11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures

Abstract

In this chapter, we describe some approximate methods for estimating the forces in indeterminate structures. We start with multi-span beams subjected to gravity loading. Next, we treat rigid frame structures under gravity loading. Then, we consider rigid frame structures under lateral loading. For this case, we distinguish between short and tall buildings. For short buildings, we first describe the portal method, an empirical procedure, for estimating the shear forces in the columns, and then present an approximate stiffness approach which is more exact but less convenient to apply. For tall buildings, we model them as beams and use beam theory to estimate the forces in the columns. With all the approximate methods, our goal is to use simple hand calculation-based methods to estimate the forces which are needed for preliminary design and also for checking computer-based analysis methods.

11.1 Introduction

The internal forces in a statically indeterminate structure depend on the member cross-sectional properties. We demonstrated this dependency with the examples presented in the previous two chapters. However, in order to design a structure, one needs the internal forces. Therefore, when starting the design process, it is necessary to estimate a sufficient number of force quantities so that the structure is reduced to a statically determinate structure for which the distribution of internal forces is independent of the material properties. For bending type structures, such as multi-span beams and frames, the approximations are usually introduced by assuming moment releases at certain locations. The choice of the release locations is based on an understanding of the behavior of the structure for the particular loading under consideration. For indeterminate trusses, we assume the magnitude of certain forces. A typical case for a truss would be when there are two diagonals in a bay. We usually assume the transverse shear is divided equally between the two diagonals.

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11.2 Multi-span Beams: Gravity Loading

11.2.1 Basic Data-Moment Diagrams

Figures 11.1, 11.2, and 11.3 show moment diagrams due to a uniform distributed loading for a range of beam geometries and support conditions. These results are presented in Chaps. 9 and 10. They provide the basis for assuming the location of moment releases (points of zero moment) for different combinations of span lengths and loading distributions. We utilize this information to develop various strategies for generating approximate solutions for multi-span beams.

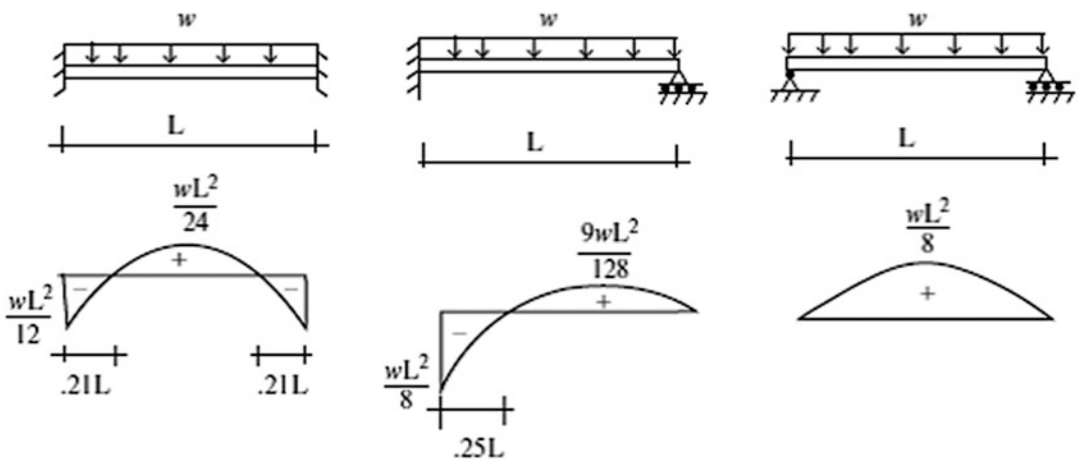


Fig. 11.1 Moment diagrams for single-span beams

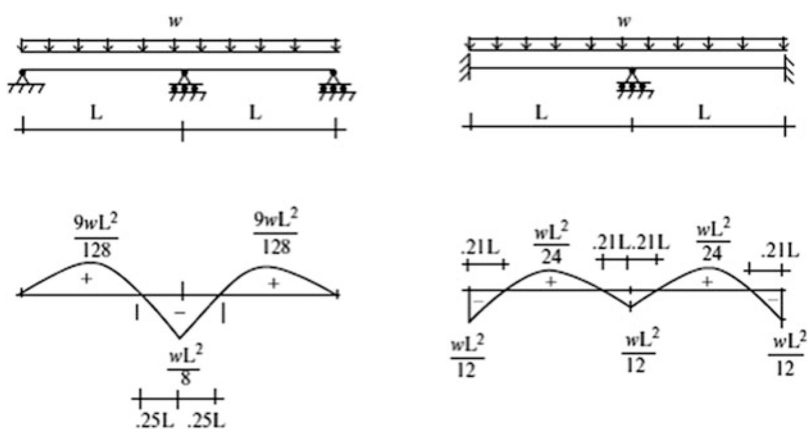


Fig. 11.2 Moment diagrams for two-span beams

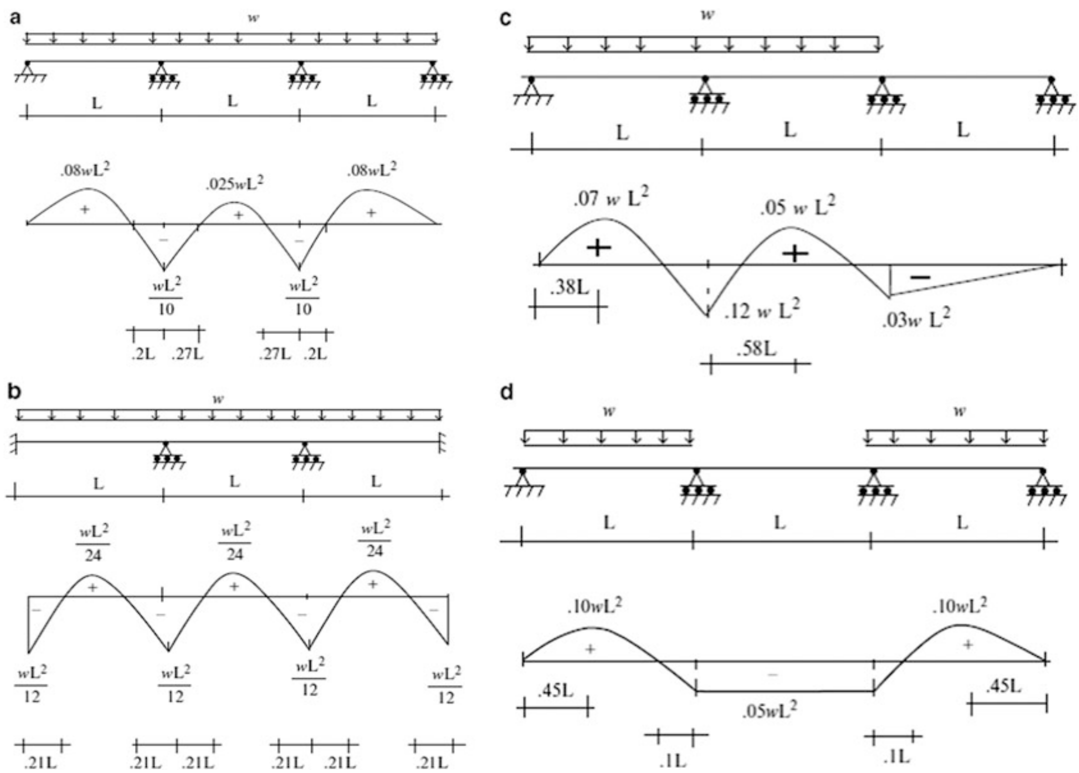
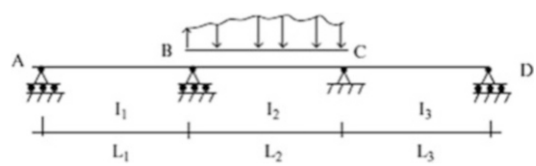


Fig. 11.3 Moment diagrams for three-span beams. (**a**) Simply supported. (**b**) Fixed at each end. (**c**) Partial loading. (**d**) Partial loading symmetrical

Fig. 11.4 Multi-span beam



11.2.2 Quantitative Reasoning Based on Relative Stiffness

Consider the multi-span beam shown in Fig. 11.4. Our objective is to estimate the peak positive and negative moments in span BC. As a first step, we estimate the end moments for this span using the member distribution factors which are related to the relative stiffness factors for the members. We consider node B. The distribution factors for members BA and BC are as follows (see Sect. 10.6):

$$
\begin{aligned}
\mathrm{DF}_{\mathrm{BA}} &= \frac{I_1/L_1}{((I_1/L_1)+(I_2/L_2))} \\
\mathrm{DF}_{\mathrm{BC}} &= \frac{I_2/L_2}{((I_1/L_1)+(I_2/L_2))}
\end{aligned}
\tag{11.1}
$$

Note that when I is constant for all spans, the relative stiffness parameters reduce to the inverse of the span length. Given the initial unbalanced moment at B, we distribute it according to

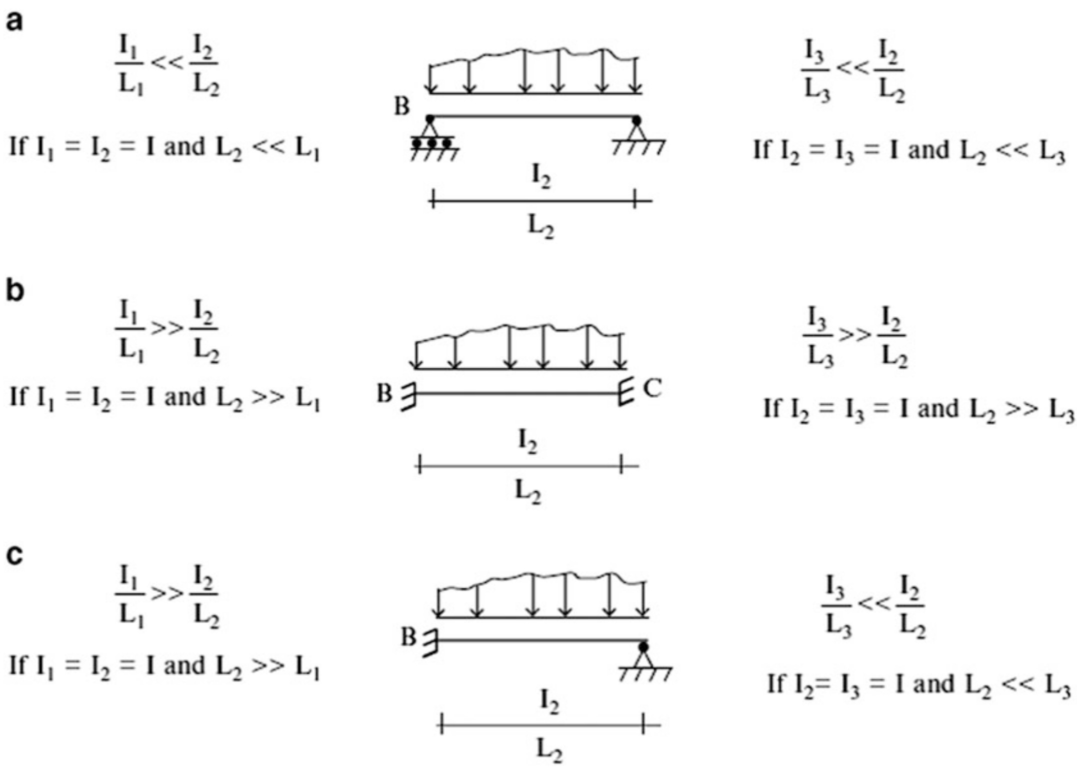


Fig. 11.5 Summary of approximate models for extreme values of L_1/L_2 and L_3/L_2. (**a**) Hinged model. (**b**) Clamped end model. (**c**) Clamped/hinged model

$$\begin{aligned} \Delta M_{\mathrm{BA}} &= -\mathrm{DF}_{\mathrm{BA}}\left(\mathrm{FEM}\big|_{\mathrm{B}}\right) \\ \Delta M_{\mathrm{BC}} &= -\mathrm{DF}_{\mathrm{BC}}\left(\mathrm{FEM}\big|_{\mathrm{B}}\right) \end{aligned} \tag{11.2}$$

We consider no carry-over movement to the other ends.

If $\frac{I_1}{L_1}$ is small in comparison to $\frac{I_2}{L_2}$, then $\mathrm{DF_{BA}}$ will be small in comparison to $\mathrm{DF_{BC}}$. It follows that only a small portion of the unbalanced nodal moment at node B will be distributed to member BA. The opposite case is where I_1/L_1 is large in comparison to I_2/L_2. Now $\mathrm{DF_{BC}}$ is small vs. $\mathrm{DF_{BA}}$. Essentially all of the unbalanced nodal moment is distributed to member BA. The final end moment in member BC is close to its initial value (the initial fixed end moment) since there is relatively little distribution.

When I is constant for all the spans, the relative stiffness parameters reduce to the inverse of the span lengths. In this case, one compares the ratio of adjacent span lengths. The limiting cases for extreme values of these ratios are listed in Fig. 11.5.

11.3 Multistory Rigid Frames: Gravity Loading

Gravity type loading is usually the dominant loading for multistory frames. It consists of both dead and live loading. Consider the frame shown in Fig. 11.6. We suppose the loading is a uniform gravity load, w. Our objective here is to determine the positive and negative moments in beam AB.

