Chapter 11
Macromechanics of Composites

Laminated fibrous composites are made by bonding together two or more laminae. These composites, frequently referred to as laminates, are different from sheet laminates made by bonding flat sheets of materials. In a laminated fibrous composite, the individual unidirectional laminae or plies are oriented in such a manner that the resulting structural component has the desired mechanical and/or physical characteristics in different directions. Thus, one exploits the inherent anisotropy of fibrous composites to design a composite material with appropriate properties.

In Chap. 10 we treated the micromechanics of fibrous composites, that is, how to obtain the composite properties when the properties of the matrix and the fiber and their geometric arrangements are known. While micromechanics is very useful in analyzing the composite behavior, we use the information obtained from a micromechanical analysis of a thin unidirectional lamina (or in the case of a lack of such analytical information, we must determine experimentally the properties of a lamina) as input for a macromechanical analysis of a laminated composite. Figure 11.1 shows this concept schematically (McCullough 1971). Once we have determined, analytically or otherwise, the characteristics of a fibrous lamina, we ignore its detailed microstructural nature and simply treat it as a homogeneous, orthotropic sheet. A laminated fibrous composite is made by stacking a number of such orthotropic fiber reinforced plies or sheets at specific orientations to get composite laminate with the desired characteristics. We then use the well-established theory of laminated plates or shells to analyze macromechanically such laminated composites.

To appreciate the significance of such a macromechanical analysis, we first review the basic ideas of the elastic constants of a bulk isotropic material and a lamina, a lamina as an orthotropic sheet, and finally the use of classical laminated plate theory to analyze macromechanically the laminated composites. The reader is referred to some standard texts on elasticity (Love 1952; Timoshenko and Goodier 1951; Nye 1985) for a detailed review of elasticity.

The original version of this book was revised. An erratum can be found at DOI 10.1007/978-0-387-74365-3_16

K.K. Chawla, *Composite Materials: Science and Engineering*,
DOI 10.1007/978-0-387-74365-3_11, © Springer Science+Business Media New York 2012

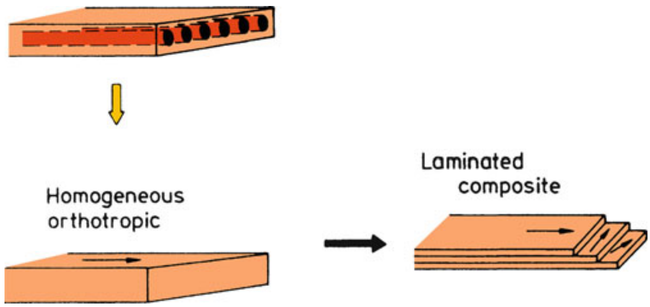


Fig. 11.1 Macromechanical analysis of laminate composites. A unidirectional ply is treated as a homogeneous, orthotropic material. Many such plies are stacked in an appropriate order (following laminated plate or shell theory) to make the composite [reprinted from McCullough (1971), courtesy of Marcel Dekker, Inc.]

11.1 Elastic Constants of an Isotropic Material

Stress is defined as force per unit area of a body. We can represent the stress acting at a point in a solid by the stress components acting on the surfaces of an elemental cube at that point. There are nine stress components (three normal and six shear) acting on the front faces of an elemental cube. The component σ_{ij} represents the force per unit area in the i direction on a face whose normal is the j direction. Rotational equilibrium requires that $\sigma_{ij} = \sigma_{ji}$. Thus, we are left with six stress components; $i = j$ gives the normal stresses while $i \neq j$ gives three shear stresses.

The displacement of a point in a deformed body with respect to its original position in the undeformed state can be represented by a vector **u** with components u_1, u_2, and u_3; these components are the projections of **u** on the x_1, x_2, and x_3 axes. *Strain* is defined as the ratio of change in length to original length. We can define the strain components in terms of the first derivatives of the displacement components as follows:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Similar to stress components, when the subscripts $i = j$, we have normal strains while $i \neq j$ gives shear strains. It should be noted, however, that ε_{ij} for $i \neq j$ gives half of the engineering shear strain, γ_{ij}; that is,

$$\gamma_{ij} = 2\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}.$$

The relationship between stress and strain in linear elasticity is described by Hooke's law. Hooke's law states that, for small strains, stress is linearly proportional to strain. For the simple case of a uniaxial stress applied to an isotropic solid, we can write Hooke's law as

$$\sigma = E\varepsilon, \tag{11.1}$$

where σ is the uniaxial stress, ε is the strain in the applied stress direction, and E is Young's modulus. We have omitted the indices in this simple unidirectional case.

In its most generalized form, Hooke's law can be written in the indicial or tensorial notation as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{11.2}$$

where C_{ijkl} are the elastic constants or stiffnesses. Equation (11.2), when written out in an expanded form, will have 81 elastic constants. It is general practice to use a contracted matrix notation for writing stresses, strains, and elastic constants. The contracted notation is especially useful for matrix algebra operations.

We use C_{mn} for C_{ijkl}, σ_m for σ_{ij}, and ε_n for ε_{kl} as per the following scheme:

ij or *kl*	11	22	33	23	31	12
m or *n*	1	2	3	4	5	6

Then Eq. (11.2) can be rewritten as

$$\sigma_m = C_{mn}\varepsilon_n. \tag{11.3}$$

It can be shown from symmetry considerations that $C_{mn} = C_{nm}$.

Conversely, we can write

$$\varepsilon_m = S_{mn}\sigma_n, \tag{11.4}$$

where S_{mn}, the compliance matrix, is the inverse of the stiffness matrix C_{mn}.

In the expanded form, we have

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}. \tag{11.5}$$

Note that σ_4, σ_5, and σ_6 now represent the shear stresses while ε_4, ε_5, and ε_6 represent the engineering shear strains. The dashed line along the diagonal indicates that the matrix is symmetric. Equation (11.5) thus gives 21 independent elastic constants in the most general case, i.e., with no symmetry elements present.

For most materials, the number of independent elastic constants is further reduced because of the various symmetry elements present. For example, only three elastic constants are independent for the cubic system. For isotropic materials where elastic properties are independent of direction, only two constants are independent. For isotropic materials, (11.5) reduces to

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ \diagdown & C_{11} & C_{12} & 0 & 0 & 0 \\ & \diagdown & C_{11} & 0 & 0 & 0 \\ & & \diagdown & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ & & & \diagdown & \frac{C_{11}-C_{12}}{2} & 0 \\ & & & & \diagdown & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix}. \tag{11.6}$$

In terms of the compliance matrix, for isotropic materials, we can write

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ \diagdown & S_{11} & S_{12} & 0 & 0 & 0 \\ & \diagdown & S_{11} & 0 & 0 & 0 \\ & & \diagdown & 2(S_{11}-S_{12}) & 0 & 0 \\ & & & \diagdown & 2(S_{11}-S_{12}) & 0 \\ & & & & \diagdown & 2(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix}. \tag{11.7}$$

Only C_{11} and C_{12} (or S_{11} and S_{12}) are the independent constants for an isotropic material. Engineers frequently use elastic constants such as Young's modulus E, Poisson's ratio ν, shear modulus G, and bulk modulus K. Only two of these four constants are independent because E, G, ν, and K are interrelated:

$$E = 2G(1+\nu) \quad \text{and} \quad K = \frac{E}{3(1-2\nu)}.$$

The relationships between these engineering constants and compliances are as follows:

$$E = \frac{1}{S_{11}}, \qquad v = -\frac{S_{12}}{S_{11}}, \qquad G = \frac{1}{2(S_{11} - S_{12})}$$

and the compliances are related to the stiffnesses as follows:

$$S_{11} = \frac{C_{11} + C_{12}}{(C_{11} - C_{12})(C_{11} + 2C_{12})},$$
$$S_{12} = -\frac{C_{12}}{(C_{11} - C_{12})(C_{11} + 2C_{12})}.$$

11.2 Elastic Constants of a Lamina

We can make a laminated composite by stacking a sufficiently large number of laminae. A *lamina*, the unit building block of a composite, can be considered to be in a state of generalized plane stress. Plane stress condition implies that the through thickness stress components are zero (hence, the term plane stress). Thus $\sigma_3 = \sigma_4 = \sigma_5 = 0$. Then (11.6) and (11.7) are reduced, for an isotropic lamina, to

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ 0 & 0 & \frac{C_{11} - C_{12}}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}, \tag{11.8}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{11} & 0 \\ 0 & 0 & 2(S_{11} - S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}. \tag{11.9}$$

Note that the subscript 6 indicates the inplane shear component. Equations (11.8) and (11.9) describe the stress–strain relationships for an isotropic lamina, for example, an aluminum sheet. A fiber reinforced lamina, however, is not an isotropic material. It is an orthotropic material; that is, it has three mutually perpendicular axes of symmetry. Relationships become slightly more complicated when we have orthotropy rather than isotropy.

Recall that a fiber reinforced lamina or ply is a thin sheet (~0.1 mm) containing oriented fibers. Generally, the fibers are oriented unidirectionally as in a prepreg but fibers in the form of a woven fabric may also be used. Several such thin laminae are stacked in a specific order of fiber orientation, cured, and bonded into a laminated composite. Because the behavior of a laminated composite depends on the characteristics of individual laminae, and with due regard to their directionality, we now discuss the elastic behavior of an orthotropic lamina.

