Chapter 3
Plasmas as Fluids

3.1 Introduction

In a plasma the situation is much more complicated than that in the last chapter; the **E** and **B** fields are not prescribed but are determined by the positions and motions of the charges themselves. One must solve a self-consistent problem; that is, find a set of particle trajectories and field patterns such that the particles will generate the fields as they move along their orbits and the fields will cause the particles to move in those exact orbits. And this must be done in a time-varying situation. It sounds very hard, but it is not.

We have seen that a typical plasma density might be 10^{18} ion–electron pairs per m^3. If each of these particles follows a complicated trajectory and it is necessary to follow each of these, predicting the plasma's behavior would be a hopeless task. Fortunately, this is not usually necessary because, surprisingly, the majority—perhaps as much as 80 %—of plasma phenomena observed in real experiments can be explained by a rather crude model. This model is that used in fluid mechanics, in which the identity of the individual particle is neglected, and only the motion of fluid elements is taken into account. Of course, in the case of plasmas, the fluid contains electrical charges. In an ordinary fluid, frequent collisions between particles keep the particles in a fluid element moving together. It is surprising that such a model works for plasmas, which generally have infrequent collisions. But we shall see that there is a reason for this.

In the greater part of this book, we shall be concerned with what can be learned from the fluid theory of plasmas. A more refined treatment—the kinetic theory of plasmas—requires more mathematical calculation than is appropriate for an introductory course. An introduction to kinetic theory is given in Chap. 7.

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In some plasma problems, neither fluid theory nor kinetic theory is sufficient to describe the plasma's behavior. Then one has to fall back on the tedious process of following the individual trajectories. Modern computers can do this, although they have only enough memory to store the position and velocity components for about 10^6 particles if all three dimensions are involved. Nonetheless, computer simulation plays an important role in filling the gap between theory and experiment in those instances where even kinetic theory cannot come close to explaining what is observed.

3.2 Relation of Plasma Physics to Ordinary Electromagnetics

3.2.1 Maxwell's Equations

In vacuum:

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \sigma \tag{3.1}$$

$$\nabla \cdot \mathbf{E} = -\dot{\mathbf{B}} \tag{3.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \dot{\mathbf{E}} \right) \tag{3.4}$$

In a medium:

$$\nabla \cdot \mathbf{D} = \sigma \tag{3.5}$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{3.6}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.7}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \dot{\mathbf{D}} \tag{3.8}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{3.9}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{3.10}$$

In Eqs. (3.5) and (3.8), $\boldsymbol{\sigma}$ and $\mathbf{j}$ stand for the "free" charge and current densities. The "bound" charge and current densities arising from polarization and magnetization of the medium are included in the definition of the quantities **D** and **H** in terms of ε and μ. In a plasma, the ions and electrons comprising the plasma are the equivalent of the "bound" charges and currents. Since these charges move in a complicated way, it is impractical to try to lump their effects into two constants ε and μ. Consequently, in plasma physics, one generally works with the vacuum equations (3.1)–(3.4), in which $\boldsymbol{\sigma}$ and $\mathbf{j}$ include *all* the charges and currents, both external and internal.

Note that we have used **E** and **B** in the vacuum equations rather than their counterparts **D** and **H**, which are related by the constants ε_0 and μ_0. This is because the forces $q\mathbf{E}$ and $\mathbf{j} \times \mathbf{B}$ depend on **E** and **B** rather than **D** and **H**, and it is not necessary to introduce the latter quantities as long as one is dealing with the vacuum equations.

3.2.2 *Classical Treatment of Magnetic Materials*

Since each gyrating particle has a magnetic moment, it would seem that the logical thing to do would be to consider a plasma as a magnetic material with a permeability μ_m. (We have put a subscript *m* on the permeability to distinguish it from the adiabatic invariant μ.) To see why this is *not* done in practice, let us review the way magnetic materials are usually treated.

The ferromagnetic domains, say, of a piece of iron have magnetic moments μ_i, giving rise to a bulk magnetization

$$\mathbf{M} = \frac{1}{V}\sum_i \boldsymbol{\mu}_i \tag{3.11}$$

per unit volume. This has the same effect as a bound current density equal to

$$\mathbf{j}_b = \nabla \times \mathbf{M} \tag{3.12}$$

In the vacuum equation (3.4), we must include in **j** both this current and the "free," or externally applied, current $\mathbf{j}_f$:

$$\mu_0^{-1}\nabla \times \mathbf{B} = \mathbf{j}_f + \mathbf{j}_b + \varepsilon_0\dot{\mathbf{E}} \tag{3.13}$$

We wish to write Eq. (3.13) in the simple form

$$\nabla \times \mathbf{H} = \mathbf{j}_f + \varepsilon_0\dot{\mathbf{E}} \tag{3.14}$$

by including $\mathbf{j}_b$ in the definition of **H**. This can be done if we let

$$\mathbf{H} = \mu_0^{-1}\mathbf{B} - \mathbf{M} \tag{3.15}$$

To get a simple relation between **B** and **H**, we assume **M** to be proportional to **B** or **H**:

$$\mathbf{M} = \chi_m\mathbf{H} \tag{3.16}$$

The constant χ_m is the magnetic susceptibility. We now have

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} \equiv \mu_m\mathbf{H} \tag{3.17}$$

This simple relation between **B** and **H** is possible because of the linear form of Eq. (3.16).

In a plasma with a magnetic field, each particle has a magnetic moment μ_α, and the quantity **M** is the sum of all these μ_α's in 1 m^3. But we now have

$$\mu_\alpha = \frac{mv_{\perp\alpha}^2}{2B} \propto \frac{1}{B} \qquad \boldsymbol{M} \propto \frac{1}{B}$$

The relation between **M** and **H** (or **B**) is no longer linear, and we cannot write $\mathbf{B} = \mu_m\mathbf{H}$ with μ_m constant. It is therefore not useful to consider a plasma as a magnetic medium.

3.2.3 *Classical Treatment of Dielectrics*

The polarization **P** per unit volume is the sum over all the individual moments $\mathbf{p}_i$ of the electric dipoles. This gives rise to a bound charge density

$$\sigma_b = -\nabla \cdot \mathbf{P} \tag{3.18}$$

In the vacuum equation (3.1), we must include both the bound charge and the free charge:

$$\varepsilon_0\nabla \cdot \mathbf{E} = \left(\sigma_f + \sigma_b\right) \tag{3.19}$$

We wish to write this in the simple form

$$\nabla \cdot \mathbf{D} = \sigma_f \tag{3.20}$$

by including $\boldsymbol{\sigma}_b$ in the definition of **D**. This can be done by letting

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \equiv \varepsilon\,\mathbf{E} \tag{3.21}$$

If **P** is linearly proportional to **E**,

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \tag{3.22}$$

then ε is a constant given by

$$\varepsilon = (1 + \chi_e)\varepsilon_0 \tag{3.23}$$

There is no a priori reason why a relation like Eq. (3.22) cannot be valid in a plasma, so we may proceed to try to get an expression for ε in a plasma.