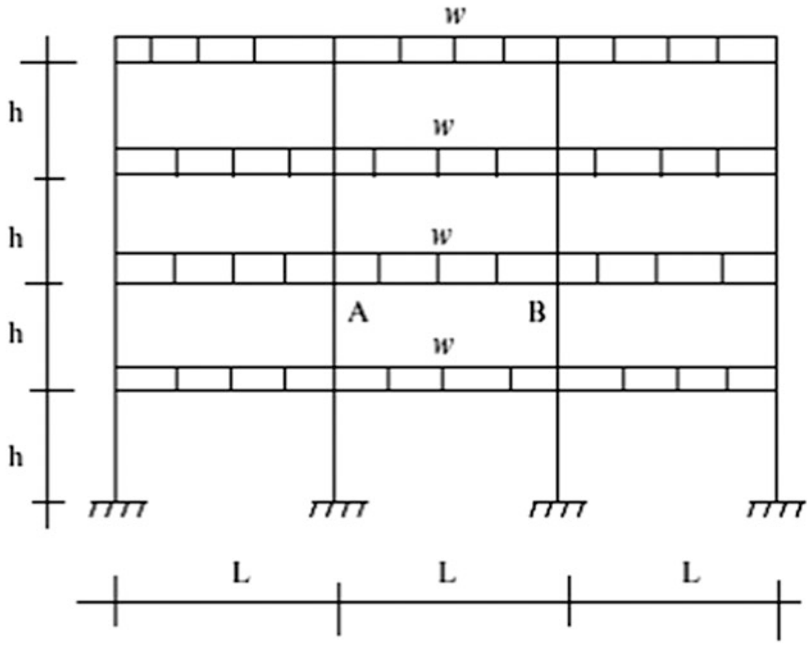


Fig. 11.6 Multistory frame—gravity loading

One can estimate moments at the ends and at the center by assuming moment releases in the beams. Assuming moment releases at 0.1 L leads to

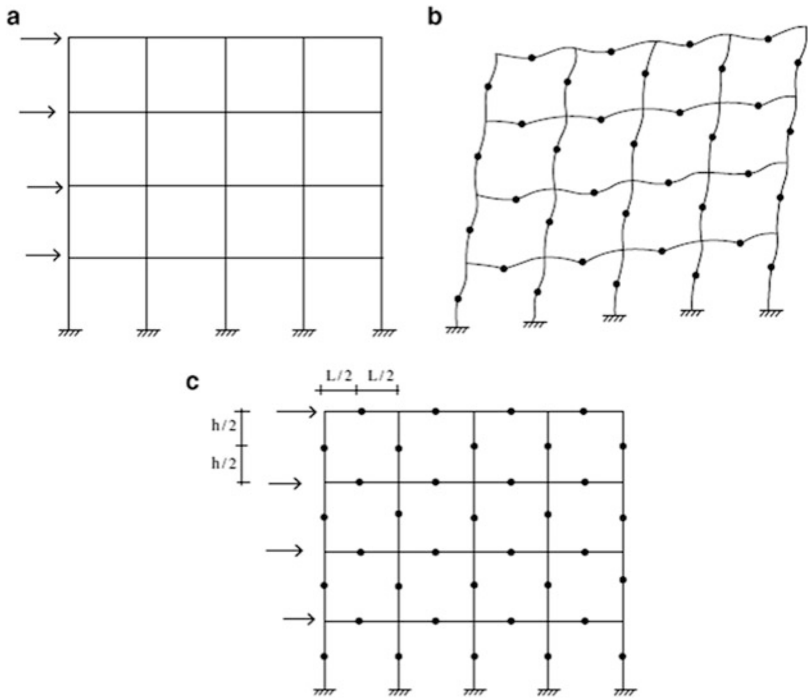
$$M^{+}_{\text{center}} = \frac{w(0.8L)^2}{8} = 0.08wL^2$$
$$M_{\text{A}} = M_{\text{B}} = w\frac{(0.1L)^2}{2} + w(0.4L)(0.1L) = 0.045wL^2$$

11.4 Multistory Rigid Frames: Lateral Loading

Consider the rigid frame shown in Fig. 11.7a. Under a lateral loading, the frame develops inflection points (points where the bending moment is equal to zero) in the columns and beams. Most of the approximate methods published in the literature are based on the assumption that the inflection points occur at mid-height of the columns and mid-span of the beams, as indicated in Fig. 11.7c. This assumption, coupled with an assumption concerning how the column axial and shear forces are distributed within a story, is sufficient to allow us to compute estimates for the end moments, the axial forces, and the shear forces in the columns.

In what follows, we present two different approaches for estimating the forces in the columns. The first approach (11.4.1–11.4.3) estimates the column shears in a story, and is applicable mainly for *low-rise rigid frames*. The second approach (11.5) estimates the axial forces in the columns. Because of the nature of the underlying assumptions, the latter procedure is appropriate only for *tall, narrow rigid frames*. Both procedures are derived using the idealized model of the structure shown in Fig. 11.7c, i.e., with inflection points at mid-height of the columns and mid-span of the beams.

Fig. 11.7 Multistory rigid frame. (**a**) Initial position. (**b**) Deflected Position. (**c**) Assumed location of inflection points



11.4.1 Portal Method

The portal method is an empirical procedure for estimating the forces in low-rise rigid frames subjected to lateral loads. In addition to assuming inflection points in the columns and beams, *the shear in the exterior columns is assumed to be one-half the shear in the interior columns*, which is taken to be equal for all the interior columns. We use this method to generate the first estimate of the member forces. Of particular interest are the end moments in the columns.

Example 11.1 Application of the Portal Method

Given: The rigid frame shown in Fig. E11.1a.

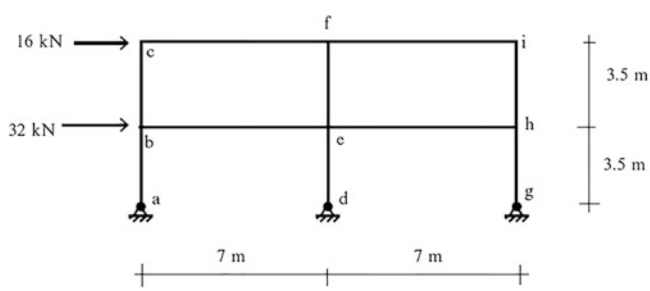


Fig. E11.1a

Determine: The axial force, shear force, and bending moment in the beams and columns using the Portal method.

Solution: The portal method assumes the exterior column shear, V_E, is equal to one-half the interior column shear force V_I, which is taken to be equal for all the interior columns.

$$V_E = \frac{1}{2} V_I$$

Summing the column shear forces for this structure leads to an expression for the total story shear.

$$\begin{aligned} V_T &= 2V_E + V_I \\ &= 2V_I \end{aligned}$$

Then,

$$V_I = \frac{1}{2} V_T$$

$$V_E = \frac{1}{2} V_I = \frac{1}{4} V_T$$

We range over the stories and generate the column shear for each story. The calculations are summarized below.

Story	V_T (kN)	V_I (kN)	V_E (kN)
Top	16	8	4
Bottom	48	24	12

Given the column shear forces, one can determine the column end moments using the assumption that there are inflection points at certain locations in the columns. For this structure, since the base is pinned, the inflection points for the first story are at the base. The inflection points for the second story are taken at mid-height. The free body diagrams for the various segments are shown below along with the final results. Once the column end moments are known, we can determine the end moments and shear forces in the beams and lastly, the axial forces in the columns using equilibrium equations (Figs. E11.1b, E11.1c, E11.1d, E11.1e, E11.1f, E11.1g, E11.1h).

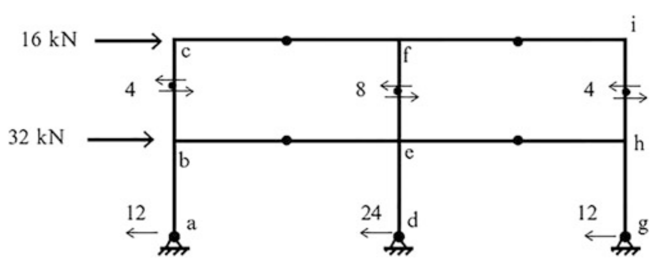


Fig. E11.1b Shear distribution for the columns (kN)

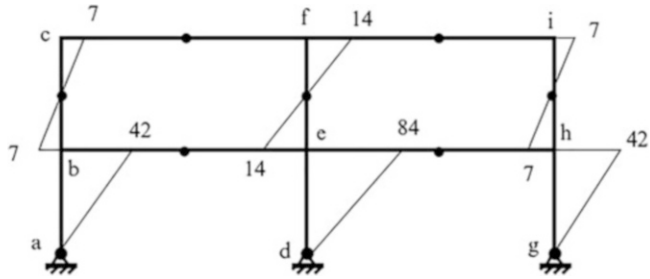


Fig. E11.1c Bending moment distribution for the columns (kN m)

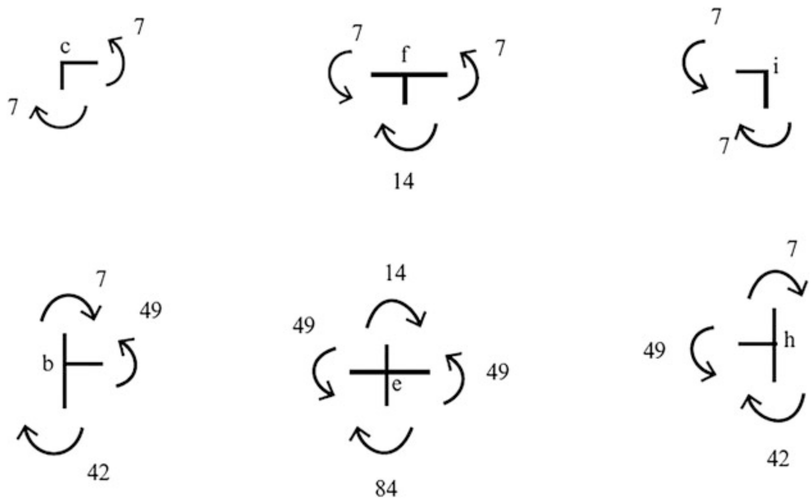


Fig. E11.1d Moments at the joints (kN m)

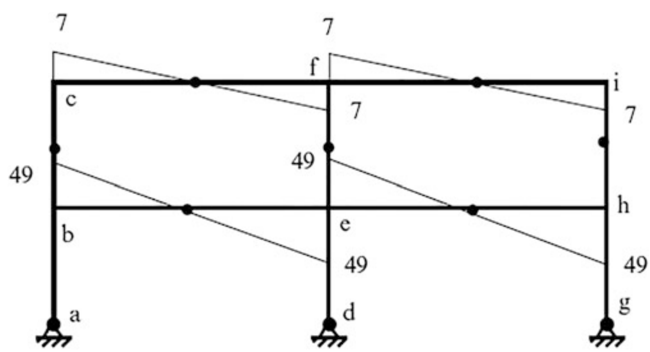


Fig. E11.1e Bending moment distribution for the beams (kN m)

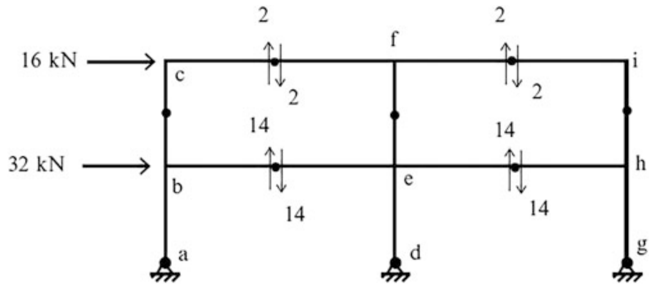


Fig. E11.1f Shear distribution for the beams (kN)

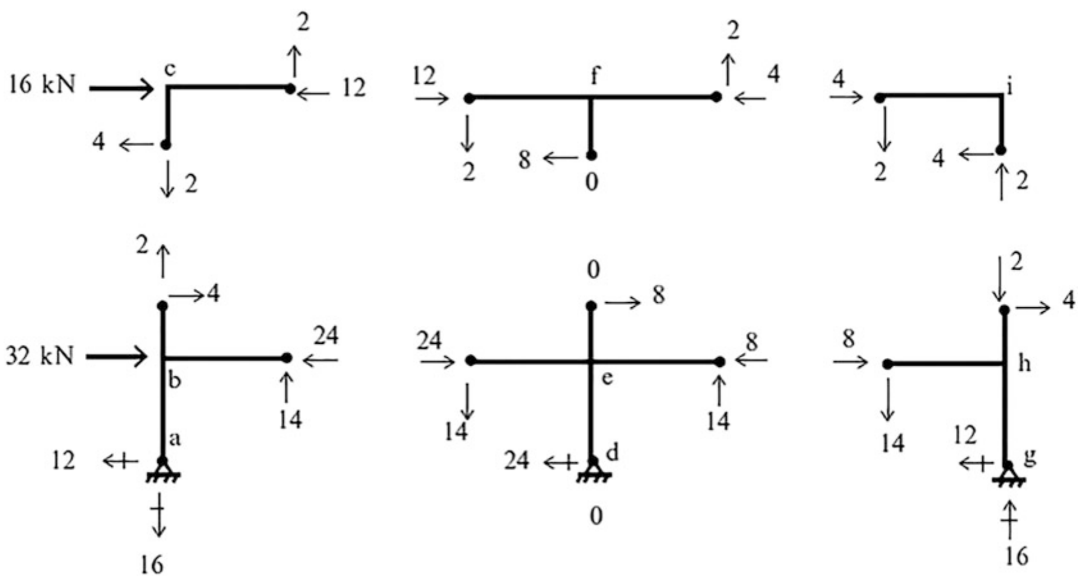


Fig. E11.1g Axial and shear forces (kN)

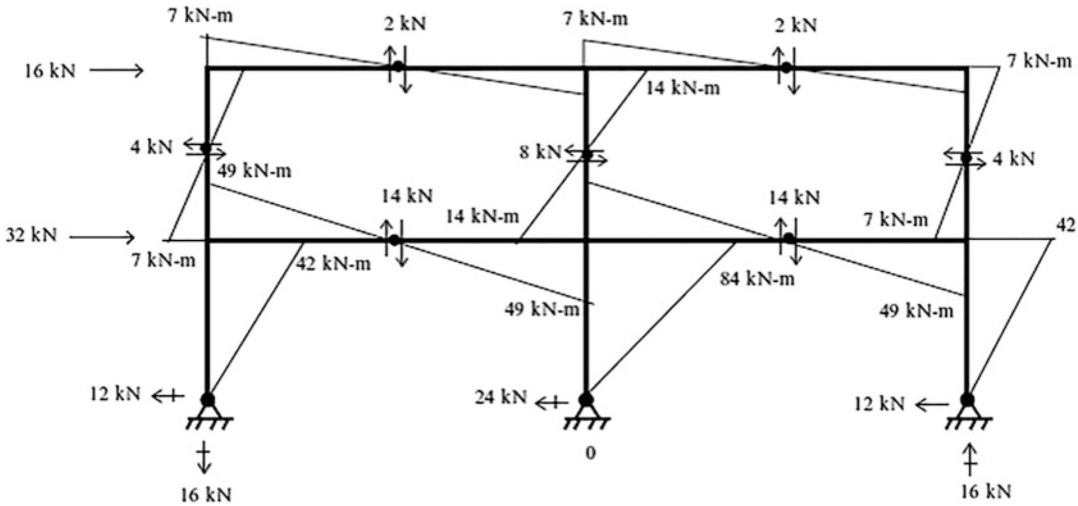


Fig. E11.1h Reactions, shear forces, and moment distribution

Example 11.2 Application of the Portal Method

Given: The rigid frame shown in Fig. E11.2a.

