

Chapter 5

Diffusion and Resistivity



5.1 Diffusion and Mobility in Weakly Ionized Gases

The infinite, homogeneous plasmas assumed in the previous chapter for the equilibrium conditions are, of course, highly idealized. Any realistic plasma will have a density gradient, and the plasma will tend to diffuse toward regions of low density. The central problem in controlled thermonuclear reactions is to impede the rate of diffusion by using a magnetic field. Before tackling the magnetic field problem, however, we shall consider the case of diffusion in the absence of magnetic fields. A further simplification results if we assume that the plasma is weakly ionized, so that charge particles collide primarily with neutral atoms rather than with one another. The case of a fully ionized plasma is deferred to a later section, since it results in a nonlinear equation for which there are few simple illustrative solutions. In any case, partially ionized gases are not rare: High-pressure arcs and ionospheric plasmas fall into this category, and most of the early work on gas discharges involved fractional ionizations between 10^{-3} and 10^{-6} , when collisions with neutral atoms are dominant.

The picture, then, is of a nonuniform distribution of ions and electrons in a dense background of neutrals (Fig. 5.1). As the plasma spreads out as a result of pressure-gradient and electric field forces, the individual particles undergo a random walk, colliding frequently with the neutral atoms. We begin with a brief review of definitions from atomic theory.

5.1.1 Collision Parameters

When an electron, say, collides with a neutral atom, it may lose any fraction of its initial momentum, depending on the angle at which it rebounds. In a head-on

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Fig. 5.1 Diffusion of gas atoms by random collisions

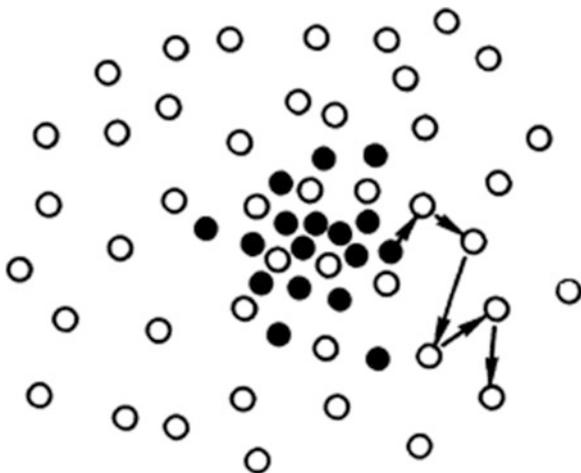
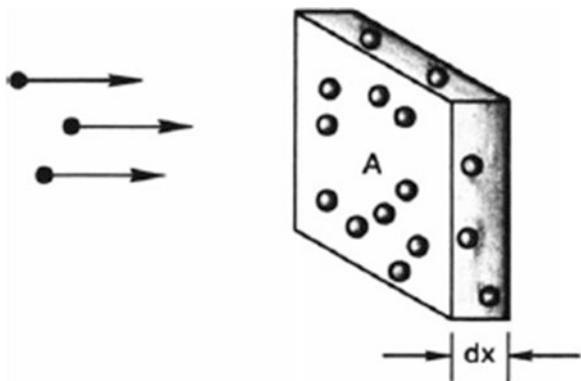


Fig. 5.2 Illustration of the definition of cross section



collision with a heavy atom, the electron can lose twice its initial momentum, since its velocity reverses sign after the collision. The probability of momentum loss can be expressed in terms of the equivalent cross section σ that the atoms would have if they were perfect absorbers of momentum.

In Fig. 5.2, electrons are incident upon a slab of area A and thickness dx containing n_n neutral atoms per m^3 . The atoms are imagined to be opaque spheres of cross-sectional area σ ; that is, every time an electron

comes within the area blocked by the atom, the electron loses all of its momentum. The number of atoms in the slab is

$$n_n A dx$$

The fraction of the slab blocked by atoms is

$$n_n A \sigma dx / A = n_n \sigma dx$$

If a flux Γ of electrons is incident on the slab, the flux emerging on the other side is

$$\Gamma' = \Gamma(1 - n_n \sigma dx)$$

Thus the change of Γ with distance is

$$d\Gamma/dx = -n_n \sigma \Gamma$$

or

$$\Gamma = \Gamma_0 e^{-n_n \sigma x} \equiv \Gamma_0 e^{-x/\lambda_m} \quad (5.1)$$

In a distance λ_m , the flux would be decreased to $1/e$ of its initial value. The quantity λ_m is the *mean free path* for collisions:

$$\boxed{\lambda_m = 1/n_n \sigma} \quad (5.2)$$

After traveling a distance λ_m , a particle will have had a good probability of making a collision. The mean time between collisions, for particles of velocity v , is given by

$$\tau = \lambda_m/v$$

and the mean frequency of collisions is

$$\tau^{-1} = v/\lambda_m = n_n \sigma v \quad (5.3)$$

If we now average over particles of all velocities v in a Maxwellian distribution, we have what is generally called the collision frequency ν :

$$\nu = n_n \overline{\sigma v} \quad (5.4)$$

5.1.2 Diffusion Parameters

The fluid equation of motion including collisions is, for any species,

$$m n \frac{d\mathbf{v}}{dt} = mn \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \pm en \mathbf{E} - \nabla p - mn \nu \mathbf{v} \quad (5.5)$$

where again the \pm indicates the sign of the charge. The averaging process used to compute ν is such as to make Eq. (5.5) correct; we need not be concerned with the details of this computation. The quantity ν must, however, be assumed to be a constant in order for Eq. (5.5) to be useful. We shall consider a steady state in which $\partial \mathbf{v} / \partial t = 0$. If \mathbf{v} is sufficiently small (or ν sufficiently large), a fluid element will not move into regions of different \mathbf{E} and ∇p in a collision time, and the convective

derivative dv/dt will also vanish. Setting the left-hand side of Eq. (5.5) to zero, we have, for an isothermal plasma,

$$\begin{aligned}\mathbf{v} &= \frac{1}{mn\nu}(\pm en\mathbf{E} - KT\nabla n) \\ &= \pm \frac{e}{m\nu}\mathbf{E} - \frac{KT}{m\nu} \frac{\nabla n}{n}\end{aligned}\quad (5.6)$$

The coefficients above are called the *mobility* and the *diffusion coefficient*:

$$\boxed{\mu \equiv |q|/m\nu} \quad \text{Mobility} \quad (5.7)$$

$$\boxed{D \equiv KT/m\nu} \quad \text{Diffusion coefficient} \quad (5.8)$$

These will be different for each species. Note that D is measured in m^2/s . The transport coefficients μ and D are connected by the *Einstein relation*:

$$\boxed{\mu = |q|D/KT} \quad (5.9)$$

With the help of these definitions, the flux Γ_j of the j th species can be written

$$\boxed{\Gamma_j = n\mathbf{v}_j = \pm\mu_j n\mathbf{E} - D_j\nabla n} \quad (5.10)$$

Fick's law of diffusion is a special case of this, occurring when either $\mathbf{E} = 0$ or the particles are uncharged, so that $\mu = 0$:

$$\boxed{\Gamma = -D\nabla n} \quad \text{Fick's law} \quad (5.11)$$

This equation merely expresses the fact that diffusion is a random-walk process, in which a net flux from dense regions to less dense regions occurs simply because more particles start in the dense region. This flux is obviously proportional to the gradient of the density. In plasmas, Fick's law is not necessarily obeyed. Because of the possibility of organized motions (plasma waves), a plasma may spread out in a manner which is not truly random.

5.2 Decay of a Plasma by Diffusion

5.2.1 Ambipolar Diffusion

We now consider how a plasma created in a container decays by diffusion to the walls. Once ions and electrons reach the wall, they recombine there. The density near the wall, therefore, is essentially zero. The fluid equations of motion and

continuity govern the plasma behavior; but if the decay is slow, we need only keep the time derivative in the continuity equation. The time derivative in the equation of motion, Eq. (5.5), will be negligible if the collision frequency ν is large. We thus have

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{\Gamma}_j = 0 \quad (5.12)$$

with $\mathbf{\Gamma}_j$ given by Eq. (5.10). It is clear that if $\mathbf{\Gamma}_i$ and $\mathbf{\Gamma}_e$ were not equal, a serious charge imbalance would soon arise. If the plasma is much larger than a Debye length, it must be quasineutral; and one would expect that the rates of diffusion of ions and electrons would somehow adjust themselves so that the two species leave at the same rate. How this happens is easy to see. The electrons, being lighter, have higher thermal velocities and tend to leave the plasma first. A positive charge is left behind, and an electric field is set up of such a polarity as to retard the loss of electrons and accelerate the loss of ions. The required \mathbf{E} field is found by setting $\mathbf{\Gamma}_i = \mathbf{\Gamma}_e = \mathbf{\Gamma}$. From Eq. (5.10), we can write

$$\mathbf{\Gamma} = \mu_i n \mathbf{E} - D_i \nabla n = -\mu_e n \mathbf{E} - D_e \nabla n \quad (5.13)$$

$$\mathbf{E} = \frac{D_i - D_e}{\mu_i + \mu_e} \frac{\nabla n}{n} \quad (5.14)$$

The common flux $\mathbf{\Gamma}$ is then given by

$$\begin{aligned} \mathbf{\Gamma} &= \mu_i \frac{D_i - D_e}{\mu_i + \mu_e} \nabla n - D_i \nabla n \\ &= \frac{\mu_i D_i - \mu_i D_e - \mu_i D_i - \mu_e D_i}{\mu_i + \mu_e} \nabla n \\ &= - \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} \nabla n \end{aligned} \quad (5.15)$$

This is Fick's law with a new diffusion coefficient

$$\boxed{D_a \equiv \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}} \quad (5.16)$$

called the *ambipolar diffusion coefficient*. If this is constant, Eq. (5.12) becomes simply

$$\partial n / \partial t = D_a \nabla^2 n \quad (5.17)$$

The magnitude of D_a can be estimated if we take $\mu_e \gg \mu_i$. That this is true can be seen from Eq. (5.7). Since ν is proportional to the thermal velocity, which is

proportional to $m^{-1/2}$, μ is proportional to $m^{-1/2}$. Equations (5.16) and (5.9) then give

$$D_a \approx D_i + \frac{\mu_i}{\mu_e} D_e = D_i + \frac{T_e}{T_i} D_i \quad (5.18)$$

For $T_e = T_i$, we have

$$D_a \approx 2D_i \quad (5.19)$$

The effect of the ambipolar electric field is to enhance the diffusion of ions by a factor of two, and the diffusion rate of the two species together is primarily controlled by the slower species.