3.2.4 *The Dielectric Constant of a Plasma*

We have seen in Sect. 2.5 that a fluctuating **E** field gives rise to a polarization current $\mathbf{j}_p$. This leads, in turn, to a polarization charge given by the equation of continuity:

$$\frac{\partial \sigma_p}{\partial t} + \nabla \cdot \mathbf{j}_p = 0 \tag{3.24}$$

This is the equivalent of Eq. (3.18), except that, as we noted before, a polarization effect does not arise in a plasma unless the electric field is time varying. Since we have an explicit expression for $\mathbf{j}_p$ but not for $\boldsymbol{\sigma}_p$, it is easier to work with the fourth Maxwell equation, Eq. (3.4):

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j}_f + \mathbf{j}_p + \varepsilon_0 \dot{\mathbf{E}} \right) \tag{3.25}$$

We wish to write this in the form

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j}_f + \varepsilon \dot{\mathbf{E}} \right) \tag{3.26}$$

This can be done if we let

$$\varepsilon = \varepsilon_0 + \frac{j_p}{\dot{E}} \tag{3.27}$$

From Eq. (2.67) for $\mathbf{j}_p$, we have

$$\varepsilon = \varepsilon_0 + \frac{\rho}{B^2} \qquad \text{or} \qquad \varepsilon_R \equiv \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{\mu_0 \rho c^2}{B^2} \tag{3.28}$$

This is the *low-frequency plasma dielectric constant for transverse motions*. The qualifications are necessary because our expression for $\mathbf{j}_p$ is valid only for $\omega^2 \ll \omega_c^2$ and for **E** perpendicular to **B**. The general expression for ε, of course, is very complicated and hardly fits on one page.

Note that as $\rho \to 0$, ε_R approaches its vacuum value, unity, as it should. As $B \to \infty$, $\boldsymbol{\varepsilon}_R$ also approaches unity. This is because the polarization drift $\mathbf{v}_p$ then vanishes, and the particles do not move in response to the transverse electric field. In a usual laboratory plasma, the second term in Eq. (3.28) is large compared with unity. For instance, if $n = 10^{16}\ \text{m}^{-3}$ and $B = 0.1$ T we have (for hydrogen)

$$\frac{\mu_0 \rho c^2}{B^2} = \frac{\left(4\pi \times 10^{-7}\right)\left(10^{16}\right)\left(1.67 \times 10^{-27}\right)\left(9 \times 10^{16}\right)}{(0.1)^2} = 189$$

This means that the electric fields due to the particles in the plasma greatly alter the fields applied externally. A plasma with large ε shields out alternating fields, just as a plasma with small λ_D shields out dc fields.

Problems

3.1 Derive the uniform-plasma low-frequency dielectric constant, Eq. (3.28), by reconciling the time derivative of the equation $\nabla \cdot \mathbf{D} = \nabla \cdot (\varepsilon \mathbf{E}) = 0$ with that of the vacuum Poisson equation (3.1), with the help of equations (3.24) and (2.67).

3.2 If the ion cyclotron frequency is denoted by Ω_c and the ion plasma frequency is defined by

$$\Omega_p = \left(ne^2/\varepsilon_0 M\right)^{1/2}$$

where M is the ion mass, under what circumstances is the dielectric constant $\boldsymbol{\varepsilon}$ approximately equal to Ω_p^2/Ω_c^2?

3.3 The Fluid Equation of Motion

Maxwell's equations tell us what **E** and **B** are for a given state of the plasma. To solve the self-consistent problem, we must also have an equation giving the plasma's response to given **E** and **B**. In the fluid approximation, we consider the plasma to be composed of two or more *interpenetrating fluids*, one for each species. In the simplest case, when there is only one species of ion, we shall need two equations of motion, one for the positively charged ion fluid and one for the negatively charged electron fluid. In a partially ionized gas, we shall also need an equation for the fluid of neutral atoms. The neutral fluid will interact with the ions and electrons only through collisions. The ion and electron fluids will interact with each other even in the absence of collisions, because of the **E** and **B** fields they generate.

3.3.1 The Convective Derivative

The equation of motion for a single particle is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3.29}$$

Assume first that there are no collisions and no thermal motions. Then all the particles in a fluid element move together, and the average velocity **u** of the particles in the element is the same as the individual particle velocity **v**. The fluid equation is obtained simply by multiplying Eq. (3.29) by the density n:

$$mn\frac{d\mathbf{u}}{dt} = qn(\mathbf{E} + \mathbf{u}\times\mathbf{B}) \tag{3.30}$$

This is, however, not a convenient form to use. In Eq. (3.29), the time derivative is to be taken *at the position of the particles*. On the other hand, we wish to have an equation for fluid elements *fixed in space*, because it would be impractical to do otherwise. Consider a drop of cream in a cup of coffee as a fluid element. As the coffee is stirred, the drop distorts into a filament and finally disperses all over the cup, losing its identity. A fluid element at a fixed spot in the cup, however, retains its identity although particles continually go in and out of it.

To make the transformation to variables in a fixed frame, consider $\mathbf{G}(x, t)$ to be any property of a fluid in one-dimensional x space. The change of **G** with time *in a frame moving with the fluid* is the sum of two terms:

$$\frac{d\mathbf{G}(x,t)}{dt} = \frac{\partial\mathbf{G}}{\partial t} + \frac{\partial\mathbf{G}}{\partial x}\frac{dx}{\partial t} = \frac{\partial\mathbf{G}}{\partial t} + u_x\frac{\partial\mathbf{G}}{\partial x} \tag{3.31}$$

The first term on the right represents the change of **G** at a fixed point in space, and the second term represents the change of **G** as the observer moves with the fluid into a region in which **G** is different. In three dimensions, Eq. (3.31) generalizes to

$$\frac{d\mathbf{G}}{dt} = \frac{\partial\mathbf{G}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{G} \tag{3.32}$$

This is called the *convective derivative* and is sometimes written $D\mathbf{G}/Dt$. Note that $(\mathbf{u}\cdot\nabla)$ is a *scalar* differential operator. Since the sign of this term is sometimes a source of confusion, we give two simple examples.