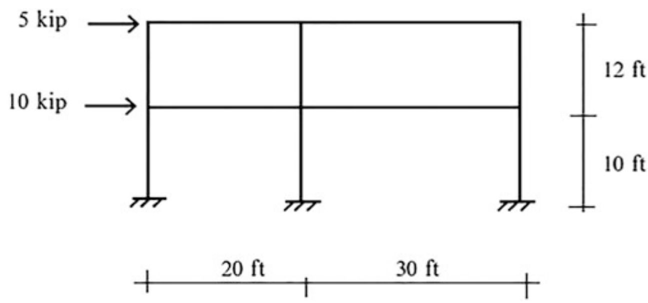


Fig. E11.2a

Determine: The reactions and the bending moments in the beams and columns using the Portal method.

Solution: The Portal method assumes the exterior column shear, V_E, is equal to one-half the interior column shear force V_I. The calculations are summarized in the table below. Note that, since the base is fixed, we assume inflection points at mid-height for the first story (Figs. E11.2b, E11.2c, E11.2d, E11.2e, E11.2f, E11.2g, E11.2h).

Story	V_T (kip)	V_I (kip)	V_E (kip)
Top	5	2.5	1.25
Bottom	15	7.5	3.75

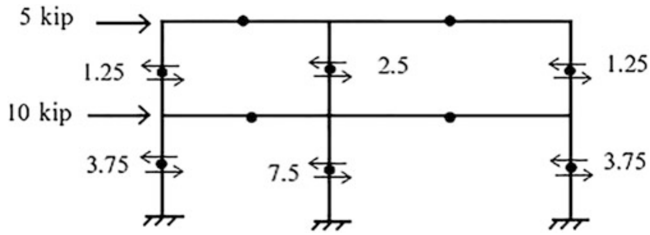


Fig. E11.2b Shear distribution for the columns (kip)

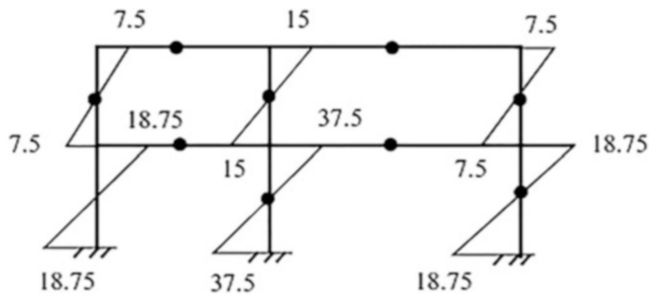


Fig. E11.2c Bending moment distribution for the columns (kip ft)

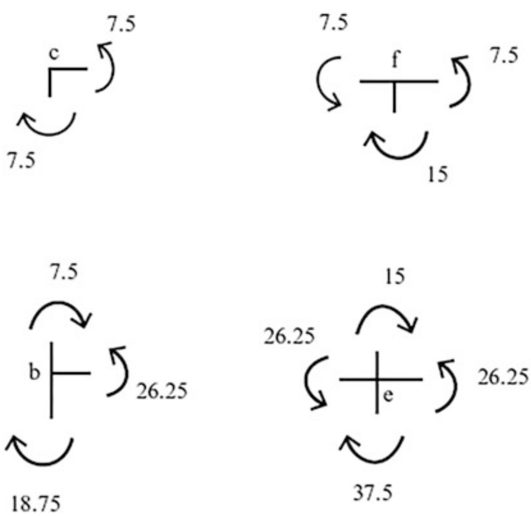


Fig. E11.2d Moments at the joints (kip ft)

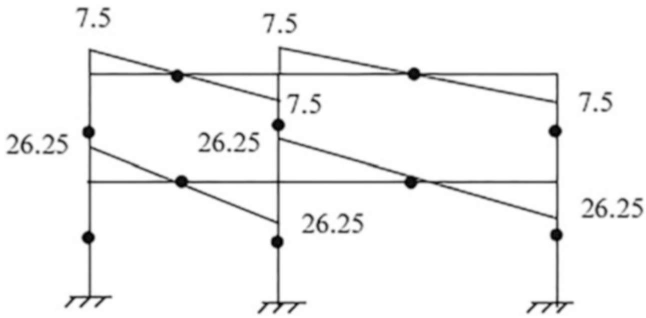


Fig. E11.2e Bending moment distribution for the beams (kip ft)

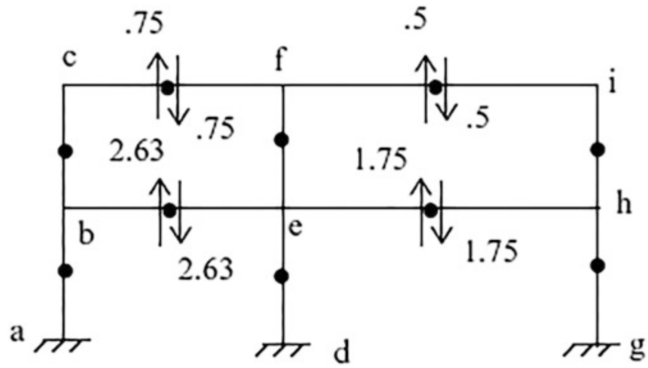


Fig. E11.2f Shear distribution for the beams (kip)

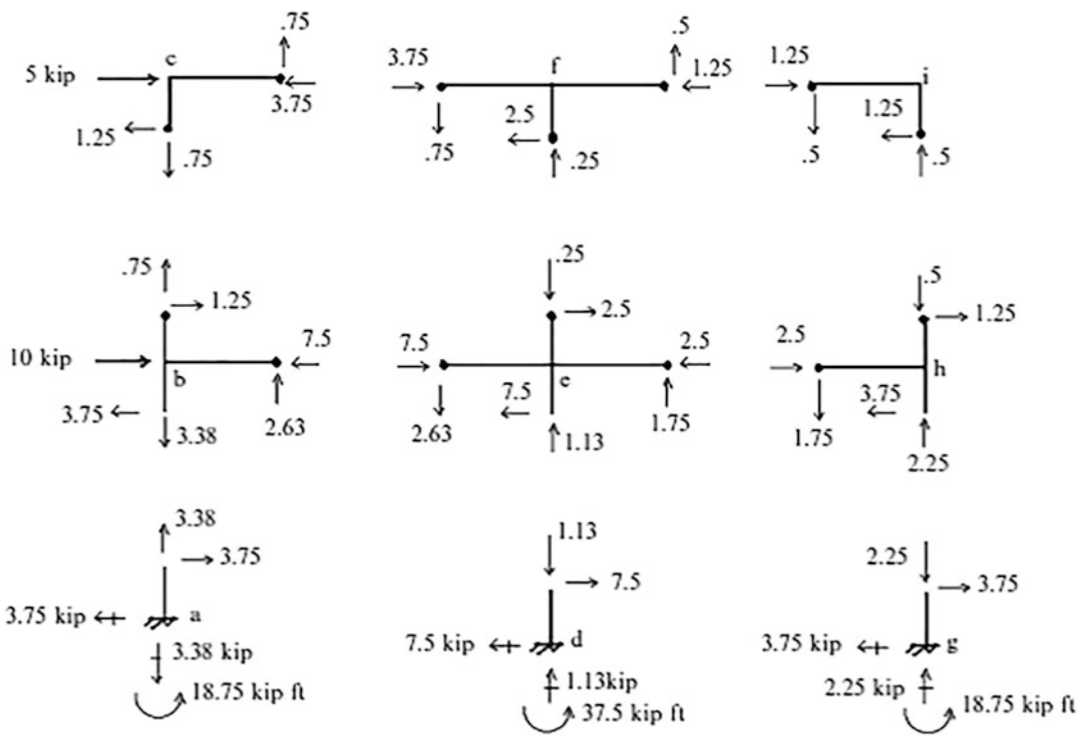


Fig. E11.2g Axial and shear forces (kip)

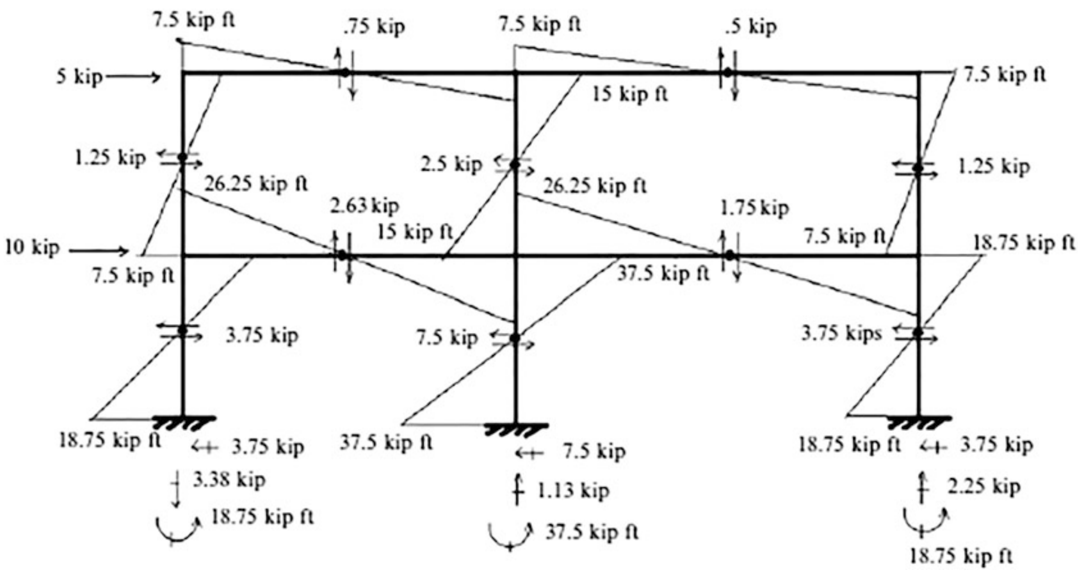


Fig. E11.2h Reactions, shear forces and moment distribution

11.4.2 Shear Stiffness Method: Low-Rise Rigid Frames

In this approach, we model a frame as a set of substructures, which resist the lateral shear in the stories through shearing action. First, we idealize the frame as a rigid frame with moment releases at the midpoints of the columns and beams such as shown in Fig. 11.8. We consider a segment bounded by floor $i + 1$ and floor i. For convenience, we assume I_b and L are *constant* in a story. We allow for different values of I for the exterior and interior columns. We define V_T as the sum of the lateral loads acting on floor $i + 1$, and all the floors above floor $i + 1$. This quantity represents the :*total transverse shear* for the story. Next, we define Δu as the *differential lateral displacement between floor i and floor i + 1*. We assume the floor beams are *rigid* with respect to axial deformation so that all points on a floor experience the same lateral displacement. Lastly, we assume the floors do not move in the vertical direction, and insert lines as indicated in Fig. 11.9. Our objective in this section is to establish an expression for the column shear forces in a story as a function of the total transverse shear for the story.

