Chapter 6
Equilibrium and Stability

6.1 Introduction

If we look only at the motions of individual particles, it would be easy to design a magnetic field which will confine a collisionless plasma. We need only make sure that the lines of force do not hit the vacuum wall and arrange the symmetry of the system in such a way that all the particle drifts $\mathbf{v}_E$, $\mathbf{v}_{\nabla B}$, and so forth are parallel to the walls. From a macroscopic fluid viewpoint, however, it is not easy to see whether a *plasma* will be confined in a magnetic field designed to contain individual particles. No matter how the external fields are arranged, the plasma can generate internal fields which affect its motion. For instance, charge bunching can create **E** fields which can cause $\mathbf{E} \times \mathbf{B}$ drifts to the wall. Currents in the plasma can generate **B** fields which cause grad-B drifts outward.

We can arbitrarily divide the problem of confinement into two parts: the problem of equilibrium and the problem of stability. A tight-rope walker can easily find an equilibrium, but it is not stable unless he holds a drooping rod. The difference between equilibrium and stability can also be illustrated by a mechanical analogy. Figure 6.1 shows various cases of a marble resting on a hard surface. An equilibrium is a state in which all the forces are balanced, so that a time-independent solution is possible. The equilibrium is stable or unstable according to whether small perturbations are damped or amplified. In case (F), the marble is in a stable equilibrium as long as it is not pushed too far. Once it is moved beyond a threshold, it is in an unstable state. This is called an "explosive instability." In case (G), the marble is in an unstable state, but it cannot make very large excursions. Such an in stability is not very dangerous if the nonlinear limit to the amplitude of the motion is small. The situation with a plasma is, of course, much more complicated than what is seen in Fig. 6.1; to achieve equilibrium requires balancing the forces on each fluid element.

The original version of this chapter was revised. An erratum to this chapter can be found at https://doi.org/10.1007/978-3-319-22309-4_11

F.F. Chen, *Introduction to Plasma Physics and Controlled Fusion*,
DOI 10.1007/978-3-319-22309-4_6

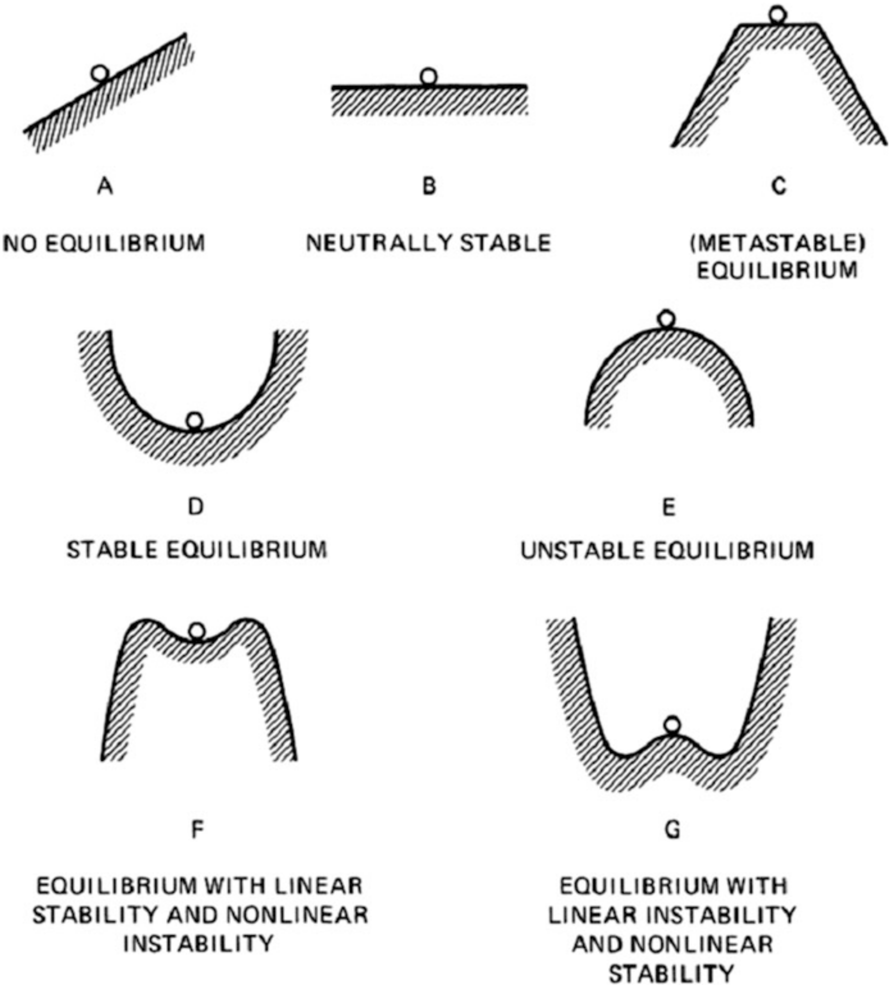

Fig. 6.1 Mechanical analogy of various types of equilibrium

Of the two problems, equilibrium and stability, the latter is easier to treat. One can linearize the equations of motion for small deviations from an equilibrium state. We then have linear equations, just as in the case of plasma waves. The equilibrium problem, on the other hand, is a nonlinear problem like that of diffusion. In complex magnetic geometries, the calculation of equilibria is a tedious process.

6.2 Hydromagnetic Equilibrium

Although the general problem of equilibrium is complicated, several physical concepts are easily gleaned from the MHD equations. For a steady state with $\partial/\partial t = 0$ and $\mathbf{g} = 0$, the plasma must satisfy (cf. Eq. (5.85))

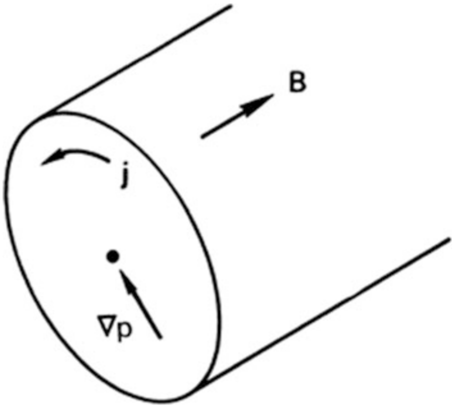

Fig. 6.2 The $\mathbf{j} \times \mathbf{B}$ force of the diamagnetic current balances the pressure-gradient force in steady state

$$\nabla p = \mathbf{j} \times \mathbf{B} \tag{6.1}$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{6.2}$$

From the simple equation (6.1), we can already make several observations.

(a) Equation (6.1) states that there is a balance of forces between the pressure-gradient force and the Lorentz force. How does this come about? Consider a cylindrical plasma with ∇p directed toward the axis (Fig. 6.2). To counteract the outward force of expansion, there must be an azimuthal current in the direction shown. The magnitude of the required current can be found by taking the cross product of Eq. (6.1) with $\mathbf{B}$:

$$\mathbf{j}_\perp = \frac{\mathbf{B} \times \nabla p}{B^2} = (KT_i + KT_e)\frac{\mathbf{B} \times \nabla n}{B^2} \tag{6.3}$$

This is just the diamagnetic current found previously in Eq. (3.69)! From a single-particle viewpoint, the diamagnetic current arises from the Larmor gyration velocities of the particles, which do not average to zero when there is a density gradient. From an MHD fluid viewpoint, the diamagnetic current is generated by the ∇p force across $\mathbf{B}$; the resulting current is just sufficient to balance the forces on each element of fluid and stop the motion.