Figure 3.1 shows an electric water heater in which the hot water has risen to the top and the cold water has sunk to the bottom. Let $G(x, t)$ be the temperature T; ∇G is then upward. Consider a fluid element near the edge of the tank. If the heater element is turned on, the fluid element is heated as it moves, and we have $dT/dt > 0$. If, in addition, a paddle wheel sets up a flow pattern as shown, the temperature in a *fixed* fluid element is lowered by the convection of cold water from the bottom. In this case, we have $\partial T/\partial x > 0$ and $u_x > 0$, so that $\mathbf{u}\cdot\nabla T > 0$. The temperature change in the fixed element, $\partial T/\partial t$, is given by a balance of these effects,

$$\frac{\partial T}{\partial t} = \frac{dT}{dt} - \mathbf{u}\cdot\nabla T \tag{3.33}$$

It is quite clear that $\partial T/\partial t$ can be made zero, at least for a short time.

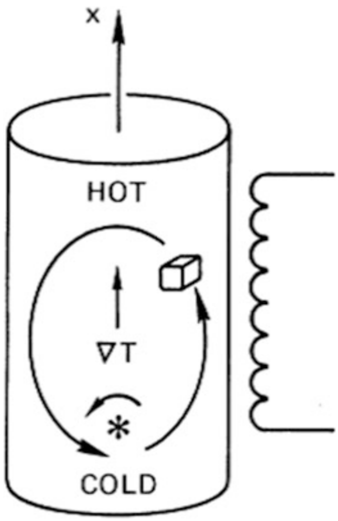

Fig. 3.1 Motion of fluid elements in a hot water heater

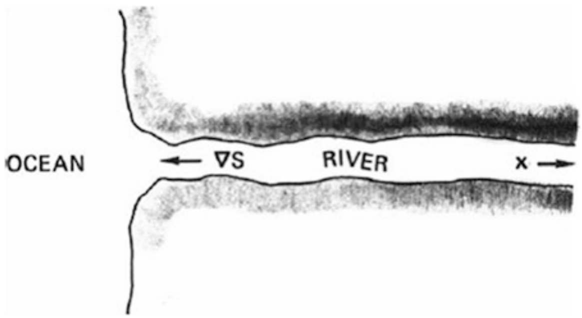

Fig. 3.2 Direction of the salinity gradient at the mouth of a river

As a second example we may take G to be the salinity S of the water near the mouth of a river (Fig. 3.2). If $\boldsymbol{x}$ is the upstream direction, there is normally a gradient of S such that $\partial S/\partial x < 0$. When the tide comes in, the entire interface between salt and fresh water moves upstream, and $u_x > 0$. Thus

$$\frac{\partial S}{\partial t} = -u_x \frac{\partial S}{\partial x} > 0 \tag{3.34}$$

meaning that the salinity increases at any given point. Of course, if it rains, the salinity decreases everywhere, and a negative term dS/dt is to be added to the middle part of Eq. (3.34).

As a final example, take G to be the density of cars near a freeway entrance at rush hour. A driver will see the density around him increasing as he approaches the crowded freeway. This is the convective term $(\mathbf{u}\cdot\nabla)G$. At the same time, the local streets may be filling with cars that enter from driveways, so that the density will increase even if the observer does not move. This is the $\partial G/\partial t$ term. The total increase seen by the observer is the sum of these effects.

In the case of a plasma, we take $\mathbf{G}$ to be the fluid velocity $\mathbf{u}$ and write Eq. (3.30) as

$$mn\left[\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=qn(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \tag{3.35}$$

where $\partial\mathbf{u}/\partial t$ is the time derivative in a fixed frame.

3.3.2 The Stress Tensor

When thermal motions are taken into account, a pressure force has to be added to the right-hand side of Eq. (3.35). This force arises from the random motion of particles in and out of a fluid element and does not appear in the equation for a single particle. Let a fluid element $\Delta x\ \Delta y\ \Delta z$ be centered at $(x_0, \frac{1}{2}\Delta y, \frac{1}{2}\Delta z)$ (Fig. 3.3). For simplicity, we shall consider only the x component of motion through the faces A and B. The number of particles per second passing through the face A with velocity v_x is

$$\Delta n_v\, v_x \Delta y \Delta z$$

where Δn_v is the number of particles per m^3 with velocity v_x:

$$\Delta n_v=\Delta v_x\iint f\left(v_x,v_y,v_z\right)dv_y dv_z$$

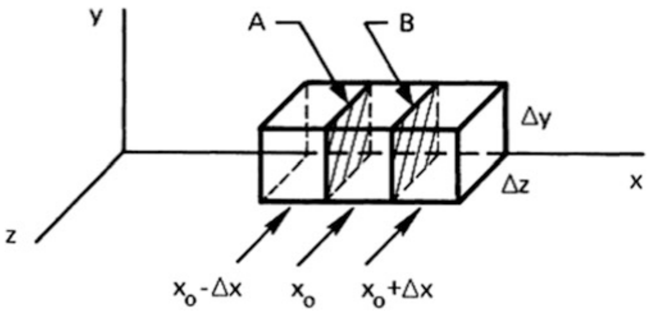

Fig. 3.3 Origin of the elements of the stress tensor

Each particle carries a momentum mv_x. The density n and temperature KT in each cube is assumed to have the value associated with the cube's center. The momentum P_{A+} carried into the element at x_0 through A is then

$$P_{A+} = \Sigma\, \Delta n_v\, mv_x^2\, \Delta y\, \Delta z = \Delta y\, \Delta z \left[m\overline{v_x^2}\, \frac{1}{2} n \right]_{x_0 - \Delta x} \tag{3.36}$$

The sum over Δn_v results in the average $\overline{v_x^2}$ over the distribution. The factor 1/2 comes from the fact that only half the particles in the cube at $x_0 - \Delta x$ are going *toward* face A. Similarly, the momentum carried out through face B is

$$P_{B+} = \Delta y\, \Delta z \left[m\overline{v_x^2}\, \frac{1}{2} n \right]_{x_0}$$

Thus the net gain in x momentum from right-moving particles is

$$\begin{aligned} P_{A+} - P_{B+} &= \Delta y\, \Delta z \tfrac{1}{2} m \left(\left[n\overline{v_x^2} \right]_{x_0 - \Delta x} - \left[n\overline{v_x^2} \right]_{x_0} \right) \\ &= \Delta y\, \Delta z \tfrac{1}{2} m (-\Delta x) \frac{\partial}{\partial x} \left(n\overline{v_x^2} \right) \end{aligned} \tag{3.37}$$

This result will be just doubled by the contribution of left-moving particles, since they carry negative x momentum and also move in the opposite direction relative to the gradient of $n\overline{v_x^2}$. The total change of momentum of the fluid element at x_0 is therefore

$$\frac{\partial}{\partial t}(nmu_x) \Delta x\, \Delta y\, \Delta z = -m \frac{\partial}{\partial x} \left(n\overline{v_x^2} \right) \Delta x\, \Delta y\, \Delta z \tag{3.38}$$