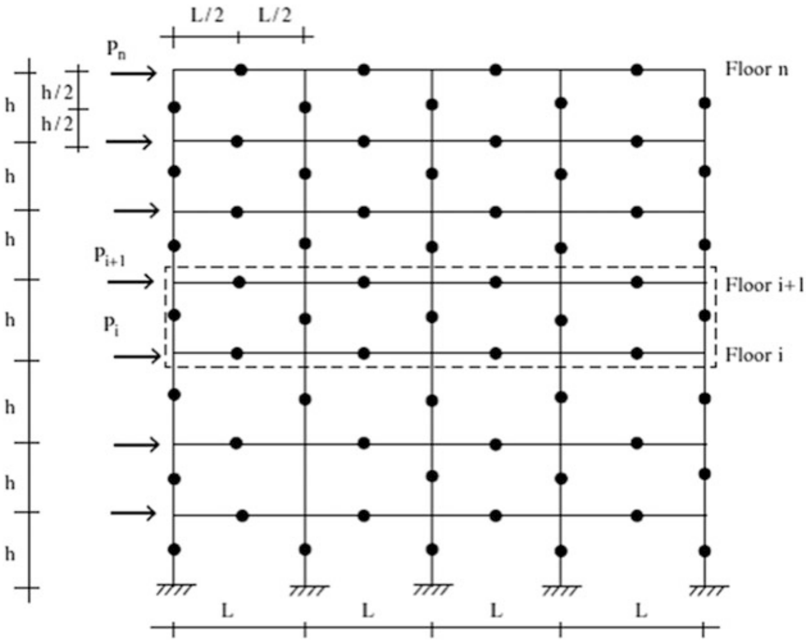


Fig. 11.8 Low-rise, frame

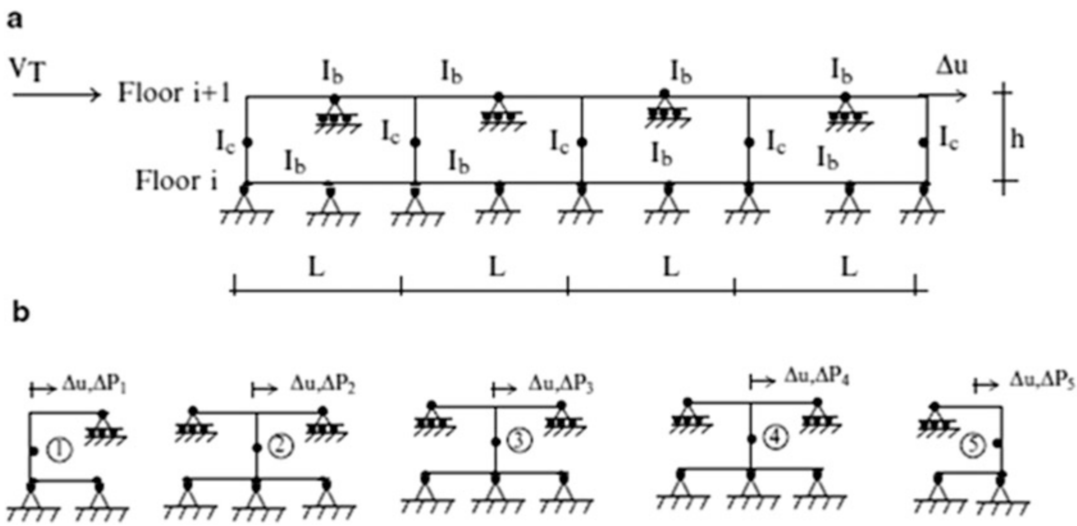


Fig. 11.9 (**a**) Idealized model for a story in a low-rise frame. (**b**) Sub-elements of the idealized model—low-rise frame

We visualize the model to consist of the sub-elements shown in Fig. 11.9b. Each sub-element experiences the same Δu. The resistance force ΔP_i for sub-element i depends on the stiffness of the element.

$$\Delta P_i = k_i\, \Delta u \equiv V_i \tag{11.3}$$

Then, summing the forces over the number of sub-elements leads to

$$V_{\mathrm{T}} = \sum V_i = \left(\sum k_i\right)\Delta u = k_{\mathrm{T}}\Delta u \tag{11.4}$$

Noting (11.3) and (11.4), the shear carried by sub-element i is given by

$$V_i = \Delta P_i = \left(\frac{k_i}{\sum k_i}\right)V_{\mathrm{T}} = \left(\frac{k_i}{k_{\mathrm{T}}}\right)V_{\mathrm{T}} \tag{11.5}$$

According to (11.5), the shear in a column depends on the ratio of the shear stiffness of the corresponding sub-element to the total story shear stiffness.

Using the slope-deflection equations :presented in Sect. 10.3, one can derive the following expressions for the sub-element shear stiffness factors (Fig. 11.10):

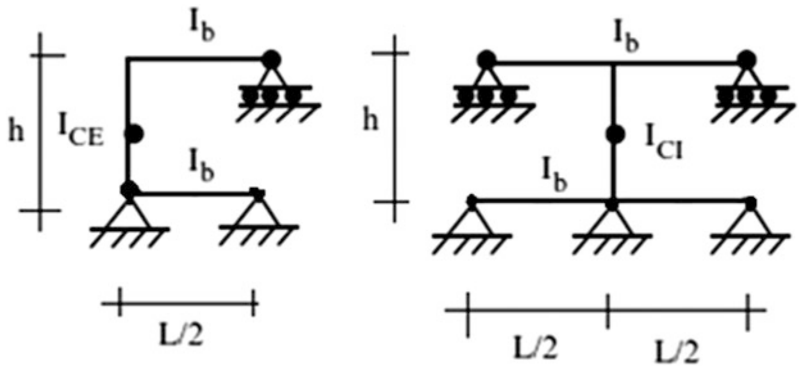


Fig. 11.10 Typical sub-elements. (**a**) Exterior. (**b**) Interior

Exterior Element: Upper Story

$$k_{\mathrm{E}} = \frac{12EI_{\mathrm{CE}}}{h^3}\left\{\frac{1}{1 + (I_{\mathrm{CE}}/h)/(I_{\mathrm{b}}/L)}\right\} = \frac{12EI_{\mathrm{CE}}}{h^3}f_{\mathrm{E}} \tag{11.6}$$

Interior Element: Upper Story

$$k_{\mathrm{I}} = \frac{12EI_{\mathrm{CI}}}{h^3}\left\{\frac{1}{1 + (1/2)((I_{\mathrm{CI}}/h)/(I_{\mathrm{b}}/L))}\right\} = \frac{12EI_{\mathrm{CI}}}{h^3}f_{\mathrm{I}} \tag{11.7}$$

where the dimensionless factor $(I_c/h)/(I_b/L)$ accounts for the flexibility of the beam.

Values of k_E/k_I for a range of values of $(I_{CI}/h)/(I_b/L)$ and I_{CE}/I_{CI} are tabulated in (Table 11.1).

Table 11.1 Stiffness ratios: upper stories

	$\frac{k_E}{k_I}$	
$(I_{CI}/h)/(I_b/L)$	$\frac{I_{CE}}{I_{CI}} = 1/2$	$\frac{I_{CE}}{I_{CI}} = 1$
0	0.5	1
0.25	0.5	0.9
0.5	0.5	0.83
1.0	0.5	0.75
1.5	0.5	0.75
1.5	0.5	0.7
2.0	0.5	0.67

Noting (11.5), we observe that the ratio of the shear in the external column to the shear in the interior column is given by

$$\frac{V|_{\mathrm{E-col}}}{V|_{\mathrm{I-col}}} = \frac{k_{\mathrm{E}}}{k_{\mathrm{I}}} \tag{11.8}$$

The derivation listed above applies for the upper stories, and needs to be modified for the bottom story. Figure 11.11 shows the idealized model used to estimate the story stiffness for the case where the base is fixed. The sub-elements are illustrated in Fig. 11.12 and the corresponding story stiffness factors are defined by (11.9) and (11.10).

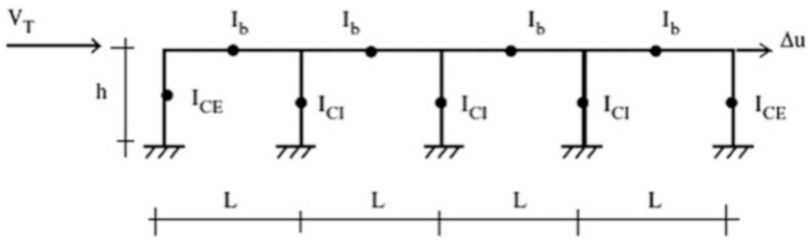


Fig. 11.11 Transverse shear model for bottom story—fixed support

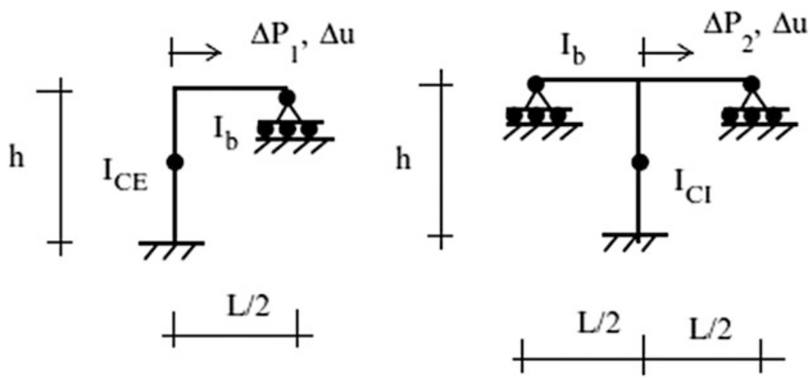


Fig. 11.12 Typical :sub-elements for base story—fixed support. (**a**) Exterior. (**b**) Interior

Exterior Element: Base Story (Fixed Support)

$$k_{\mathrm{BE}} = \frac{12EI_{\mathrm{CE}}}{h^3}\left\{\frac{1+\frac{1}{6}\left(\frac{I_{\mathrm{CE}}/h}{\mathrm{I_b}/\mathrm{L}}\right)}{1+\frac{2}{3}\left(\frac{I_{\mathrm{CE}}/h}{I_{\mathrm{b}}/\mathrm{L}}\right)}\right\} = \frac{12EI_{\mathrm{CE}}}{h^3}f_{\mathrm{BE}} \tag{11.9}$$

Interior Element: Base Story (Fixed Support)

$$k_{\mathrm{BI}} = \frac{12EI_{\mathrm{CI}}}{h^3}\left\{\frac{1+\frac{1}{12}\left(\frac{I_{\mathrm{CI}}/h}{I_{\mathrm{b}}/L}\right)}{1+\frac{1}{3}\left(\frac{I_{\mathrm{CI}}/h}{I_{\mathrm{b}}/L}\right)}\right\} = \frac{12EI_{\mathrm{CI}}}{h^3}f_{\mathrm{BI}} \tag{11.10}$$

When the base is hinged, we use the expressions listed in (11.11) and (11.12). In this case, we do not assume an inflection point at mid-height of the first story (Fig. 11.13).

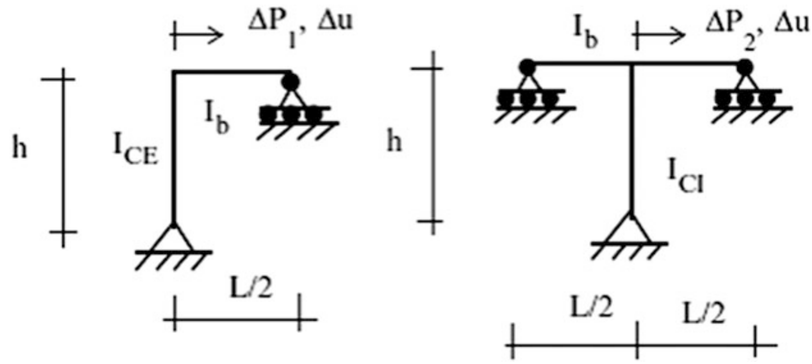


Fig. 11.13 Typical sub-elements for base story—hinged support. (**a**) Exterior. (**b**) Interior

Exterior Element:: Base Story (Hinged Support)

$$k_{\mathrm{BE}} = \frac{3EI_{\mathrm{CE}}}{h^3}\left\{\frac{1}{1+(1/2)((I_{\mathrm{CE}}/h)/(I_{\mathrm{b}}/L))}\right\} = \frac{3EI_{\mathrm{CE}}}{h^3}f_{\mathrm{BE}} \tag{11.11}$$

Interior Element: Base Story (Hinged Support)

$$k_{\mathrm{BI}} = \frac{3EI_{\mathrm{CI}}}{h^3}\left\{\frac{1}{1+(1/4)((I_{\mathrm{CI}}/h)/(I_{\mathrm{b}}/L))}\right\} = \frac{3EI_{\mathrm{CI}}}{h^3}f_{\mathrm{BI}} \tag{11.12}$$

The base shears are related by

$$V\big|_{\mathrm{E-col}} = V\big|_{\mathrm{I-col}}\frac{k_{\mathrm{BE}}}{k_{\mathrm{BI}}} \tag{11.13}$$

Values of $k_{\mathrm{BE}}/k_{\mathrm{BI}}$ for a range of $(I_{\mathrm{CI}}/h)/(I_{\mathrm{b}}/L)$ for both hinged and fixed supports are listed in Table 11.2.

Table 11.2 Stiffness ratios: lowest story

	Hinged support		Fixed support	
	$\frac{k_E}{k_I}$		$\frac{k_E}{k_I}$	
$(I_{\mathrm{CI}}/h)/(I_{\mathrm{b}}/L)$	$I_{\mathrm{CE}} = 1/2\, I_{\mathrm{CI}}$	$I_{\mathrm{CE}} = I_{\mathrm{CI}}$	$I_{\mathrm{CE}} = 1/2\, I_{\mathrm{CI}}$	$I_{\mathrm{CE}} = I_{\mathrm{CI}}$
0	0.5	1	0.5	1
0.25	0.5	0.944	0.5	0.948
1.0	0.5	0.9	0.5	0.91
1.0	0.5	0.833	0.5	0.862
1.5	0.5	0.786	0.5	0.833
2.0	0.5	0.75	0.5	0.816

Example 11.3 Approximate Analysis Based on the Shear Stiffness Method

Given: The frame shown in Fig. E11.3a.

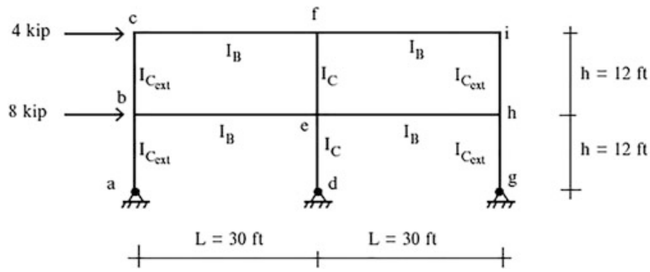


Fig. E11.3a

Determine: The column shear forces using the shear stiffness model and the member properties defined as cases A, B, C, and D.