(b) Equation (6.1) obviously tells us that $\mathbf{j}$ and $\mathbf{B}$ are each perpendicular to ∇p. This is not a trivial statement when one considers that the geometry may be very complicated. Imagine a toroidal plasma in which there is a smooth radial density gradient so that the surfaces of constant density (actually, constant p) are nested tori (Fig. 6.3). Since $\mathbf{j}$ and $\mathbf{B}$ are perpendicular to ∇p, they must lie on the surfaces of constant p. In general, the lines of force and of current may be twisted this way and that, but they must not cross the constant-p surfaces.

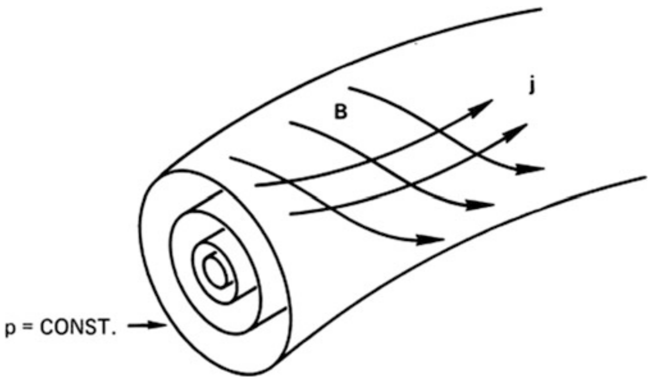

Fig. 6.3 Both the **j** and **B** vectors lie on constant-pressure surfaces

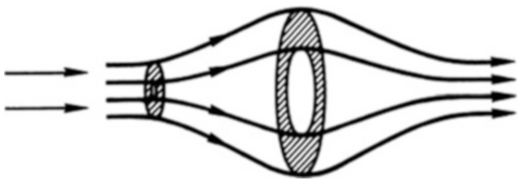

Fig. 6.4 Expansion of a plasma streaming into a mirror

(c) Consider the component of Eq. (6.1) along **B**. It says that

$$\partial p/\partial s = 0 \tag{6.4}$$

where s is the coordinate along a line of force. For constant KT, this means that in hydromagnetic equilibrium the density is constant along a line of force. At first sight, it seems that this conclusion must be in error. For, consider a plasma injected into a magnetic mirror (Fig. 6.4). As the plasma streams through, following the lines of force, it expands and then contracts; and the density is clearly not constant along a line of force. However, this situation does not satisfy the conditions of a static equilibrium. The $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term, which we neglected along the way, does not vanish here. We must consider a static plasma with $\mathbf{v} = 0$. In that case, particles are trapped in the mirror, and there are more particles trapped near the midplane than near the ends because the mirror ratio is larger there. This effect just compensates for the larger cross section at the midplane, and the net result is that the density is constant along a line of force.

6.3 The Concept of β

We now substitute Eq. (6.2) into Eq. (6.1) to obtain

$$\nabla p = \mu_0^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0^{-1}\left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \tfrac{1}{2}\nabla B^2\right] \tag{6.5}$$

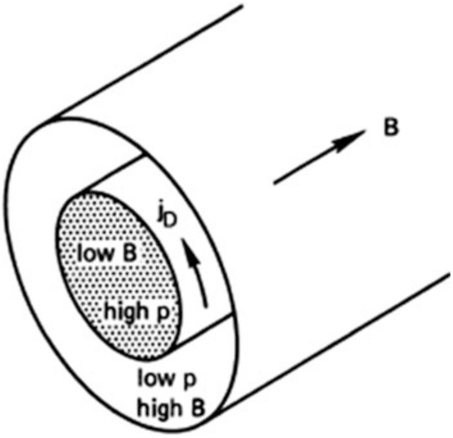

Fig. 6.5 In a finite-β plasma, the diamagnetic current significantly decreases the magnetic field, keeping the sum of the magnetic and particle pressures a constant

or

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} \tag{6.6}$$

In many interesting cases, such as a straight cylinder with axial field, the right-hand side vanishes; **B** does not vary along **B**. In many other cases, the right-hand side is small. Equation (6.6) then says that

$$p + \frac{B^2}{2\mu_0} = \text{ constant} \tag{6.7}$$

Since $B^2/2\,\mu_0$ is the magnetic field pressure, the sum of the particle pressure and the magnetic field pressure is a constant. In a plasma with a density gradient (Fig. 6.5), the magnetic field must be low where the density is high, and vice versa. The decrease of the magnetic field inside the plasma is caused, of course, by the diamagnetic current. The size of the diamagnetic effect is indicated by the ratio of the two terms in Eq. (6.7). This ratio is usually denoted by β:

$$\beta \equiv \frac{\sum nKT}{B^2/2\mu_0} = \frac{\text{Particle pressure}}{\text{Magnetic field pressure}} \tag{6.8}$$

Up to now we have implicitly considered low-β plasmas, in which β is between 10^{-3} and 10^{-6}. The diamagnetic effect, therefore, is very small. This is the reason we could assume a uniform field B_0 in the treatment of plasma waves. If β is low, it does not matter whether the denominator of Eq. (6.8) is evaluated with the vacuum field or the field in the presence of plasma. If β is high, the local value of B can be greatly reduced by the plasma. In that case, it is customary to use the vacuum value of B in the definition of β. High-β plasmas are common in space and MHD energy conversion research. Fusion reactors will have to have β well in excess of 1 % in order to be economical, since the energy produced is proportional to n^2, while the cost of the magnetic container increases with some power of B.

In principle, one can have a $\beta = 1$ plasma in which the diamagnetic current generates a field exactly equal and opposite to an externally generated uniform field. There are then two regions: a region of plasma without field, and a region of field without plasma. If the external field lines are straight, this equilibrium would likely be unstable, since it is like a blob of jelly held together with stretched rubber bands. It remains to be seen whether a $\beta = 1$ plasma of this type can ever be achieved. In some magnetic configurations, the vacuum field has a null inside the plasma; the local value of β would then be infinite there. This happens, for instance, when fields are applied only near the surface of a large plasma. It is then customary to define β as the ratio of maximum particle pressure to maximum magnetic pressure; in this sense, it is not possible for a magnetically confined plasma to have $\beta > 1$.

6.4 Diffusion of Magnetic Field into a Plasma

A problem which often arises in astrophysics is the diffusion of a magnetic field into a plasma. If there is a boundary between a region with plasma but no field and a region with field but no plasma (Fig. 6.6), the regions will stay separated if the plasma has no resistivity, for the same reason that flux cannot penetrate a superconductor. Any emf that the moving lines of force generate will create an infinite current, and this is not possible. As the plasma moves around, therefore, it pushes the lines of force and can bend and twist them. This may be the reason for the filamentary structure of the gas in the Crab nebula. If the resistivity is finite, however, the plasma can move through the field and vice versa. This diffusion takes a certain amount of time, and if the motions are slow enough, the lines of force

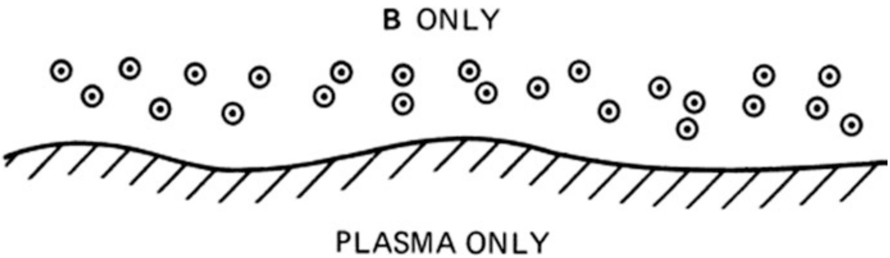

Fig. 6.6 In a perfectly conducting plasma, regions of plasma and magnetic field can be separated by a sharp boundary. Currents on the surface exclude the field from the plasma

need not be distorted by the gas motions. The diffusion time is easily calculated from these equations (cf. Eq. (5.91)):