Let the velocity $\boldsymbol{v_x}$ of a particle be decomposed into two parts,

$$v_x = u_x + v_{xr} \qquad u_x = \overline{v}_x$$

where u_x is the fluid velocity and v_{xr} is the random thermal velocity. For a one-dimensional Maxwellian distribution, we have from Eq. (1.7)

$$\frac{1}{2} m\overline{v_{xr}^2} = \frac{1}{2} KT \tag{3.39}$$

Equation (3.38) now becomes

$$\frac{\partial}{\partial t}(nmu_x) = -m \frac{\partial}{\partial x} \left[n \left(\overline{u_x^2} + 2\overline{uv}_{xr} + \overline{v_{xr}^2} \right) \right] = -m \frac{\partial}{\partial x} \left[n \left(u_x^2 + \frac{KT}{m} \right) \right]$$

We can cancel two terms by partial differentiation:

$$mn\frac{\partial u_x}{\partial t} + mu_x\frac{\partial n}{\partial t} = -mu_x\frac{\partial (nu_x)}{\partial x} - mnu_x\frac{\partial u_x}{\partial x} - \frac{\partial}{\partial x}(nKT) \tag{3.40}$$

The equation of mass conservation[1]

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu_x) = 0 \tag{3.41}$$

allows us to cancel the terms nearest the equal sign in Eq. (3.40). Defining the pressure

$$\boxed{p \equiv nKT} \tag{3.42}$$

we have finally

$$mn\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial p}{\partial x} \tag{3.43}$$

This is the usual pressure-gradient force. Adding the electromagnetic forces and generalizing to three dimensions, we have the fluid equation

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla p \tag{3.44}$$

What we have derived is only a special case: the transfer of x momentum by motion in the x direction; and we have assumed that the fluid is isotropic, so that the same result holds in the y and z directions. But it is also possible to transfer y momentum by motion in the x direction, for instance. Suppose, in Fig. 3.3, that u_y is zero in the cube at $x = x_0$ but is positive on both sides. Then as particles migrate across the faces $\boldsymbol{A}$ and $\boldsymbol{B}$, they bring in more positive y momentum than they take out, and the fluid element gains momentum in the y direction. This *shear stress* cannot be represented by a scalar p but must be given by a tensor $\mathbf{P}$, the stress tensor, whose components $\boldsymbol{P}_{ij} = mn\overline{v_i v_j}$ specify both the direction of motion and the component of momentum involved. In the general case the term $-\nabla p$ is replaced by $-\nabla\cdot\mathbf{P}$.

We shall not give the stress tensor here except for the two simplest cases. When the distribution function is an isotropic Maxwellian, $\mathbf{P}$ is written

[1] If the reader has not encountered this before, it is derived in Sect. 3.3.5.

$$\mathbf{P} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \tag{3.45}$$

$\nabla \cdot \mathbf{P}$ is just ∇p. In Sect. 1.3, we noted that a plasma could have two temperatures $T_\perp$ and $T_\parallel$ in the presence of a magnetic field. In that case, there would be two pressures $p_\perp = nKT_\perp$ and $p_\parallel = nKT_\parallel$. The stress tensor is then

$$\mathbf{P} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix} \tag{3.46}$$

where the coordinate of the third row or column is the direction of **B**. This is still diagonal and shows isotropy in a plane perpendicular to **B**.

In an ordinary fluid, the off-diagonal elements of **P** are usually associated with viscosity. When particles make collisions, they come off with an average velocity in the direction of the fluid velocity **u** at the point where they made their last collision. This momentum is transferred to another fluid element upon the next collision. This tends to equalize **u** at different points, and the resulting resistance to shear flow is what we intuitively think of as viscosity. The longer the mean free path, the farther momentum is carried, and the larger is the viscosity. In a plasma there is a similar effect which occurs even in the absence of collisions. The Larmor gyration of particles (particularly ions) brings them into different parts of the plasma and tends to equalize the fluid velocities there. The Larmor radius rather than the mean free path sets the scale of this kind of collisionless viscosity. It is a finite-Larmor-radius effect which occurs in addition to collisional viscosity and is closely related to the $\mathbf{v}_E$ drift in a nonuniform **E** field (Eq. (2.58)).

3.3.3 Collisions

If there is a neutral gas, the charged fluid will exchange momentum with it through collisions. The momentum lost per collision will be proportional to the relative velocity $\mathbf{u} - \mathbf{u}_0$, where $\mathbf{u_0}$ is the velocity of the neutral fluid. If τ, the mean free time between collisions, is approximately constant, the resulting force term can be roughly written as $-mn(\mathbf{u} - \mathbf{u_0})/\tau$. The equation of motion (3.44) can be generalized to include anisotropic pressure and neutral collisions as follows:

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla \cdot \mathbf{P} - \frac{mn(\mathbf{u} - \mathbf{u}_0)}{\tau} \tag{3.47}$$

Collisions between charged particles have not been included; these will be treated in Chap. 5.

3.3.4 Comparison with Ordinary Hydrodynamics

Ordinary fluids obey the Navier–Stokes equation

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p+\rho\nu\nabla^{2}\mathbf{u} \tag{3.48}$$

This is the same as the plasma equation (3.47) except for the absence of electromagnetic forces and collisions between species (there being only one species). The viscosity term $\rho\nu\nabla^2\mathbf{u}$, where ν is the kinematic viscosity coefficient, is just the collisional part of $\nabla\cdot\mathbf{P}-\nabla p$ in the absence of magnetic fields. Equation (3.48) describes a fluid in which there are frequent collisions between particles. Equation (3.47), on the other hand, was derived without any explicit statement of the collision rate. Since the two equations are identical except for the **E** and **B** terms, can Eq. (3.47) really describe a plasma species? The answer is a guarded yes, and the reasons for this will tell us the limitations of the fluid theory.

In the derivation of Eq. (3.47), we did actually assume implicitly that there were many collisions. This assumption came in Eq. (3.39) when we took the velocity distribution to be Maxwellian. Such a distribution generally comes about as the result of frequent collisions. However, this assumption was used only to take the average of v_{xr}^2. Any other distribution with the same average would give us the same answer. The fluid theory, therefore, is not very sensitive to deviations from the Maxwellian distribution, but there are instances in which these deviations are important. Kinetic theory must then be used.

There is also an empirical observation by Irving Langmuir which helps the fluid theory. In working with the electrostatic probes which bear his name, Langmuir discovered that the electron distribution function was far more nearly Maxwellian than could be accounted for by the collision rate. This phenomenon, called *Langmuir's paradox*, has been attributed at times to high-frequency oscillations. There has been no satisfactory resolution of the paradox, but this seems to be one of the few instances in plasma physics where nature works in our favor.