Case A: $I_{C_{ext}} = \frac{1}{2} I_C \text{ and } I_B = 4I_C \Rightarrow \frac{I_{C_{ext}}/h}{I_B/L} = 0.31 \frac{I_C/h}{I_B/L} = 0.62.$

Case B: $I_{C_{ext}} = I_C \text{ and } I_B = 4I_C \Rightarrow \frac{I_{C_{ext}}/h}{I_B/L} = \frac{I_C/h}{I_B/L} = 0.625.$

Case C: $I_{C_{ext}} = I_C \text{ and } I_B = 2I_C \Rightarrow \frac{I_{C_{ext}}/h}{I_B/L} = \frac{I_C/h}{I_B/L} = 1.25.$

Case D: $I_{C_{ext}} = \frac{1}{2} I_C \text{ and } I_B = 2I_C \Rightarrow \frac{I_{C_{ext}}/h}{I_B/L} = 0.625 \frac{I_C/h}{I_B/L} = 1.25.$

Solution: Using (11.6), (11.7), and (11.11), (11.12) the sub-element stiffnesses are:

	Case A	Case B	Case C	Case D
Top story	$f_E = 0.762$ $f_I = 0.762$ $\frac{k_E}{k_I} = 0.5$	$f_E = 0.615$ $f_I = 0.762$ $\frac{k_E}{k_I} = 0.808$	$f_E = 0.444$ $f_I = 0.615$ $\frac{k_E}{k_I} = 0.722$	$f_E = 0.615$ $f_I = 0.615$ $\frac{k_E}{k_I} = 0.5$
Bottom story	$f_{BE} = 0.865$ $f_{BI} = 0.865$ $\frac{k_{BE}}{k_{BI}} = 0.5$	$f_{BE} = 0.762$ $f_{BI} = 0.865$ $\frac{k_{BE}}{k_{BI}} = 0.881$	$f_{BE} = 0.615$ $f_{BI} = 0.762$ $\frac{k_{BE}}{k_{BI}} = 0.808$	$f_{BE} = 0.762$ $f_{BI} = 0.762$ $\frac{k_{BE}}{k_{BI}} = 0.5$

Noting that

$$\frac{V_E}{V_I} = \frac{k_E}{k_I}$$

we express the total shear as

$$V_{Total} = 2V_E + V_I = \left[2\left(\frac{k_E}{k_I}\right) + 1\right] V_I$$

Once I is specified for the interior and exterior columns, we can evaluate the ratio, k_E/k_I, and then V_I. The computations corresponding to Cases A, B, C, and D are summarized below. We also list the results predicted by the portal method. Note that the portal method agrees exactly with the stiffness method when $I_{C_{exterior}} = \frac{1}{2} I_{C_{interior}}$ (Cases A and D).

		Stiffness method									
		Case A		Case B		Case C		Case D		Portal method	
Story	V_T (kip)	V_E (kip)	V_I (kip)	V_E (kip)	V_I (kip)	V_E (kip)	V_I (kip)	V_E (kip)	V_I (kip)	V_E (kip)	V_I (kip)
Top	4	1	2	1.235	1.53	1.18	1.64	1	2	1	2
Bottom	12	3	6	3.83	4.34	3.7	4.59	3	6	3	6

11.4.3 Low-Rise Rigid Frames with Bracing

11.4.3.1 Lateral Load

A rigid frame resists lateral loading through bending action of the columns. When a bracing system is combined with the frame, both of these systems participate in carrying the lateral load. From a stiffness perspective, the load is distributed according to the relative stiffness, i.e., the stiffer element carries more load. For low-rise frames, the transverse shear stiffness is the controlling parameter. Figure 11.14 illustrates the structural scheme for a one-story structure.. A similar arrangement is used for multistory structures. Of particular interest is the distribution of lateral load between the rigid frame and the brace.

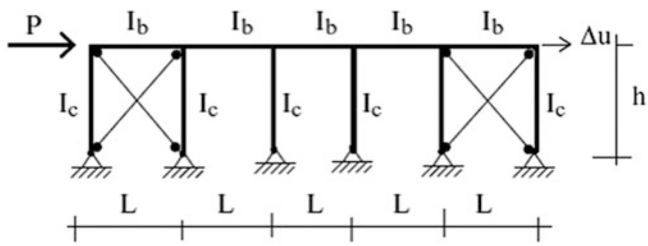


Fig. 11.14 Rigid frame with bracing

The individual systems are defined in Fig. 11.15. We assume all sub-elements experience the same lateral displacement Δu, and express the lateral loads carried by each structural system as

$$\begin{aligned} P_{\text{frame}} &= k_{\text{frame}} \Delta u \\ P_{\text{brace}} &= k_{\text{brace}} \Delta u \end{aligned} \tag{11.14}$$

where k_{frame}, k_{brace} denote the frame and brace stiffness factors. Summing these forces, we write

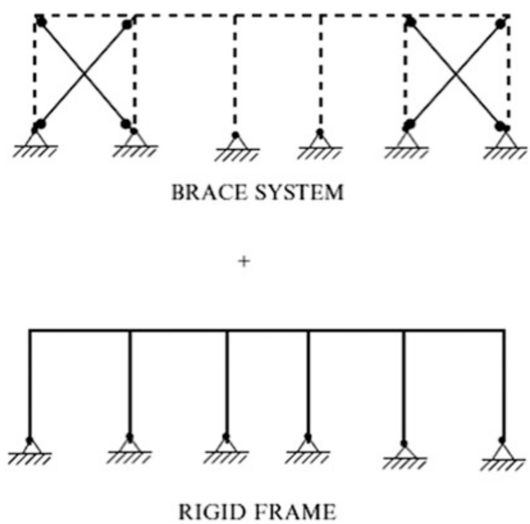


Fig. 11.15 Individual systems

$$P = P_{\text{frame}} + P_{\text{brace}} = (k_{\text{brace}} + k_{\text{frame}})\Delta u = k_{\text{T}} \Delta u \tag{11.15}$$

Solving for Δu and back substituting in (11.14) results in

$$P_{\text{frame}} = \frac{k_{\text{frame}}}{k_{\text{T}}} P \qquad P_{\text{brace}} = \frac{k_{\text{brace}}}{k_{\text{T}}} P \tag{11.16}$$

Also,

$$\frac{P_{\text{frame}}}{P_{\text{brace}}} = \frac{k_{\text{frame}}}{k_{\text{brace}}} \tag{11.17}$$

According to (11.16), the lateral force carried. by a system depends on its relative stiffness. Increasing k_{brace} shifts load onto the bracing system.

Considering a single story, the lateral load required to introduce an inter-story lateral displacement Δu is equal to

$$P_{\text{frame}} = k_{\text{frame}} \Delta u \tag{11.18}$$

where k_{frame} is estimated by combining (11.11) and (11.12).

$$k_{\text{frame}} = \frac{3E}{h^3} \left\{ \frac{2I_{\text{CE}}}{1 + (1/2)((I_{\text{CE}}/h)/(I_{\text{b}}/L))} + \sum_{\text{intercol}} \frac{I_{\text{CI}}}{1 + (1/4)((I_{\text{CI}}/h)/(I_{\text{b}}/L))} \right\} \tag{11.19}$$

Once the member properties are known, one can evaluate k_{frame}. We need to develop a similar expression for a brace.

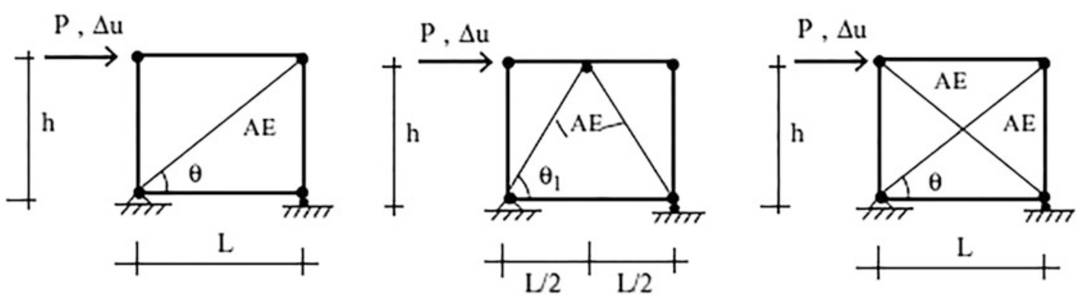


Fig. 11.16 Diagonal bracing systems. (**a**) Single. (**b**) Chevron. (**c**) X Brace

Typical bracing schemes are shown in Fig. 11.16. The lateral load is carried equally by the diagonal members. One determines k_{brace} using structural mechanics concepts such as deformation and equilibrium. The analytical expressions for the different schemes are

$$\begin{aligned} k_{\text{brace(single)}} &= \frac{AE}{h}\left(\sin\theta\cos^2\theta\right) \\ k_{\text{brace}_{\text{(chevron)}}} &= \frac{2AE}{h}\left(\sin\theta_1\cos^2\theta_1\right) \\ k_{\text{brace(x brace)}} &= \frac{2AE}{h}\left(\sin\theta\cos^2\theta\right) \end{aligned} \tag{11.20}$$

The diagonal forces reverse when the lateral load reverses, which occurs for wind and earthquake loading.

Example 11.4 Shear Force Distribution

Given: The one-story frame defined in Fig. E11.4a.

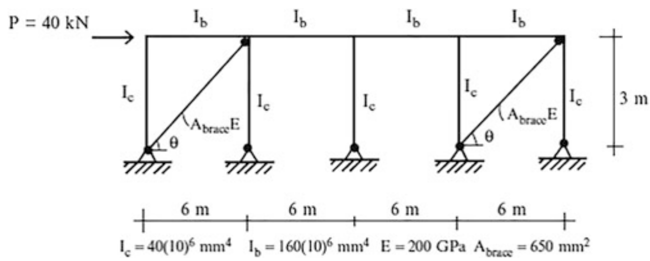


Fig. E11.4a

Determine: The column shears and the diagonal brace forces.
Solution: Using (11.19), the frame stiffness is

$$k_{\text{frame}} = \frac{3EI_c}{h^3}\left(\frac{2}{1+(1/2)((I_c/h)/(I_b/L))} + \frac{3}{1+(1/4)((I_c/h)/(I_b/L))}\right)$$
$$= \frac{3EI_c}{h^3}\left(\frac{8}{5}+\frac{8}{3}\right) = 12.8\frac{EI_c}{h^3}$$

The brace stiffness follows from (11.20). Note that there are two braces.

$$k_{\text{brace}} = 2\frac{A_{\text{brace}}E}{h}\left(\sin\theta\cos^2\theta\right) = \frac{A_{\text{brace}}E}{h}2(0.447)(0.894)^2 = 0.714\frac{A_{\text{brace}}E}{h}$$

Noting (11.16), the individual forces are related by

$$P_{\text{frame}} = \left(\frac{k_{\text{frame}}}{k_{\text{brace}}}\right)P_{\text{brace}} = 17.9\left(\frac{I_c}{A_{\text{brace}}h^2}\right)P_{\text{brace}}$$

Summing the forces leads to

$$P_{\text{frame}} + P_{\text{brace}} = P \Rightarrow \left(17.9\frac{I_c}{A_{\text{brace}}h^2}+1\right)P_{\text{brace}} = 40$$

Then

$$P_{\text{brace}} = \frac{40}{1+(17.9)\left(\left(40(10)^6\right)/\left((650)(3000)^2\right)\right)} = 35.6\,\text{kN}$$
$$P_{\text{frame}} = 4.36\,\text{kN}$$

The force in each diagonal brace is given by

$$F_b = \frac{P_{\text{brace}}}{2\cos\theta} = 19.9\,\text{kN}$$

We evaluate the column shear forces using the corresponding stiffness factors defined by (11.11) and (11.12).

$$k_{\mathrm{E}} = \frac{3EI_{\mathrm{c}}}{h^3}\left(\frac{4}{5}\right)$$
$$k_{\mathrm{I}} = \frac{3EI_{\mathrm{c}}}{h^3}\left(\frac{8}{9}\right)$$
$$\frac{V_{\mathrm{E}}}{V_{\mathrm{I}}} = \frac{k_{\mathrm{E}}}{k_{\mathrm{I}}} = \frac{4/5}{8/9} = \frac{9}{10}$$
$$\therefore V_{\mathrm{E}} = 0.9V_{\mathrm{I}}$$

Summing the shears,

$$2V_{\mathrm{E}} + 3V_{\mathrm{I}} = 4.36 \Rightarrow 1.8V_{\mathrm{I}} + 3V_{\mathrm{I}} = 4.36$$

Then

$$V_{\mathrm{I}} = 0.91\,\mathrm{kN}$$
$$V_{\mathrm{E}} = 0.82\,\mathrm{kN}$$

Note that the brace carries the major portion of the story shear.

Example 11.5 Shear Force Distribution

Given: The braced rigid frame defined in Fig. E11.5a. $A_{\mathrm{b}} = 0.8$ in.^{2}, $E = 29,000$ ksi, and $I = 150$ in.^{4}.