5.2.2 Diffusion in a Slab

The diffusion equation (5.17) can easily be solved by the method of separation of variables. We let

$$n(\mathbf{r}, t) = T(t)S(\mathbf{r}) \quad (5.20)$$

whereupon Eq. (5.17), with the subscript on D_a understood, becomes

$$S \frac{dT}{dt} = DT \nabla^2 S \quad (5.21)$$

$$\frac{1}{T} \frac{dT}{dt} = \frac{D}{S} \nabla^2 S \quad (5.22)$$

Since the left side is a function of time alone and the right side a function of space alone, they must both be equal to the same constant, which we shall call $-1/\tau$. The function T then obeys the equation

$$\frac{dT}{dt} = -\frac{T}{\tau} \quad (5.23)$$

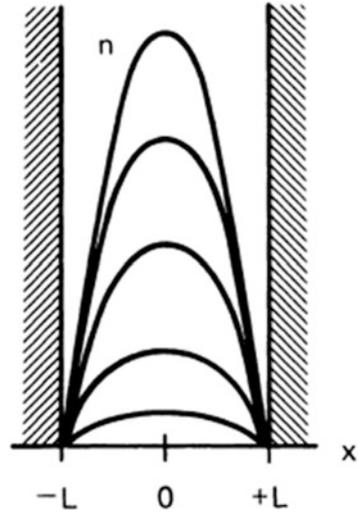
with the solution

$$T = T_0 e^{-t/\tau} \quad (5.24)$$

The spatial part S obeys the equation

$$\nabla^2 S = -\frac{1}{D\tau} S \quad (5.25)$$

Fig. 5.3 Density of a plasma at various times as it decays by diffusion to the walls



In slab geometry, this becomes

$$\frac{d^2S}{dx^2} = -\frac{1}{D\tau}S \tag{5.26}$$

with the solution

$$S = A \cos \frac{x}{(D\tau)^{1/2}} + B \sin \frac{x}{(D\tau)^{1/2}} \tag{5.27}$$

We would expect the density to be nearly zero at the walls (Fig. 5.3) and to have one or more peaks in between. The simplest solution is that with a single maximum. By symmetry, we can reject the odd (sine) term in Eq. (5.27). The boundary conditions $S = 0$ at $x = \pm L$ then requires

$$\frac{L}{(D\tau)^{1/2}} = \frac{\pi}{2}$$

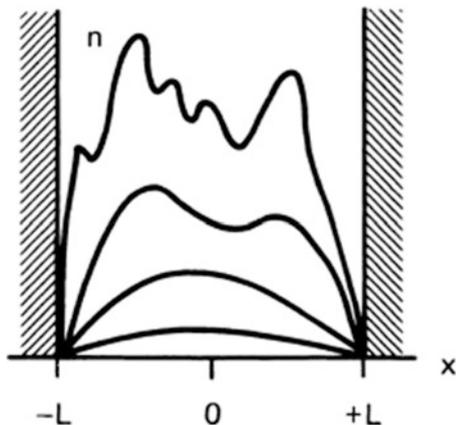
or

$$\tau = \left(\frac{2L}{\pi}\right)^2 \frac{1}{D} \tag{5.28}$$

Combining Eqs. (5.20), (5.24), (5.27), and (5.28), we have

$$n = n_0 e^{-t/\tau} \cos \frac{\pi x}{2L} \tag{5.29}$$

Fig. 5.4 Decay of an initially nonuniform plasma, showing the rapid disappearance of the higher-order diffusion modes



This is called the *lowest diffusion mode*. The density distribution is a cosine, and the peak density decays exponentially with time. The time constant τ increases with L and varies inversely with D , as one would expect.

There are, of course, higher diffusion modes with more than one peak. Suppose the initial density distribution is as shown by the top curve in Fig. 5.4. Such an arbitrary distribution can be expanded in a Fourier series:

$$n = n_0 \left(\sum_l a_l \cos \frac{(l + \frac{1}{2})\pi x}{L} + \sum_m b_m \sin \frac{m\pi x}{L} \right) \quad (5.30)$$

We have chosen the indices so that the boundary condition at $x = \pm L$ is automatically satisfied. To treat the time dependence, we can try a solution of the form

$$n = n_0 \left(\sum_l a_l e^{-t/\tau_l} \cos \frac{(l + \frac{1}{2})\pi x}{L} + \sum_m b_m e^{-t/\tau_m} \sin \frac{m\pi x}{L} \right) \quad (5.31)$$

Substituting this into the diffusion equation (5.17), we see that each cosine term yields a relation of the form

$$-\frac{1}{\tau_l} = -D \left[\left(l + \frac{1}{2} \right) \frac{\pi}{L} \right]^2 \quad (5.32)$$

and similarly for the sine terms. Thus the decay time constant for the l th mode is given by

$$\tau_l = \left[\frac{L}{(l + \frac{1}{2})\pi} \right]^2 \frac{1}{D} \quad (5.33)$$

The fine-grained structure of the density distribution, corresponding to large l numbers, decays faster, with a smaller time constant τ_l . The plasma decay will

proceed as indicated in Fig. 5.4. First, the fine structure will be washed out by diffusion. Then the lowest diffusion mode, the simple cosine distribution of Fig. 5.3, will be reached. Finally, the peak density continues to decay while the plasma density profile retains the same shape.

5.2.3 Diffusion in a Cylinder

The spatial part of the diffusion equation, Eq. (5.25), reads, in cylindrical geometry,

$$\frac{d^2S}{dr^2} + \frac{1}{r} \frac{dS}{dr} + \frac{1}{D\tau} S = 0 \quad (5.34)$$

This differs from Eq. (5.26) by the addition of the middle term, which merely accounts for the change in coordinates. The need for the extra term is illustrated simply in Fig. 5.5. If a slice of plasma in (a) is moved toward larger x without being allowed to expand, the density would remain constant. On the other hand, if a shell of plasma in (b) is moved toward larger r with the shell thickness kept constant, the density would necessarily decrease as $1/r$. Consequently, one would expect the solution to Eq. (5.34) to be like a damped cosine (Fig. 5.6). This function is called a *Bessel function of order zero*, and Eq. (5.34) is called Bessel's equation (of order zero). Instead of the symbol \cos , it is given the symbol J_0 . The function $J_0(r/[D\tau]^{1/2})$ is a solution to Eq. (5.34), just as $\cos [x/(D\tau)^{1/2}]$ is a solution to Eq. (5.26). Both $\cos kx$ and $J_0(kr)$ are expressible in terms of infinite series and may be found in mathematical tables. Unfortunately, Bessel functions are not yet found in most hand calculators.

To satisfy the boundary condition $n = 0$ at $r = a$, we must set $a/(D\tau)^{1/2}$ equal to the first zero of J_0 ; namely, 2.4. This yields the decay time constant τ . The plasma again decays exponentially, since the temporal part of the diffusion equation, Eq. (5.23), is unchanged. We have described the lowest diffusion mode in a

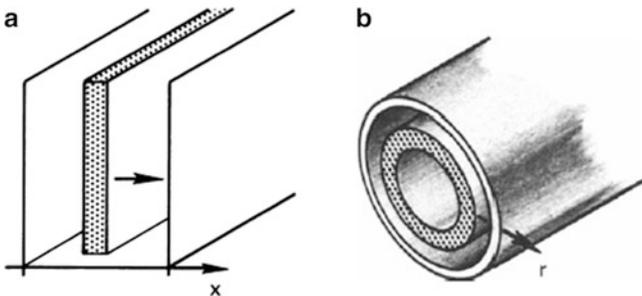
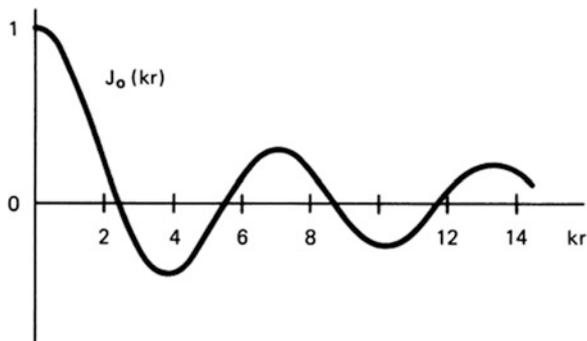


Fig. 5.5 Motion of a plasma slab in rectilinear and cylindrical geometry, illustrating the difference between a cosine and a Bessel function

Fig. 5.6 The Bessel function of order zero



cylinder. Higher diffusion modes, with more than one maximum in the cylinder, will be given in terms of Bessel functions of higher order, in direct analogy to the case of slab geometry.

5.3 Steady State Solutions

In many experiments, a plasma is maintained in a steady state by continuous ionization or injection of plasma to offset the losses. To calculate the density profile in this case, we must add a source term to the equation of continuity:

$$\frac{\partial n}{\partial t} - D \nabla^2 n = Q(\mathbf{r}) \quad (5.35)$$

The sign is chosen so that when Q is positive, it represents a source and contributes to positive $\partial n/\partial t$. In steady state, we set $\partial n/\partial t = 0$ and are left with a Poisson-type equation for $n(\mathbf{r})$.

5.3.1 Constant Ionization Function

In many weakly ionized gases, ionization is produced by energetic electrons in the tail of the Maxwellian distribution. In this case, the source term Q is proportional to the electron density n . Setting $Q = Zn$, where Z is the “ionization function,” we have

$$\nabla^2 n = -(Z/D)n \quad (5.36)$$

This is the same equation as that for S , Eq. (5.25). Consequently, the density profile is a cosine or Bessel function, as in the case of a decaying plasma, only in this case the density remains constant. The plasma is maintained against diffusion losses by whatever heat source keeps the electron temperature at its constant value and by a small influx of neutral atoms to replenish those that are ionized.

5.3.2 Plane Source

We next consider what profile would be obtained in slab geometry if there is a localized source on the plane $x=0$. Such a source might be, for instance, a slit-collimated beam of ultraviolet light strong enough to ionize the neutral gas. The steady state diffusion equation is then

$$\frac{d^2n}{dx^2} = -\frac{Q}{D}\delta(0) \quad (5.37)$$

Except at $x=0$, the density must satisfy $\partial^2n/\partial x^2=0$. This obviously has the solution (Fig. 5.7)

$$n = n_0 \left(1 - \frac{|x|}{L}\right) \quad (5.38)$$

The plasma has a linear profile. The discontinuity in slope at the source is characteristic of δ -function sources.

5.3.3 Line Source

Finally, we consider a cylindrical plasma with a source located on the axis. Such a source might, for instance, be a beam of energetic electrons producing ionization along the axis. Except at $r=0$, the density must satisfy

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) = 0 \quad (5.39)$$

Fig. 5.7 The triangular density profile resulting from a plane source under diffusion

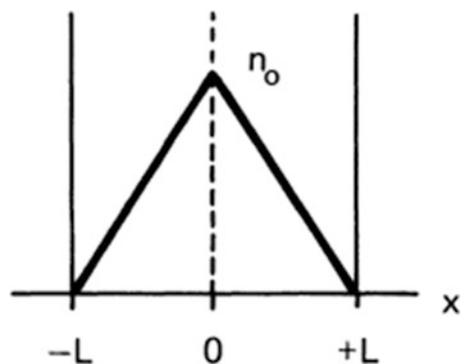
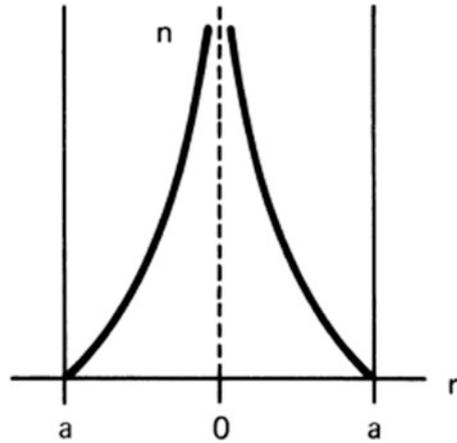


Fig. 5.8 The logarithmic density profile resulting from a line source under diffusion



The solution that vanishes at $r = a$ is

$$n = n_0 \ln(a/r) \quad (5.40)$$

The density becomes infinite at $r = 0$ (Fig. 5.8); it is not possible to determine the density near the axis accurately without considering the finite width of the source.