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{6.9}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} \tag{6.10}$$

For simplicity, let us assume that the plasma is at rest and the field lines are moving into it. Then $\mathbf{v} = 0$, and we have

$$\partial \mathbf{B}/\partial t = -\nabla \times \eta \mathbf{j} \tag{6.11}$$

Since $\mathbf{j}$ is given by Eq. (6.2), this becomes

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{\eta}{\mu_0}\nabla \times (\nabla \times \mathbf{B}) = -\frac{\eta}{\mu_0}\left[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}\right] \tag{6.12}$$

Since $\nabla \cdot \mathbf{B} = 0$, we obtain a diffusion equation of the type encountered in Chap. 5:

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0}\nabla^2 \mathbf{B} \tag{6.13}$$

This can be solved by the separation of variables, as usual. To get a rough estimate, let us take L to be the scale length of the spatial variation of $\mathbf{B}$. Then we have

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0 L^2}\mathbf{B} \tag{6.14}$$

$$\mathbf{B} = \mathbf{B}_0 e^{\pm t/\tau} \tag{6.15}$$

where

$$\tau = \mu_0 L^2/\eta \tag{6.16}$$

This is the characteristic time for magnetic field penetration into a plasma.

The time τ can also be interpreted as the time for annihilation of the magnetic field. As the field lines move through the plasma, the induced currents cause ohmic heating of the plasma. This energy comes from the energy of the field. The energy lost per m^3 in a time τ is $\eta j^2 \tau$. Since

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B} \approx \frac{B}{L} \tag{6.17}$$

from Maxwell's equation with displacement current neglected, the energy dissipation is

$$\eta j^2 \tau = \eta \left(\frac{B}{\mu_0 L} \right)^2 \frac{\mu_0 L^2}{\eta} = \frac{B^2}{\mu_0} = 2 \left(\frac{B^2}{2\mu_0} \right) \tag{6.18}$$

Thus τ is essentially the time it takes for the field energy to be dissipated into Joule heat.

Problems

6.1. Suppose that an electromagnetic instability limits β to $(m/M)^{1/2}$ in a D–D reactor. Let the magnetic field be limited to 20 T by the strength of materials. If $KT_e = KT_i = 20$ keV, find the maximum plasma density that can be contained.

6.2. In laser-fusion experiments, absorption of laser light on the surface of a pellet creates a plasma of density $n = 10^{27}$ m^{-3} and temperature $T_e \simeq T_i \simeq 10^4$ eV. Thermoelectric currents can cause spontaneous magnetic fields as high as 10^3 T.

(a) Show that $\omega_c \tau_{ei} \gg 1$ in this plasma, and hence electron motion is severely affected by the magnetic field.
(b) Show that $\beta \gg 1$, so that magnetic fields cannot effectively confine the plasma.
(c) How do the plasma and field move so that the seemingly contradictory conditions (a) and (b) can both be satisfied?

6.3. A cylindrical plasma column of radius a contains a coaxial magnetic field $\mathbf{B} = B_0 \hat{\mathbf{z}}$ and has a pressure profile

$$p = p_0 \cos^2(\pi r / 2a)$$

(a) Calculate the maximum value of p_0.
(b) Using this value of p_0, calculate the diamagnetic current $\mathbf{j}(r)$ and the total field $\mathbf{B}(r)$.
(c) Show $j(r)$, $B(r)$, and $p(r)$ on a graph.
(d) If the cylinder is bent into a torus with the lines of force closing upon themselves after a single turn, this equilibrium, in which the macroscopic forces are everywhere balanced, is obviously disturbed. Is it possible to redistribute the pressure $p(r, \theta)$ in such a way that the equilibrium is restored?

6.4 Consider an infinite, straight cylinder of plasma with a square density profile created in a uniform field B_0 (Fig. P6.4). Show that B vanishes on the axis if $\beta = 1$, by proceeding as follows.

(a) Using the MHD equations, find $\mathbf{j}_\perp$ in steady state for $KT = \text{constant}$.

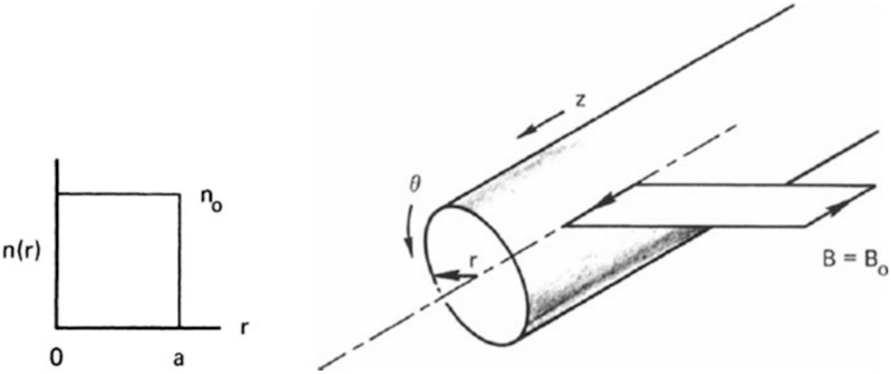

Fig. P6.4

(b) Using $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ and Stokes' theorem, integrate over the area of the loop shown to obtain

$$B_{ax} - B_0 = \mu_0 \sum KT \int_0^\infty \frac{\partial n / \partial r}{B(r)} dr, \quad B_{ax} \equiv B_{r=0}$$

(c) Do the integral by noting that $\partial n / \partial r$ is a δ-function, so that $B(r)$ at $r = a$ is the average between B_{ax} and B_0.

6.5 A diamagnetic loop is a device used to measure plasma pressure by detecting the diamagnetic effect (Fig. P6.5). As the plasma is created, the diamagnetic current increases, B decreases inside the plasma, and the flux Φ enclosed by the loop decreases, inducing a voltage, which is then time-integrated by an RC circuit (Fig. P6.5).

(a) Show that

$$\int_{\text{loop}} V\,dt = -N\,\Delta\Phi = -N \int \mathbf{B}_d \cdot d\mathbf{S}, \qquad B_d \equiv B - B_0$$

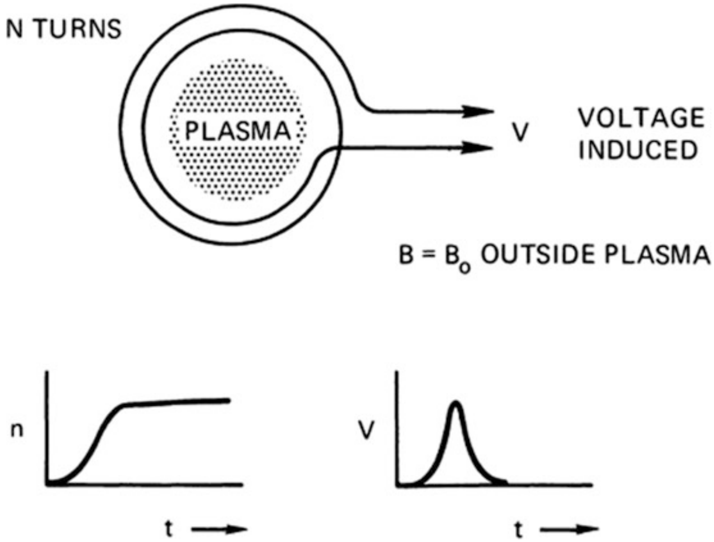

Fig. P6.5

(b) Use the technique of the previous problem to find $B_d(r)$, but now assume $n(r) = n_0 \exp[-(r/r_0)^2]$. To do the integral, assume $\beta \ll 1$, so that B can be approximated by B_0 in the integral.