Another reason the fluid model works for plasmas is that the magnetic field, when there is one, can play the role of collisions in a certain sense. When a particle is accelerated, say by an **E** field, it would continuously increase in velocity if it were allowed to free-stream. When there are frequent collisions, the particle comes to a limiting velocity proportional to **E**. The electrons in a copper wire, for instance, drift together with a velocity $\mathbf{v}=\mu\mathbf{E}$, where μ is the mobility. A magnetic field also limits free-streaming by forcing particles to gyrate in Larmor orbits. The electrons in a plasma also drift together with a velocity proportional to **E**, namely, $\mathrm{v}_E=\mathbf{E}\times\mathbf{B}/B^2$. In this sense, a collisionless plasma behaves like a collisional fluid. Of course, particles do free-stream *along* the magnetic field, and the fluid picture is not particularly suitable for motions in that direction. *For motions perpendicular to* **B**, *the fluid theory is a good approximation.*

3.3.5 Equation of Continuity

The conservation of matter requires that the total number of particles N in a volume V can change only if there is a net flux of particles across the surface S bounding that volume. Since the particle flux density is $n\mathbf{u}$, we have, by the divergence theorem,

$$\frac{\partial N}{\partial t} = \int_V \frac{\partial n}{\partial t} dV = -\oint n\mathbf{u} \cdot d\mathbf{S} = -\int_V \nabla \cdot (n\mathbf{u}) dV \tag{3.49}$$

Since this must hold for any volume V, the integrands must be equal:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \tag{3.50}$$

There is one such *equation of continuity* for each species. Any sources or sinks of particles are to be added to the right-hand side.

3.3.6 Equation of State

One more relation is needed to close the system of equations. For this, we can use the thermodynamic equation of state relating p to n:

$$p = C\rho^{\gamma} \tag{3.51}$$

where C is a constant and γ is the ratio of specific heats C_p/C_v. The term ∇p is therefore given by

$$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n} \tag{3.52}$$

For isothermal compression, we have

$$\nabla p = \nabla(nKT) = KT\nabla n$$

so that, clearly, $\gamma = 1$. For adiabatic compression, KT will also change, giving γ a value larger than one. If N is the number of degrees of freedom, γ is given by

$$\gamma = (2 + N)/N \tag{3.53}$$

The validity of the equation of state requires that heat flow be negligible; that is, that thermal conductivity be low. Again, this is more likely to be true in directions perpendicular to $\mathbf{B}$ than parallel to it. Fortunately, most basic phenomena can be described adequately by the crude assumption of Eq. (3.51).

3.3.7 *The Complete Set of Fluid Equations*

For simplicity, let the plasma have only two species: ions and electrons; extension to more species is trivial. The charge and current densities are then given by

$$\sigma = n_i q_i + n_e q_e \tag{3.54}$$
$$\mathbf{j} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e$$

Since single-particle motions will no longer be considered, we may now use **v** instead of **u** for the fluid velocity. We shall neglect collisions and viscosity. Equations (3.1)–(3.4), (3.44), (3.50), and (3.51) form the following set:

$$\varepsilon_0 \nabla \cdot \mathbf{E} = n_i q_i + n_e q_e \tag{3.55}$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{3.56}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0} \tag{3.57}$$

$$\boldsymbol{\mu}_0^{-1} \nabla \times \mathbf{B} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e + \varepsilon_0 \dot{\mathbf{E}} \tag{3.58}$$

$$m_j n_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + \left(\mathbf{v}_j \cdot \nabla \right) \mathbf{v}_j \right] = q_j n_j \left(\mathbf{E} + \mathbf{v}_j \times \mathbf{B} \right) - \nabla p_j \qquad j = i, e \tag{3.59}$$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \left(n_j \mathbf{v}_j \right) = 0 \qquad j = i, e \tag{3.60}$$

$$p_j = C_j n_j^{\gamma_j} \qquad j = i, e \tag{3.61}$$

There are 16 scalar unknowns: n_i, n_e, p_i, p_e, $\mathbf{v}_i$, $\mathbf{v}_e$, **E**, and **B**. There are apparently 18 scalar equations if we count each vector equation as three scalar equations. However, two of Maxwell's equations are superfluous, since Eqs. (3.55) and (3.57) can be recovered from the divergences of Eqs. (3.58) and (3.56) (Problem 3.3). The simultaneous solution of this set of 16 equations in 16 unknowns gives a self-consistent set of fields and motions in the fluid approximation.

3.4 Fluid Drifts Perpendicular to B

Since a fluid element is composed of many individual particles, one would expect the fluid to have drifts perpendicular to **B** if the individual guiding centers have such drifts. However, since the ∇p term appears only in the fluid equations, there is a drift associated with it which the fluid elements have but the particles do not have. For each species, we have an equation of motion

$$mn\left[\underset{\textcircled{1}}{\frac{\partial \mathbf{v}}{\partial t}}+\underset{\textcircled{2}}{(\mathbf{v}\cdot\nabla)\mathbf{v}}\right]=qn(\mathbf{E}+\underset{\textcircled{3}}{\mathbf{v}\times\mathbf{B}})-\nabla p \tag{3.62}$$

Consider the ratio of term ① to term ③:

$$\frac{\textcircled{1}}{\textcircled{3}}\approx\left|\frac{mni\omega v_\perp}{qnv_\perp B}\right|\approx\frac{\omega}{\omega_c}$$

Here we have taken $\partial/\partial t=i\omega$ and are concerned only with $\mathbf{v}_\perp$. For drifts slow compared with the time scale of ω_c, we may neglect term ①. We shall also neglect the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term and show *a posteriori* that this is all right. Let **E** and **B** be uniform, but let n and p have a gradient. This is the usual situation in a magnetically confined plasma column (Fig. 3.4). Taking the cross product of Eq. (3.62) with **B**, we have (neglecting the left-hand side)

$$\begin{aligned}0&=qn[\mathbf{E}\times\mathbf{B}+(\mathbf{v}_\perp\times\mathbf{B})\times\mathbf{B}]-\nabla p\times\mathbf{B}\\&=qn[\mathbf{E}\times\mathbf{B}+\cancel{\mathbf{B}(\mathbf{v}_\perp\cdot\mathbf{B})}-\mathbf{v}_\perp B^2]-\nabla p\times\mathbf{B}\end{aligned}$$

Therefore,

$$\mathbf{v}_\perp=\frac{\mathbf{E}\times\mathbf{B}}{B^2}-\frac{\nabla p\times\mathbf{B}}{qnB^2}\equiv\mathbf{v}_E+\mathbf{v}_D \tag{3.63}$$

where

$$\boxed{\mathbf{v}_E\equiv\frac{\mathbf{E}\times\mathbf{B}}{B^2}}\qquad \mathbf{E}\times\mathbf{B}\quad\text{drift} \tag{3.64}$$

$$\boxed{\mathbf{v}_D\equiv-\frac{\nabla p\times\mathbf{B}}{qnB^2}}\qquad \text{Diamagnetic drift} \tag{3.65}$$

The drift $\mathbf{v}_\mathbf{E}$ is the same as for guiding centers, but there is now a new drift $\mathbf{v}_D$, called the diamagnetic drift. Since $\mathbf{v}_D$ is perpendicular to the direction of the gradient, our neglect of $(\mathbf{v}\cdot\nabla)\mathbf{v}$ is justified if $\mathbf{E}=0$. If $\mathbf{E}=-\nabla\phi\neq 0$, $(\mathbf{v}\cdot\nabla)\mathbf{v}$ is still zero if $\nabla\phi$ and ∇p are in the same direction; otherwise, there could be a more complicated solution involving $(\mathbf{v}\cdot\nabla)\mathbf{v}$.