5.4 Recombination

When an ion and an electron collide, particularly at low relative velocity, they have a finite probability of recombining into a neutral atom. To conserve momentum, a third body must be present. If this third body is an emitted photon, the process is called *radiative recombination*. If it is a particle, the process is called *three-body recombination*. The loss of plasma by recombination can be represented by a negative source term in the equation of continuity. It is clear that this term will be proportional to $n_e n_i = n^2$. In the absence of the diffusion terms, the equation of continuity then becomes

$$\partial n / \partial t = -\alpha n^2 \quad (5.41)$$

The constant of proportionality α is called the *recombination coefficient* and has units of m^3/s . Equation (5.41) is a nonlinear equation for n . This means that the straightforward method for satisfying initial and boundary conditions by linear superposition of solutions is not available. Fortunately, Eq. (5.41) is such a simple nonlinear equation that the solution can be found by inspection. It is

$$\frac{1}{n(\mathbf{r}, t)} = \frac{1}{n_0(\mathbf{r})} + \alpha t \quad (5.42)$$

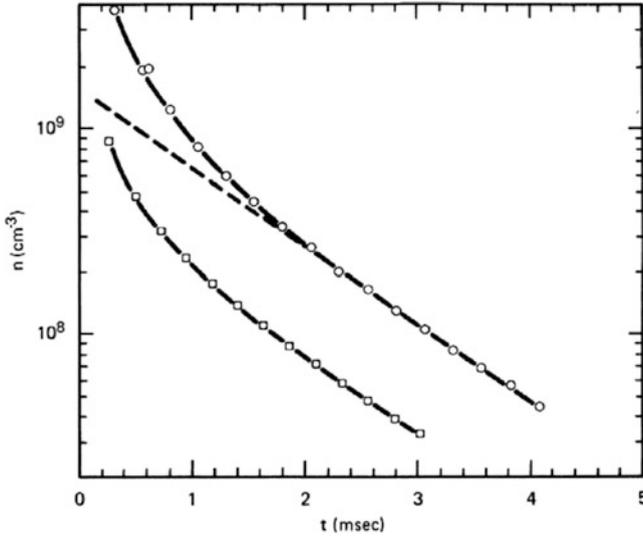


Fig. 5.9 Density decay curves of a weakly ionized plasma under recombination and diffusion (From S. C. Brown, *Basic Data of Plasma Physics*, John Wiley and Sons, New York, 1959)

where $n_0(\mathbf{r})$ is the initial density distribution. It is easily verified that this satisfies Eq. (5.41). After the density has fallen far below its initial value, it decays *reciprocally* with time:

$$n \propto 1/at \tag{5.43}$$

This is a fundamentally different behavior from the case of diffusion, in which the time variation is exponential.

Figure 5.9 shows the results of measurements of the density decay in the afterglow of a weakly ionized H plasma. When the density is high, recombination, which is proportional to n^2 , is dominant, and the density decays reciprocally. After the density has reached a low value, diffusion becomes dominant, and the decay is thenceforth exponential.

5.5 Diffusion Across a Magnetic Field

The rate of plasma loss by diffusion can be decreased by a magnetic field; this is the problem of confinement in controlled fusion research. Consider a *weakly ionized* plasma in a magnetic field (Fig. 5.10). Charged particles will move along \mathbf{B} by diffusion and mobility according to Eq. (5.10), since \mathbf{B} does not affect motion in the parallel direction. Thus we have, for each species,

Fig. 5.10 A charged particle in a magnetic field will gyrate about the same line of force until it makes a collision

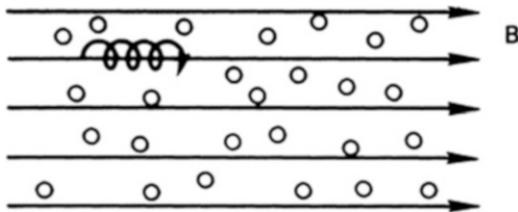
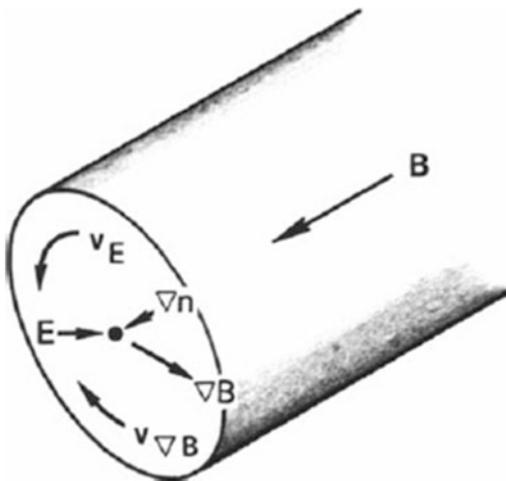


Fig. 5.11 Particle drifts in a cylindrically symmetric plasma column do not lead to losses



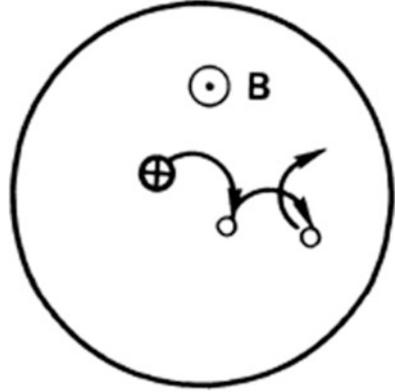
$$\Gamma_z = \pm \mu n E_z - D \frac{\partial n}{\partial z} \quad (5.44)$$

If there were no collisions, particles would not diffuse at all in the perpendicular direction—they would continue to gyrate about the same line of force. There are, of course, particle drifts across \mathbf{B} because of electric fields or gradients in \mathbf{B} , but these can be arranged to be parallel to the walls. For instance, in a perfectly symmetric cylinder (Fig. 5.11), the gradients are all in the radial direction, so that the guiding center drifts are in the azimuthal direction. The drifts would then be harmless.

When there are collisions, particles migrate across \mathbf{B} to the walls along the gradients. They do this by a random-walk process (Fig. 5.12). When an ion, say, collides with a neutral atom, the ion leaves the collision traveling in a different direction. It continues to gyrate about the magnetic field in the same direction, but its phase of gyration is changed discontinuously. (The Larmor radius may also change, but let us suppose that the ion does not gain or lose energy on the average.)

The guiding center, therefore, shifts position in a collision and undergoes a random walk. The particles will diffuse in the direction opposite ∇n . The step length in the random walk is no longer λ_m , as in magnetic-field-free diffusion, but has instead the magnitude of the Larmor radius r_L . Diffusion across \mathbf{B} can therefore be slowed down by decreasing r_L ; that is, by increasing B .

Fig. 5.12 Diffusion of gyrating particles by collisions with neutral atoms



To see how this comes about, we write the perpendicular component of the fluid equation of motion for either species as follows:

$$mn \frac{d\mathbf{v}_\perp}{dt} = \pm en(\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B}) - KT\nabla n - mn\nu\mathbf{v} = 0 \quad (5.45)$$

We have again assumed that the plasma is isothermal and that ν is large enough for the $d\mathbf{v}_\perp/dt$ term to be negligible. The x and y components are

$$\begin{aligned} mn\nu v_x &= \pm enE_x - KT \frac{\partial n}{\partial x} \pm env_y B \\ mn\nu v_y &= \pm enE_y - KT \frac{\partial n}{\partial y} \mp env_x B \end{aligned} \quad (5.46)$$

Using the definitions of μ and D , we have

$$\begin{aligned} v_x &= \pm \mu E_x - \frac{D}{n} \frac{\partial n}{\partial x} \pm \frac{\omega_c}{\nu} v_y \\ v_y &= \pm \mu E_y - \frac{D}{n} \frac{\partial n}{\partial y} \mp \frac{\omega_c}{\nu} v_x \end{aligned} \quad (5.47)$$

Substituting for v_x , we may solve for v_y :

$$v_y(1 + \omega_c^2 \tau^2) = \pm \mu E_y - \frac{D}{n} \frac{\partial n}{\partial y} - \omega_c^2 \tau^2 \frac{E_x}{B} \pm \omega_c^2 \tau^2 \frac{KT}{eB} \frac{1}{n} \frac{\partial n}{\partial x} \quad (5.48)$$

where $\tau = \nu^{-1}$. Similarly, v_x is given by

$$v_x(1 + \omega_c^2 \tau^2) = \pm \mu E_x - \frac{D}{n} \frac{\partial n}{\partial x} + \omega_c^2 \tau^2 \frac{E_y}{B} \mp \omega_c^2 \tau^2 \frac{KT}{eB} \frac{1}{n} \frac{\partial n}{\partial y} \quad (5.49)$$

The last two terms of these equations contain the $\mathbf{E} \times \mathbf{B}$ and diamagnetic drifts:

$$\begin{aligned} v_{Ex} &= \frac{E_y}{B} & v_{Ey} &= -\frac{E_x}{B} \\ v_{Dx} &= \mp \frac{KT}{eB} \frac{1}{n} \frac{\partial n}{\partial y} & v_{Dy} &= \pm \frac{KT}{eB} \frac{1}{n} \frac{\partial n}{\partial x} \end{aligned} \quad (5.50)$$

The first two terms can be simplified by defining the perpendicular mobility and diffusion coefficients:

$$\mu_{\perp} = \frac{\mu}{1 + \omega_c^2 \tau^2} \quad \boxed{D_{\perp} = \frac{D}{1 + \omega_c^2 \tau^2}} \quad (5.51)$$

With the help of Eqs. (5.50) and (5.51), we can write Eqs. (5.48) and (5.49) as

$$\mathbf{v}_{\perp} = \pm \mu_{\perp} \mathbf{E} - D_{\perp} \frac{\nabla n}{n} + \frac{\mathbf{v}_E + \mathbf{v}_D}{1 + (\nu^2/\omega_c^2)} \quad (5.52)$$

From this, it is evident that the perpendicular velocity of either species is composed of two parts. First, there are usual \mathbf{v}_E and \mathbf{v}_D drifts *perpendicular* to the gradients in potential and density. These drifts are slowed down by collisions with neutrals; the drag factor $1 + (\nu^2/\omega_c^2)$ becomes unity when $\nu \rightarrow 0$. Second, there are the mobility and diffusion drifts *parallel* to the gradients in potential and density. These drifts have the same form as in the $B = 0$ case, but the coefficients μ and D are reduced by the factor $1 + \omega_c^2 \tau^2$.

The product $\omega_c \tau$ is an important quantity in magnetic confinement. When $\omega_c^2 \tau^2 \ll 1$, the magnetic field has little effect on diffusion. When $\omega_c^2 \tau^2 \gg 1$, the magnetic field significantly retards the rate of diffusion across \mathbf{B} . The following alternative forms for $\omega_c \tau$ can easily be verified:

$$\omega_c \tau = \omega_c / \nu = \mu B \cong \lambda_m / r_L \quad (5.53)$$

In the limit $\omega_c^2 \tau^2 \gg 1$, we have

$$D_{\perp} = \frac{KT}{m\nu} \frac{1}{\omega_c^2 \tau^2} = \frac{KT\nu}{m\omega_c^2} \quad (5.54)$$

Comparing with Eq. (5.8), we see that the role of the collision frequency ν has been reversed. In diffusion parallel to \mathbf{B} , D is proportional to ν^{-1} , since collisions retard the motion. In diffusion perpendicular to \mathbf{B} , D_{\perp} is proportional to ν , since collisions are needed for cross-field migration. The dependence on m has also been reversed. Keeping in mind that ν is proportional to $m^{-1/2}$, we see that $D \propto m^{-1/2}$, while $D_{\perp} \propto m^{1/2}$. In parallel diffusion, electrons move faster than ions because of their

higher thermal velocity; in perpendicular diffusion, electrons escape more slowly because of their smaller Larmor radius.

Disregarding numerical factors of order unity, we may write Eq. (5.8) as

$$D = KT/mv \sim v_{th}^2 \tau \sim \lambda_m^2 / \tau \tag{5.55}$$

This form, the square of a length over a time, shows that diffusion is a random-walk process with a step length λ_m . Equation (5.54) can be written

$$D_{\perp} = \frac{KTv}{m\omega_c^2} \sim v_{th}^2 \frac{r_L^2}{v_{th}^2} v \sim \frac{r_L^2}{\tau} \tag{5.56}$$

This shows that perpendicular diffusion is a random-walk process with a step length r_L , rather than λ_m .

5.5.1 Ambipolar Diffusion Across B

Because the diffusion and mobility coefficients are anisotropic in the presence of a magnetic field, the problem of ambipolar diffusion is not as straightforward as in the $B = 0$ case. Consider the particle fluxes perpendicular to \mathbf{B} (Fig. 5.13). Ordinarily, since $\Gamma_{e\perp}$ is smaller than $\Gamma_{i\perp}$, a transverse electric field would be set up so as to aid electron diffusion and retard ion diffusion. However, this electric field can be short-circuited by an imbalance of the fluxes *along* \mathbf{B} . That is, the negative charge resulting from $\Gamma_{e\perp} < \Gamma_{i\perp}$ can be dissipated by electrons escaping along the field lines. Although the total diffusion must be ambipolar, the perpendicular part of the losses need not be ambipolar. The ions can diffuse out primarily radially, while the electrons diffuse out primarily along \mathbf{B} . Whether or not this in fact happens depends on the particular experiment. In short plasma columns with the field lines terminating on conducting plates, one would expect the ambipolar electric field to be short-circuited out. Each species then diffuses radially at a different rate. In long, thin plasma columns terminated by insulating plates, one would expect the radial diffusion to be ambipolar because escape along \mathbf{B} is arduous.