For the case of an orthotropic material with the coordinate axes parallel to the symmetry axes of the material, the array of elastic constants is given by

$$[S_{ij}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix}. \quad (11.10)$$

A similar expression can be written for C_{ij}. We now take into account the fact that a lamina is a thin orthotropic material, that is, through thickness components are zero. We can obtain the compliance matrix for an orthotropic lamina by simply eliminating the terms involving the z axis:

$$[S_{ij}] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ & & S_{66} \end{bmatrix}. \quad (11.11)$$

We can rewrite in full form Hooke's law for a thin orthotropic lamina, with natural and geometric axes coinciding, as follows:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}. \quad (11.12)$$

Conversely,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}. \quad (11.13)$$

It is customary to use the symbol Q_{ij} rather than C_{ij} for thin material. The Q_{ij} are called *reduced stiffnesses*. The relationships between Q_{ij} and S_{ij} can easily be shown to be

$$\begin{aligned} Q_{11} &= \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}, \\ Q_{12} &= -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}, \\ Q_{22} &= \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}, \\ Q_{66} &= \frac{1}{S_{66}}. \end{aligned} \tag{11.14}$$

Also,

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}\,(i, j = 1, 2, 6).$$

Note that three-dimensional orthotropy requires nine independent elastic constants [Eq. (11.10)], while bidimensional orthotropy requires only four [Eq. (11.11) or (11.14)]. For an isotropic material (two- or three-dimensional), one just needs two independent elastic constants [Eqs. (11.6) through (11.9)].

It is worth emphasizing that Eqs. (11.12) and (11.13), showing terms with indices 16 and 26 to be zero, represent a special case of orthotropy when the principal material axes of symmetry (the fiber direction [1] and the direction transverse to it [2]) coincide with the geometric directions. If this is not so, that is, if the material symmetry axes and the geometric axes do not coincide, then we have the more general case of orthotropy. In the case of two-dimensional general orthotropy, we shall have a fully populated elastic constant matrix and the stress–strain relationships become

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix}, \tag{11.15}$$

where the $\bar{Q}_{ij}$ matrix is called the transformed reduced stiffness matrix because it is obtained by transforming Q_{ij} (specially orthotropic) to $\bar{Q}_{ij}$ (generally orthotropic), the subscripts x and y represent the geometric axes x and y, and the subscript s refers to inplane shear components. We can perform the transformation of axes as will be shown presently and obtain $\bar{Q}_{ij}$ from Q_{ij}.

Figure 11.2 shows the situation for a unidirectional composite lamina where the two sets of axes do not coincide. The properties in the 1–2 or material system of axes are known, and we wish to determine them in the x–y or geometric system or

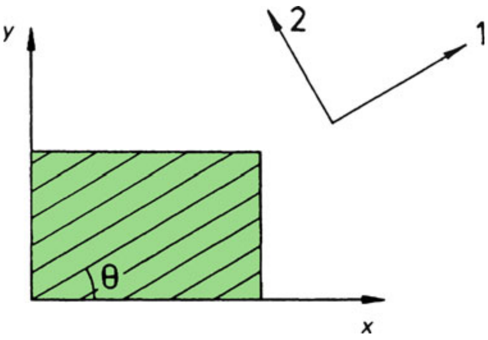


Fig. 11.2 An off-axis unidirectional lamina

Table 11.1 Direction cosines

Direction	x	y	
1	$a_{11} = m$	$a_{12} = n$	$m = \cos\,\theta$
2	$a_{21} = -n$	$a_{22} = m$	$m = \sin\,\theta$

vice versa. In order to carry out the transformation of axes, we need to bring in the concept of direction cosines, a_{ij}. Table 11.1 gives the direction cosines for the transformation of axes shown in Fig. 11.2. Angle θ is positive when the x–y axes are rotated counterclockwise with respect to the 1–2 axes. This transformation of axes is carried out easily in the matrix form (see Appendix A). For stresses we can write

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = [T]_\sigma \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix} \tag{11.16}$$

while for strains,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = [T]_\varepsilon \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix}, \tag{11.17}$$

where $[T]_\sigma$ and $[T]_\varepsilon$ are the transformation matrices for stress and strain transformations, respectively, and are given by

$$[T]_\sigma = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}, \tag{11.18}$$

$$[T]_\varepsilon = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix}, \tag{11.19}$$

where $m = \cos\theta$ and $n = \sin\theta$. This method of using different transformation matrices for stress and stain transformations avoids the need of putting the factor 1/2 before the engineering shear strains to convert them to tensorial strain components suitable for transformation. Multiplying both sides of Eq. (11.16) by $[T]_\sigma^{-1}$and remembering that $[T]_\sigma [T]_\sigma^{-1} = [I]$, the identity matrix, we get

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix} = [T]_\sigma^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}. \tag{11.20}$$

$[T]_\sigma^{-1}$ can be obtained from $[T]_\sigma$ by simply substituting $-\theta$ for θ. Appendix A gives the procedure for obtaining the inverse of a given matrix. In this particular case, substituting $-\theta$ for θ in Eq. (11.18) results in

$$[T]_\sigma^{-1} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix}, \tag{11.21}$$

where $m = \cos\theta$ and $n = \sin\theta$. Substituting Eq. (11.13) in Eq. (11.20), we obtain

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix} = [T]_\sigma^{-1} [Q] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}. \tag{11.22}$$

If we now substitute Eq. (11.17) in Eq. (11.22), we arrive at

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix} = [T]_\sigma^{-1} [Q][T]_\varepsilon \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix} = \left|\bar{Q}\right| \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix}, \tag{11.23}$$

where

$$\left[\bar{Q}\right] = [T]_\sigma^{-1} [Q][T]_\varepsilon. \tag{11.24}$$

$[\bar{Q}]$ is the stiffness matrix for a generally orthotropic lamina whose components in expanded form are written as follows ($m = \cos\theta$, $n = \sin\theta$):

$$\begin{aligned} \bar{Q}_{11} &= Q_{11} m^4 + 2(Q_{12} + 2Q_{66}) m^2 n^2 + Q_{22} n^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) m^2 n^2 + Q_{12}(m^4 + n^4) \\ \bar{Q}_{22} &= Q_{11} n^4 + 2(Q_{12} + 2Q_{66}) m^2 n^2 + Q_{22} m^4 \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) m^3 n + (Q_{12} - Q_{22} + 2Q_{66}) mn^3 \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) mn^3 + (Q_{12} - Q_{22} + 2Q_{66}) m^3 n \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) m^2 n^2 + Q_{66}(m^4 + n^4) \end{aligned} . \tag{11.25}$$

Note that although $\bar{Q}_{ij}$ is a completely filled matrix, only four of its components are independent: $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are linear combinations of the other four.

A corresponding stress–strain relationship in terms of compliances of a generally orthotropic lamina can be obtained:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix}. \tag{11.26}$$

In a generally orthotropic lamina wherein we have nonzero 16 and 26 terms, a unidirectional normal stress σ_x has both normal as well as shear strains as responses and vice versa; that is, there is a coupling between the normal and shear components. In the case of a specially orthotropic lamina where the 16 and 26 terms are zero, we have normal stresses producing normal strains and shear stresses producing shear strains and vice versa. Thus, in the specially orthotropic case, there is no coupling between the normal and shear components. We present more information about such coupling effects in Sect. 11.5.

11.3 Relationships Between Engineering Constants and Reduced Stiffnesses and Compliances

Consider the thin lamina shown in Fig. 11.3 with the natural or material axes coinciding with the geometric axes. The conventional engineering constants in this case are Young's moduli in direction 1 (E_1) and direction 2 (E_2), the principal shear modulus G_6, and the principal Poisson's ratio v_1. The Poisson's ratio v_1, when the lamina is strained in direction 1, is equal to $-\varepsilon_2/\varepsilon_1$. The Poisson's ratio v_2, when the lamina is strained in direction 2, is equal to $-\varepsilon_1/\varepsilon_2$.

We wish to relate these five conventional engineering constants to the four independent elastic constants, the reduced stiffnesses Q_{ij}. Let us consider that σ_1 is the only nonzero component in (11.13). Then we can write

$$\sigma_1 = Q_{11}\varepsilon_1 + Q_{12}\varepsilon_2,$$

$$\sigma_2 = Q_{12}\varepsilon_1 + Q_{22}\varepsilon_2 = 0.$$

Solving for ε_1 and ε_2, we get

$$\varepsilon_1 = \frac{Q_{22}}{Q_{11}Q_{22} - Q_{12}^2}\sigma_1$$

and

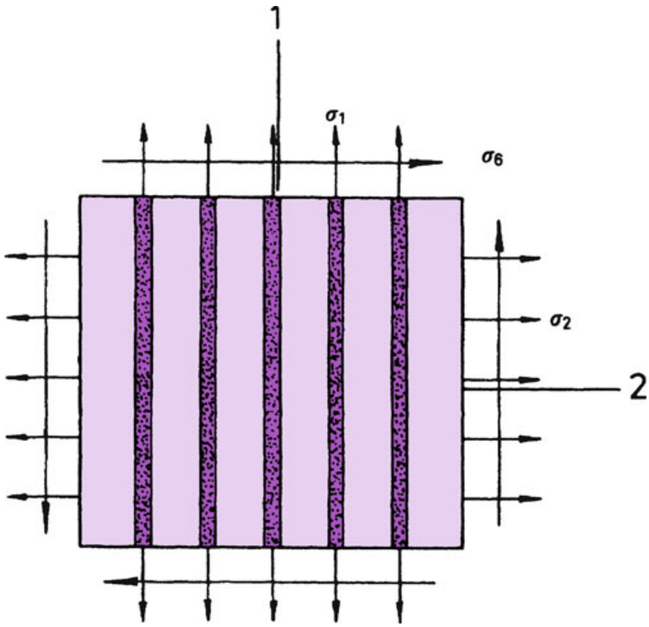


Fig. 11.3 A thin lamina with natural (or material) axes coinciding with the geometric axes

$$\varepsilon_2 = \frac{Q_{12}}{Q_{11}Q_{22} - Q_{12}^2}\sigma_1.$$

By definition, we have $E_1 = \sigma_1/\varepsilon_1$. Thus,

$$E_1 = \frac{Q_{11}Q_{22} - Q_{12}^2}{Q_{22}} \tag{11.27}$$

and

$$v_1 = -\frac{\varepsilon_2}{\varepsilon_1} = \frac{Q_{12}}{Q_{22}}. \tag{11.28}$$