With the help of Eq. (3.52), we can write the diamagnetic drift as

$$\mathbf{v}_D=\pm\frac{\gamma KT}{eB}\frac{\hat{\mathbf{z}}\times\nabla n}{n} \tag{3.66}$$

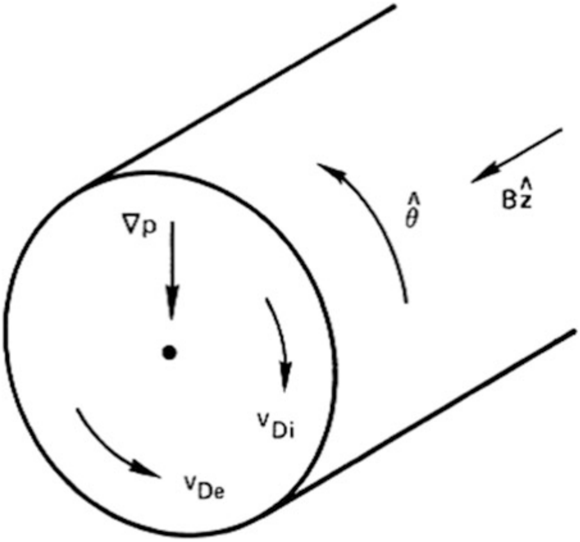

Fig. 3.4 Diamagnetic drifts in a cylindrical plasma

In particular, for an isothermal plasma in the geometry of Fig. 3.4, in which $\nabla n = n'\hat{\mathbf{r}}$, we have the following formulas familiar to experimentalists who have worked with Q-machines[2]:

$$\mathbf{v}_{Di} = \frac{KT_i}{eB}\frac{n'}{n}\hat{\boldsymbol{\theta}} \qquad \left(n' \equiv \frac{\partial n}{\partial r} < 0\right) \tag{3.67}$$

$$\mathbf{v}_{De} = -\frac{KT_e}{eB}\frac{n'}{n}\hat{\boldsymbol{\theta}}$$

The magnitude of $\mathbf{v}_D$ is easily computed from the formula

$$\boxed{v_D = \frac{KT(eV)}{B(T)}\frac{1}{\Lambda}\frac{\mathrm{m}}{\sec}} \tag{3.68}$$

where Λ is the density scale length $|n'/n|$ in m.

The physical reason for this drift can be seen from Fig. 3.5. Here we have drawn the orbits of ions gyrating in a magnetic field. There is a density gradient toward the left, as indicated by the density of orbits. Through any fixed volume element there are more ions moving downward than upward, since the downward-moving ions come from a region of higher density. There is, therefore, a fluid drift perpendicular to ∇n and $\mathbf{B}$, *even though the guiding centers are stationary*. The diamagnetic drift reverses sign with q because the direction of gyration reverses. The magnitude of $\mathbf{v}_D$

[2] A $\boldsymbol{Q}$-machine produces a quiescent plasma by thermal ionization of Cs or K atoms impinging on hot tungsten plates. Diamagnetic drifts were first measured in $\boldsymbol{Q}$-machines.

Fig. 3.5 Origin of the diamagnetic drift

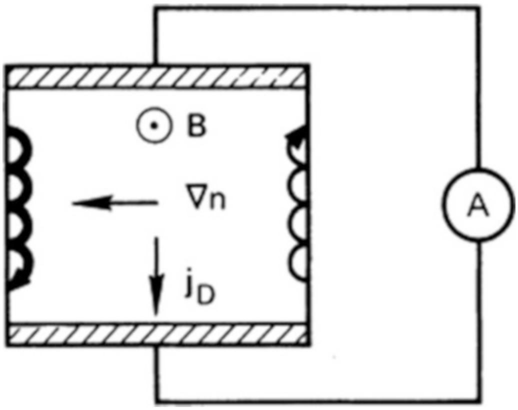

Fig. 3.6 Particle drifts in a bounded plasma, illustrating the relation to fluid drifts

does not depend on mass because the $m^{-1/2}$ dependence of the velocity is cancelled by the $m^{1/2}$ dependence of the Larmor radius—less of the density gradient is sampled during a gyration if the mass is small.

Since ions and electrons drift in opposite directions, there is a diamagnetic current. For $\gamma = Z = 1$, this is given by

$$\mathbf{j}_D = ne(\mathbf{v}_{Di} - \mathbf{v}_{De}) = (KT_i + KT_e)\frac{\mathbf{B} \times \nabla n}{B^2} \tag{3.69}$$

In the particle picture, one would not expect to measure a current if the guiding centers do not drift. In the fluid picture, the current $\mathbf{j}_D$ flows wherever there is a pressure gradient. These two viewpoints can be reconciled if one considers that all experiments must be carried out in a finite-sized plasma. Suppose the plasma were in a rigid box (Fig. 3.6). If one were to calculate the current from the single-particle picture, one would have to take into account the particles at the edges which have

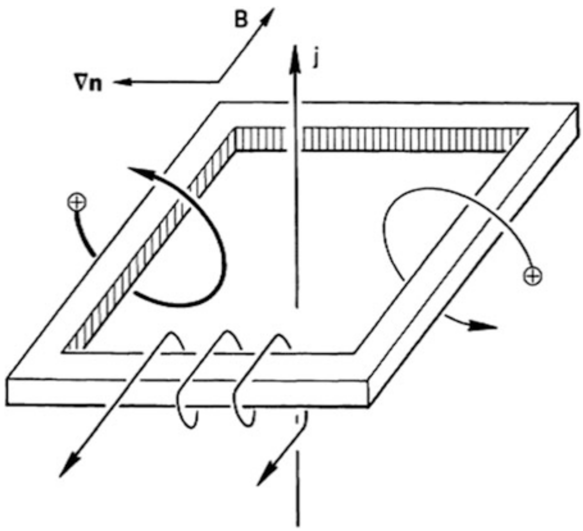

Fig. 3.7 Measuring the diamagnetic current in an inhomogeneous plasma

cycloidal paths. Since there are more particles on the left than on the right, there is a net current downward, in agreement with the fluid picture.