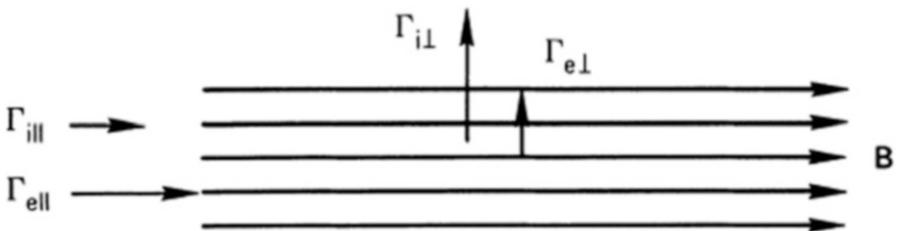


Fig. 5.13 Parallel and perpendicular particle fluxes in a magnetic field

Mathematically, the problem is to solve simultaneously the equations of continuity (5.12) for ions and electrons. It is not the fluxes Γ_j , but the divergences $\nabla \cdot \Gamma_j$ which must be set equal to each other. Separating $\nabla \cdot \Gamma_j$ into perpendicular and parallel components, we have

$$\begin{aligned}\nabla \cdot \Gamma_i &= \nabla_{\perp} \cdot (\mu_{i\perp} n \mathbf{E}_{\perp} - D_{i\perp} \nabla n) + \frac{\partial}{\partial z} \left(\mu_i n E_z - D_i \frac{\partial n}{\partial z} \right) \\ \nabla \cdot \Gamma_e &= \nabla_{\perp} \cdot (-\mu_{e\perp} n \mathbf{E}_{\perp} - D_{e\perp} \nabla n) + \frac{\partial}{\partial z} \left(-\mu_e n E_z - D_e \frac{\partial n}{\partial z} \right)\end{aligned}\tag{5.57}$$

The equation resulting from setting $\nabla \cdot \Gamma_i = \nabla \cdot \Gamma_e$ cannot easily be separated into one-dimensional equations. Furthermore, the answer depends sensitively on the boundary conditions at the ends of the field lines. Unless the plasma is so long that parallel diffusion can be neglected altogether, there is no simple answer to the problem of ambipolar diffusion across a magnetic field.

5.5.2 Experimental Checks

Whether or not a magnetic field reduces transverse diffusion in accordance with Eq. (5.51) became the subject of numerous investigations. The first experiment performed in a tube long enough that diffusion to the ends could be neglected was that of Lehnert and Hoh in Sweden. They used a helium positive column about 1 cm in diameter and 3.5 m long (Fig. 5.14). In such a plasma, the electrons are continuously lost by radial diffusion to the walls and are replenished by ionization of the neutral gas by the electrons in the tail of the velocity distribution. These fast electrons, in turn, are replenished by acceleration in the longitudinal electric field. Consequently, one would expect E_z to be roughly proportional to the rate of transverse diffusion. Two probes set in the wall of the discharge tube were used to measure E_z as B was varied. The ratio of $E_z(B)$ to $E_z(0)$ is shown as a function of B in Fig. 5.15. At low B fields, the experimental points follow closely the predicted curve, calculated on the basis of Eq. (5.52). At a critical field B_c of about 0.2 T, however, the experimental points departed from theory and, in fact, showed an

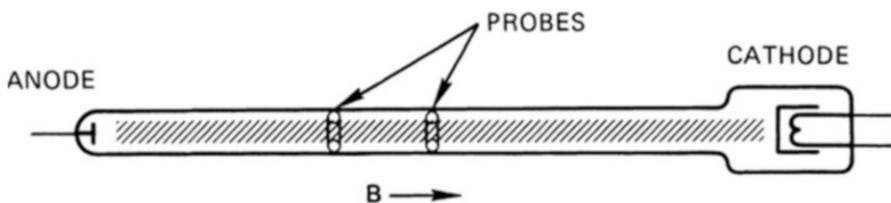
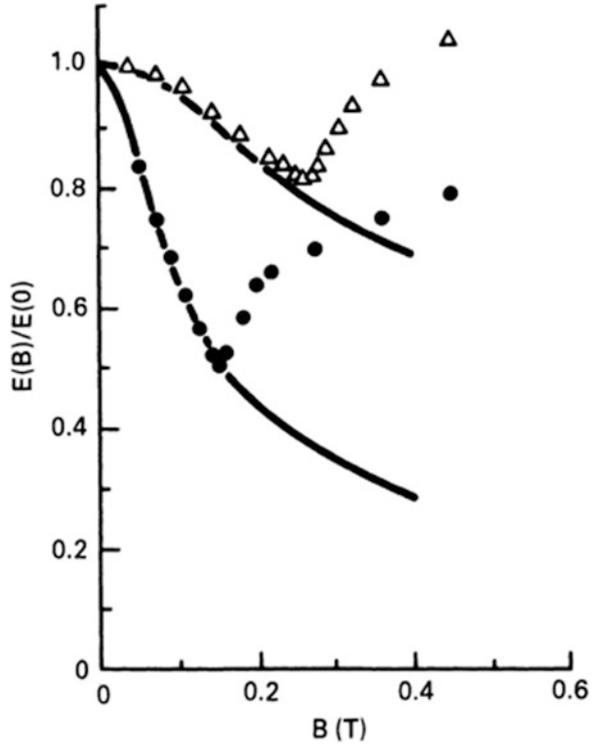


Fig. 5.14 The Lehnert-Hoh experiment to check the effect of a magnetic field on diffusion in a weakly ionized gas

Fig. 5.15 The normalized longitudinal electric field measured as a function of B at two different pressures. Theoretical curves are shown for comparison (From F. C. Hoh and B. Lehnert, *Phys. Fluids* 3, 600 (1960))



increase of diffusion with B . The critical field B_c increased with pressure, suggesting that a critical value of $\omega_c \tau$ was involved and that something went wrong with the “classical” theory of diffusion when $\omega_c \tau$ was too large.

The trouble was soon found by Kadomtsev and Nedospasov in the U.S.S.R. These theorists discovered that an instability should develop at high magnetic fields; that is, a plasma wave would be excited by the E_z field, and that this wave would cause enhanced radial losses. The theory correctly predicted the value of B_c . The wave, in the form of a helical distortion of the plasma column, was later seen directly in an experiment by Allen, Paulikas, and Pyle at Berkeley. This helical instability of the positive column was the first instance in which “anomalous diffusion” across magnetic fields was definitively explained, but the explanation was applicable only to weakly ionized gases. In the fully ionized plasmas of fusion research, anomalous diffusion proved to be a much tougher problem to solve.

Problems

5.1. The electron–neutral collision cross section for 2-eV electrons in He is about $6\pi a_0^2$, where $a_0 = 0.53 \times 10^{-8}$ cm is the radius of the first Bohr orbit of the hydrogen atom. A positive column with no magnetic field has $p = 1$ Torr of He (at room temperature) and $KT_e = 2$ eV.

- (a) Compute the electron diffusion coefficient in m^2/s , assuming that $\overline{\sigma v}$ averaged over the velocity distribution is equal to σv for 2-eV electrons.
- (b) If the current density along the column is $2 \text{ kA}/\text{m}^2$ and the plasma density is 10^{16} m^{-3} , what is the electric field along the column?

5.2. A weakly ionized plasma slab in plane geometry has a density distribution

$$n(x) = n_0 \cos(\pi x/2L) \quad -L \leq x \leq L$$

The plasma decays by both diffusion and recombination. If $L = 0.03 \text{ m}$, $D = 0.4 \text{ m}^2/\text{s}$, and $\alpha = 10^{-15} \text{ m}^3/\text{s}$, at what density will the rate of loss by diffusion be equal to the rate of loss by recombination?

- 5.3. A weakly ionized plasma is created in a cubical aluminum box of length L on each side. It decays by ambipolar diffusion.
- (a) Write an expression for the density distribution in the lowest diffusion mode.
 - (b) Define what you mean by the decay time constant and compute it if $D_a = 10^{-3} \text{ m}^2/\text{s}$.
- 5.4. A long, cylindrical positive column has $B = 0.2 \text{ T}$, $KT_i = 0.1 \text{ eV}$, and other parameters the same as in Problem 5.1. The density profile is

$$n(r) = n_0 J_0\left(r/[D\tau]^{1/2}\right)$$

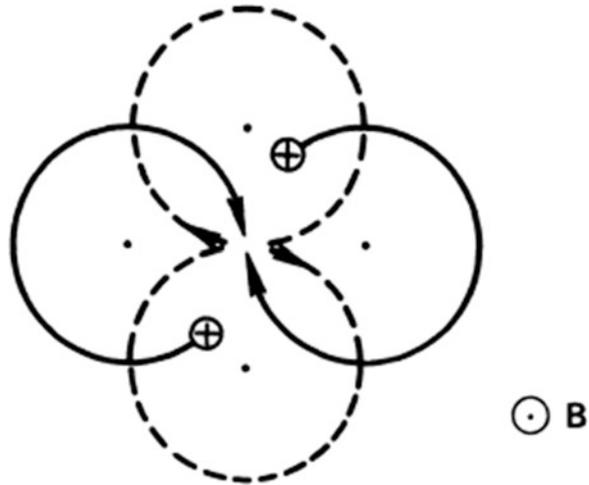
with the boundary condition $n = 0$ at $r = a = 1 \text{ cm}$. Note: $J_0(z) = 0$ at $z = 2.4$.

- (a) Show that the ambipolar diffusion coefficient to be used above can be approximated by $D_{\perp e}$.
 - (b) Neglecting recombination and losses from the ends of the column, compute the confinement time τ .
- 5.5. For the density profile of Fig. 5.7, derive an expression for the peak density n_0 in terms of the source strength ϱ in ion-electron pairs per m^2 .
- 5.6. You do a recombination experiment in a weakly ionized gas in which the main loss mechanism is recombination. You create a plasma of density 10^{20} m^{-3} by a sudden burst of ultraviolet radiation and observe that the density decays to half its initial value in 10 ms. What is the value of the recombination coefficient α ? Give units.

5.6 Collisions in Fully Ionized Plasmas

When the plasma is composed of ions and electrons alone, all collisions are Coulomb collisions between charged particles. However, there is a distinct difference between (a) collisions between like particles (ion-ion or electron-electron collisions) and

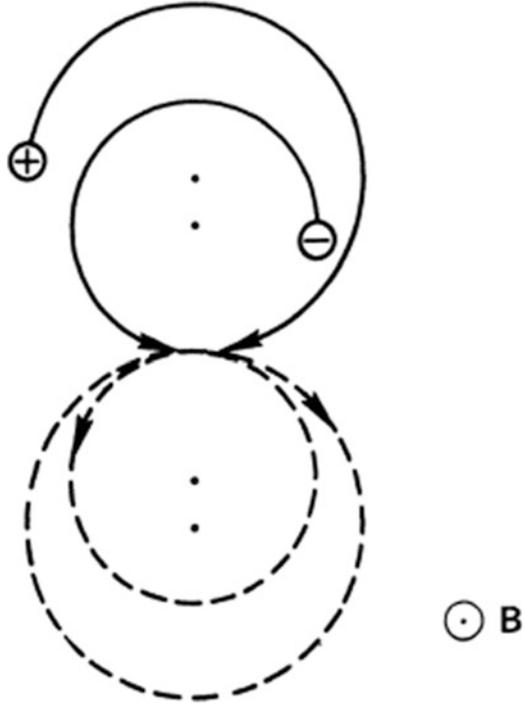
Fig. 5.16 Shift of guiding centers of two like particles making a 90° collision



(b) collisions between unlike particles (ion–electron or electron–ion collisions). Consider two identical particles colliding (Fig. 5.16). If it is a head-on collision, the particles emerge with their velocities reversed; they simply interchange their orbits, and the two guiding centers remain in the same places. The result is the same as in a glancing collision, in which the trajectories are hardly disturbed. The worst that can happen is a 90° collision, in which the velocities are changed 90° in direction. The orbits after collision will then be the dashed circles, and the guiding centers will have shifted. However, it is clear that the “center of mass” of the two guiding centers remains stationary. For this reason, *collisions between like particles give rise to very little diffusion*. This situation is to be contrasted with the case of ions colliding with neutral atoms. In that case, the final velocity of the neutral is of no concern, and the ion random-walks away from its initial position. In the case of ion–ion collisions, however, there is a detailed balance in each collision; for each ion that moves outward, there is another that moves inward as a result of the collision.