If we repeat this procedure with σ_2 as the only nonzero stress component in Eq. (11.13), we obtain

$$E_2 = \frac{\sigma_2}{\varepsilon_2} = \frac{Q_{11}Q_{22} - Q_{12}^2}{Q_{11}} \tag{11.29}$$

and

$$v_2 = -\frac{\varepsilon_1}{\varepsilon_2} = \frac{Q_{12}}{Q_{11}}. \tag{11.30}$$

If we consider that σ_6 is the only nonzero component, we can get

$$G_6 = \frac{\sigma_6}{\varepsilon_6} = Q_{66}. \tag{11.31}$$

Note that only four of the five constants are independent because

$$v_1 E_2 = v_2 E_1 \tag{11.32}$$

or

$$\frac{E_1}{E_2} = \frac{v_1}{v_2}. \tag{11.33}$$

We can solve Eqs. (11.27)–(11.30) for the Q_{ij} to give

$$Q_{11} = \frac{E_1}{1 - v_1 v_2} \qquad Q_{22} = \frac{E_2}{1 - v_1 v_2} \qquad Q_{12} = \frac{v_1 E_2}{1 - v_1 v_2} = \frac{v_2 E_1}{1 - v_1 v_2}$$

and $Q_{66} = G_6$ is given by Eq. (11.31). Similarly, we can show that the relationships between compliances and engineering constants are as follows:

$$S_{11} = \frac{1}{E_1} \qquad S_{22} = \frac{1}{E_2} \qquad S_{12} = -\frac{v_1}{E_1} = -\frac{v_2}{E_2} \qquad S_{66} = \frac{1}{G_6}.$$

11.4 Variation of Lamina Properties with Orientation

In Sect. 11.2 we obtained the relationships between Q_{ij} and $\bar{Q}_{ij}$. It is of interest to obtain similar relationships for conventional engineering constants referred to geometric axes x–y (E_x, E_y, G_s, and v_x) in terms of engineering constants referred to material symmetry axes 1–2 (E_1, E_2, G_6, and v_1). Consider Eqs. (11.16) and (11.18) and let σ_x be the only nonzero stress component. Then

$$\sigma_1 = \sigma_x m^2, \tag{11.34a}$$

$$\sigma_2 = \sigma_x n^2, \tag{11.34b}$$

$$\sigma_6 = -\sigma_x mn. \tag{11.34c}$$

By Hooke's law, we can write for the strains in a lamina

$$\varepsilon_1 = \frac{1}{E_1}(\sigma_1 - v_1 \sigma_2), \tag{11.35a}$$

$$\varepsilon_2 = \frac{1}{E_2}(\sigma_2 - \nu_2\sigma_2), \tag{11.35b}$$

$$\varepsilon_6 = \frac{\sigma_6}{G_6}. \tag{11.35c}$$

From Eqs. (11.34a–11.34c) and (11.35a–11.35c), we get

$$\varepsilon_1 = \sigma_x\left(\frac{m^2}{E_1} - \nu_1\frac{n^2}{E_1}\right) = \sigma_x\left(\frac{m^2}{E_1} - \frac{\nu_2}{E_2}n^2\right), \tag{11.36a}$$

$$\varepsilon_2 = \sigma_x\left(\frac{n^2}{E_2} - \nu_2\frac{m^2}{E_2}\right) = \sigma_x\left(\frac{n^2}{E_2} - \frac{\nu_1}{E_1}m^2\right), \tag{11.36b}$$

$$\varepsilon_6 = -\frac{\sigma_x mn}{G_6}. \tag{11.36c}$$

Because we have the strain transformation given by Eq. (11.17), we can write the inverse of Eq. (11.17) as

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{bmatrix} = [T]_\varepsilon^{-1}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{bmatrix},$$

where $[T]_\varepsilon^{-1}$ can be obtained by substituting $-\theta$ for θ in Eq. (11.19). In the expanded form, we can write the above expression as

$$\varepsilon_x = m^2\varepsilon_1 + n^2\varepsilon_2 - mn\varepsilon_6, \tag{11.37a}$$

$$\varepsilon_y = n^2\varepsilon_1 + m^2\varepsilon_2 + mn\varepsilon_6, \tag{11.37b}$$

$$\varepsilon_s = 2(\varepsilon_1 - \varepsilon_2)mn + \varepsilon_6(m^2 - n^2). \tag{11.37c}$$

Substituting Eqs. (11.36a–11.36c) in (11.37a–11.37c), we obtain

$$\varepsilon_x = \sigma_x\left[\frac{m^4}{E_1} + \frac{n^4}{E_2} + \left(\frac{1}{G_6} - \frac{2\nu_1}{E_1}\right)m^2n^2\right], \tag{11.38a}$$

$$\varepsilon_y = -\sigma_x\left[\frac{\nu_1}{E_1} - \left(\frac{1}{E_1} + \frac{2\nu_1}{E_1} + \frac{1}{E_2} - \frac{1}{G_6}\right)m^2n^2\right], \tag{11.38b}$$

$$\varepsilon_s = -\sigma_x(2mn)\left[\frac{\nu_1}{E_1} + \frac{1}{E_2} - \frac{1}{2G_6} - m^2\left(\frac{1}{E_1} + \frac{2\nu_1}{E_1} + \frac{1}{E_2} - \frac{1}{G_6}\right)\right]. \tag{11.38c}$$

Now, by definitions, $E_x = \sigma_x/E_x$. Combining this with Eq. (11.38a), we obtain

$$\frac{1}{E_x} = \frac{m^4}{E_1} + \frac{n^4}{E_2} + \left(\frac{1}{G_6} - \frac{2\nu_1}{E_1}\right)m^2n^2. \tag{11.39}$$

E_y can be obtained from E_x by substituting θ + 90° for θ in Eq. (11.39):

$$\frac{1}{E_y} = \frac{n^4}{E_1} + \frac{m^4}{E_2} + \left(\frac{1}{G_6} - \frac{2\nu_1}{E_1}\right)m^2n^2. \tag{11.40}$$

Now, $\nu_x = -\varepsilon_y/\varepsilon_x$ when σ_x is the applied stress. Then, from Eqs. (11.38b) and (11.39), we obtain

$$\frac{\nu_x}{E_x} = -\frac{\varepsilon_y}{E_x\varepsilon_x} = -\frac{\varepsilon_y}{\sigma_x} = \frac{\nu_1}{E_1} - \left(\frac{1}{E_1} + \frac{2\nu_1}{E_1} + \frac{1}{E_2} - \frac{1}{G_6}\right)m^2n^2$$

or

$$\nu_x = E_x\left[\frac{\nu_1}{E_1} - \left(\frac{1}{E_1} + \frac{2\nu_1}{E_1} + \frac{1}{E_2} - \frac{1}{G_6}\right)m^2n^2\right]. \tag{11.41}$$

Similarly, it can be shown that

$$\nu_y = E_y\left[\frac{\nu_2}{E_2} - \left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_1}{E_1} - \frac{1}{G_6}\right)m^2n^2\right]. \tag{11.42}$$

Taking σ_s to be the only nonzero stress component, noting that $\varepsilon_s = \sigma_s/G_s$, and applying Hooke's law, we can obtain

$$\frac{1}{G_s} = \frac{1}{G_6} + 4m^2n^2\left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G_6}\right). \tag{11.43}$$

Equations (11.39)–(11.43) give the expressions for the variations of elastic constants of a unidirectional composite with fiber orientation. These are very useful expressions. Figure 11.4 shows the variations of E_x, E_y, G_s, ν_x, and ν_y with fiber orientation θ for a 50 % V_f carbon/epoxy composite. The other relevant data used in Fig. 11.4 are $E_1 = 240$ GPa, $E_2 = 8$ GPa, $G_6 = 6$ GPa, and $\nu_1 = 0.26$.

11.5 Analysis of Laminated Composites

Now that we have discussed the analysis of an individual lamina, we proceed to discuss the macroscopic analysis of laminated composites (Jones 1975; Christensen 1979; Tsai and Hahn 1980; Halpin 1984; Daniel and Ishai 1994).