The reader may not be satisfied with this explanation because it was necessary to specify reflecting walls. If the walls were absorbing or if they were removed, one would find that electric fields would develop because more of one species—the one with larger Larmor radius—would collected than the other. Then the guiding centers would drift, and the simplicity of the model would be lost. Alternatively, one could imagine trying to measure the diamagnetic current with a current probe (Fig. 3.7). This is just a transformer with a core of magnetic material. The primary winding is the plasma current threading the core, and the secondary is a multiturn winding all around the core. Let the whole thing be infinitesimally thin, so it does not intercept any particles. It is clear from Fig. 3.7 that a net upward current would be measured, there being higher density on the left than on the right, so that the diamagnetic current is a real current. From this example, one can see that it can be quite tricky to work with the single-particle picture. The fluid theory usually gives the right results when applied straightforwardly, even though it contains "fictitious" drifts like the diamagnetic drift.

What about the grad-B and curvature drifts which appeared in the single-particle picture? The curvature drift also exists in the fluid picture, since the centrifugal force is felt by all the particles in a fluid element as they move around a bend in the magnetic field. A term $\overline{F}_{cf} = \overline{nmv_{\|}^2}/R_c = nKT_{\|}/R_c$ has to be added to the right-hand side of the fluid equation of motion. This is equivalent to a gravitational force Mng, with $\boldsymbol{g} = KT_{\|}/MR_c$, and leads to a drift $\mathbf{v}_g = (m/q)(\mathbf{g} \times \mathbf{B})/B^2$, as in the particle picture (Eq. (2.18)).

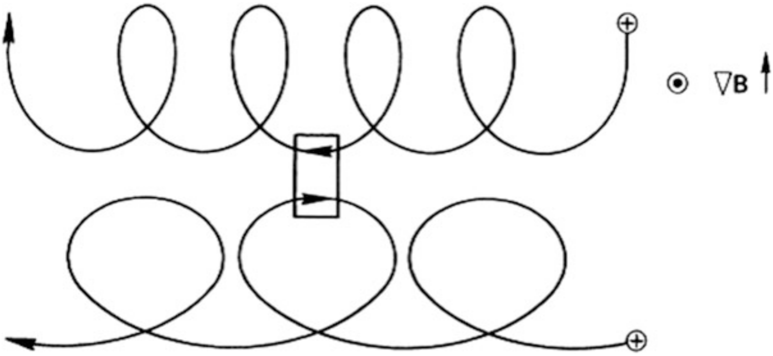

Fig. 3.8 In a nonuniform B field the guiding centers drift but the fluid elements do not

The grad-B drift, however, does not exist for fluids. It can be shown on thermodynamic grounds that a magnetic field does not affect a Maxwellian distribution. This is because the Lorentz force is perpendicular to **v** and cannot change the energy of any particle. The most probable distribution $f(\mathbf{v})$ in the absence of **B** is also the most probable distribution in the presence of **B**. If $f(\mathbf{v})$ remains Maxwellian in a nonuniform **B** field, and there is no density gradient, then the net momentum carried into any fixed fluid element is zero. There is no fluid drift even though the individual guiding centers have drifts; the particle drifts in any fixed fluid element cancel out. To see this pictorially, consider the orbits of two particles moving through a fluid element in a nonuniform **B** field (Fig. 3.8). Since there is no **E** field, the Larmor radius changes only because of the gradient in **B**; there is no acceleration, and the particle energy remains constant during the motion. If the two particles have the same energy, they will have the same velocity and Larmor radius while inside the fluid element. There is thus a perfect cancellation between particle pairs when their velocities are added to give the fluid velocity.

When there is a nonuniform **E** field, it is not easy to reconcile the fluid and particle pictures. Then the finite-Larmor-radius effect of Sect. 2.4 causes both a guiding center drift and a fluid drift, but these are not the same; in fact, they have opposite signs! The particle drift was calculated in Chap. 2, and the fluid drift can be calculated from the off-diagonal elements of **P**. It is extremely difficult to explain how the finite-Larmor-radius effects differ. A simple picture like Fig. 3.6 will not work because one has to take into account subtle points like the following: In the presence of a density gradient, the density of guiding centers is not the same as the density of particles!

Problems

3.3 Show that Eqs. (3.55) and (3.57) are redundant in the set of Maxwell's equations.

3.4 Show that the expression for $\mathbf{j}_D$ on the right-hand side of Eq. (3.69) has the dimensions of a current density.

3.5 Show that if the current calculated from the particle picture (Fig. 3.6) agrees with that calculated from the diamagnetic drift for one width of the box, then it will agree for all widths.

3.6 An isothermal plasma is confined between the planes $x = \pm a$ in a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. The density distribution is

$$n = n_0\left(1 - x^2/a^2\right)$$

(a) Derive an expression for the electron diamagnetic drift velocity $\mathbf{v}_{De}$ as a function of x.
(b) Draw a diagram showing the density profile and the direction of $\mathbf{v}_{De}$ on both sides of the midplane if $\mathbf{B}$ is out of the paper.
(c) Evaluate v_{De} at $x = a/2$ if $B = 0.2$ T, $KT_e = 2$ eV, and $a = 4$ cm.

3.7 A cylindrically symmetric plasma column in a uniform $\mathbf{B}$ field has

$$n(r) = n_0\exp\left(-r^2/r_0^2\right) \quad \text{and} \quad n_i = n_e = n_0\exp(e\phi/KT_e).$$

(The latter is the Boltzmann relation, Eq. (3.73).)
(a) Show that $\mathbf{v}_E$ and $\mathbf{v}_{De}$ are equal and opposite.
(b) Show that the plasma rotates as a solid body.
(c) In the frame which rotates with velocity $\mathbf{v}_E$, some plasma waves (drift waves) propagate with a phase velocity $\mathbf{v}_\phi = 0.5\mathbf{v}_{De}$. What is $\mathbf{v}_\phi$ in the lab frame? On a diagram of the $r - \theta$ plane, draw arrows indicating the relative magnitudes and directions of $\mathbf{v}_E$, $\mathbf{v}_{De}$, and $\mathbf{v}_\phi$ in the lab frame.

3.8 (a) For the plasma of Problem 3.7, find the diamagnetic current density j_D as a function of radius.
(b) Evaluate j_D in A/m^2 for $B = 0.4$ T, $n_0 = 10^{16}$ m^{-3}, $KT_e = KT_i = 0.25$ eV, $r = r_0 = 1$ cm.
(c) In the lab frame, is this current carried by ions or by electrons or by both?