When two particles of opposite charge collide, however, the situation is entirely different (Fig. 5.17). The worst case is now the 180° collision, in which the particles emerge with their velocities reversed. Since they must continue to gyrate about the lines of force in the proper sense, both guiding centers will move in the same direction. *Unlike-particle collisions give rise to diffusion*. The physical picture is somewhat different for ions and electrons because of the disparity in mass. The electrons bounce off the nearly stationary ions and random-walk in the usual fashion. The ions are slightly jostled in each collision and move about as a result of frequent bombardment by electrons. Nonetheless, because of the conservation of momentum in each collision, the rates of diffusion are the same for ions and electrons, as we shall show.

Fig. 5.17 Shift of guiding centers of two oppositely charged particles making a 180° collision



5.6.1 Plasma Resistivity

The fluid equations of motion including the effects of charged-particle collisions may be written as follows (cf. Eq. (3.47)):

$$\begin{aligned}
 Mn \frac{d\mathbf{v}_i}{dt} &= en(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i - \nabla \cdot \boldsymbol{\pi}_i + \mathbf{P}_{ie} \\
 mn \frac{d\mathbf{v}_e}{dt} &= -en(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e + \mathbf{P}_{ei}
 \end{aligned}
 \tag{5.58}$$

The terms \mathbf{P}_{ie} and \mathbf{P}_{ei} represent, respectively, the momentum gain of the ion fluid caused by collisions with electrons, and vice versa. The stress tensor \mathbf{P}_j has been split into the isotropic part p_j and the anisotropic viscosity tensor $\boldsymbol{\pi}_j$. Like-particle collisions, which give rise to stresses within each fluid individually, are contained in $\boldsymbol{\pi}_j$. Since these collisions do not give rise to much diffusion, we shall ignore the terms $\nabla \cdot \boldsymbol{\pi}_j$. As for the terms \mathbf{P}_{ei} and \mathbf{P}_{ie} , which represent the friction between the two fluids, the conservation of momentum requires

$$\mathbf{P}_{ie} = -\mathbf{P}_{ei}
 \tag{5.59}$$

We can write \mathbf{P}_{ei} in terms of the collision frequency in the usual manner:

$$\mathbf{P}_{ei} = mn(\mathbf{v}_i - \mathbf{v}_e)\nu_{ei} \quad (5.60)$$

and similarly for \mathbf{P}_{ie} . Since the collisions are Coulomb collisions, one would expect \mathbf{P}_{ei} to be proportional to the Coulomb force, which is proportional to e^2 (for singly charged ions). Furthermore, \mathbf{P}_{ei} must be proportional to the density of electrons n_e and to the density of scattering centers n_i , which, of course, is equal to n_e . Finally, \mathbf{P}_{ei} should be proportional to the relative velocity of the two fluids. On physical grounds, then, we can write \mathbf{P}_{ei} as

$$\mathbf{P}_{ei} = \eta e^2 n^2 (\mathbf{v}_i - \mathbf{v}_e) \quad (5.61)$$

where η is a constant of proportionality. Comparing this with Eq. (5.60), we see that

$$\boxed{\nu_{ei} = \frac{ne^2}{m}\eta} \quad (5.62)$$

The constant η is the *specific resistivity* of the plasma; that this jibes with the usual meaning of resistivity will become clear shortly.

5.6.2 Mechanics of Coulomb Collisions

When an electron collides with a neutral atom, no force is felt until the electron is close to the atom on the scale of atomic dimensions; the collisions are like billiard-ball collisions. When an electron collides with an ion, the electron is gradually deflected by the long-range Coulomb field of the ion. Nonetheless, one can derive an effective cross section for this kind of collision. It will suffice for our purposes to give an order-of-magnitude estimate of the cross section. In Fig. 5.18, an electron of velocity \mathbf{v} approaches a fixed ion of charge e . In the absence of Coulomb forces, the electron would have a distance of closest approach r_0 , called the *impact parameter*. In the presence of a Coulomb attraction, the electron will be deflected by an angle χ , which is related to r_0 . The Coulomb force is

$$F = -\frac{e^2}{4\pi\epsilon_0 r^2} \quad (5.63)$$

This force is felt during the time the electron is in the vicinity of the ion; this time is roughly

$$T \approx r_0/v \quad (5.64)$$

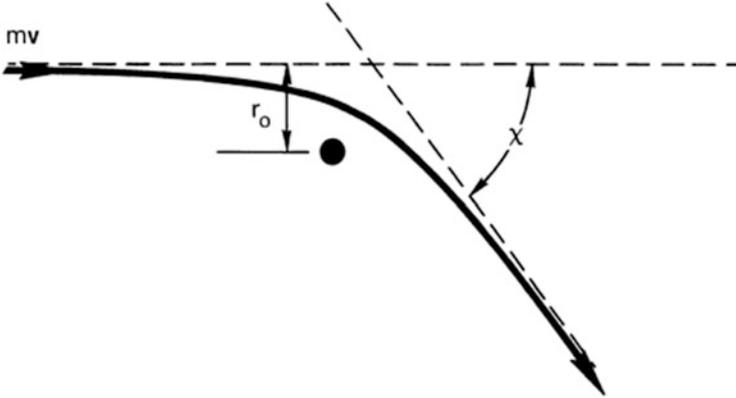


Fig. 5.18 Orbit of an electron making a Coulomb collision with an ion

The change in the electron's momentum is therefore approximately

$$\Delta(mv) = |FT| \approx \frac{e^2}{4\pi\epsilon_0 r_0 v} \quad (5.65)$$

We wish to estimate the cross section for large-angle collisions, in which $\chi \geq 90^\circ$. For a 90° collision, the change in mv is of the order of mv itself. Thus

$$\Delta(mv) \cong mv \cong e^2/4\pi\epsilon_0 r_0 v, \quad r_0 = e^2/4\pi\epsilon_0 m v^2 \quad (5.66)$$

The cross section is then

$$\sigma = \pi r_0^2 = e^4/16\pi\epsilon_0^2 m^2 v^4 \quad (5.67)$$

The collision frequency is, therefore,

$$\nu_{ei} = n\sigma v = ne^4/16\pi\epsilon_0^2 m^2 v^3 \quad (5.68)$$

and the resistivity is

$$\eta = \frac{m}{ne^2} \nu_{ei} = \frac{e^2}{16\pi\epsilon_0^2 m v^3} \quad (5.69)$$

For a Maxwellian distribution of electrons, we may replace v^2 by KT_e/m for our order-of-magnitude estimate:

$$\eta \approx \frac{\pi e^2 m^{1/2}}{(4\pi\epsilon_0)^2 (KT_e)^{3/2}} \quad (5.70)$$

Equation (5.70) is the resistivity based on large-angle collisions alone. In practice, because of the long range of the Coulomb force, small-angle collisions are much more frequent, and the cumulative effect of many small-angle deflections turns out to be larger than the effect of large-angle collisions. It was shown by Spitzer that Eq. (5.70) should be multiplied by a factor $\ln \Lambda$:

$$\eta \approx \frac{\pi e^2 m^{1/2}}{(4\pi\epsilon_0)^2 (KT_e)^{3/2}} \ln \Lambda \quad (5.71)$$

where

$$\Lambda = \overline{\lambda_D/r_0} = 12\pi n \lambda_D^3 \quad (5.72)$$

This factor represents the maximum impact parameter, in units of r_0 as given by Eq. (5.66), averaged over a Maxwellian distribution. The maximum impact parameter is taken to be λ_D because Debye shielding suppresses the Coulomb field at larger distances. Although Λ depends on n and KT_e , its logarithm is insensitive to the exact values of the plasma parameters. Typical values of $\ln \Lambda$ are given below.

KT_e (eV)	n (m^{-3})	$\ln \Lambda$	
0.2	10^{15}	9.1	(Q-machine)
2	10^{17}	10.2	(lab plasma)
100	10^{19}	13.7	(typical torus)
10^4	10^{21}	16.0	(fusion reactor)
10^3	10^{27}	6.8	(laser plasma)

It is evident that $\ln \Lambda$ varies only a factor of two as the plasma parameters range over many orders of magnitude. For most purposes, it will be sufficiently accurate to let $\ln \Lambda = 10$ regardless of the type of plasma involved.

5.6.3 Physical Meaning of η

Let us suppose that an electric field \mathbf{E} exists in a plasma and that the current that it drives is all carried by the electrons, which are much more mobile than the ions. Let $B = 0$ and $KT_e = 0$, so that $\nabla \cdot \mathbf{P}_e = 0$. Then, in steady state, the electron equation of motion (5.58) reduces to

$$en\mathbf{E} = \mathbf{P}_{ei} \quad (5.73)$$

Since $\mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e)$, Eq. (5.61) can be written

$$\mathbf{P}_{ei} = \eta en \mathbf{j} \quad (5.74)$$

so that Eq. (5.73) becomes

$$\mathbf{E} = \eta \mathbf{j} \quad (5.75)$$

This is simply Ohm's law, and the constant η is just the specific resistivity. The expression for η in a plasma, as given by Eq. (5.71) or Eq. (5.69), has several features which should be pointed out.

- (a) In Eq. (5.71), we see that η is *independent of density* (except for the weak dependence in $\ln \Lambda$). This is a rather surprising result, since it means that if a field \mathbf{E} is applied to a plasma, the current \mathbf{j} , as given by Eq. (5.75), is independent of the number of charge carriers. The reason is that although j increases with n_e , the frictional drag against the ions increases with n_i . Since $n_e = n_i$ these two effects cancel. This cancellation can be seen in Eqs. (5.68) and (5.69). The collision frequency ν_{ei} is indeed proportional to n , but the factor n cancels out in η . A fully ionized plasma behaves quite differently from a weakly ionized one in this respect. In a weakly ionized plasma, we have $\mathbf{j} = -nev_e$, $\mathbf{v}_e = -\mu_e \mathbf{E}$, so that $\mathbf{j} = ne\mu_e \mathbf{E}$. Since μ_e depends only on the density of *neutrals*, the current is proportional to the plasma density n .
- (b) Equation (5.71) shows that η is proportional to $(KT_e)^{-3/2}$. As a plasma is heated, the Coulomb cross section decreases, and the resistivity drops rather rapidly with increasing temperature. Plasmas at thermonuclear temperatures (tens of keV) are essentially collisionless; this is the reason so much theoretical research is done on collisionless plasmas. Of course, there must always be some collisions; otherwise, there wouldn't be any fusion reactions either. An easy way to heat a plasma is simply to pass a current through it. The I^2R (or $j^2\eta$) losses then turn up as an increase in electron temperature. This is called *ohmic heating*. The $(KT_e)^{-3/2}$ dependence of η , however, does not allow this method to be used up to thermonuclear temperatures. The plasma becomes such a good conductor at temperatures above 1 keV that ohmic heating is a very slow process in that range.
- (c) Equation (5.68) shows that ν_{ei} varies as v^{-3} . The fast electrons in the tail of the velocity distribution make very few collisions. The current is therefore carried mainly by these electrons rather than by the bulk of the electrons in the main body of the distribution. The strong dependence on v has another interesting consequence. If an electric field is suddenly applied to a plasma, a phenomenon known as *electron runaway* can occur. A few electrons which happen to be moving fast in the direction of $-\mathbf{E}$ when the field is applied will have gained so much energy before encountering an ion that they can make only a glancing collision. This allows them to pick up more energy from the electric field and decrease their collision cross section even further. If E is large enough, the cross section falls so fast that these runaway electrons never make a collision. They form an accelerated electron beam detached from the main body of the distribution.