(c) Show that $\int V dt = \frac{1}{2} N \pi r_0^2 \beta B_0$, with β defined as in Eq. (6.8).

6.5 Classification of Instabilities

In the treatment of plasma waves, we assumed an unperturbed state which was one of perfect thermodynamic equilibrium: The particles had Maxwellian velocity distributions, and the density and magnetic field were uniform. In such a state of highest entropy, there is no free energy available to excite waves, and we had to consider waves that were excited by external means. We now consider states that are not in perfect thermodynamic equilibrium, although they are in equilibrium in the sense that all forces are in balance and a time-independent solution is possible. The free energy which is available can cause waves to be self-excited; the equilibrium is then an unstable one. An instability is always a motion which decreases the free energy and brings the plasma closer to true thermodynamic equilibrium.

Instabilities may be classified according to the type of free energy available to drive them. There are four main categories.

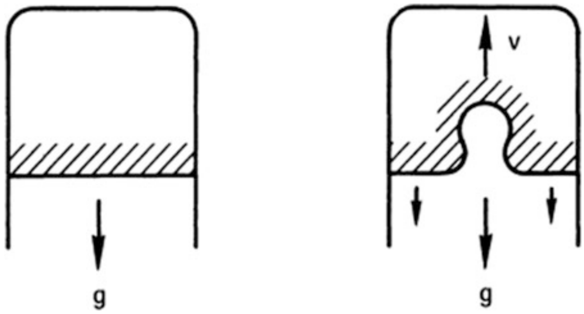

Fig. 6.7 Hydrodynamic Rayleigh–Taylor instability of a heavy fluid supported by a light one

6.5.1 Streaming instabilities

In this case, either a beam of energetic particles travels through the plasma, or a current is driven through the plasma so that the different species have drifts relative to one another. The drift energy is used to excite waves, and oscillation energy is gained at the expense of the drift energy in the unperturbed state.

6.5.2 Rayleigh–Taylor instabilities

In this case, the plasma has a density gradient or a sharp boundary, so that it is not uniform. In addition, an external, non-electromagnetic force is applied to the plasma. It is this force which drives the instability. An analogy is available in the example of an inverted glass of water (Fig. 6.7). Although the plane interface between the water and air is in a state of equilibrium in that the weight of the water is supported by the air pressure, it is an unstable equilibrium. Any ripple in the surface will tend to grow at the expense of potential energy in the gravitation al field. This happens whenever a heavy fluid is supported by a light fluid, as is well known in hydrodynamics.

6.5.3 Universal instabilities

Even when there are no obvious driving forces such as an electric or a gravitational field, a plasma is not in perfect thermodynamic equilibrium as long as it is confined. The plasma pressure tends to make the plasma expand, and the expansion energy can drive an instability. This type of free energy is always present in any finite plasma, and the resulting waves are called *universal instabilities*.

6.5.4 Kinetic instabilities

In fluid theory the velocity distributions are assumed to be Maxwellian. If the distributions are in fact not Maxwellian, there is a deviation from thermodynamic equilibrium; and instabilities can be driven by the anisotropy of the velocity distribution. For instance, if $T_{\|}$ and $T_{\perp}$ are different, an instability called the modified Harris instability can arise. In mirror devices, there is a deficit of particles with large $v_{\|}/v_{\perp}$ because of the loss cone; this anisotropy gives rise to a "loss cone instability."

In the succeeding sections, we shall give a simple example of each of these types of instabilities. The instabilities driven by anisotropy cannot be described by fluid theory and a detailed treatment of them is beyond the scope of this book.

Not all instabilities are equally dangerous for plasma confinement. A high-frequency instability near ω_p, for instance, cannot affect the motion of heavy ions. Low-frequency instabilities with $\omega \ll \Omega_c$, however, can cause anomalous ambipolar losses via $\mathbf{E} \times \mathbf{B}$ drifts. Instabilities with $\omega \approx \Omega_c$ not efficiently transport particles across $\mathbf{B}$ but are dangerous in mirror machines, where particles are lost by diffusion in *velocity* space into the loss cone.

6.6 Two-Stream Instability

As a simple example of a streaming instability, consider a uniform plasma in which the ions are stationary and the electrons have a velocity $\mathbf{v}_0$ relative to the ions. That is, the observer is in a frame moving with the "stream" of ions. Let the plasma be cold ($KT_e = KT_i = 0$), and let there be no magnetic field ($B_0 = 0$). The linearized equations of motion are then

$$Mn_0 \frac{\partial \mathbf{v}_{i1}}{\partial t} = en_0 \mathbf{E}_1 \tag{6.19}$$

$$mn_0 \left[\frac{\partial \mathbf{v}_{e1}}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_{e1} \right] = -en_0 \mathbf{E}_1 \tag{6.20}$$

The term $(\mathbf{v}_{e1} \cdot \nabla)\mathbf{v}_\mathbf{0}$ in Eq. (6.20) has been dropped because we assume $\mathbf{v}_0$ to be uniform. The $(\mathbf{v}_\mathbf{0} \cdot \nabla)\mathbf{v}_1$ term does not appear in Eq. (6.19) because we have taken $\mathbf{v}_{i0} = 0$. We look for electrostatic waves of the form

$$\mathbf{E}_1 = Ee^{i(kx-\omega t)}\hat{\mathbf{x}} \tag{6.21}$$

where $\hat{\mathbf{x}}$ is the direction of $\mathbf{v}_0$ and $\mathbf{k}$. Equations (6.19) and (6.20) become

$$-i\omega Mn_0 \mathbf{v}_{i1} = en_0 \mathbf{E}_1, \qquad \mathbf{v}_{i1} = \frac{ie}{M\omega} E\hat{\mathbf{x}} \tag{6.22}$$

$$mn_0(-i\omega + ikv_0)\mathbf{v}_{e1} = -en_0\mathbf{E}_1 \qquad \mathbf{v}_{e1} = -\frac{ie}{m}\frac{E\hat{\mathbf{x}}}{\omega - kv_0} \tag{6.23}$$

The velocities $\mathbf{v}_{j1}$ are in the x direction, and we may omit the subscript x. The ion equation of continuity yields

$$\frac{\partial n_{i1}}{\partial t} + n_0\nabla\cdot\mathbf{v}_{i1} = 0 \qquad n_{i1} = \frac{k}{\omega}n_0 v_{i1} = \frac{ien_0 k}{M\omega^2}E \tag{6.24}$$

Note that the other terms in $\nabla\cdot(n\mathbf{v}_i)$ vanish because $\nabla n_0 = \mathbf{v}_{0i} = 0$. The electron equation of continuity is

$$\frac{\partial n_{e1}}{\partial t} + n_0\nabla\cdot\mathbf{v}_{e1} + (\mathbf{v}_0\cdot\nabla)n_{e1} = 0 \tag{6.25}$$

$$(-i\omega + ikv_0)n_{e1} + ikn_0 v_{e1} = 0 \tag{6.26}$$

$$n_{e1} = \frac{kn_0}{\omega - kv_0}v_{e1} = -\frac{iekn_0}{m(\omega - kv_0)^2}E \tag{6.27}$$

Since the unstable waves are high-frequency plasma oscillations, we may not use the plasma approximation but must use Poisson's equation:

$$\varepsilon_0\nabla\cdot\mathbf{E}_1 = e(n_{i1} - n_{e1}) \tag{6.28}$$

$$ik\varepsilon_0 E = e(ien_0 kE)\left[\frac{1}{M\omega^2} + \frac{1}{m(\omega - kv_0)^2}\right] \tag{6.29}$$