3.9 In the preceding problem, by how much does the diamagnetic current reduce B on the axis? Hint: You may use Ampere's circuital law over an appropriate path.

3.10 In 2013, the Voyager 1 spacecraft left the heliosphere, the region dominated by solar winds, and entered outer space. The plasma frequency jumped from 2.2 to 2.6 kHz. What was the change in plasma density?

3.5 Fluid Drifts Parallel to B

The z component of the fluid equation of motion is

$$mn\left[\frac{\partial v_z}{\partial t} + (\mathbf{v}\cdot\nabla)v_z\right] = qnE_z - \frac{\partial p}{\partial z} \tag{3.70}$$

The convective term can often be neglected because it is much smaller than the $\partial v_z/\partial t$ term. We shall avoid complicated arguments here and simply consider cases in which v_z is spatially uniform. Using Eq. (3.52), we have

$$\frac{\partial v_z}{\partial t} = \frac{q}{m}E_z - \frac{\gamma KT}{mn}\frac{\partial n}{\partial z} \tag{3.71}$$

This shows that the fluid is accelerated along **B** under the combined electrostatic and pressure gradient forces. A particularly important result is obtained by applying Eq. (3.71) to massless electrons. Taking the limit $m \to 0$ and specifying $q = -e$ and $\mathbf{E} = -\nabla\phi$, we have[3]

$$qE_z = e\frac{\partial \phi}{\partial z} = \frac{\gamma KT_e}{n}\frac{\partial n}{\partial z} \tag{3.72}$$

Electrons are so mobile that their heat conductivity is almost infinite. We may then assume isothermal electrons and take $\gamma = 1$. Integrating, we have

$$e\phi = KT_e \ln n + C$$

or

$$\boxed{n = n_0 \exp\left(e\phi/KT_e\right)} \tag{3.73}$$

This is just the *Boltzmann relation* for electrons.

What this means physically is that electrons, being light, are very mobile and would be accelerated to high energies very quickly if there were a net force on them. Since electrons cannot leave a region *en masse* without leaving behind a large ion charge, the electrostatic and pressure gradient forces on the electrons must be closely in balance. This condition leads to the Boltzmann relation. Note that Eq. (3.73) *applies to each line of force separately*. Different lines of force may be charged to different potentials arbitrarily unless a mechanism is provided for the electrons to move across **B**. The conductors on which lines of force terminate can provide such a mechanism, and the experimentalist has to take these end effects into account carefully.

Figure 3.9 shows graphically what occurs when there is a local density clump in the plasma. Let the density gradient be toward the center of the diagram, and suppose KT is constant. There is then a pressure gradient toward the center. Since the plasma is quasineutral, the gradient exists for both the electron and ion fluids.

[3] Why can't $v_z \to \infty$ keeping mv_z constant? Consider the energy!

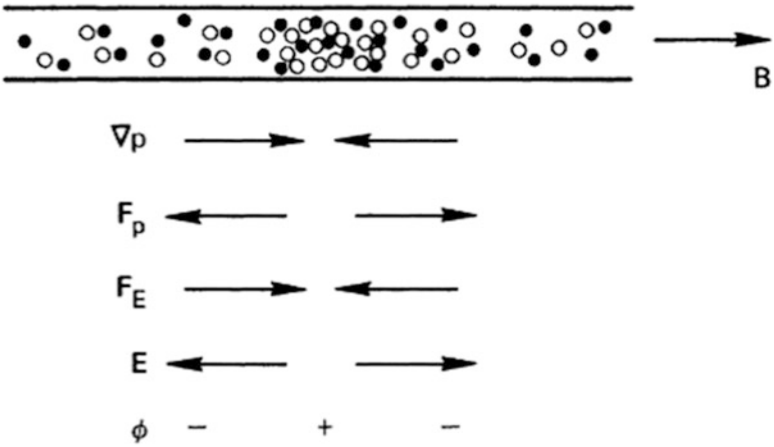

Fig. 3.9 Physical reason for the Boltzmann relation between density and potential

Consider the pressure gradient force $\mathbf{F}_p$ on the electron fluid. It drives the mobile electrons away from the center, leaving the ions behind. The resulting positive charge generates a field **E** whose force $\mathbf{F}_E$ on the electrons opposes $\mathbf{F}_p$. Only when $\mathbf{F}_E$ is equal and opposite to $\mathbf{F}_p$ is a steady state achieved. If **B** is constant, **E** is an electrostatic field $\mathrm{E} = -\nabla\phi$, and ϕ must be large at the center, where n is large. This is just what Eq. (3.73) tells us. The deviation from strict neutrality adjusts itself so that there is just enough charge to set up the **E** field required to balance the forces on the electrons.

3.6 The Plasma Approximation

The previous example reveals an important characteristic of plasmas that has wide application. We are used to solving for **E** from Poisson's equation when we are given the charge density $\boldsymbol{\sigma}$. In a plasma, the opposite procedure is generally used. **E** is found from the equations of motion, and Poisson's equation is used only to find $\boldsymbol{\sigma}$. The reason is that a plasma has an overriding tendency to remain neutral. If the ions move, the electrons will follow. **E** must adjust itself so that the orbits of the electrons and ions preserve neutrality. The charge density is of secondary importance; it will adjust itself so that Poisson's equation is satisfied. This is true, of course, only for low-frequency motions in which the electron inertia is not a factor.

In a plasma, it is usually possible to assume $n_i = n_e$ and $\nabla \cdot \mathrm{E} \neq 0$ at the same time. We shall call this the *plasma approximation*. It is a fundamental trait of plasmas, one which is difficult for the novice to understand. *Do not use Poisson's equation to obtain* **E** *unless it is unavoidable*! In the set of fluid equations (3.55)–(3.61), we may now eliminate Poisson's equation and also eliminate one of the unknowns by setting $n_i = n_e = n$.

The *plasma approximation* is almost the same as the condition of quasineutrality discussed earlier but has a more exact meaning. Whereas quasineutrality refers to a general tendency for a plasma to be neutral in its state of rest, the plasma approximation is a mathematical shortcut that one can use even for wave motions. As long as these motions are slow enough that both ions and electrons have time to move, it is a good approximation to replace Poisson's equation by the equation $n_i = n_e$. Of course, if only one species can move and the other cannot follow, such as in high-frequency electron waves, then the plasma approximation is not valid, and **E** must be found from Maxwell's equations rather than from the ion and electron equations of motion. We shall return to the question of the validity of the plasma approximation when we come to the theory of ion waves. At that time, it will become clear why we had to use Poisson's equation in the derivation of Debye shielding.