5.6.4 Numerical Values of η

Exact computations of η which take into account the ion recoil in each collision and are properly averaged over the electron distribution were first given by Spitzer. The following result for hydrogen is sometimes called the Spitzer resistivity:

$$\eta_{\parallel} = 5.2 \times 10^{-5} \frac{Z \ln \Lambda}{T^{3/2}(\text{eV})} \text{ohm-m} \tag{5.76}$$

Here Z is the ion charge number, which we have taken to be 1 elsewhere in this book. Since the dependence on M is weak, these values can also be used for other gases. The subscript \parallel means that this value of η is to be used for motions parallel to \mathbf{B} . For motions perpendicular to \mathbf{B} , one should use η_{\perp} given by

$$\eta_{\perp} = 2.0\eta_{\parallel} \tag{5.77}$$

This does not mean that conductivity along \mathbf{B} is only two times better than conductivity across \mathbf{B} . A factor like $\omega_c^2 \tau^2$ still has to be taken into account. The factor 2.0 comes from a difference in weighting of the various velocities in the electron distribution. In perpendicular motions, the slow electrons, which have small Larmor radii, contribute more to the resistivity than in parallel motions.

For $KT_e = 100 \text{ eV}$, Eq. (5.76) yields

$$\eta = 5 \times 10^{-7} \text{ohm-m}$$

This is to be compared with various metallic conductors:

copper	$\eta = 2 \times 10^{-8} \text{ohm-m}$
stainless steel	$\eta = 7 \times 10^{-7} \text{ohm-m}$
mercury	$\eta = 10^{-6} \text{ohm-m}$

A 100-eV plasma, therefore, has a conductivity like that of stainless steel.

5.6.5 Pulsed Currents

When a steady-state current is drawn between two electrodes aligned along the magnetic field, electrons are the dominant current carrier, and sheaths are set up at the cathode to limit the current to the set value. When the current is pulsed, however, it takes time to set up the current distribution. It was shown by Stenzel and Urrutia that this time is controlled by whistler waves (R -waves), which must travel the length of the device to communicate the voltage information.

5.7 The Single-Fluid MHD Equations

We now come to the problem of diffusion in a fully ionized plasma. Since the dissipative term \mathbf{P}_{ei} contains the difference in velocities $\mathbf{v}_i - \mathbf{v}_e$, it is simpler to work with a linear combination of the ion and electron equations such that $\mathbf{v}_i - \mathbf{v}_e$ is the unknown rather than \mathbf{v}_i or \mathbf{v}_e separately. Up to now, we have regarded a plasma as composed of two interpenetrating fluids. The linear combination we are going to choose will describe the plasma as a single fluid, like liquid mercury, with a mass density ρ and an electrical conductivity $1/\eta$. These are the equations of magneto-hydrodynamics (MHD).

For a quasineutral plasma with singly charged ions, we can define the mass density ρ , mass velocity \mathbf{v} , and current density \mathbf{j} as follows:

$$\rho \equiv n_i M + n_e m \approx n(M + m) \quad (5.78)$$

$$\mathbf{v} \equiv \frac{1}{\rho}(n_i M \mathbf{v}_i + n_e m \mathbf{v}_e) \approx \frac{M \mathbf{v}_i + m \mathbf{v}_e}{M + m} \quad (5.79)$$

$$\mathbf{j} \equiv e(n_i \mathbf{v}_i - n_e \mathbf{v}_e) \approx ne(\mathbf{v}_i - \mathbf{v}_e) \quad (5.80)$$

In the equation of motion, we shall add a term $Mn\mathbf{g}$ for a gravitational force. This term can be used to represent any nonelectromagnetic force applied to the plasma. The ion and electron equations can be written

$$Mn \frac{\partial \mathbf{v}_i}{\partial t} = en(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i + Mn\mathbf{g} + \mathbf{P}_{ie} \quad (5.81)$$

$$mn \frac{\partial \mathbf{v}_e}{\partial t} = -en(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e + mn\mathbf{g} + \mathbf{P}_{ei} \quad (5.82)$$

For simplicity, we have neglected the viscosity tensor $\boldsymbol{\pi}$, as we did earlier. This neglect does not incur much error if the Larmor radius is much smaller than the scale length over which the various quantities change. We have also neglected the $(\mathbf{v} \cdot \nabla)\mathbf{v}$ terms because the derivation would be unnecessarily complicated otherwise. This simplification is more difficult to justify. To avoid a lengthy discussion, we shall simply say that \mathbf{v} is assumed to be so small that this quadratic term is negligible.

We now add Eqs. (5.81) and (5.82), obtaining

$$n \frac{\partial}{\partial t}(M \mathbf{v}_i + m \mathbf{v}_e) = en(\mathbf{v}_i - \mathbf{v}_e) \times \mathbf{B} - \nabla p + n(M + m)\mathbf{g} \quad (5.83)$$

The electric field has cancelled out, as have the collision terms $\mathbf{P}_{ei} = -\mathbf{P}_{ie}$. We have introduced the notation

$$p = p_i + p_e \quad (5.84)$$

for the total pressure. With the help of Eqs. (5.78)–(5.80), Eq. (5.83) can be written simply

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g} \quad (5.85)$$

This is the single-fluid equation of motion describing the mass flow. The electric field does not appear explicitly because the fluid is neutral. The three body forces on the right-hand side are exactly what one would have expected.

A less obvious equation is obtained by taking a different linear combination of the two-fluid equations. Let us multiply Eq. (5.81) by m and Eq. (5.82) by M and subtract the latter from the former. The result is

$$\begin{aligned} Mmn \frac{\partial}{\partial t} (\mathbf{v}_i - \mathbf{v}_e) &= en(M + m)\mathbf{E} + en(m\mathbf{v}_i + M\mathbf{v}_e) \times \mathbf{B} - m\nabla p_i \\ &\quad + M\nabla p_e - (M + m)\mathbf{P}_{ei} \end{aligned} \quad (5.86)$$

With the help of Eqs. (5.78), (5.80), and (5.61), this becomes

$$\begin{aligned} \frac{Mmn}{e} \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{n} \right) &= e\rho \mathbf{E} - (M + m)ne\eta \mathbf{j} - m\nabla p_i + M\nabla p_e + en(m\mathbf{v}_i + M\mathbf{v}_e) \\ &\quad \times \mathbf{B} \end{aligned} \quad (5.87)$$

The last term can be simplified as follows:

$$\begin{aligned} m\mathbf{v}_i + M\mathbf{v}_e &= M\mathbf{v}_i + m\mathbf{v}_e + M(\mathbf{v}_e - \mathbf{v}_i) + m(\mathbf{v}_i - \mathbf{v}_e) \\ &= \frac{\rho}{n} \mathbf{v} - (M - m) \frac{\mathbf{j}}{ne} \end{aligned} \quad (5.88)$$

Dividing Eq. (5.87) by $e\rho$, we now have

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} - \eta \mathbf{j} = \frac{1}{e\rho} \left[\frac{Mmn}{e} \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{n} \right) + (M - m) \mathbf{j} \times \mathbf{B} + m\nabla p_i - M\nabla p_e \right] \quad (5.89)$$

The $\partial/\partial t$ term can be neglected in slow motions, where inertial (i.e., cyclotron frequency) effects are unimportant. In the limit $m/M \rightarrow 0$, Eq. (5.89) then becomes

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{en} (\mathbf{j} \times \mathbf{B} - \nabla p_e) \quad (5.90)$$

This is our second equation, called the *generalized Ohm's law*. It describes the electrical properties of the conducting fluid. The $\mathbf{j} \times \mathbf{B}$ term is called the *Hall current* term. It often happens that this and the last term are small enough to be neglected; Ohm's law is then simply

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} \quad (5.91)$$

Equations of continuity for mass ρ and charge σ are easily obtained from the sum and difference of the ion and electron equations of continuity. The set of MHD equations is then as follows:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g} \quad (5.85)$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} \quad (5.91)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (5.92)$$

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (5.93)$$

Together with Maxwell's equations, this set is often used to describe the equilibrium state of the plasma. It can also be used to derive plasma waves, but it is considerably less accurate than the two-fluid equations we have been using. For problems involving resistivity, the simplicity of the MHD equations outweighs their disadvantages. The MHD equations have been used extensively by astrophysicists working in cosmic electrodynamics, by hydrodynamicists working on MHD energy conversion, and by fusion theorists working with complicated magnetic geometries.

5.8 Diffusion of Fully Ionized Plasmas

In the absence of gravity, Eqs. (5.85) and (5.91) for a steady state plasma become

$$\mathbf{j} \times \mathbf{B} = \nabla p \quad (5.94)$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} \quad (5.95)$$

The parallel component of the latter equation is simply

$$E_{\parallel} = \eta_{\parallel} j_{\parallel}$$

which is the ordinary Ohm's law. The perpendicular component is found by taking the cross-product with \mathbf{B} :

$$\mathbf{E} \times \mathbf{B} + (\mathbf{v}_{\perp} \times \mathbf{B}) \times \mathbf{B} = \eta_{\perp} \mathbf{j} \times \mathbf{B} = \eta_{\perp} \nabla p$$

$$\mathbf{E} \times \mathbf{B} - \mathbf{v}_{\perp} B^2 = \eta_{\perp} \nabla p$$

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{\eta_\perp}{B^2} \nabla p \quad (5.96)$$

The first term is just the $\mathbf{E} \times \mathbf{B}$ drift of both species together. The second term is the diffusion velocity in the direction of $-\nabla p$. For instance, in an axisymmetric cylindrical plasma in which \mathbf{E} and ∇p are in the radial direction, we would have

$$v_\theta = -\frac{E_r}{B} \quad v_r = -\frac{\eta_\perp}{B^2} \frac{\partial p}{\partial r} \quad (5.97)$$

The flux associated with diffusion is

$$\mathbf{\Gamma}_\perp = n\mathbf{v}_\perp = -\frac{\eta_\perp n(KT_i + KT_e)}{B^2} \nabla n \quad (5.98)$$

This has the form of Fick's law, Eq. (5.11), with the diffusion coefficient

$$D_\perp = \frac{\eta_\perp n \Sigma K T}{B^2} \quad (5.99)$$

This is the so-called “classical” diffusion coefficient for a fully ionized gas.

Note that D_\perp is proportional to $1/B^2$, just as in the case of weakly ionized gases. This dependence is characteristic of classical diffusion and can ultimately be traced back to the random-walk process with a step length r_\perp . Equation (5.99), however, differs from Eq. (5.54) for a partially ionized gas in three essential ways. First, D_\perp is *not a constant* in a fully ionized gas; it is proportional to n . This is because the density of scattering centers is not fixed by the neutral atom density but is the plasma density itself. Second, since η is proportional to $(KT)^{-3/2}$, D_\perp decreases with increasing temperature in a fully ionized gas. The opposite is true in a partially ionized gas. The reason for the difference is the velocity dependence of the Coulomb cross section. Third, *diffusion is automatically ambipolar* in a fully ionized gas (as long as like-particle collisions are neglected). D_\perp in Eq. (5.99) is the coefficient for the entire fluid; no ambipolar electric field arises, because both species diffuse at the same rate. This is a consequence of the conservation of momentum in ion-electron collisions. This point is somewhat clearer if one uses the two-fluid equations (see Problem 5.15).