The dispersion relation is found upon dividing by $ik\varepsilon_0 E$:

$$1 = \omega_p^2\left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - kv_0)^2}\right] \tag{6.30}$$

Let us see if oscillations with real k are stable or unstable. Upon multiplying through by the common denominator, one would obtain a fourth-order equation for ω. If all the roots ω_j are real, each root would indicate a possible oscillation

$$\mathbf{E}_1 = Ee^{i(kx - \omega_j t)}\hat{\mathbf{x}}$$

If some of the roots are complex, they will occur in complex conjugate pairs. Let these complex roots be written

$$\omega_j = \alpha_j + i\gamma_j \tag{6.31}$$

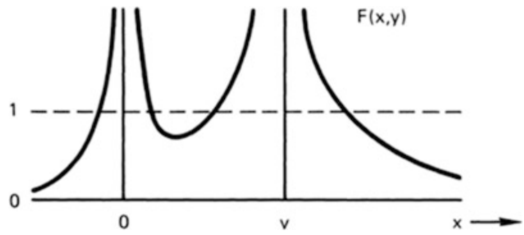

Fig. 6.8 The function $F(x, y)$ in the two-stream instability, when the plasma is stable

where α and γ are Re(ω) and Im(ω), respectively. The time dependence is now given by

$$\mathbf{E}_1 = Ee^{i\left(kx-\alpha_j t\right)}e^{\gamma_j t}\hat{\mathbf{x}} \tag{6.32}$$

Positive Im(ω) indicates an exponentially growing wave; negative Im(ω) indicates a damped wave. Since the roots ω_j occur in conjugate pairs, one of these will always be unstable unless all the roots are real. The damped roots are not self-excited and are not of interest.

The dispersion relation (Eq. (6.30)) can be analyzed without actually solving the fourth-order equation. Let us define

$$x \equiv \omega/\omega_p \quad y \equiv kv_0/\omega_p \tag{6.33}$$

Then Eq. (6.30) becomes

$$1 = \frac{m/M}{x^2} + \frac{1}{(x-y)^2} \equiv F(x, y). \tag{6.34}$$

For any given value of y, we can plot $F(x, y)$ as a function of x. This function will have singularities at $x = 0$ and $x = y$ (Fig. 6.8). The intersections of this curve with the line $F(x, y) = 1$ give the values of x satisfying the dispersion relation. In the example of Fig. 6.8, there are four intersections, so there are four real roots ω_j. However, if we choose a smaller value of y, the graph would look as shown in Fig. 6.9. Now there are only two intersections and, therefore, only two real roots. The other two roots must be complex, and one of them must correspond to an unstable wave. Thus, for sufficiently small kv_0, the plasma is unstable. For any given v_0, the plasma is always unstable to long-wavelength oscillations (small k and y). The maximum growth rate predicted by Eq. (6.30) is, for $m/M \ll 1$ (cf. Problem 6.6),

$$\text{Im}\left(\frac{\omega}{\omega_p}\right) \approx \left(\frac{m}{M}\right)^{1/3} \tag{6.35}$$

Since a small value of kv_0 is required for instability, one can say that for a given k, v_0 has to be sufficiently small for instability. This does not make much physical sense, since v_0 is the source of energy driving the instability. The difficulty comes from our use of the fluid equations. Any real plasma has a finite temperature, and

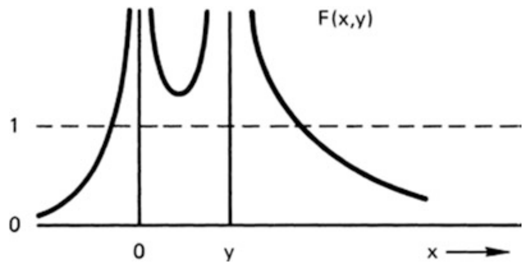

Fig. 6.9 The function $F(x, y)$ in the two-stream instability, when the plasma is unstable

thermal effects should be taken into account by a kinetic-theory treatment. A phenomenon known as Landau damping (Chap. 7) will then occur for $v_0 \lesssim v_{th}$, and no instability is predicted if v_0 is too small.

This "Buneman" instability, as it is sometimes called, has the following physical explanation. The natural frequency of oscillations in the electron fluid is ω_p, and the natural frequency of oscillations in the ion fluid is $\Omega_p = (m/M)^{1/2}\omega_p$. Because of the Doppler shift of the ω_p oscillations in the moving electron fluid, these two frequencies can coincide in the laboratory frame if kv_0 has the proper value. The density fluctuations of ions and electrons can then satisfy Poisson's equation. Moreover, the electron oscillations can be shown to have *negative energy*. That is to say, the total kinetic energy of the electrons is less when the oscillation is present than when it is absent. In the undisturbed beam, the kinetic energy per m^3 is $\frac{1}{2}mn_0v_0^2$. When there is an oscillation, the kinetic energy is $\frac{1}{2}m(n_0 + n_1)(v_0 + v_1)^2$. When this is averaged over space, it turns out to be less than $\frac{1}{2}mn_0v_0^2$ because of the phase relation between n_1 and v_1 required by the equation of continuity. Consequently, the electron oscillations have negative energy, and the ion oscillations have positive energy. Both waves can grow together while keeping the total energy of the system constant. An instability of this type is used in klystrons to generate microwaves. Velocity modulation due to $\mathbf{E}_1$ causes the electrons to form bunches. As these bunches pass through a microwave resonator, they can be made to excite the natural modes of the resonator and produce microwave power.

Problems

6.6. (a) Derive the dispersion relation for a two-stream instability occurring when there are two cold electron streams with equal and opposite $\mathbf{v}_0$ in a background of fixed ions. Each stream has a density $\frac{1}{2}n_0$.

(b) Calculate the maximum growth rate.

6.7. A plasma consists of two uniform streams of protons with velocities $+v_0\hat{\mathbf{x}}$ and $-v_0\hat{\mathbf{x}}$, and respective densities $\frac{2}{3}n_0$ and $\frac{1}{3}n_0$. There is a neutralizing electron fluid with density n_0 and with $v_{0e} = 0$. All species are cold, and there is no magnetic field. Derive a dispersion relation for streaming instabilities in this system.

6.8. A cold electron beam of density δn_0 and velocity u is shot into a cold plasma of density n_0 at rest.

(a) Derive a dispersion relation for the high-frequency beam-plasma instability that ensues.

(b) The maximum growth rate γ_m is difficult to calculate, but one can make a reasonable guess if $\delta \ll 1$ by analogy with the electron–ion Buneman instability. Using the result given without proof in Eq. (6.35), give an expression for γ_m in terms of δ.

6.9. Let two cold, counter-streaming ion fluids have densities $\frac{1}{2}n_0$ and velocities $\pm v_0\hat{\boldsymbol{y}}$ in a magnetic field $B_0\hat{\boldsymbol{z}}$ and a cold neutralizing electron fluid. The field B_0 is strong enough to confine electrons but not strong enough to affect ion orbits.

(a) Obtain the following dispersion relation for electrostatic waves propagating in the $\pm\hat{y}$ direction in the frequency range $\Omega_c^2 << \omega^2 << \omega_c^2$:

$$\frac{\Omega_p^2}{2(\omega - kv_0)^2} + \frac{\Omega_p^2}{2(\omega + kv_0)^2} = \frac{\omega_p^2}{\omega_c^2} + 1$$

(b) Calculate the dispersion $\omega(k)$, growth rate $\gamma(k)$, and the range of wave numbers of the unstable waves.