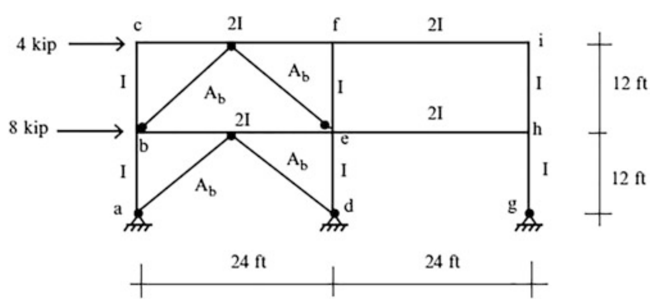


Fig. E11.5a

Determine: The lateral forces carried by the frame and brace systems.
Solution:

$P_{\mathrm{frame}} + P_{\mathrm{brace}} = 4$ kip Upper floor
$P_{\mathrm{frame}} + P_{\mathrm{brace}} = 12$ kip Lower floor

Frame:
Upper story sub-element: Equations (11.6) *and* (11.7)

$$\left.\begin{aligned} k_E &= \frac{6EI_c}{h^3} \\ k_I &= \frac{8EI_c}{h^3} \end{aligned}\right\} \Rightarrow k_{\text{frame}} = \frac{EI_c}{h^3}(2(6)+8) = \frac{20EI_c}{h^3} = \frac{20(29{,}000)150}{(12\times 12)^3} = 29.14\,\text{kip/in.}$$

Base story sub-element: Equations (11.11) *and* (11.12)

$$\left.\begin{aligned} k_E &= \frac{2EI_c}{h^3} \\ k_I &= \frac{2.4EI_c}{h^3} \end{aligned}\right\} \Rightarrow k_{\text{frame}} = \frac{EI_c}{h^3}(2(2)+2.4) = \frac{6.4EI_c}{h^3} = 9.32\,\text{kip/in.}$$

Brace:

$$k_{\text{brace}} = \frac{2EA_b}{h}\sin\theta_1(\cos\theta_1)^2 = 0.707\frac{EA_b}{h} = \frac{0.707(0.8)(29{,}000)}{12(12)} = 113.9\,\text{kip/in.}$$

Shear distributions

$$P_{\text{frame}} = \left(\frac{k_{\text{frame}}}{k_{\text{brace}}}\right)P_{\text{brace}} \Rightarrow \begin{aligned} P_{\text{frame}} &= \left(\frac{29.14}{113.9}\right)P_{\text{brace}} = 0.256P_{\text{brace}} \quad \text{Upper floor} \\ P_{\text{frame}} &= \left(\frac{9.32}{113.9}\right)P_{\text{brace}} = 0.082P_{\text{brace}} \quad \text{Lower floor} \end{aligned}$$

Then

$$\begin{aligned} (0.256+1)P_{\text{brace}} &= 4\,\text{kip} \Rightarrow P_{\text{brace}} = 3.18\,\text{kip} \quad \text{Upper floor} \\ (0.082+1)P_{\text{brace}} &= 12\,\text{kip} \Rightarrow P_{\text{brace}} = 11.09\,\text{kip} \quad \text{Lower floor} \end{aligned}$$

Therefore

$$\begin{aligned} P_{\text{frame}} &= 0.256P_{\text{brace}} = 0.81\,\text{kip} \quad \text{Upper floor} \\ P_{\text{frame}} &= 0.082P_{\text{brace}} = 0.91\,\text{kip} \quad \text{Lower floor} \end{aligned}$$

Example 11.6 Shear Force Distribution

Given: The braced frame defined in Fig. E11.6a.

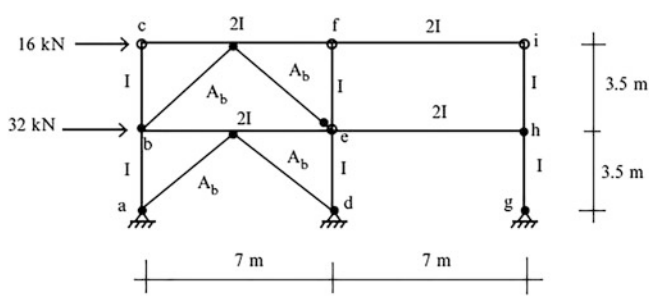
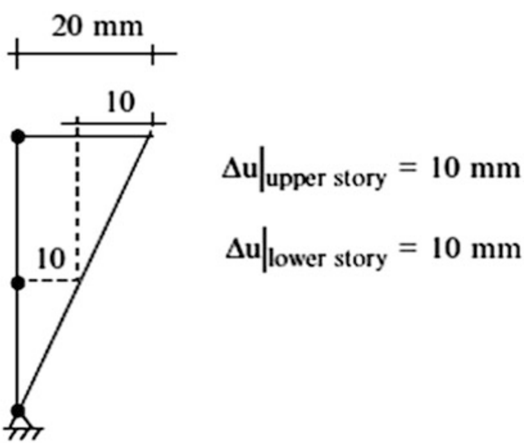


Fig. E11.6a

Determine: The required brace area A_b, to limit the inter-story displacement to 10 mm for each story. Assume $E = 200$ GPa.
Solution:

$$P_{brace} = 16 \text{ Upper floor}$$
$$P_{brace} = 16 + 32 = 48 \text{ Lower floor}$$

$$k_{brace} = \frac{2EA_b}{h}\sin\theta_1(\cos\theta_1)^2 = 0.707\frac{EA_b}{h}$$
$$P_{brace} = k_{brace}u$$



$$\text{Upper floor } 16 = 0.707\frac{EA_b}{h}\Delta u_{upper} \Rightarrow A_b \geq \frac{16(3500)}{0.707(10)(200)} = 39.6\,\text{mm}^2$$

$$\text{Lower floor } 48 = 0.707\frac{EA_b}{h}\Delta u_{upper} \Rightarrow A_b \geq \frac{48(3500)}{0.707(10)(200)} = 118.8\,\text{mm}^2$$

The value for the lower floor controls the design.

$$\therefore A_{b_{required}} = 118.8\,\text{mm}^2$$

11.5 High-Rise Rigid Frames: The Cantilever Method

The approximate procedure described above is applicable for low-rise rigid frames, which behave as "shear type" frames, i.e., the floors displace laterally but do not rotate. One determines the axial forces in the columns using the shear forces in the floor beams. High-rise frames behave more like a cantilever beam. As illustrated in Fig. 11.17b, the floors rotate as rigid planes. Their behavior is similar to what is assumed for the cross section of a beam in the formulation of the bending theory of beams; the floors experience both a translation and a rotation. Just as for beams, the rotational component produces axial strain in the columns. The column shears and moments are found from equilibrium considerations, given the axial forces in the columns. In what follows, we describe an idealized structural model that is used to establish the distribution of column axial forces in a story. This approach is called the "Cantilever Method."

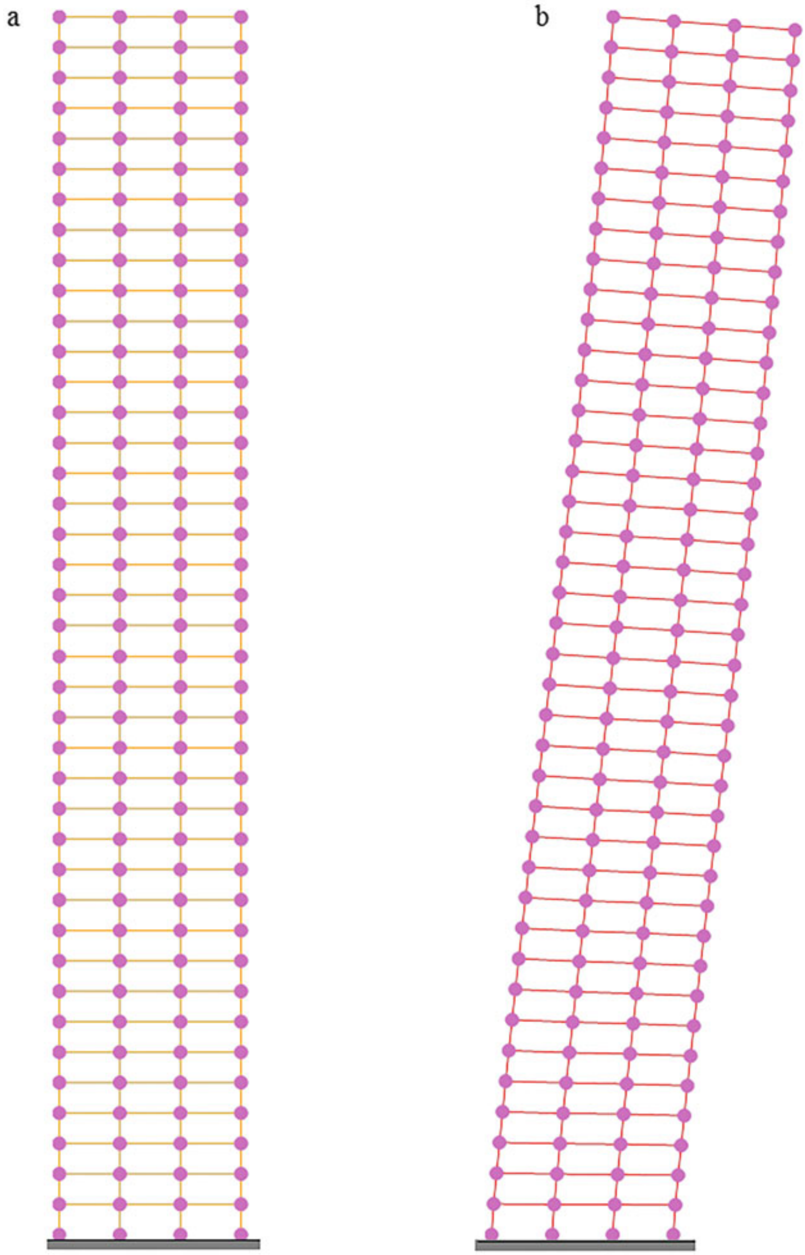


Fig. 11.17 Tall building model—lateral deflections

One normally applies this method to estimate the axial forces in the columns at the base of the building, i.e., where the bending moment due to lateral loading is a maximum.

We consider the typical tall building shown in Fig. 11.18. Given the lateral load, we can determine the bending moment and transverse shear at mid-height of each story. We denote these quantities as M_{i+1} and $V_{\mathrm{T}_{i+1}}$.

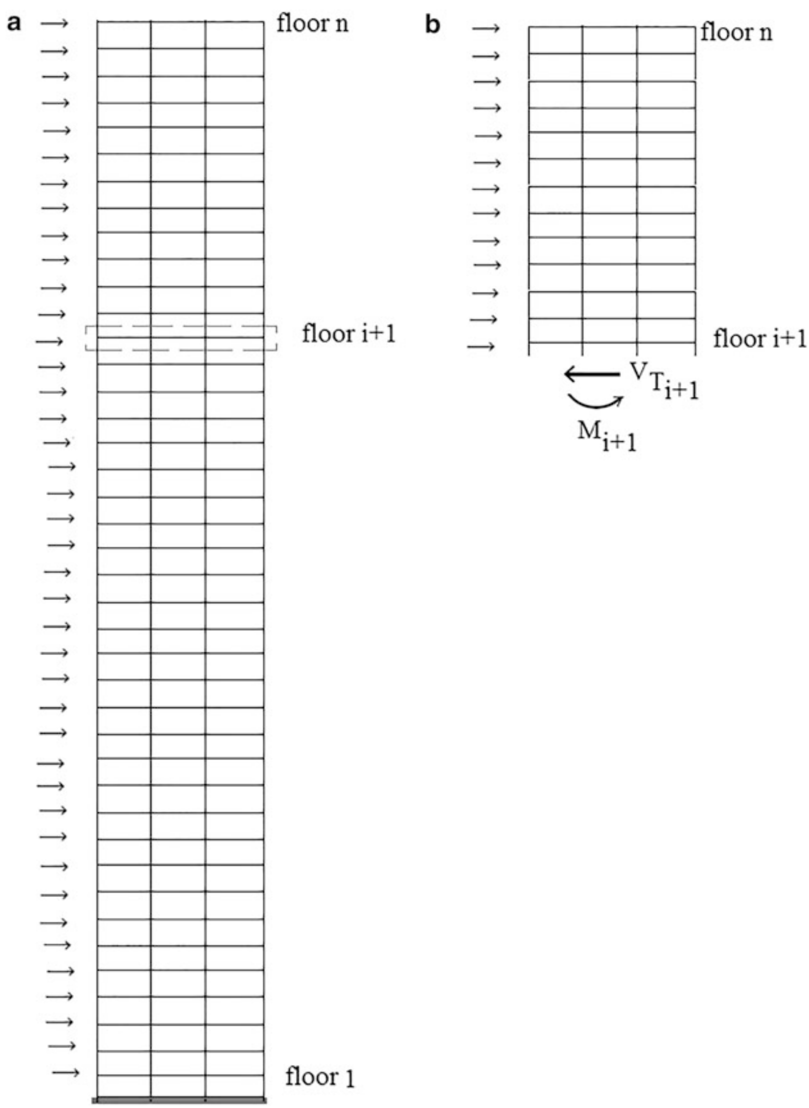


Fig. 11.18 Tall building model (**a**) under lateral loading (**b**) Segment of building above floor *i*

Now, we isolate a segment of the building consisting of floors $i + 1$, i, and the columns connecting these floors. Figure 11.19 shows this segment. The floors are assumed to be rigid plates, and the columns are represented as axial springs. Floor $i + 1$ experiences a rotation, $\Delta\beta$, with respect to floor i due to the moment M_{i+1}.