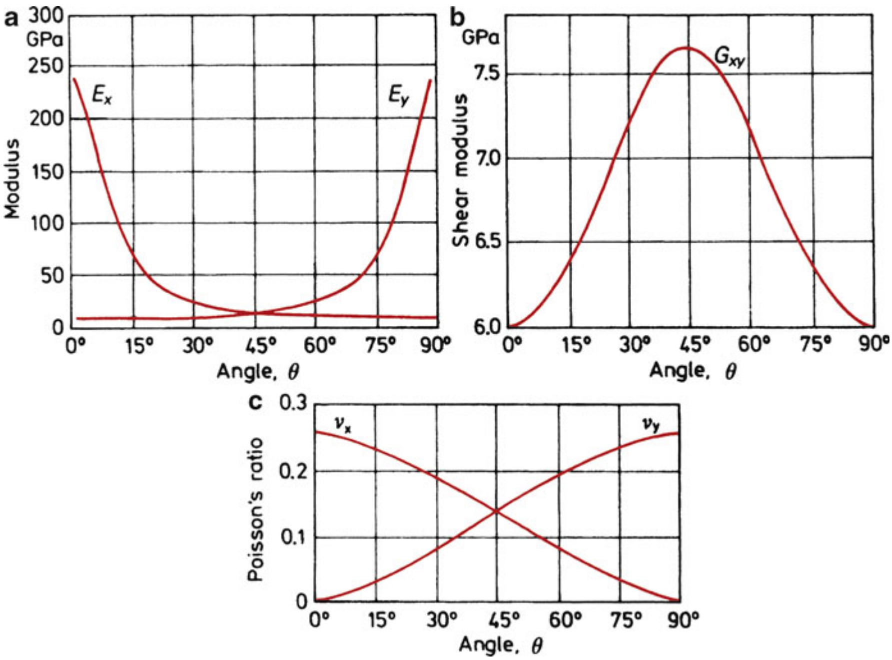


Fig. 11.4 Variation of elastic constants with fiber orientation angle for a 50 v/o carbon/epoxy composite: (**a**) longitudinal and transverse Young's moduli (E_x and E_y), (**b**) shear modulus (G_{xy}), and (**c**) Poisson's ratio (ν_x and ν_y)

In this analysis, the individual identities of fiber and matrix are ignored. Each individual, fiber reinforced lamina (also called ply) is treated as a homogeneous, orthotropic sheet, and the laminated composite is analyzed using the classical theory of laminated plates.

It would be in order at this point to describe how a multidirectional laminate can be defined by using a code to designate the stacking sequence of laminae. Figure 11.5 shows an example of a stacking sequence. The stacking sequence shown in Fig. 11.5 can be described by the following code:

$$[0_2/90_2/-45_3/45_3]_s.$$

This code says that starting from the bottom of the laminate, that is, at $z = -h/2$, we have two plies at 0° orientation; then two plies at 90° orientation; followed by a group of three plies at −45° orientation; and lastly, a group of three plies at +45° orientation. The subscript *s* indicates that the laminate is symmetric with respect to the midplane ($z = 0$); that is, the top half of the laminate is a mirror image of the bottom half. It is not necessary that a laminate composite be symmetrical. If the top

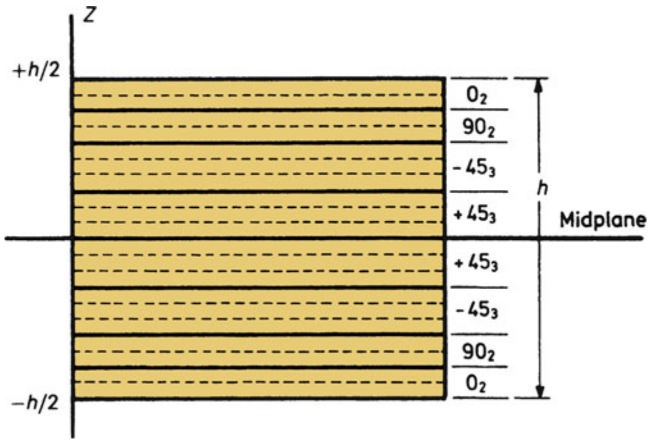


Fig. 11.5 A laminate composite with the stacking sequence given by $[0_2/90_2/{-45_3}/45_3]_s$

half has the sequence opposite to that of the bottom half, then we shall have an asymmetric laminate. In any event, we may represent the total stacking sequence of the laminate shown in Fig. 11.5 in the following way:

$$[0_2/90_2/-45_3/45_6/-45_3/90_2/0_2]_T,$$

where the subscript T indicates that the code represents the total or the whole thickness of the laminate. Note that we have merged the two middle groups of the same ply orientation into one group. If the laminate composite consists of an odd number of laminae, the midplane will lie in the central ply.

11.5.1 Basic Assumptions

We assume that the laminate thickness is small compared to its lateral dimensions. Therefore, stresses acting on the interlaminar planes in the interior of the laminate, that is, away from the free edges, are negligibly small (we shall see later that the situation is different at the free edges). We also assume that there exists a perfect bond between any two laminae. That being so, the laminae cannot slide over each other, and we have continuous displacements across the bond. We make yet another important assumption: namely, a line originally straight and perpendicular to the laminate midplane remains so after deformation. Actually, this follows from the perfect bond assumption, which does not allow sliding between the laminae.

Finally, we have the so-called Kirchhoff assumption, which states that inplane displacements are linear functions of the thickness, and therefore the interlaminar shear strains, ε_{xz} and ε_{yz}, are negligible. With these assumptions we can reduce the laminate behavior to a two-dimensional analysis of the laminate midplane.

We have the following strain–displacement relationships:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \quad \varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \\ \varepsilon_z &= \frac{\partial w}{\partial z} \quad \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{aligned} \tag{11.44}$$

Here, u, v, and w are the displacements in the x, y, and z directions, respectively. For $i \neq j$, the ε_{ij} represent engineering shear strain components equal to twice the tensorial shear components. As per Kirchhoff's assumption, the inplane displacements are linear functions of the thickness coordinate, z. Then

$$u = u_0(x,y) + zF_1(x,y), \qquad v = v_0(x,y) + zF_2(x,y), \tag{11.45}$$

where u_0 and v_0 are displacements of the midplane and F_1 and F_2 are functions to be determined; see below. It also follows from Kirchhoff's assumptions that interlaminar shear strains ε_{xz} and ε_{yz} are zero. Therefore, from Eqs. (11.44) and (11.45) we obtain

$$\begin{aligned} \varepsilon_{xz} &= F_1(x,y) + \frac{\partial w}{\partial x} = 0 \\ \varepsilon_{yz} &= F_2(x,y) + \frac{\partial w}{\partial y} = 0 \end{aligned}.$$

It therefore follows that

$$F_1(x,y) = -\frac{\partial w}{\partial x} \quad \text{and} \quad F_2(x,y) = -\frac{\partial w}{\partial y}. \tag{11.46}$$

The normal strain in the thickness direction, ε_z, is negligible: thus we can write

$$w = w(x,y).$$

That is, the vertical displacement of any point does not change in the thickness direction.

Substituting Eq. (11.46) in Eq. (11.45), we obtain

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_x^0 + zK_x, \tag{11.47a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} = \varepsilon_y^0 + zK_y, \tag{11.47b}$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} = \varepsilon_{xy}^0 + zK_{xy}. \tag{11.47c}$$

Denoting ε_{xy} by ε_s and K_{xy} by K_s, as per our notation, we can rewrite the expression for ε_{xy} as

$$\varepsilon_s = \varepsilon_s^0 + zK_s. \tag{11.47d}$$

Here, ε_x^0, ε_y^0, and ε_s^0 are the midplane strains, while K_x, K_y, and K_s are the plate curvatures. We can represent these quantities in a compact form as follows:

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{bmatrix} = \begin{bmatrix} \partial u_0/\partial x \\ \partial v_0/\partial y \\ \partial u_0/\partial y + \partial v_0/\partial x \end{bmatrix} \tag{11.48}$$

and

$$\begin{bmatrix} K_x \\ K_y \\ K_s \end{bmatrix} = -\begin{bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x \partial y \end{bmatrix}. \tag{11.49}$$

Equation (11.47a–11.47d) can be put into the following form:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{bmatrix} + z\begin{bmatrix} K_x \\ K_y \\ K_s \end{bmatrix}. \tag{11.50}$$

11.5.2 Constitutive Relationships for Laminated Composites

Consider a composite made of n stacked layers or plies; see Fig. 11.6a. Let h be the thickness of the laminated composite. Then we can write, for the kth layer, the following constitutive relationship:

$$[\sigma]_k = \left[\bar{Q}\right]_k [\varepsilon]_k. \tag{11.51}$$

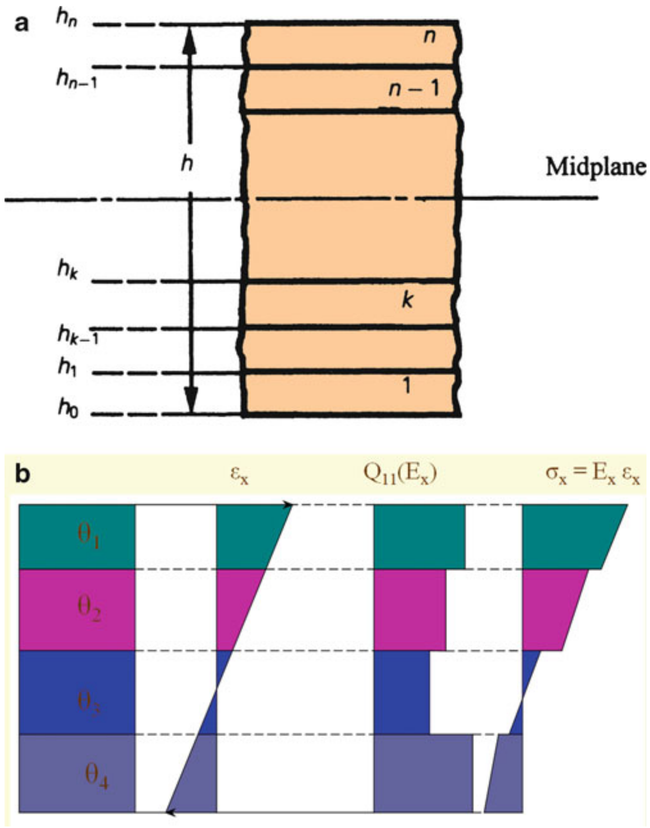


Fig. 11.6 (**a**) A laminated composite made up of *n* stacked plies. (**b**) Variation of strain and stress through the thickness in a four-ply laminated composite

