)}.$$

□

4.4.1 Rational Spectral Density Estimation

An alternative to the spectral density estimator of Definition 4.2.2 is the estimator obtained by fitting an ARMA model to the data and then computing the spectral density of the fitted model. The spectral density shown in Figure 4-14 can be regarded as such an estimate, obtained by fitting an AR(2) model to the mean-corrected sunspot data.

Provided that there is an ARMA model that fits the data satisfactorily, this procedure has the advantage that it can be made systematic by selecting the model according (for example) to the AICC criterion (see Section 5.5.2). For further information see Brockwell and Davis (1991), Section 10.6.

Problems

4.1 Show that

$$\int_{-\pi}^{\pi} e^{i(k-h)\lambda}\, d\lambda = \begin{cases} 2\pi, & \text{if } k = h, \\ 0, & \text{otherwise.} \end{cases}$$

4.2 If $\{Z_t\} \sim \mathrm{WN}\big(0, \sigma^2\big)$, apply Corollary 4.1.1 to compute the spectral density of $\{Z_t\}$.

4.3 Show that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are orthonormal in the sense of (4.2.3).

4.4 Use Corollary 4.1.1 to establish whether or not the following function is the autocovariance function of a stationary process $\{X_t\}$:

$$\gamma(h) = \begin{cases} 1 & \text{if } h = 0, \\ -0.5 & \text{if } h = \pm 2, \\ -0.25 & \text{if } h = \pm 3, \\ 0 & \text{otherwise.} \end{cases}$$

4.5 If $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary processes with autocovariance functions $\gamma_X(\cdot)$ and $\gamma_Y(\cdot)$ and spectral distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, respectively, show that the process $\{Z_t = X_t + Y_t\}$ is stationary with autocovariance function $\gamma_Z = \gamma_X + \gamma_Y$ and spectral distribution function $F_Z = F_X + F_Y$.

4.6 Let $\{X_t\}$ be the process defined by

$$X_t = A\cos(\pi t/3) + B\sin(\pi t/3) + Y_t,$$

where $Y_t = Z_t + 2.5Z_{t-1}$, $\{Z_t\} \sim \mathrm{WN}\big(0, \sigma^2\big)$, A and B are uncorrelated with mean 0 and variance ν^2, and Z_t is uncorrelated with A and B for each t. Find the autocovariance function and spectral distribution function of $\{X_t\}$.

4.7 Let $\{X_t\}$ denote the sunspot series filed as SUNSPOTS.TSM and let $\{Y_t\}$ denote the mean-corrected series $Y_t = X_t - 46.93$, $t = 1, \ldots, 100$. Use ITSM to find the Yule–Walker AR(2) model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}\left(0, \sigma^2\right),$$

i.e., find ϕ_1, ϕ_2, and σ^2. Use ITSM to plot the spectral density of the fitted model and find the frequency at which it achieves its maximum value. What is the corresponding period?

4.8 (a) Use ITSM to compute and plot the spectral density of the stationary series $\{X_t\}$ satisfying

$$X_t - 0.99X_{t-3} = Z_t, \quad \{Z_t\} \sim \text{WN}(0, 1).$$

(b) Does the spectral density suggest that the sample paths of $\{X_t\}$ will exhibit approximately oscillatory behavior? If so, then with what period?

(c) Use ITSM to simulate a realization of $X_1, \ldots, X_{60}$ and plot the realization. Does the graph of the realization support the conclusion of part (b)? Save the generated series as X.TSM by clicking on the window displaying the graph, then on the red EXP button near the top of the screen. Select `Time Series` and `File` in the resulting dialog box and click `OK`. You will then be asked to provide the file name, X.TSM.

(d) Compute the spectral density of the filtered process

$$Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$$

and compare the numerical values of the spectral densities of $\{X_t\}$ and $\{Y_t\}$ at frequency $\omega = 2\pi/3$ radians per unit time. What effect would you expect the filter to have on the oscillations of $\{X_t\}$?

(e) Open the project X.TSM and use the option `Smooth>Moving Ave.` to apply the filter of part (d) to the realization generated in part (c). Comment on the result.

4.9 The spectral density of a real-valued time series $\{X_t\}$ is defined on $[0, \pi]$ by

$$f(\lambda) = \begin{cases} 100, & \text{if } \pi/6 - 0.01 < \lambda < \pi/6 + 0.01, \\ 0, & \text{otherwise,} \end{cases}$$

and on $[-\pi, 0]$ by $f(\lambda) = f(-\lambda)$.

(a) Evaluate the ACVF of $\{X_t\}$ at lags 0 and 1.

(b) Find the spectral density of the process $\{Y_t\}$ defined by

$$Y_t := \nabla_{12} X_t = X_t - X_{t-12}.$$

(c) What is the variance of Y_t?

(d) Sketch the power transfer function of the filter ∇_{12} and use the sketch to explain the effect of the filter on sinusoids with frequencies (i) near zero and (ii) near $\pi/6$.

4.10 Suppose that $\{X_t\}$ is the noncausal and noninvertible ARMA(1,1) process satisfying

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \text{WN}\left(0, \sigma^2\right),$$

where $|\phi| > 1$ and $|\theta| > 1$. Define $\tilde{\phi}(B) = 1 - \frac{1}{\phi}B$ and $\tilde{\theta}(B) = 1 + \frac{1}{\theta}B$ and let $\{W_t\}$ be the process given by

$$W_t := \tilde{\theta}^{-1}(B)\tilde{\phi}(B)X_t.$$

(a) Show that $\{W_t\}$ has a constant spectral density function.

(b) Conclude that $\{W_t\} \sim \mathrm{WN}\big(0, \sigma_w^2\big)$. Give an explicit formula for σ_w^2 in terms of ϕ, θ, and σ^2.

(c) Deduce that $\tilde{\phi}(B)X_t = \tilde{\theta}(B)W_t$, so that $\{X_t\}$ is a causal and invertible ARMA(1,1) process relative to the white noise sequence $\{W_t\}$.