We position a reference axis at point O and define x_i as the X coordinate for spring i. The corresponding axial stiffness is k_i. The origin of the reference axis is located such that

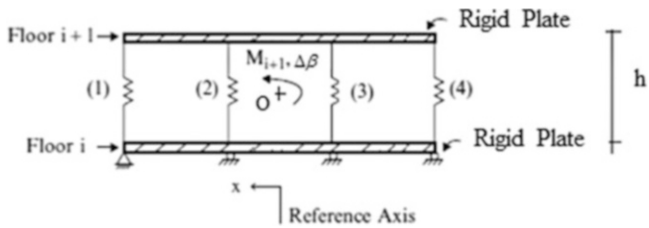


Fig. 11.19 Column-beam model for a story bounded by floors i and $i + 1$

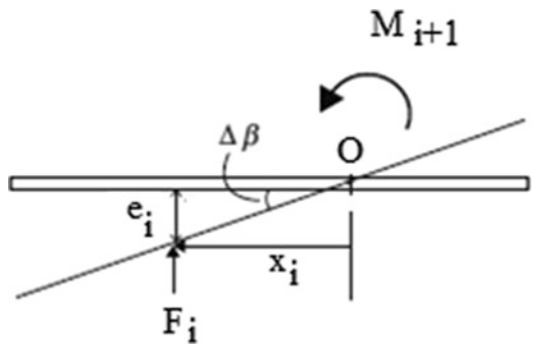


Fig. 11.20 Deformation due to relative rotation

$$\sum k_i x_i = 0 \tag{11.21}$$

Note that the axial stiffness is equal to the column stiffness,

$$k_i = \frac{A_i E}{h} \tag{11.22}$$

where A_i is the cross-sectional axis and h is the column height. Then, when E is constant, (11.21) can be written as

$$\sum A_i x_i = 0 \tag{11.23}$$

In this case, one can interpret the reference axis as equivalent to the centroidal axis for the column areas in the story.

We suppose the floors rotate about O and define $\Delta\beta$ as the relative rotation between adjacent floors. The deformation introduced in spring i follows from Fig. 11.20.

$$\begin{aligned} e_i &= x_i \Delta\beta \\ F_i &= k_i e_i = k_i x_i \Delta\beta \end{aligned} \tag{11.24}$$

Summing moments about 0, and equating the result to the applied moment, M_{i+1} results in

$$M_{i+1} = \left(\sum k_i x_i^2\right) \Delta\beta \tag{11.25}$$

Here, M_{i+1} represents the moment due to the lateral loads applied *on and above* floor $i + 1$. We solve for $\Delta\beta$ and then back substitute in the expression for F_i. The result is

$$F_i = k_i x_i \frac{M_{i+1}}{\left(\sum k_i x_i^2\right)} = x_i A_i \left(\frac{E}{h \sum k_i x_i^2}\right) M_{i+1} \tag{11.26}$$

We see that the column force distribution is proportional to the distance from the reference axis and the relative column cross-sectional area. One does not need to specify the actual areas, only the ratio of areas.

One should note that this result is based on the assumption that the *floor acts as a rigid plate*. Stiff belt-type trusses are frequently incorporated at particular floors throughout the height so that the high-rise frame behaves consistent with this hypothesis.

Example 11.7 Approximate analysis based on the cantilever method

Given: The symmetrical 42-story plane frame shown in Fig. E11.7a. Assume the building is supported on two caissons located at the edges of the base. Consider the base to be rigid.
Determine: The axial forces in the caissons.

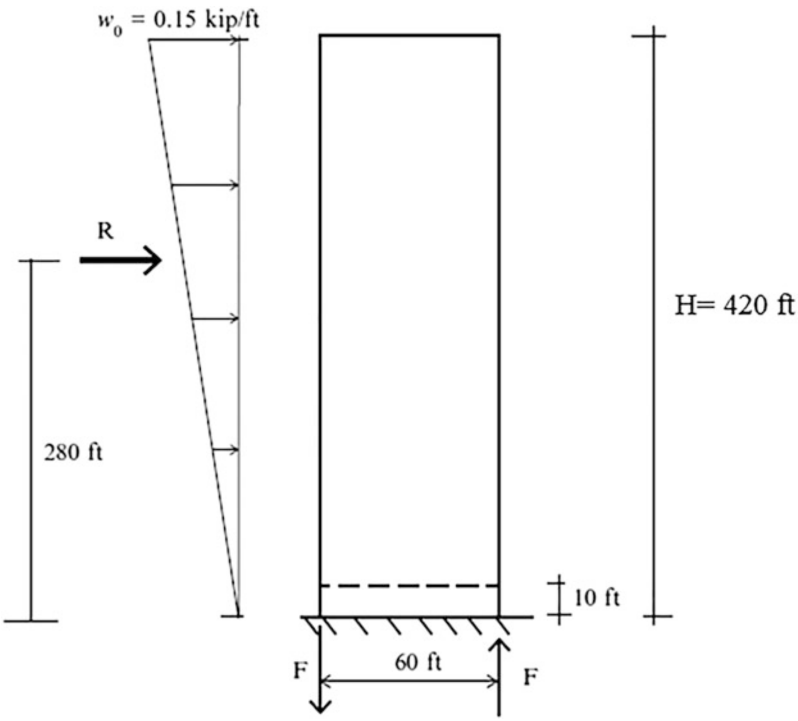


Fig. E11.7a

Solution: The Moment at the base is given by

$$R = \frac{w_0 H}{2} = 210 w_0$$

$$M = \frac{2H}{3} R = 58,800 w_0 = 8820 \,\text{kip ft}$$

This moment is resisted by the pair of caisson forces which are equivalent to a couple.

$$60F = 8820$$

$$F = 147\,\text{kip}$$

Example 11.8 Approximate Analysis Based on the Cantilever Method

Given: The symmetrical plane frame shown in Fig. E11.8a.
Determine: The column axial forces in the bottom story for the distribution of column areas shown.

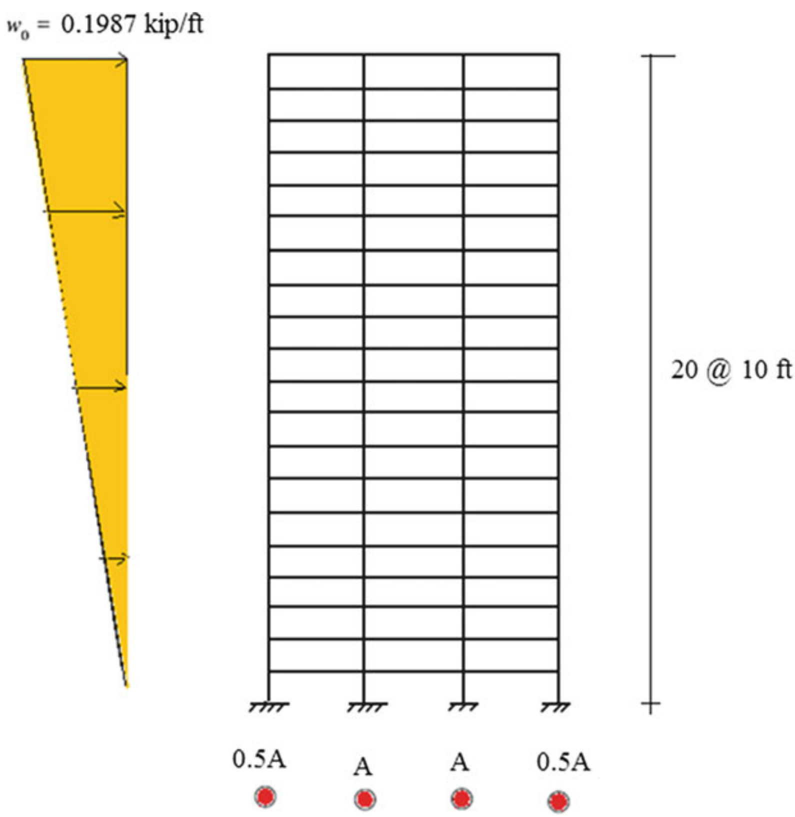


Fig. E11.8a

Solution: The rotational stiffness for the story is:

$$K_x = \sum k_i {x_i}^2 = \frac{E}{h}\sum A_i {x_i}^2 = 1100\left(\frac{EA}{h}\right)$$

The bending moment at mid-height of the first story is 2550 kip ft. Then, substituting for M_{i+1} and K_x in (11.26) leads to the axial forces in the columns (Fig. E11.8b),

$$F_2 = (10)\frac{AE}{h}\left(\frac{2550}{1100(AE/h)}\right) = 23.2\,\text{kip}\downarrow$$

$$F_1 = (30)\frac{.5AE}{h}\left(\frac{2550}{1100(AE/h)}\right) = 34.8\,\text{kip}\downarrow$$

$$F_3 = -F_2 = 23.2\,\text{kip}\uparrow$$
$$F_4 = -F_1 = 34.8\,\text{kip}\uparrow$$

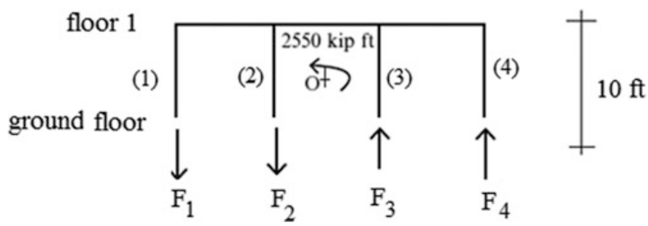


Fig. E11.8b

This computation is repeated for successive stories. Once all the column axial forces are known, one can compute the column shears by assuming inflection points at the midpoints of the columns and beams and applying static equilibrium conditions. The procedure is similar to that followed in Example 11.2.

11.6 Summary

11.6.1 Objectives of the Chapter

Our goals in this chapter are

- To describe some approximate methods for estimating the bending moment distribution in multi-span beams and multi-bay frames subjected to gravity loading.
- To present approximate methods for analyzing multistory rigid frames subjected to lateral loading.

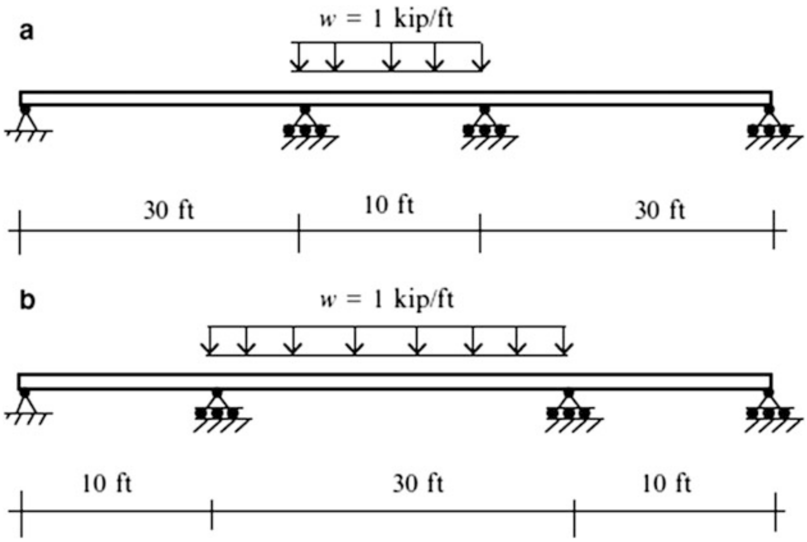
11.6.2 Key Concepts

- Reasoning in a qualitative sense about the behavior using the concept of relative stiffness provides the basis for a method to estimate the bending moment distribution in multi-span beams.
- Two methods are described for analyzing low-rise rigid frame structures.
- The Portal method: The Portal method assumes the shear forces in the interior columns are equal to a common value, and the shears in the exterior columns are equal to ½ this value. This is an empirical-based procedure.
- The shear stiffness method: The shear stiffness method uses simplified structural models to estimate the shear forces in the columns given the total shear force for a story. This procedure predicts that the shear force in a particular column is proportioned to the relative stiffness. It follows that a stiff column attracts more load than a flexible column.

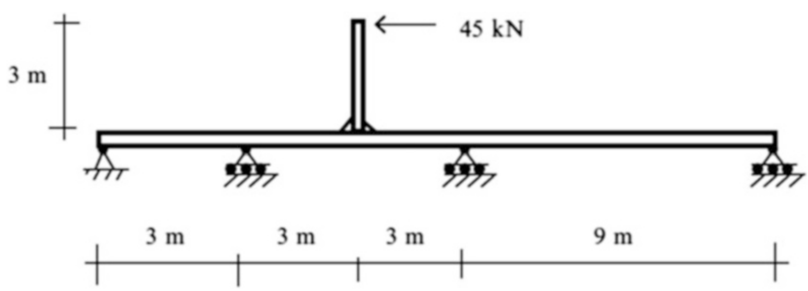
- High-rise rigid frames are modeled as equivalent cantilever beams. The floor slabs are considered rigid and the bending rigidity is generated through the axial action of the columns. One starts with the bending moment at the midpoint between a set of floors and determines the axial forces in the columns. According to this method, the axial force depends on the axial rigidity of the column and the distance from the centroidal axis.

11.7 Problems

Problem 11.1. Estimate the bending .moment distribution for the cases listed below.. Use qualitative reasoning based on relative stiffness. Assume I as constant.



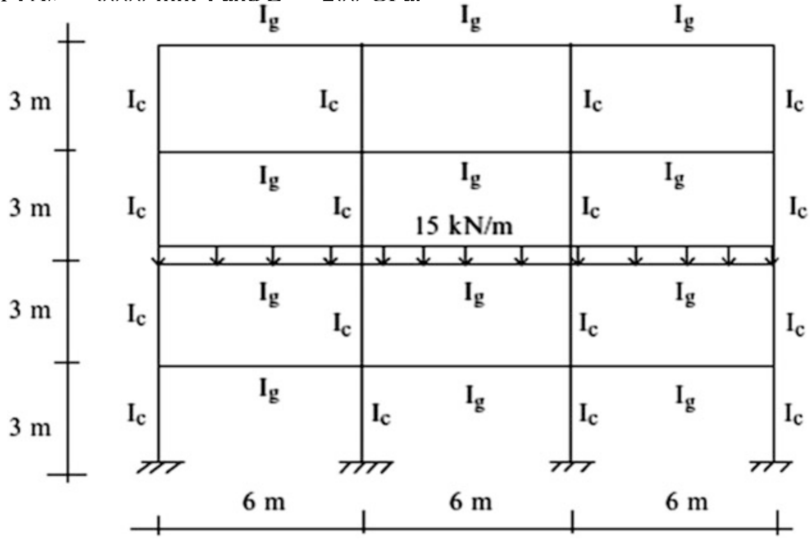
Problem 11.2. Estimate the bending moment distribution. Use qualitative reasoning based on relative stiffness. Assume I as constant.