From the theory of laminated plates, we have the strain–displacement relationships given by Eq. (11.50). We can rewrite Eq. (11.48) as

$$[\varepsilon] = \left[\varepsilon^0\right] + z[K]. \tag{11.52}$$

Substituting Eq. (11.52) in Eq. (11.51), for the *k*th ply we get

$$[\sigma]_k = \left[\bar{Q}\right]_k\left[\varepsilon^0\right] + z\left[\bar{Q}\right]_k[K]. \tag{11.53}$$

It is worth emphasizing that strains vary through the thickness of laminated composite linearly with thickness; the stresses do not. This is easy to visualize if we recall that stiffness of each ply will be different, being a function of the fiber orientation. This is shown schematically in a four-ply laminated composite in Fig. 11.6b. Because the stresses in a laminated composite vary from ply to ply, it is convenient to define laminate force and moment resultants as shown in Fig. 11.7.

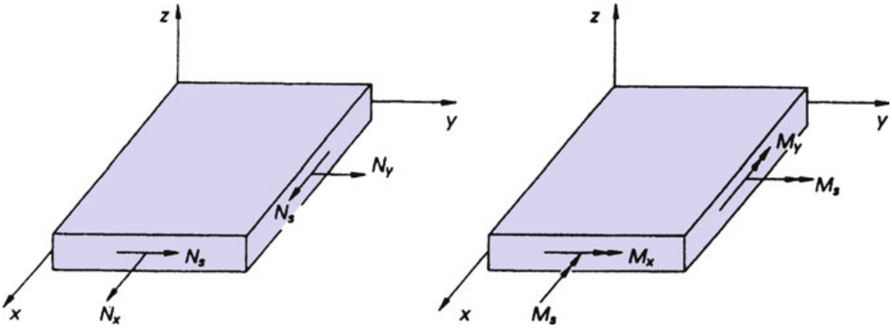


Fig. 11.7 Force (*N*) and moment (*M*) resultants in a laminated composite

These resultants of stresses and moments acting on a laminate cross section, defined as follows, provide us with a statically equivalent system of forces and moments acting at the midplane of the laminated composite. In the most general case, such a composite will have σ_x, σ_y, σ_z, σ_{xy}, σ_{yz}, and σ_{zx} as the six stress components. Our laminated composite, however, is in a state of plane stress. Thus, we shall have only three stress components: σ_x, σ_y, and σ_{xy} $(=\sigma_s)$. Accordingly, we define the three corresponding stress resultants as

$$\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_x \, \mathrm{d}z \\ N_y &= \int_{-h/2}^{h/2} \sigma_y \, \mathrm{d}z \,. \\ N_s &= \int_{-h/2}^{h/2} \sigma_s \, \mathrm{d}z \end{aligned} \tag{11.54}$$

These stress or force resultants have the dimensions of force per unit length and are positive in the same direction as the corresponding stress components. These resultants give the total force per unit length acting at the midplane. Additionally, moments are applied at the midplane, which are equivalent to the moments produced by the stresses with respect to the midplane. We define the moment resultants as

$$\begin{aligned} M_x &= \int_{-h/2}^{h/2} \sigma_x z \, \mathrm{d}z \\ M_y &= \int_{-h/2}^{h/2} \sigma_y z \, \mathrm{d}z \quad . \\ M_{xy} = M_s &= \int_{-h/2}^{h/2} \sigma_s z \, \mathrm{d}z \end{aligned} \tag{11.55}$$

This system of three stress resultants Eq. (11.54) and three moment resultants Eq. (11.55) is statically equivalent to actual stress distribution through the thickness of the composite laminate.

From Eqs. (11.52) and (11.53), we can write for the stress resultants a summation over the n plies:

$$\begin{bmatrix} N_x \\ N_y \\ N_s \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix} \mathrm{d}z$$
$$= \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{bmatrix} \mathrm{d}z + \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_x \end{bmatrix} z\,\mathrm{d}z \right) \tag{11.56}$$

Note that $[\varepsilon^0]$ and $[K]$ are not functions of z and in a given ply $[\bar{Q}]$ is not a function of z. Thus, we can simplify the preceding expression to

$$\begin{bmatrix} N_x \\ N_y \\ N_s \end{bmatrix} = \sum_{k=1}^{n} \left(\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{bmatrix} \int_{h_{k-1}}^{h_k} \mathrm{d}z + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_x \end{bmatrix} \int_{h_{k-1}}^{h_k} z\,\mathrm{d}z \right). \tag{11.57}$$

We can rewrite Eq. (11.57) as

$$\begin{bmatrix} N_x \\ N_y \\ N_s \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_s \end{bmatrix} \tag{11.58}$$

or

$$[N] = [A]\left[\varepsilon^0\right] + [B][K], \tag{11.59}$$

where

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) \tag{11.60}$$

and

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k (h_k^2 - h_{k-1}^2). \tag{11.61}$$

Similarly, from Eqs. (11.52) and (11.55), we can write for the moment resultants

$$\begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} \tag{11.62}$$

or

$$[M] = [B]\left[\varepsilon^0\right] + [D][K], \tag{11.63}$$

where

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k (h_k^3 - h_{k-1}^3) \tag{11.64}$$

and the B_{ij} are given by (11.61).

We may combine Eqs. (11.59) and (11.63) and write the constitutive equations for the laminate composite in a more compact form. Thus,

$$\left[\frac{N}{M}\right] = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]\left[\frac{\varepsilon^0}{K}\right]. \tag{11.65}$$

To appreciate the significance of the preceding expressions, let us examine the expression for N_x:

$$N_x = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\varepsilon_s^0 + B_{11}K_x + B_{12}K_y + B_{16}K_s.$$

We note that the stress resultant is a function of the midplane tensile strains (ε_x^0 and ε_y^0), the midplane shear strain $\left(\varepsilon_s^0\right)$, the bending curvatures (K_x and K_y), and the twisting (K_s). This is a much more complex situation than that observed in a homogeneous plate where tensile loads result in only tensile strains. In a laminated plate we have coupling between tensile and shear, tensile and bending, and tensile and twisting effects. Specifically, the terms A_{16} and A_{26} bring in the tension–shear coupling, while the terms B_{16} and B_{26}

represent the tension–twisting coupling. The D_{16} and D_{26} terms in a similar expression for M_x represent flexure–twisting coupling.

Under certain conditions, the stress and moment resultants become uncoupled. It is instructive to examine the conditions under which some of these simplifications can result. The A_{ij} terms are the sum of ply $\bar{Q}_{ij}$ times the ply thickness, Eq. (11.60). Thus, the A_{ij} will be zero if the positive contributions of some laminae are nullified by the negative contributions of others. Now the Q_{ij} terms of a ply are derived from orthotropic stiffnesses and, because of the form of transformation equations (11.25), $\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{22}$, and $\bar{Q}_{66}$ are always positive. This means that A_{11}, A_{12}, A_{22}, and A_{66} are always positive. $\bar{Q}_{16}$ and $\bar{Q}_{26}$, however, are zero for 0° and 90° orientations and can be positive or negative for θ between 0° and 90°. In fact, $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are odd functions of θ; that is, for equal positive and negative orientations, they will be equal in magnitude but opposite in sign. In particular, $\bar{Q}_{16}$ and $\bar{Q}_{26}$ for a $+\theta$ orientation are equal to but opposite in sign to $\bar{Q}_{16}$ and $\bar{Q}_{26}$ values for a $-\theta$ orientation. Thus, *if for each* $+\theta$ *ply, we have another identical ply of the same thickness at* $-\theta$*, then we shall have what is called a specially orthotropic laminate with respect to in-plane stresses and strains; that is,* $A_{16} = A_{26} = 0$*. The relative position of such plies in the stacking sequence does not matter.*

The B_{ij} terms are sums of terms involving $\bar{Q}_{ij}$ and differences of the square of z terms for the top (h_k) and bottom (h_{k-1}) of each ply. Thus, the B_{ij} terms are even functions of h_k, which means that they are zero if the laminate composite is symmetric with respect to thickness. In other words, *the* B_{ij} *are zero if we have for each ply above the midplane a ply identical in properties and orientation and at an equal distance below the midplane*. Such a laminate is called a *symmetric laminate* and will have B_{ij} identically zero. This simplifies the constitutive equations, and symmetric laminates are considerably easier to analyze. Additionally, because of the absence of bending–stretching coupling in symmetric laminates, they do not have the problem of warping encountered in nonsymmetric laminates and caused by in-plane forces induced by thermal contractions occurring during the curing of the resin matrix. Symmetric laminates will only experience tensile strains at the midplane but no flexure. The reader should realize that the origin of the [B] matrix lies not in the intrinsic orthotropy of the laminae, but in the heterogeneous (nonsymmetric) stacking sequence of the plies. Thus, a two-ply composite consisting of isotropic materials such as aluminum and steel will show a nonzero [B].