Finally, we wish to point out that *there is no transverse mobility* in a fully ionized gas. Equation (5.96) for \mathbf{v}_\perp contains no component along \mathbf{E} which depends on \mathbf{E} . If a transverse \mathbf{E} field is applied to a uniform plasma, both species drift together with the $\mathbf{E} \times \mathbf{B}$ velocity. Since there is no relative drift between the two species, they do not collide, and there is no drift in the direction of \mathbf{E} . Of course, there are collisions due to thermal motions, and this simple result is only an approximate one. It comes from our neglect of (a) like-particle collisions, (b) the electron mass, and (c) the last two terms in Ohm's law, Eq. (5.90).

5.9 Solutions of the Diffusion Equation

Since D_{\perp} is not a constant in a fully ionized gas, let us define a quantity A which is constant:

$$A \equiv \eta KT / B^2 \quad (5.100)$$

We have assumed that KT and B are uniform, and that the dependence of η on n through the $\ln \Lambda$ factor can be ignored. For the case $T_i = T_e$, we then have

$$D_{\perp} = 2nA \quad (5.101)$$

With Eq. (5.98), the equation of continuity (5.92) can now be written

$$\begin{aligned} \partial n / \partial t &= \nabla \cdot (D_{\perp} \nabla n) = A \nabla \cdot (2n \nabla n) \\ \partial n / \partial t &= A \nabla^2 n^2 \end{aligned} \quad (5.102)$$

This is a nonlinear equation for n , for which there are very few simple solutions.

5.9.1 Time Dependence

If we separate the variables by letting

$$n = T(t)S(\mathbf{r})$$

we can write Eq. (5.102) as

$$\frac{1}{T^2} \frac{dT}{dt} = \frac{A}{S} \nabla^2 S^2 = -\frac{1}{\tau} \quad (5.103)$$

where $-1/\tau$ is the separation constant. The spatial part of this equation is difficult to solve, but the temporal part is the same equation that we encountered in recombination, Eq. (5.41). The solution, therefore, is

$$\frac{1}{T} = \frac{1}{T_0} + \frac{t}{\tau} \quad (5.104)$$

At large times t , the density decays as $1/t$, as in the case of recombination. This reciprocal decay is what would be expected of a fully ionized plasma diffusing classically. The exponential decay of a weakly ionized gas is a distinctly different behavior.

5.9.2 Time-Independent Solutions

There is one case in which the diffusion equation can be solved simply. Imagine a long plasma column (Fig. 5.19) with a source on the axis which maintains a steady state as plasma is lost by radial diffusion and recombination. The density profile outside the source region will be determined by the competition between diffusion and recombination. The density falloff distance will be short if diffusion is small and recombination is large, and will be long in the opposite case. In the region outside the source, the equation of continuity is

$$-A\nabla^2 n^2 = -\alpha n^2 \tag{5.105}$$

This equation is linear in n^2 and can easily be solved. In cylindrical geometry, the solution is a Bessel function. In plane geometry, Eq. (5.105) reads

$$\frac{d^2 n^2}{dx^2} = \frac{\alpha}{A} n^2 \tag{5.106}$$

with the solution

$$n^2 = n_0^2 \exp\left[-(\alpha/A)^{1/2}x\right] \tag{5.107}$$

The scale distance is

$$l = (A/\alpha)^{1/2} \tag{5.108}$$

Since A changes with magnetic field while α remains constant, the change of l with B constitutes a check of classical diffusion. This experiment was actually tried on a Q -machine, which provides a fully ionized plasma. Unfortunately, the presence of asymmetric $\mathbf{E} \times \mathbf{B}$ drifts leading to another type of loss—by convection—made the experiment inconclusive.

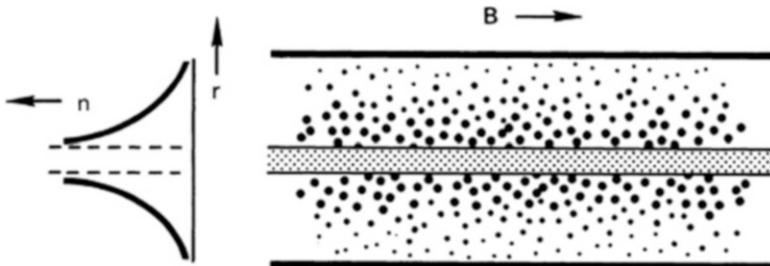


Fig. 5.19 Diffusion of a fully ionized cylindrical plasma across a magnetic field

Finally, we wish to point out a scaling law which is applicable to any fully ionized steady state plasma maintained by a constant source Q in a uniform B field. The equation of continuity then reads

$$-A \nabla^2 n^2 = -\eta K T \nabla^2 (n^2 / B^2) = Q \quad (5.109)$$

Since n and B occur only in the combination n/B , the density profile will remain unchanged as B is changed, and the density itself will increase linearly with B :

$$n \propto B \quad (5.110)$$

One might have expected the equilibrium density n to scale as B^2 , since $D_{\perp} \propto B^{-2}$; but one must remember that D_{\perp} is itself proportional to n .

5.10 Bohm Diffusion and Neoclassical Diffusion

Although the theory of diffusion via Coulomb collisions had been known for a long time, laboratory verification of the $1/B^2$ dependence of D_{\perp} in a fully ionized plasma eluded all experimenters until the 1960s. In almost all previous experiments, D_{\perp} scaled as B^{-1} , rather than B^{-2} , and the decay of plasmas was found to be exponential, rather than reciprocal, with time. Furthermore, the absolute value of D_{\perp} was far larger than that given by Eq. (5.99). This anomalously poor magnetic confinement was first noted in 1946 by Bohm, Burhop, and Massey, who were developing a magnetic arc for use in uranium isotope separation. Bohm gave the semiempirical formula

$$D_{\perp} = \frac{1}{16} \frac{K T_e}{e B} \equiv D_B \quad (5.111)$$

This formula was obeyed in a surprising number of different experiments. Diffusion following this law is called *Bohm diffusion*. Since D_B is independent of density, the decay is exponential with time. The time constant in a cylindrical column of radius R and length L can be estimated as follows:

$$\tau \approx \frac{N}{dN/dt} = \frac{n\pi R^2 L}{\Gamma_r 2\pi R L} = \frac{nR}{2\Gamma_r}$$

where N is the total number of ion-electron pairs in the plasma. With the flux Γ_r given by Fick's law and Bohm's formula, we have

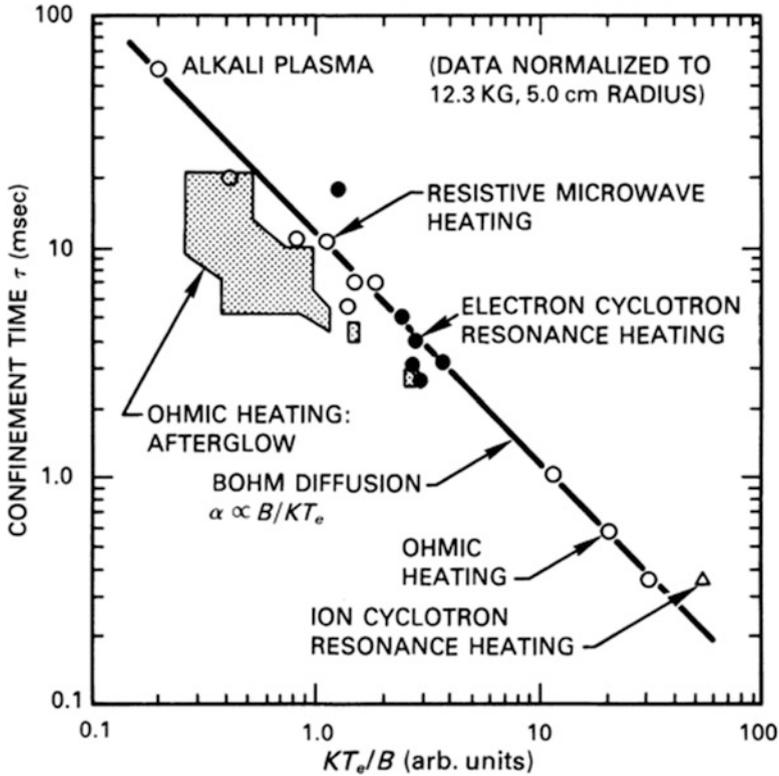


Fig. 5.20 Summary of confinement time measurements taken on various types of discharges in the Model C Stellarator, showing adherence to the Bohm diffusion law (Courtesy of D. J. Grove, Princeton University Plasma Physics Laboratory, sponsored by the U.S. Atomic Energy Commission)

$$\tau \approx \frac{nR}{2D_B \partial n / \partial r} \approx \frac{nR}{2D_B n / R} = \frac{R^2}{2D_B} \equiv \tau_B \tag{5.112}$$

The quantity τ_B is often called the *Bohm time*.

Perhaps the most extensive series of experiments verifying the Bohm formula was done on a half-dozen devices called stellarators at Princeton. A stellarator is a toroidal magnetic container with the lines of force twisted so as to average out the grad- B and curvature drifts described in Sect. 2.2.3. Figure 5.20 shows a compilation of data taken over a decade on many different types of discharges in the Model C stellarator. The measured values of τ lie near a line representing the Bohm time τ_B . Close adherence to Bohm diffusion would have serious consequences for the controlled fusion program. Equation (5.111) shows that D_B increases, rather than decreases, with temperature, and though it decreases with B , it decreases more slowly than expected. In absolute magnitude, D_B is also much larger than D_{\perp} . For instance, for a 100-eV plasma in 1-T field, we have

$$D_B = \frac{1}{16} \frac{(10^2)(1.6 \times 10^{-19})}{(1.6 \times 10^{-19})(1)} = 6.25 \text{m}^2/\text{sec}$$

If the density is 10^{19} m^{-3} , the classical diffusion coefficient is

$$\begin{aligned} D_{\perp} &= \frac{2nKT\eta_{\perp}}{B^2} = \frac{(2)(10^{19})(10^2)(1.6 \times 10^{-19})}{(1)^2} \times \frac{(2.0)(5.2 \times 10^{-5})(10)}{(100)^{3/2}} \\ &= (320)(1.04 \times 10^{-6}) = 3.33 \times 10^{-4} \text{m}^2/\text{sec} \end{aligned}$$

The disagreement is four orders of magnitude.

Several explanations have been proposed for Bohm diffusion. First, there is the possibility of magnetic field errors. In the complicated geometries used in fusion research, it is not always clear that the lines of B either close upon themselves or even stay within the chamber. Since the mean free paths are so long, only a slight asymmetry in the magnetic coil structure will enable electrons to wander out to the walls without making collisions. The ambipolar electric field will then pull the ions out. Second, there is the possibility of asymmetric electric fields. These can arise from obstacles inserted into the plasma, from asymmetries in the vacuum chamber, or from asymmetries in the way the plasma is created or heated. The dc $\mathbf{E} \times \mathbf{B}$ drifts then need not be parallel to the walls, and ions and electrons can be carried together to the walls by $\mathbf{E} \times \mathbf{B}$ convection. The drift patterns, called *convective cells*, have been observed. Finally, there is the possibility of oscillating electric fields arising from unstable plasma waves. If these fluctuating fields are random, the $\mathbf{E} \times \mathbf{B}$ drifts constitute a collisionless random-walk process. Even if the oscillating field is a pure sine wave, it can lead to enhanced losses because the phase of the $\mathbf{E} \times \mathbf{B}$ drift can be such that the drift is always outward whenever the fluctuation in density is positive. One may regard this situation as a moving convective cell pattern. Fluctuating electric fields are often observed when there is anomalous diffusion, but in many cases, it can be shown that the fields are not responsible for all of the losses. All three anomalous loss mechanisms may be present at the same time in experiments on fully ionized plasmas.