6.7 The "Gravitational" Instability

In a plasma, a Rayleigh–Taylor instability can occur because the magnetic field acts as a light fluid supporting a heavy fluid (the plasma). In curved magnetic fields, the centrifugal force on the plasma due to particle motion along the curved lines of force acts as an equivalent "gravitational" force. To treat the simplest case, consider a plasma boundary lying in the y–z plane (Fig. 6.10). Let there be a density gradient

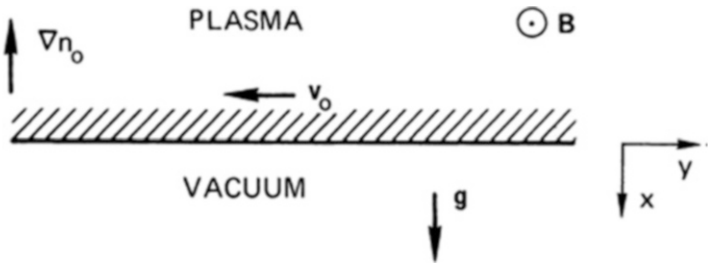

Fig. 6.10 A plasma surface subject to a gravitational instability

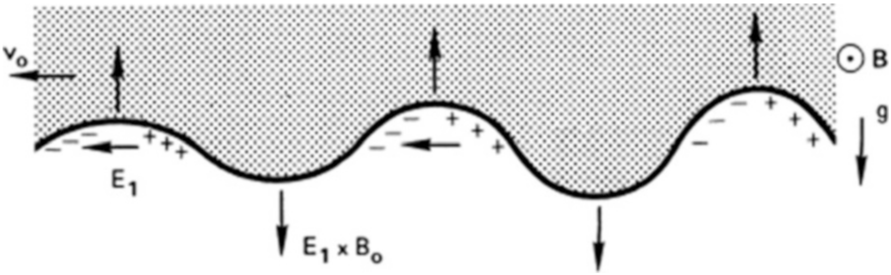

Fig. 6.11 Physical mechanism of the gravitational instability

∇n_0 in the $-x$ direction and a gravitational field $\mathbf{g}$ in the x direction. We may let $KT_i = KT_e. = 0$ for simplicity and treat the low-β case, in which $\mathbf{B}_0$ is uniform. In the equilibrium state, the ions obey the equation

$$Mn_0(\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 = en\mathbf{v}_0 \times \mathbf{B}_0 + Mn_0\mathbf{g} \tag{6.36}$$

If g is a constant, v_0 will be also; and $(\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0$ vanishes. Taking the cross product of Eq. (6.36) with $\mathbf{B}_0$, we find, as in Sect. 2.2,

$$\mathbf{v}_0 = \frac{M}{e}\frac{\mathbf{g} \times \mathbf{B}_0}{B_0^2} = -\frac{g}{\Omega_c}\hat{\mathbf{y}} \tag{6.37}$$

The electrons have an opposite drift which can be neglected in the limit $m/M \rightarrow 0$. There is no diamagnetic drift because $KT = 0$, and no $\mathbf{E}_0 \times \mathbf{B}_0$ drift because $\mathbf{E}_0 = 0$.

If a ripple should develop in the interface as the result of random thermal fluctuations, the drift $\mathbf{v}_0$ will cause the ripple to grow (Fig. 6.11). The drift of ions causes a charge to build up on the sides of the ripple, and an electric field develops which changes sign as one goes from crest to trough in the perturbation. As can be seen from Fig. 6.11, the $\mathbf{E}_1 \times \mathbf{B}_0$ drift is always upward in those regions where the surface has moved upward, and downward where it has moved downward. The ripple grows as a result of these properly phased $\mathbf{E}_1 \times \mathbf{B}_0$ drifts.

To find the growth rate, we can perform the usual linearized wave analysis for waves propagating in the y direction: $\mathbf{k} = k\hat{y}$. The perturbed ion equation of motion is

$$\begin{aligned} M(n_0 + n_1)&\left[\frac{\partial}{\partial t}(\mathbf{v}_0 + \mathbf{v}_1) + (\mathbf{v}_0 + \mathbf{v}_1) \cdot \nabla(\mathbf{v}_0 + \mathbf{v}_1)\right] \\ &= e(n_0 + n_1)[\mathbf{E}_1 + (\mathbf{v}_0 + \mathbf{v}_1) \times \mathbf{B}_0] + M(n_0 + n_1)\mathbf{g} \end{aligned} \tag{6.38}$$

We now multiply Eq. (6.36) by $1 + (n_1/n_0)$ to obtain

$$M(n_0 + n_1)(\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 = e(n_0 + n_1)\mathbf{v}_0 \times \mathbf{B}_0 + M(n_0 + n_1)\mathbf{g} \tag{6.39}$$

Subtracting this from Eq. (6.38) and neglecting second-order terms, we have the *linearized* equation

$$Mn_0\left[\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_1\right] = en_0(\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0) \tag{6.40}$$

Note that $\mathbf{g}$ has cancelled out. Information regarding $\mathbf{g}$, however, is still contained in $\mathbf{v}_0$. For perturbations of the form exp $[i(ky - \omega t)]$, we have

$$M(\omega - kv_0)\mathbf{v}_1 = ie(\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0) \tag{6.41}$$

This is the same as Eq. (4.96) except that ω is replaced by $\omega - kv_0$, and electron quantities are replaced by ion quantities. The solution, therefore, is given by Eq. (4.98) with the appropriate changes. For $E_x = 0$ and

$$\Omega_c^2 >> (\omega - kv_0)^2 \tag{6.42}$$

the solution is

$$v_{ix} = \frac{E_y}{B_0} \quad v_{iy} = -i\frac{\omega - kv_0}{\Omega_c}\frac{E_y}{B_0} \tag{6.43}$$

The latter quantity is the polarization drift in the ion frame. The corresponding quantity for electrons vanishes in the limit $m/M \rightarrow 0$. For the electrons, we therefore have

$$v_{ex} = E_y/B_0 \qquad v_{ey} = 0 \tag{6.44}$$

The perturbed equation of continuity for ions is

$$\begin{aligned} &\frac{\partial n_1}{\partial t} + \nabla \cdot (n_0\mathbf{v}_0) + (\mathbf{v}_0 \cdot \nabla)n_1 + n_1\nabla \cdot \mathbf{v}_0 \\ &\quad + (\mathbf{v}_1 \cdot \nabla)n_0 + n_0\nabla \cdot \mathbf{v}_1 + \nabla \cdot (n_1\mathbf{v}_1) = 0 \end{aligned} \tag{6.45}$$

The zeroth-order term vanishes since $\mathbf{v}_0$ is perpendicular to ∇n_0, and the $n_1\ \nabla \cdot \mathbf{v}_0$ term vanishes if $\mathbf{v}_0$ is constant. The first-order equation is, therefore,

$$-i\omega n_1 + ikv_0n_1 + v_{ix}n_0' + ikn_0v_{iy} = 0 \tag{6.46}$$

where $n_0' = \partial n_0/\partial x$. The electrons follow a simpler equation, since $\mathbf{v}_{e0} = 0$ and $v_{ey} = 0$:

$$-i\omega n_1 + v_{ex}n_0' = 0 \tag{6.47}$$

Note that we have used the plasma approximation and have assumed $n_{i1} = n_{e1}$. This is possible because the unstable waves are of low frequencies (this can be justified *a posteriori*). Equations (6.43) and (6.46) yield

$$(\omega - kv_0)n_1 + i\frac{E_y}{B_0}n_0' + ikn_0\frac{\omega - kv_0}{\Omega_c}\frac{E_y}{B_0} = 0 \tag{6.48}$$

Equations (6.44) and (6.47) yield

$$\omega n_1 + i\frac{E_y}{B_0}n_0' = 0 \qquad \frac{E_y}{B_0} = \frac{i\omega n_1}{n_0'} \tag{6.49}$$