The terms in the D_{ij} matrix are defined in terms of $\bar{Q}_{ij}$ and the difference between h_k^3 and h_{k-1}^3. The geometrical contribution $\left(h_k^3 - h_{k-1}^3\right)$ is always positive. Thus, as explained above for A_{ij}, D_{11}, D_{12}, D_{22}, and D_{66} are always positive. Recall that $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are odd functions of θ. D_{16} and D_{26} are zero for all plies oriented at 0° or 90° because these plies have $\bar{Q}_{16} = \bar{Q}_{26} = 0$. D_{16} and D_{26} can also be made zero if, for each ply oriented at $+\theta$ and at a given distance above the midplane, we have an identical ply at an equal distance below the midplane but oriented at $-\theta$. This follows from the property of the odd function of θ that is, $\bar{Q}_{16}(+\theta) = -\bar{Q}_{16}(-\theta)$, and $\bar{Q}_{26}(+\theta) = -\bar{Q}_{26}(-\theta)$, while $(h_k^3 - h_{k-1}^3)$ is the same for both plies. Note, however, that such a laminated composite does not have a midplane of symmetry; that is, $B_{ij} \neq 0$. In fact, D_{16} and D_{26} are not zero for any midplane symmetric laminate except for unidirectional laminates (0° or 90°) and crossplied laminates (0°/90°). We can make D_{16} and D_{26} arbitrarily small, however, by using a large enough number of plies stacked at $\pm\theta$. This is because the contributions of $+\theta$ plies to D_{16} and D_{26} are opposite in sign to those of $-\theta$ plies, and although their locations are different distances from the midplane, given large enough number of plies, they will practically cancel each other.

Yet another simple stacking sequence is the quasi-isotropic sequence. Such a laminated composite can be made by having plies of identical properties oriented in such a way that the angle between any two adjacent layers is $2\pi/n$, where n is the number of plies. Such a laminate has [A] independent of orientation in the plane. We call such a stacking sequence *quasi-isotropic*, because [B] and [D] are not necessarily isotropic. We provide a summary of some of these special laminates that we discussed above in Fig. 11.8. Note the coupling coefficients that go to zero for each case.

The important results of this section can be summarized as follows:

$$[\sigma]_k = \left[\bar{Q}\right]_k [\varepsilon]_k,$$

where $1 \leq k \leq n$ and $i, j = 1, 2, 6$.

$$\begin{aligned}
\varepsilon_i &= \varepsilon_i^0 + zK_i \\
N_i &= \int_{-h/2}^{h/2} \sigma_i \, \mathrm{d}z \\
M_i &= \int_{-h/2}^{h/2} \sigma_i \mathrm{d}z \\
N_i &= A_{ij}\varepsilon_j^0 + B_{ij}K_j \\
M_i &= B_{ij}\varepsilon_j^0 + D_{ij}K_j \\
A_{ij} &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) \\
B_{ij} &= \frac{1}{2}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k^2 - h_{k-1}^2) \\
D_{ij} &= \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k^3 - h_{k-1}^3)
\end{aligned}.$$

Symmetric laminates

$$\bar{Q}(z) = \bar{Q}(-z).$$

Antisymmetric laminates

$$\bar{Q}(z) = -\bar{Q}(-z).$$

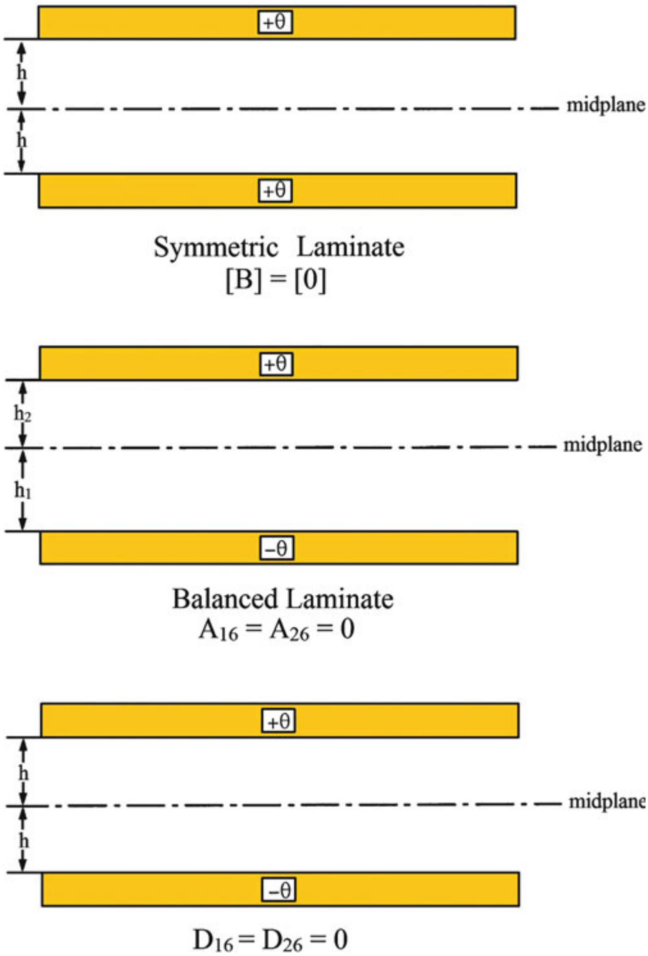
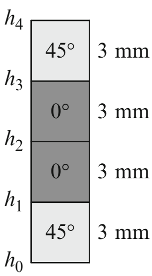


Fig. 11.8 Some special laminates

Example 11.1
A laminate is made up by stacking 0° and 45° plies as shown below:



The $[Q_{ij}]_{0^\circ}$ and $[\bar{Q}_{ij}]_{45^\circ}$ matrices are

$$[Q_{ij}]_{0^\circ} = \begin{bmatrix} 140 & 5 & 0 \\ & 5 & 0 \\ & & 5 \end{bmatrix} \text{GPa}$$

$$[\bar{Q}_{ij}]_{45^\circ} = \begin{bmatrix} 50 & 35 & 30 \\ & 50 & 30 \\ & & 35 \end{bmatrix} \text{GPa}.$$

Compute the [A], [B], and [D] matrices for this laminate.

Solution

$$(\bar{Q}_{ij})_{45^\circ} = \begin{bmatrix} 50 & 35 & 30 \\ & 50 & 30 \\ & & 35 \end{bmatrix}.$$

$$(\bar{Q}_{ij})_{0^\circ} = (Q_{ij})_{0^\circ} = \begin{bmatrix} 140 & 5 & 0 \\ & 5 & 0 \\ & & 5 \end{bmatrix}$$

Let us now compute the submatrices, [A], [B], and [D].

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{4} (\bar{Q}_{ij})_k (h_k - h_{k-1}) \\ &= (\bar{Q}_{ij})_1 (h_1 - h_0) + (\bar{Q}_{ij})_2 (h_2 - h_1) + (\bar{Q}_{ij})_3 (h_3 - h_2) \\ &\quad + (\bar{Q}_{ij})_4 (h_4 - h_3) \end{aligned}$$

$$\begin{aligned} A_{ij} &= (\bar{Q}_{ij})_1 [(-3) - (-6)] + (\bar{Q}_{ij})_2 [0 - (-3)] + (\bar{Q}_{ij})_3 [3 - 0] \\ &\quad + (\bar{Q}_{ij})_4 [6 - 3] \\ &= 3\left[(\bar{Q}_{ij})_1 + (\bar{Q}_{ij})_2 + (\bar{Q}_{ij})_3 + (\bar{Q}_{ij})_4\right] \end{aligned}$$

$$\begin{aligned} (\bar{Q}_{ij})_1 = (\bar{Q}_{ij})_4 = (\bar{Q}_{ij})_{45^\circ} \\ (\bar{Q}_{ij})_2 = (\bar{Q}_{ij})_3 = (\bar{Q}_{ij})_{0^\circ} \end{aligned}$$

$$\begin{aligned} A &= 6\left[(\bar{Q}_{ij})_{45^\circ} + (\bar{Q}_{ij})_{0^\circ}\right] \\ &= 6\left\{\begin{bmatrix} 50 & 35 & 30 \\ & 50 & 30 \\ & & 35 \end{bmatrix} + \begin{bmatrix} 140 & 5 & 0 \\ & 5 & 0 \\ & & 5 \end{bmatrix}\right\}. \\ &= \begin{bmatrix} 1140 & 240 & 180 \\ & 330 & 180 \\ & & 240 \end{bmatrix} \times 10^6 \text{ N/m} \end{aligned}$$