The scaling of D_B with KT_e and B can easily be shown to be the natural one whenever the losses are caused by $\mathbf{E} \times \mathbf{B}$ drifts, either stationary or oscillating. Let the escape flux be proportional to the $\mathbf{E} \times \mathbf{B}$ drift velocity:

$$\Gamma_{\perp} = nv_{\perp} \propto nE/B \quad (5.113)$$

Because of Debye shielding, the maximum potential in the plasma is given by

$$e\phi_{\text{max}} \approx KT_e \quad (5.114)$$

If R is a characteristic scale length of the plasma (of the order of its radius), the maximum electric field is then

$$E_{\max} \approx \frac{\phi_{\max}}{R} \approx \frac{KT_e}{eR} \quad (5.115)$$

This leads to a flux Γ_{\perp} given by

$$\Gamma_{\perp} \approx \gamma \frac{n KT_e}{R e B} \approx -\gamma \frac{KT_e}{eB} \nabla n = -D_B \nabla n \quad (5.116)$$

where γ is some fraction less than unity. Thus the fact that D_B is proportional to KT_e/eB is no surprise. The value $\gamma = \frac{1}{16}$ has no theoretical justification but is an empirical number agreeing with most experiments to within a factor of two or three.

Recent experiments on toroidal devices have achieved confinement times of order $1000\tau_B$. This was accomplished by carefully eliminating oscillations and asymmetries. However, in toroidal devices, other effects occur which enhance collisional diffusion. Figure 5.21 shows a torus with helical lines of force. The twist is needed to eliminate the unidirectional grad- B and curvature drifts. As a particle follows a B-line, it sees a larger $|B|$ near the inside wall of the torus and a smaller $|B|$ near the outside wall. Some particles are trapped by the magnetic mirror effect and do not circulate all the way around the torus. The guiding centers of these trapped particles trace out banana-shaped orbits as they make successive passes through a given cross section (Fig. 5.21). As a particle makes collisions, it becomes trapped and untrapped successively and goes from one banana orbit to another. The random-walk step length is therefore the width of the banana orbit rather than r_L , and the “classical” diffusion coefficient is increased. This is called *neoclassical diffusion*. The dependence of D_{\perp} on collision frequency ν is shown in Fig. 5.22. In the region of small ν , banana diffusion is larger than classical diffusion. In the region of large ν , there is classical diffusion, but it is modified by currents along \mathbf{B} . The theoretical rate for neoclassical diffusion has been observed experimentally by Ohkawa at La Jolla, CA, and by Chen at Princeton, NJ.

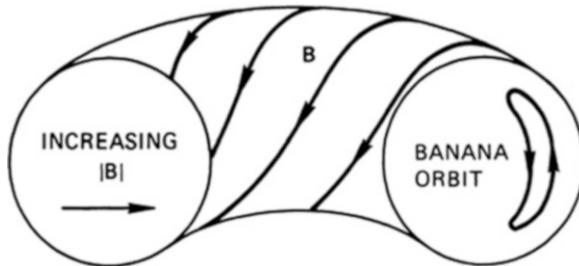
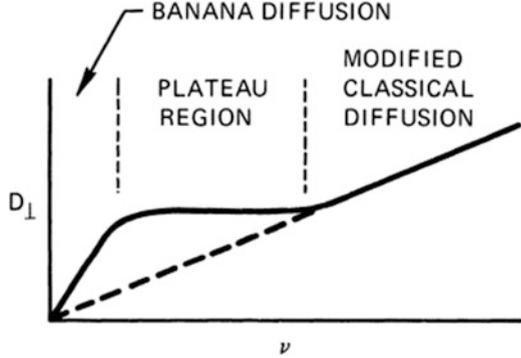


Fig. 5.21 A banana orbit of a particle confined in the twisted magnetic field of a toroidal confinement device. The “orbit” is really the locus of points at which the particle crosses the plane of the paper

Fig. 5.22 Behavior of the neoclassical diffusion coefficient with collision frequency ν



Problems

- 5.7. Show that the mean free path λ_{ei} for electron-ion collisions is proportional to T_e^2 .
- 5.8. A tokamak is a toroidal plasma container in which a current is driven in the fully ionized plasma by an electric field applied along \mathbf{B} (Fig. P5.8). How many V/m must be applied to drive a total current of 200 kA in a plasma with $KT_e = 500$ eV and a cross-sectional area of 75 cm^2 ?
- 5.9. Suppose the plasma in a fusion reactor is in the shape of a cylinder 1.2 m in diameter and 100 m long. The 5-T magnetic field is uniform except for short mirror regions at the ends, which we may neglect. Other parameters are $KT_i = 20$ keV, $KT_e = 10$ keV, and $n = 10^{21} \text{ m}^{-3}$ (at $r = 0$). The density profile is found experimentally to be approximately as sketched in Fig. P5.9.
 - (a) Assuming classical diffusion, calculate D_{\perp} at $r = 0.5$ m.
 - (b) Calculate dN/dt , the total number of ion-electron pairs leaving the central region radially per second.

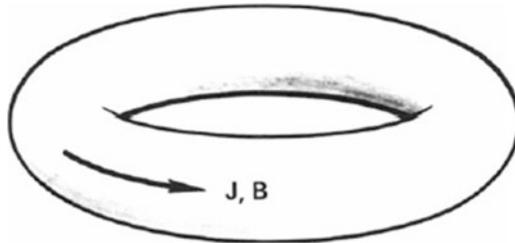


Fig. P5.8

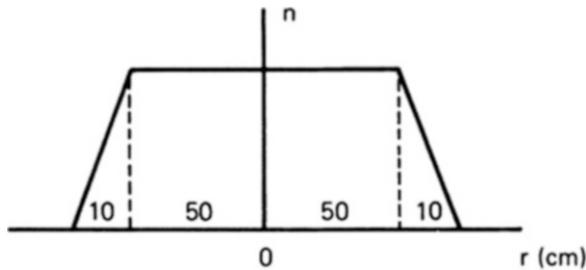


Fig. P5.9

(c) Estimate the confinement time τ by $\tau \approx -N/(dN/dt)$. Note: a rough estimate is all that can be expected in this type of problem. The profile has been simplified and is not realistic.

- 5.10. Estimate the classical diffusion time of a long plasma cylinder 10 cm in radius, with $n = 10^{21} \text{ m}^{-3}$, $KT_e = KT_i = 10 \text{ keV}$, $B = 5 \text{ T}$.
- 5.11. A cylindrical plasma column has a density distribution

$$n = n_0(1 - r^2/a^2)$$

where $a = 10 \text{ cm}$ and $n_0 = 10^{19} \text{ m}^{-3}$. If $KT_e = 100 \text{ eV}$, $KT_i = 0$, and the axial magnetic field B_0 is 1 T, what is the ratio between the Bohm and the classical diffusion coefficients perpendicular to B ?

- 5.12. A weakly ionized plasma can still be governed by Spitzer resistivity if $\nu_{ei} \gg \nu_{e0}$ where ν_{e0} is the electron-neutral collision frequency. Here are some data for the electron-neutral momentum transfer cross section σ_{e0} in square angstroms (\AA^2):

	$E = 2 \text{ eV}$	$E = 10 \text{ eV}$
Helium	6.3	4.1
Argon	2.5	13.8

For singly ionized He and A plasmas with $KT_e = 2$ and 10 eV (four cases), estimate the fractional ionization $f \equiv n_i/(n_0 + n_i)$ at which $\nu_{ei} = \nu_{e0}$, assuming that the value of $\overline{\sigma v}(T_e)$ can be crudely approximated by $\sigma(E)|\overline{v}|(E)$, where $E = KT_e$. (Hint: For ν_{e0} , use Eq. (7.11); for ν_{ei} , use Eqs. (5.62) and (5.76).

- 5.13. The plasma in a toroidal stellarator is ohmically heated by a current along \mathbf{B} of 10^5 A/m^2 . The density is uniform at $n = 10^{19} \text{ m}^{-3}$ and does not change. The Joule heat ηj^2 goes to the electrons. Calculate the rate of increase of KT_e in $\text{eV}/\mu\text{sec}$ at the time when $KT_e = 10 \text{ eV}$.
- 5.14. In a θ -pinch, a large current is discharged through a one-turn coil. The rising magnetic field inside the coil induces a surface current in the highly conducting plasma. The surface current is opposite in direction to the coil current and hence keeps the magnetic field out of the plasma. The magnetic field pressure between the coil and the plasma then compresses the plasma.

This can work only if the magnetic field does not penetrate into the plasma during the pulse. Using the Spitzer resistivity, estimate the maximum pulse length for a hydrogen θ -pinch whose initial conditions are $KT_e = 10$ eV, $n = 10^{22} \text{ m}^{-3}$, $r = 2$ cm, if the field is to penetrate only 1/10 of the way to the axis.

- 5.15. Consider an axisymmetric cylindrical plasma with $\mathbf{E} = E_r \hat{\mathbf{r}}$, $\mathbf{B} = B_z \hat{\mathbf{z}}$, and $\nabla p_i = \nabla p_e = \hat{\mathbf{r}} \partial p / \partial r$. If we neglect the $(\mathbf{v} \cdot \nabla) \mathbf{v}$ term, which is tantamount to neglecting the centrifugal force, the steady-state two-fluid equations can be written in the form (Fig. P5.14)

$$\begin{aligned} en(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i - e^2 n^2 \eta (\mathbf{v}_i - \mathbf{v}_e) &= 0 \\ -en(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e - e^2 n^2 \eta (\mathbf{v}_e - \mathbf{v}_i) &= 0 \end{aligned}$$

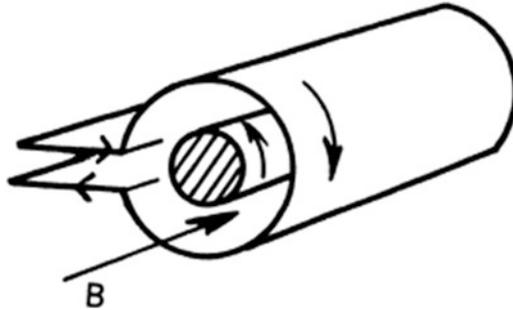


Fig. P5.14

- (a) From the θ components of these equations, show that $v_{i\theta} = v_{e\theta}$.
 (b) From the r components, show that $v_{j\theta} = v_E + v_{Dj}$ ($j = i, e$).
 (c) Find an expression for $v_{i\theta}$ showing that it does not depend on E_r .
- 5.16. Use the single-fluid MHD equation of motion and the mass continuity equation to calculate the phase velocity of an ion acoustic wave in an unmagnetized, uniform plasma with $T_e \gg T_i$.
- 5.17. Calculate the resistive damping of Alfvén waves by deriving the dispersion relation from the single-fluid equations (5.85) and (5.91) and Maxwell's equations (4.72) and (4.77). Linearize and neglect gravity, displacement current, and ∇p .
- (a) Show that

$$\frac{\omega^2}{k^2} = c^2 \epsilon_0 \left(\frac{B_0^2}{\rho_0} - i\omega\eta \right)$$

- (b) Find an explicit expression for $\text{Im}(k)$ when ω is real and η is small.

- 5.18. If a cylindrical plasma diffuses at the Bohm rate, calculate the steady state radial density profile $n(r)$, ignoring the fact that it may be unstable. Assume that the density is zero at $r = \infty$ and has a value n_0 at $r = r_0$.
- 5.19. A cylindrical column of plasma in a uniform magnetic field $\mathbf{B} = B_z \hat{\mathbf{z}}$ carries a uniform current density $\mathbf{j} = j_z \hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector parallel to the axis of the cylinder.
- Calculate the magnetic field $\mathbf{B}(r)$ produced by this plasma current.
 - Write an expression for the grad- B drift of a charged particle with $v_{||} = 0$ in terms of B_z , j_z , r , v_{\perp} , q , and m . You may assume that the field calculated in (a) is small compared to B_z (but not zero).
 - If the plasma has electrical resistivity, there is also an electric field $\mathbf{E} = E_z \hat{\mathbf{z}}$. Calculate the azimuthal electron drift due to this field, taking into account the helicity of the \mathbf{B} field.
 - Draw a diagram showing the direction of the drifts in (b) and (c) for both ions and electrons in the (r, θ) plane.