Substituting this into Eq. (6.48), we have

$$\begin{aligned}(\omega - kv_0)n_1 - \left(n_0' + kn_0\frac{\omega - kv_0}{\Omega_c}\right)\frac{\omega n_1}{n_0'} = 0 \\ \omega - kv_0 - \left(1 + \frac{kn_0}{\Omega_c}\frac{\omega - kv_0}{n_0'}\right)\omega = 0\end{aligned} \tag{6.50}$$

$$\omega(\omega - kv_0) = -v_0\Omega_c n_0'/n_0 \tag{6.51}$$

Substituting for v_0 from Eq. (6.37), we obtain a quadratic equation for ω:

$$\omega^2 - kv_0\omega - g\left(n_0'/n_0\right) = 0 \tag{6.52}$$

The solutions are

$$\omega = \frac{1}{2}kv_0 \pm \left[\frac{1}{4}k^2v_0^2 + g\left(n_0'/n_0\right)\right]^{1/2} \tag{6.53}$$

There is instability if ω is complex; that is, if

$$-gn_0'/n_0 > \frac{1}{4}k^2v_0^2 \tag{6.54}$$

From this, we see that instability requires g and n_0'/n_0 to have opposite sign. This is just the statement that the light fluid is supporting the heavy fluid; otherwise, ω is real and the plasma is stable. Since g can be used to model the effects of magnetic field curvature, we see from this that stability depends on the sign of the curvature. Configurations with field lines bending in toward the plasma tend to be stabilizing, and vice versa. For sufficiently small k (long wavelength), the growth rate is given by

$$\boxed{\gamma = \mathrm{Im}\,(\omega) \approx \left[-g\left(n_0'/n_0\right)\right]^{1/2}} \tag{6.55}$$

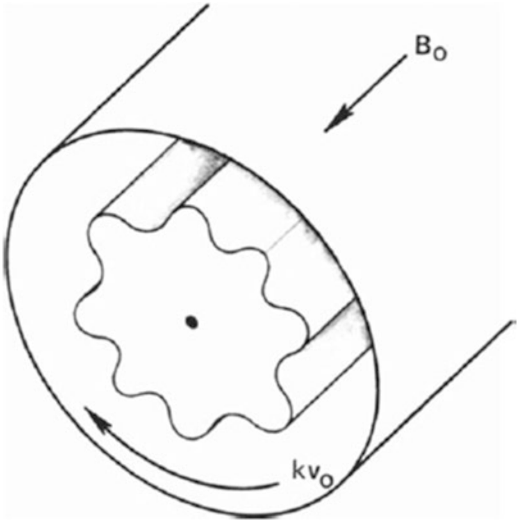

Fig. 6.12 A "flute" instability

Note that the real part of ω is $\frac{1}{2}kv_0$. Since v_0 is an ion velocity, this is a low-frequency oscillation, as previously assumed. The factor of $\frac{1}{2}$ is merely a consequence of neglecting $v_{0e.}$ The wave is stationary in the frame in which the density-weighted average of all the v_0's is zero, which in this case is the frame moving at $\frac{1}{2}v_0$. The laboratory frame has no particular significance in this case.

This instability, which has $\mathbf{k} \perp \mathbf{B}_0$, is sometimes called a "flute" instability for the following reason. In a cylinder, the waves travel in the θ direction if the forces are in the r direction. The surfaces of constant density then resemble fluted Greek columns (Fig. 6.12).

6.8 Resistive Drift Waves

A simple example of a universal instability is the resistive drift wave. In contrast to gravitational flute modes, drift waves have a small but finite component of **k** along $\mathbf{B}_0$. The constant density surfaces. Therefore, resemble flutes with a slight helical twist (Fig. 6.13). If we enlarge the cross section enclosed by the box in Fig. 6.13 and straighten it out into Cartesian geometry it would appear as in Fig. 6.14. The only driving force for the instability is the pressure gradient $KT\ \nabla n_0$ (we assume $KT =$ constant for simplicity). In this case, the zeroth-order drifts (for $\mathbf{E}_0 = 0$) are

$$\mathbf{v}_{i0} = \mathbf{v}_{Di} = \frac{KT_i}{eB_0}\frac{n_0'}{n_0}\hat{\mathbf{y}} \tag{6.56}$$

$$\mathbf{v}_{e0} = \mathbf{v}_{De} = -\frac{KT_e}{eB_0}\frac{n_0'}{n_0}\hat{\mathbf{y}} \tag{6.57}$$

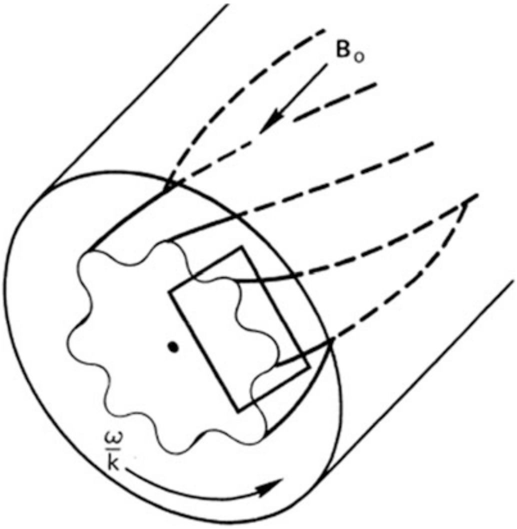

Fig. 6.13 Geometry of a drift instability in a cylinder. The region in the rectangle is shown in detail in Fig. 6.15

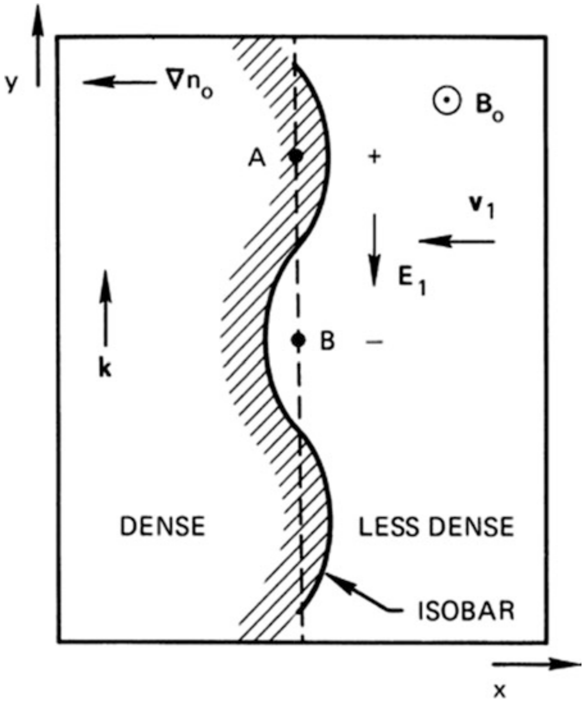

Fig. 6.14 Physical mechanism of a drift wave

From our experience with the flute instability, we might expect drift waves to have a phase velocity of the order of v_{Di} or v_{De}. We shall show that ω/k_y is approximately equal to v_{De}.