We have a symmetric laminate, therefore

$$
\begin{aligned}
&[B] = [0] \\
&3D = \sum_{k=1}^{4} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \\
&\quad = \left(\bar{Q}_{ij}\right)_1 \left(h_1^3 - h_0^3\right) + \left(\bar{Q}_{ij}\right)_2 \left(h_2^3 - h_1^3\right) + \left(\bar{Q}_{ij}\right)_3 \left(h_3^3 - h_2^3\right) \\
&\qquad + \left(\bar{Q}_{ij}\right)_4 \left(h_4^3 - h_3^3\right) \\
&\quad = \left(\bar{Q}_{ij}\right)_1 \left[(-3)^3 - (-6)^3\right] + \left(\bar{Q}_{ij}\right) \left[0 - (-3)^3\right] + \left(\bar{Q}_{ij}\right) \left[3^3 - 0\right] \\
&\qquad + \left(\bar{Q}_{ij}\right)_4 \left[6^3 - 3^3\right] \\
&\quad = \left(\bar{Q}_{ij}\right)_1 (-27 + 216) + \left(\bar{Q}_{ij}\right)_2 (0 + 27) + \left(\bar{Q}_{ij}\right)_3 (27 - 0) \\
&\qquad + \left(\bar{Q}_{ij}\right)_4 (216 - 27) \\
&3D = 189\left(\bar{Q}_{ij}\right)_1 + 27\left(\bar{Q}_{ij}\right)_2 + 27\left(\bar{Q}_{ij}\right)_3 + 189\left(\bar{Q}_{ij}\right)_4
\end{aligned}
$$

$$
\begin{aligned}
\left(\bar{Q}_{ij}\right)_1 &= \left(\bar{Q}_{ij}\right)_4 = \left(\bar{Q}_{ij}\right)_{45^\circ} \\
\left(\bar{Q}_{ij}\right)_2 &= \left(\bar{Q}_{ij}\right)_3 = \left(\bar{Q}_{ij}\right)_{0^\circ}
\end{aligned}
$$

$$
D = 126\left(\bar{Q}_{ij}\right)_{45^\circ} + 18\left(\bar{Q}_{ij}\right)_{0^\circ}
$$

$$
D = \left\{ 126 \begin{bmatrix} 50 & 35 & 30 \\ & 50 & 30 \\ & & 35 \end{bmatrix} + 18 \begin{bmatrix} 140 & 5 & 0 \\ & 5 & 0 \\ & & 5 \end{bmatrix} \right\} \times 10^9 \, \frac{\mathrm{N}}{\mathrm{m}^2} \times 10^{-9} \mathrm{m}^3
$$

$$
D = \begin{bmatrix} 8820 & 4500 & 3780 \\ & 6390 & 3780 \\ & & 4500 \end{bmatrix} \mathrm{N} \cdot \mathrm{m}.
$$

11.6 Stresses and Strains in Laminate Composites

We saw in Sect. 11.5 that strains produced in a lamina under load depend on the midplane strains, plate curvatures, and distances from the midplane. Midplane strains and plate curvatures can be expressed as functions of an applied load system, that is, in terms of stress and moment resultants. We derived the general constitutive equation (11.65) for laminate composites. We can invert Eq. (11.65) partially or fully and obtain explicit expressions for $[\varepsilon^0]$ and $[K]$. We use (11.59) and (11.63) for this purpose. Solving (11.59) for midplane strains, we obtain

$$
\left[\varepsilon^0\right] = [A]^{-1}[N] - [A]^{-1}[B][K]. \tag{11.66}
$$

Substituting Eq. (11.66) in Eq. (11.63), we obtain

$$
[M] = [B][A]^{-1}[N] - \left([B][A]^{-1}[B] - [D]\right)[K]. \tag{11.67}
$$

Combining Eqs. (11.66) and (11.67), we obtain a partially inverted form of the constitutive equation:

$$\left[\frac{\varepsilon^0}{M}\right] = \left[\begin{array}{c|c} A^* & B^* \\ \hline C^* & D^* \end{array}\right] \left[\frac{N}{K}\right], \tag{11.68}$$

where

$$\begin{aligned} [A^*] &= [A]^{-1} \\ [B^*] &= -[A]^{-1}[B] \\ [C^*] &= [B][A]^{-1} = -[B^*]^{\mathrm{T}} \\ [D^*] &= [D] - [B][A]^{-1}[B] \end{aligned}. \tag{11.69}$$

From Eqs. (11.66) and (11.69), we can write

$$\left[\varepsilon^0\right] = [A^*][N] + [B^*][K], \tag{11.70}$$

$$[M] = [C^*][N] + [D^*][K]. \tag{11.71}$$

From Eq. (11.71), we solve for $[K]$ and obtain

$$[K] = [D^*]^{-1}[M] - [D^*]^{-1}[C^*][N]. \tag{11.72}$$

Substituting this value of $[K]$ [Eq. (11.72)] in Eq. (11.70), we obtain

$$\begin{aligned} [\varepsilon^0] &= [A^*][N] + [B^*]\left([D^*]^{-1}[M] - [D^*]^{-1}[C^*][N]\right) \\ &= \left([A^*] - [B^*][D^*]^{-1}[C^*]\right)[N] + [B^*][D^*]^{-1}[M] \end{aligned}. \tag{11.73}$$

We can combine Eqs. (11.72) and (11.70) to obtain the fully inverted form:

$$\left[\frac{\varepsilon^0}{K}\right] = \left[\begin{array}{c|c} A' & B' \\ B' & D' \end{array}\right] \left[\frac{N}{M}\right] = \left[\begin{array}{c|c} A' & B' \\ C' & D' \end{array}\right] \left[\frac{N}{M}\right], \tag{11.74}$$

where

$$\begin{aligned} &[A'] = [A^*] - [B^*][D^*]^{-1}[C^*] = [A^*] + [B^*][D^*]^{-1}[B^*]^{\mathrm{T}}, \quad \left([C^*] = -[B^*]^{\mathrm{T}}\right) \\ &[B'] = [B^*][D^*]^{-1} \\ &[C'] = -[D^*]^{-1}[C^*] = [D^*]^{-1}[B^*]^{\mathrm{T}} = [B']^{\mathrm{T}} = [B'] \\ &[D'] = [D^*]^{-1} \end{aligned}.$$

Equations (11.65), (11.68), and (11.74) are useful forms of the laminate constitutive relationships. We note that each form involves using the elastic properties of the lamina (from the $\bar{Q}_{ij}$ values for each lamina) and the ply stacking sequence (z coordinate).

11.7 Interlaminar Stresses and Edge Effects

The classical lamination theory used in Sect. 11.5 to describe the laminate composite behavior is rigorously correct for an infinite laminate composite. It turns out that the assumption of a generalized plane stress state is quite valid in the interior of the laminate, that is, away from the free edges. At and near the free edges (extending a distance approximately equal to the laminate thickness) there exists, in fact, a three-dimensional state of stress. Under certain circumstances, there can be rather large interlaminar stresses present at the free edges, which can lead to delamination of plies or matrix cracking at the free edges and thereby cause failure. Researchers studied these aspects quite extensively and have clarified a number of issues. Pipes and Pagano (1970) considered a four-ply laminate, with plies oriented at $\pm\theta$ and each ply of thickness h_0, total laminate thickness $4h_0$, under a uniform axial strain as shown in Fig. 11.9. They used a finite difference method to obtain the numerical results for a carbon/epoxy composite system. The classical lamination theory states that in each ply there exists a state of plane stress with σ_x as the axial component and σ_{xy} ($=\sigma_s$) as the inplane shear stress component. As per the lamination theory, the stress components vary from layer to layer, but they are constant within each layer; see Fig. 11.6b. This is correct for an infinitely wide laminate. It is not correct for a finite-width laminate because the inplane shear stress does not vanish at the free edge surface. Figure 11.10 shows the stress distribution at the interface $z = h_0$ as a function of y/b, where $2b$ is the laminate width. The inplane shear stress σ_{xy} ($=\sigma_s$) converges to the value predicted by the lamination theory for $y/b < 0.5$, that is, away from the free edge. The axial stress

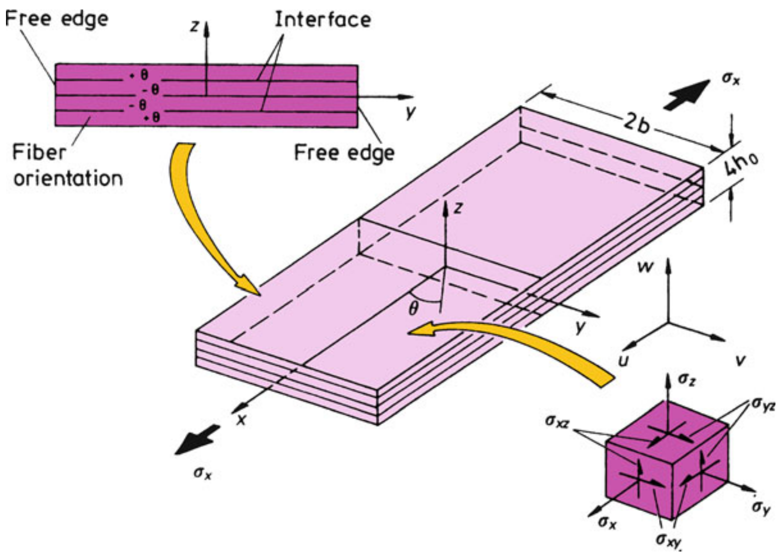


Fig. 11.9 A four-ply laminate ($\pm\theta$, thickness $4h_0$) under a uniform axial strain [from Pipes and Pagano (1970), used with permission]

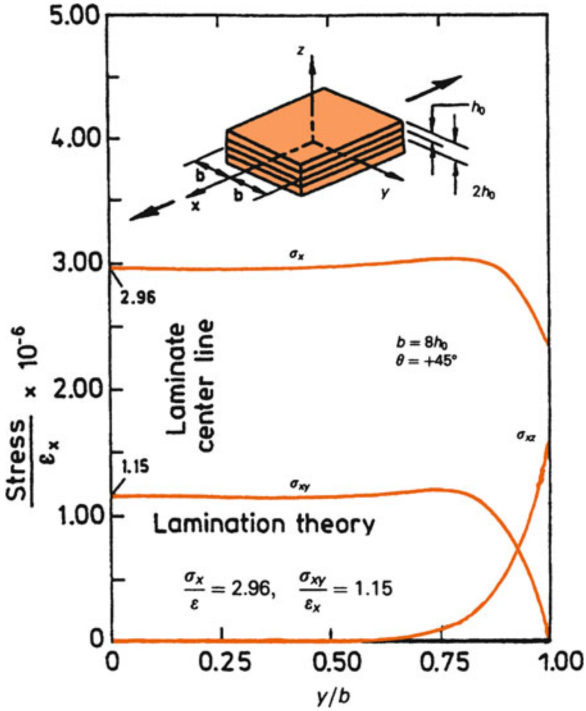


Fig. 11.10 Stress distribution at the interface $z = h_0$ as a function of y/b, where $2b$ is the laminate width. Note the high value of σ_{xy} at the free edge; σ_{xy} falls approximately to zero at $y/b = 0.5$ [from Pipes and Pagano (1970), used with permission]

component σ_x is also in accord with the lamination theory prediction for $y/b < 0.5$. The stress components σ_y, σ_z, and σ_{yz} increase near the free edge but they are quite small. The interlaminar shear stress σ_{xz}, however, has a very high value at the free edge and it falls approximately to zero at $y/b = 0.5$. As can be seen from Fig. 11.10, the perturbance owing to the free edge runs through a distance approximately equal to the laminate thickness. Thus, we may regard the interlaminar stresses as a *boundary layer phenomenon* restricted to the free edge and extending inward a distance equal to the laminate thickness. Pipes and Daniel (1971) confirmed these results experimentally. They used the Moiré technique to observe the surface displacements of a symmetric angle-ply laminate subjected to axial tension. Figure 11.11 shows that the agreement between experiment and theory is excellent.

An important result of this phenomenon of edge effects is that the laminate stacking sequence can influence the magnitude and nature of the interlaminar stresses [Pagano and Pipes 1971; Pipes and Pagano 1974; Pipes et al. 1973; Whitney 1973; Oplinger et al. 1974]. It had been observed in some earlier work that identical angle-ply laminates stacked in two different sequences had different properties: the $[\pm 15°/\pm 45°]_s$ sequence had poor mechanical properties compared to the $[\pm 45°/\pm 15°]_s$ sequence. Pagano and Pipes (1971) showed that interlaminar normal stress σ_z changed from tension to compression as the ply sequence was inverted. A tensile interlaminar stress at the free edge would initiate delamination there, which would account for the observed difference in the mechanical properties. Whitney (1973) observed the same effect in carbon/epoxy composites in fatigue testing; namely, a specimen having a stacking sequence causing a tensile interlaminar stress at the free edge showed delaminations well before the fracture, while a specimen with a stacking sequence causing compressive interlaminar stress at the free edge showed little incidence of delaminations.

We can summarize the edge effects in laminated composites as follows:

1. The classical lamination theory of plates in plane stress is valid in the laminate interior, provided the laminate is sufficiently wide (i.e., $b/4h_0 \gg 2$).
2. Interlaminar stresses are confined to narrow regions of dimensions comparable to the laminate thickness and adjoining the free edges (i.e., $y = \pm b$).
3. The ply stacking sequence in a laminate affects the magnitude as well as the sign of the interlaminar stresses, which in turn affects the mechanical performance of the laminate. Specifically, a tensile interlaminar stress at the free edge is likely to cause delaminations.

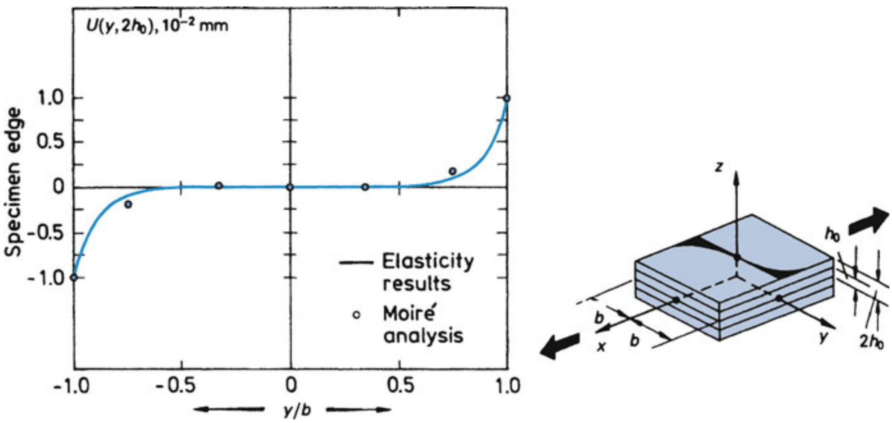


Fig. 11.11 Surface displacements of a symmetric angle-ply laminate subjected to axial tension. Experimental data points were determined by the Moiré technique [from Pipes and Daniel (1971), used with permission]

References

Christensen RM (1979) Mechanics of composite materials. Wiley, New York
Daniel IM, Ishai O (1994) Engineering mechanics of composite materials. Oxford University Press, New York
Halpin JC (1984) Primer on composite materials, 2nd edn. Technomic, Lancaster, PA
Jones RM (1975) Mechanics of composite materials. Scripta Book Co., Washington, DC
Love AEH (1952) A treatise on the mathematical theory of elasticity, 4th edn. Dover, New York
McCullough RL (1971) Concepts of fiber-resin composites. Marcel Dekker, New York, p 16
Nye JF (1985) Physical properties of crystals. Oxford University Press, London
Oplinger DW, Parker BS, Chiang FP (1974) Exp Mech 14:747
Pagano NJ, Pipes RB (1971) J Compos Mater 5:50
Pipes RB, Daniel IM (1971) J Compos Mater 5:255

Pipes RB, Kaminski BE, Pagano NJ (1973) Analysis of the test methods for high modulus fibers and composites, ASTM STP 521. ASTM, Philadelphia, PA, p 218
Pipes RB, Pagano NJ (1970) J Compos Mater 4:538
Pipes RB, Pagano NJ (1974) J Appl Mech 41:668
Timoshenko S, Goodier JN (1951) Theory of elasticity. McGraw-Hill, New York
Tsai SW, Hahn HT (1980a) Introduction to composite materials. Technomic, Westport, CT
Whitney JM (1973) Analysis of the test methods for high modulus fibers and composites, ASTM STP 521. ASTM, Philadelphia, PA, p 167

Further Reading

Bunsell AR, Renard J (2005) Fundamentals of fibre reinforced composites. Institute of Physics Pub, Bristol
Calcote LR (1969) Analysis of laminated composite structures. Van Nostrand Reinhold, New York
Daniel IM, Ishai O (2006) Engineering mechanics of composite materials, 2nd edn. Oxford University Press, New York
Herakovich CT (1998) Mechanics of fibrous composites. Wiley, New York
Hyer MW (1997) Stress analysis of fiber reinforced composite materials. McGraw-Hill, New York
Jones RM (1999) Mechanics of composite materials, 2nd edn. Taylor & Francis, Philadelphia, PA
Tsai SW, Hahn HT (1980b) Introduction to composite materials. Technomic, Westport, CT

Problems

11.1. An isotropic material is subjected to a uniaxial stress. Is the stain state also uniaxial? Write the stress and strain in matrix form.

11.2. For a symmetric laminated composite, we have, $\bar{Q}_{ij}(+z) = \bar{Q}_{ij}(-z)$, i.e., the moduli are even functions of thickness z. Starting from this definition, split the integral and show that B_{ij} is identically zero for a symmetric laminate.

11.3. An orthotropic lamina has the following characteristics: $E_{11} = 210$ GPa, $E_{22} = 8$ GPa, $G_{12} = 5$ GPa, and $\nu_{12} = 0.3$. Consider a three-ply laminate made of such laminae arranged at $\theta = \pm 60°$. Compute the submatrices $[A]$, $[B]$, and $[D]$. Take the ply thickness to be 1 mm.

11.4. Enumerate the various phenomena which can cause microcracking in a fiber composite.

11.5. A thin lamina of a composite with fibers aligned at 45° to the lamina major axis is subjected to the following stress system along its geometric axes:

$$[\sigma_i] = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 3 \end{bmatrix} \text{MPa}.$$

Compute the stress components along the material axes (i.e., σ_1, σ_2, and σ_6).

11.6. A two-ply laminate composite has the top and bottom ply orientations of 45 and 0° and thicknesses of 2 and 4 mm, respectively. The stiffness matrix for the 0° ply is

$$[Q_{ij}] = \begin{bmatrix} 20 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{GPa.}$$

Find $\left[\bar{Q}_{ij}\right]_{45}$ and then compute the matrices [A], [B], and [D] for this laminate.

11.7. A two-ply laminate composite is made of polycrystalline, isotropic aluminum and steel sheets, each 1 mm thick. The constitutive equations for the two sheets are

$$\text{Al}: \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} 70 & 26 & 0 \\ 26 & 70 & 0 \\ 0 & 0 & 26 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} \text{MPa}$$

$$\text{Steel}: \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} 210 & 60 & 0 \\ 60 & 210 & 0 \\ 0 & 0 & 78 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_1 \\ \varepsilon_6 \end{bmatrix} \text{MPa}$$

Compute the matrices [A], [B], and [D] for this laminate composite. Point out any salient features of this laminate.