Since drift waves have finite k_z, electrons can flow along $\mathbf{B}_0$ to establish a thermodynamic equilibrium among themselves (cf. discussion of Sect. 4.10). They will then obey the Boltzmann relation (Sect. 3.5):

$$n_1/n_0 = e\phi_1/KT_e \tag{6.58}$$

At point A in Fig. 6.14 the density is larger than in equilibrium, n_1 is positive, and therefore ϕ_1 is positive. Similarly, at point B, n_1 and ϕ_1 are negative. The difference in potential means there is an electric field $\mathbf{E}_1$ between A and B. Just as in the case of the flute instability, $\mathbf{E}_1$ causes a drift $\mathbf{v}_1 = \mathbf{E}_1 \times \mathbf{B}_0/B_0^2$ in the x direction. As the wave passes by, traveling in the y direction, an observer at point A will see n_1 and ϕ_1 oscillating in time. The drift $\mathbf{v}_1$ will also oscillate in time, and in fact it is $\mathbf{v}_1$ which causes the density to oscillate. Since there is a gradient ∇n_0 in the $-x$ direction, the drift $\mathbf{v}_1$ will bring plasma of different density to a fixed observer A. A drift wave, therefore, has a motion such that the fluid moves back and forth in the x direction although the wave travels in the y direction.

To be more quantitative, the magnitude of v_{1x} is given by

$$v_{1x} = E_y/B_0 = -ik_y\phi_1/B_0 \tag{6.59}$$

We shall assume v_{1x} does not vary with x and that k_z is much less than k_y; that is, the fluid oscillates incompressibly in the x direction. Consider now the number of guiding centers brought into 1 m^3 at a fixed point A; it is obviously

$$\partial n_1/\partial t = -v_{1x}\partial n_0/\partial x \tag{6.60}$$

This is just the equation of continuity for guiding centers, which, of course, do not have a fluid drift $\mathbf{v}_D$. The term $n_0\ \nabla\cdot\mathbf{v}_1$ vanishes because of our previous assumption. The difference between the density of guiding centers and the density of particles n_1 gives a correction to Eq. (6.60) which is higher order and may be neglected here. Using Eqs. (6.59) and (6.58), we can write Eq. (6.60) as

$$-i\omega n_1 = \frac{ik_y\phi_1}{B_0}n_0' = -i\omega\frac{e\phi_1}{KT_e}n_0 \tag{6.61}$$

Thus we have

$$\frac{\omega}{k_y} = -\frac{KT_e}{eB_0}\frac{n_0'}{n_0} = v_{De} \tag{6.62}$$

These waves, therefore, travel with the *electron* diamagnetic drift velocity and are called *drift waves*. This is the velocity in the y, or azimuthal, direction. In addition, there is a component of $\mathbf{k}$ in the z direction. For reasons not given here, this component must satisfy the conditions

$$k_z << k_y \quad v_{\text{th}i} << \omega/k_z << v_{\text{th}e} \tag{6.63}$$

To see why drift waves are unstable, one must realize that v_{1x} is not quite E_y/B_0 for the ions. There are corrections due to the polarization drift, Eq. (2.66), and the nonuniform **E** drift, Eq. (2.59). The result of these drifts is always to make the potential distribution ϕ_1 lag behind the density distribution n_1 (Problem 4.1). This phase shift causes $\mathbf{v}_1$ to be outward where the plasma has already been shifted outward, and vice versa; hence the perturbation grows. In the absence of the phase shift, n_1 and ϕ_1 would be 90° out of phase, as shown in Fig. 6.14, and drift waves would be purely oscillatory.

The role of resistivity comes in because the field $\mathbf{E}_1$ must not be short-circuited by electron flow along $\mathbf{B}_0$. Electron–ion collisions, together with a long distance $\frac{1}{2}\lambda_z$ between crest and trough of the wave, make it possible to have a resistive potential drop and a finite value of $\mathbf{E}_1$. The dispersion relation for resistive drift waves is approximately

$$\boxed{\omega^2 + i\sigma_{||}\left(\omega - \omega_*\right) = 0} \tag{6.64}$$

where

$$\boxed{\omega_* \equiv k_y v_{De}} \tag{6.65}$$

and

$$\sigma_{||} \equiv \frac{k_z^2}{k_y^2}\Omega_c(\omega_c\tau_{ei}) \tag{6.66}$$

If $\sigma_{||}$ is large compared with ω, Eq. (6.64) can be satisfied only if $\omega \approx \omega_*$. In that case, we may replace ω by ω_* in the first term. Solving for ω, we then obtain

$$\boxed{\omega \approx \omega_* + \left(i\omega_*^2/\sigma_{||}\right)} \tag{6.67}$$

This shows that Im(ω) is always positive and is proportional to the resistivity η. Drift waves are, therefore, unstable and will eventually occur in any plasma with a density gradient. Fortunately, the growth rate is rather small, and there are ways to stop it altogether by making $\mathbf{B}_0$ nonuniform.

Note that Eq. (6.52) for the flute instability and Eq. (6.64) for the drift instability have different structures. In the former, the coefficients are real, and ω is complex when the discriminant of the quadratic is negative; this is typical of a *reactive* instability. In the latter, the coefficients are complex, so ω is always complex; this is typical of a *dissipative* instability.

Problem

6.10 A toroidal hydrogen plasma with circular cross section has major radius $R = 50$ cm, minor radius $a = 2$ cm, $B = 1$ T, $KT_e = 10$ eV, $KT_i = 1$ eV, and $n_0 = 10^{19}\text{m}^{-3}$. Taking $n_0/n_0' \simeq a/2$ and $g \simeq (KT_e + KT_i)/MR$, estimate the growth rates of the $m = 1$ resistive drift wave and the $m = 1$ gravitational flute mode. (One can usually apply the slab-geometry formulas to cylindrical geometry by replacing k_y by m/r, where m is the azimuthal mode number.)

6.9 The Weibel Instability[1]

As an example of an instability driven by anisotropy of the distribution function, we give a physical picture (due to B. D. Fried) of the Weibel instability, in which a magnetic perturbation is made to grow. This will also serve as an example of an electromagnetic instability. Let the ions be fixed, and let the electrons be hotter in the y direction than in the x or z directions. There is then a preponderance of fast electrons in the $\pm y$ directions (Fig. 6.15), but equal numbers flow up and down, so that there is no net current. Suppose a field $\mathbf{B} = B_z\hat{\mathbf{z}} \cos kx$ spontaneously arises from noise. The Lorentz force $-e\mathbf{v} \times \mathbf{B}$ then bends the e1ectron trajectories as shown by the dashed curves, with the result that downward-moving electrons congregate at A and upward-moving ones at B. The resulting current sheets $\mathbf{j} = -en_0\mathbf{v}_e$ are phased exactly right to generate a $\mathbf{B}$ field of the shape assumed, and the perturbation grows. Though the general case requires a kinetic treatment, the simple case $v_y = v_0$, $v_x = v_z = 0$ has been solved by Fried from this physical picture, yielding a growth rate $\gamma \approx \omega_p v_0/c$.

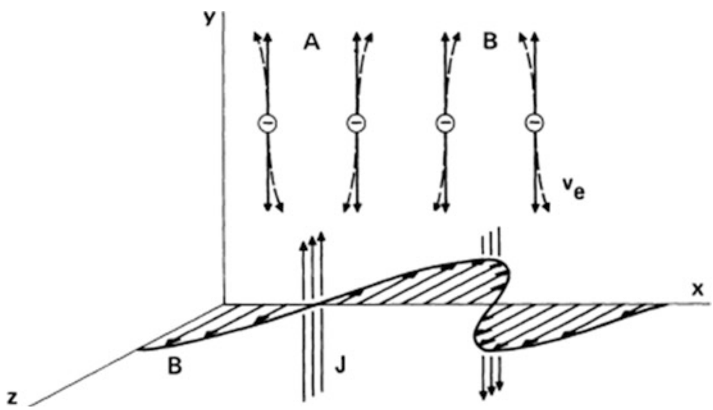

Fig. 6.15 Physical mechanism of the Weibel instability

[1] A salute to a good friend. Erich Weibel (1925–1983).