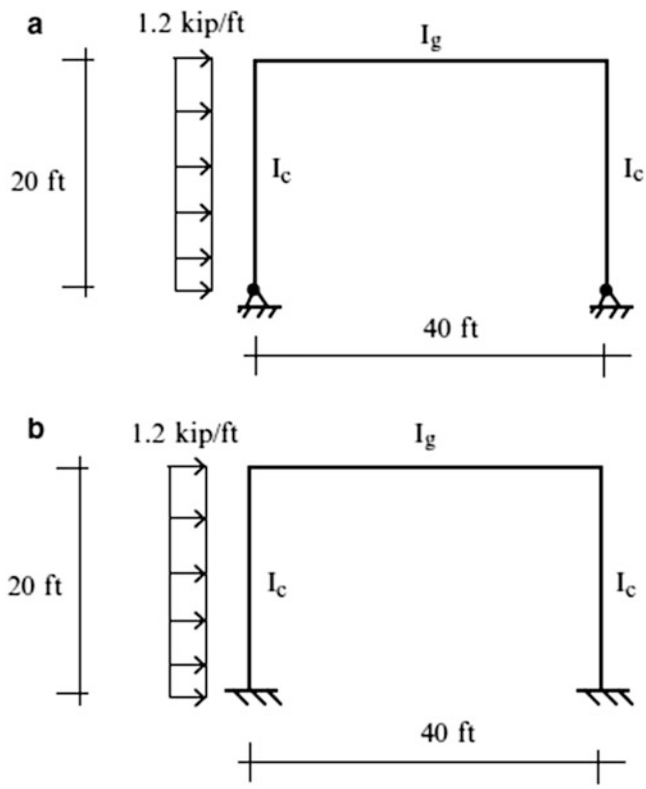
Problem 11.3. Solve Problem 11.1 cases (a) and (b) using moment distribution. Compare the approximate and exact results.

Problem 11.4. Consider the multistory steel frame shown below. Determine the maximum positive and negative moments in the beams using the following approaches:

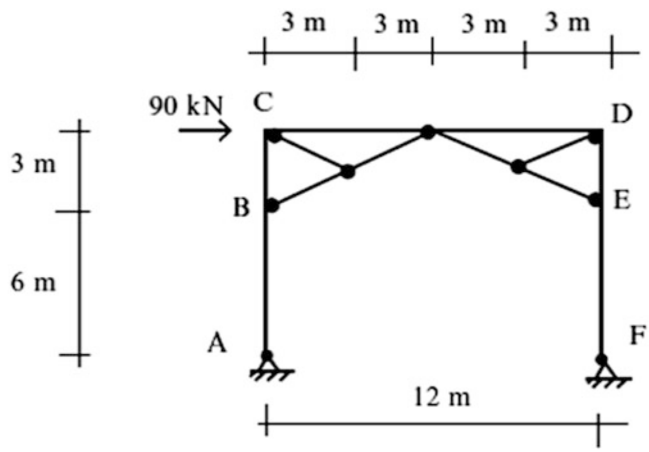
1. Assume inflection points at 0.1 L from each end of the beams.
2. Use a computer software system. Assume $I_g = 200(10)^6$ mm^4, $A_g = 16{,}000$ mm^2, $I_c = 100$ $(10)^6$ mm^4, $A_c = 6000$ mm^2, and $E = 200$ GPa.



Problem 11.5. Estimate the axial force, shear force, and bending moment distributions. Assume $I_g = 2I_c$



Problem 11.6. Members AC and FD are continuous. Estimate the bending moment distribution in AC and FD, and the axial forces in the pin-ended members. Compare your results with results generated with a computer software system. Assume $I = 100(10)^6$ mm^4, $A = 6000$ mm^2, $A_{\text{pin-ended}} = 4000$ mm^2, and $E = 200$ GPa.

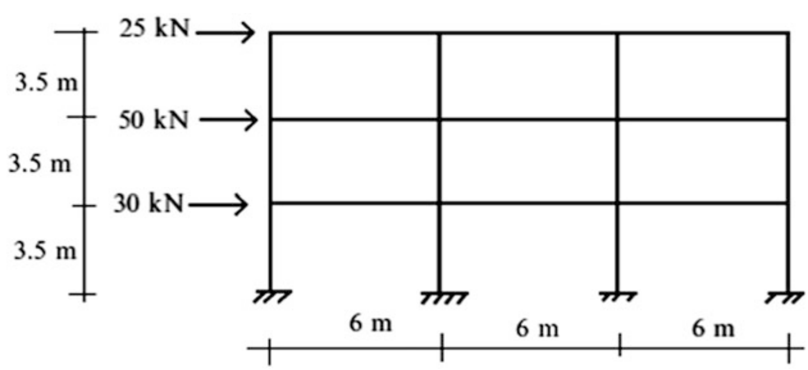


Problem 11.7. Repeat Problem 11.6 assuming fixed supports at A and F.

Problem 11.8. Consider the steel frame shown below. Determine the moment at each end of each member using

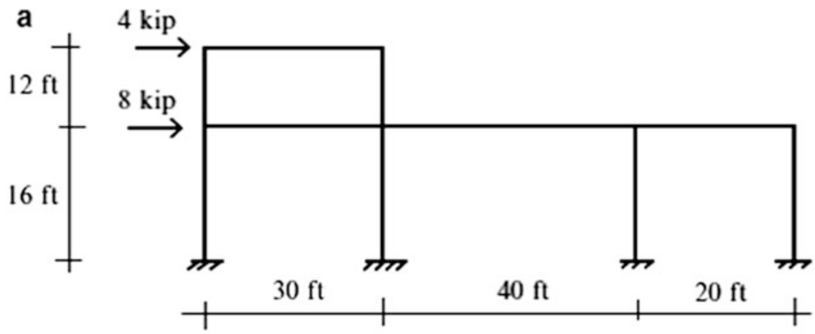
(a) The Portal method.

(b) The shear stiffness method:. Take $I_g = 300(10)^6$ mm^4, $A_g = 18{,}000$ mm^2 for all the girders, $I_c = 100(10^6)$ mm^4, and $A_c = 6000$ mm^2 for all the columns.

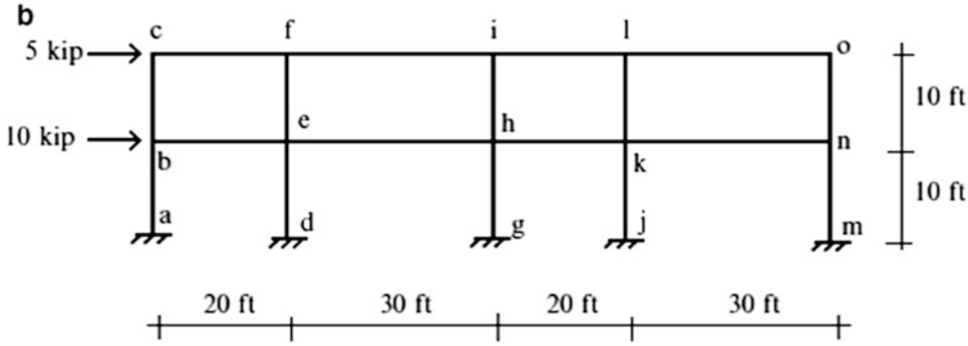


Problem 11.9. For the steel frames shown, estimate the axial force, shear force, and moments for all of the members using the Portal method. Compare your results with results generated with a computer software system. Take $I_c = 480$ in.^{4}, $A_c = 40$ in.^{2} for all the columns and $I_b = 600$ in.^{4}, $A_b = 60$ in.^{2} for all the beams.

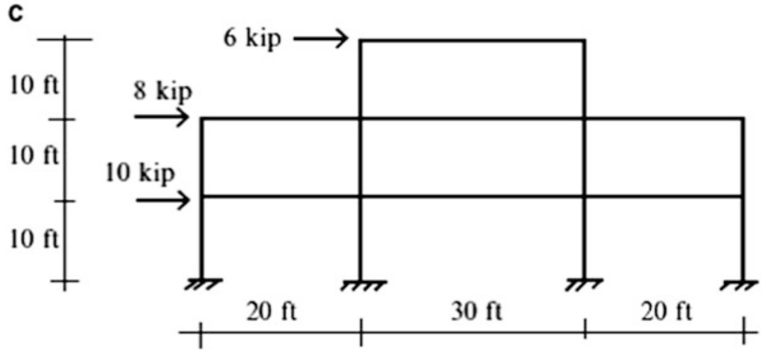
a



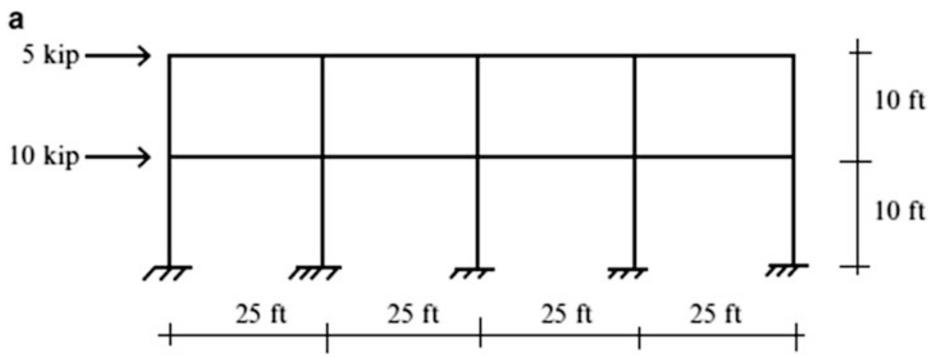
b

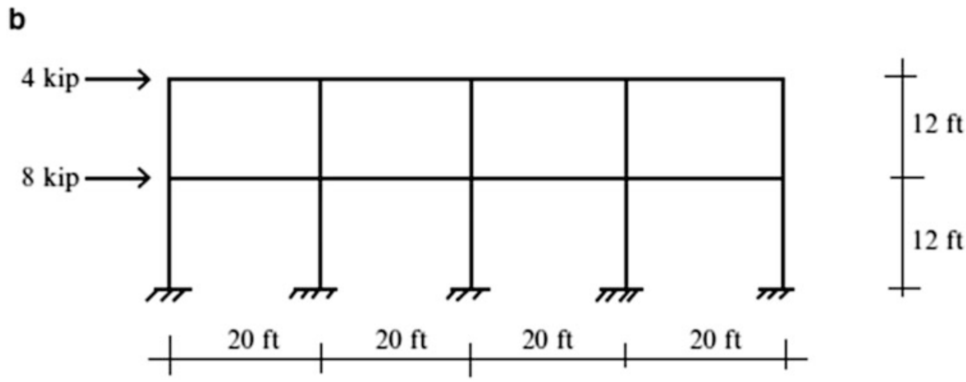


c

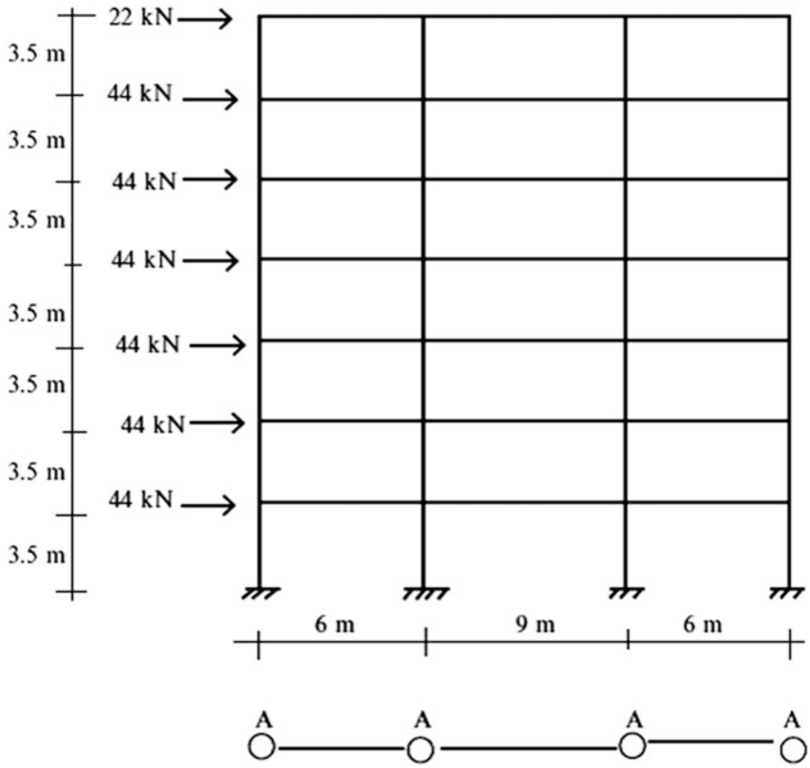


Problem 11.10. For the steel frames shown, estimate the axial force, shear force, and moments for all of the members. Use the Stiffness method. Take $I_c = 480$ in.^{4}, $A_c = 40$ in.^{2} for all the columns and $I_b = 600$ in.^{4}, $A_b = 60$ in.^{2} for all the beams.

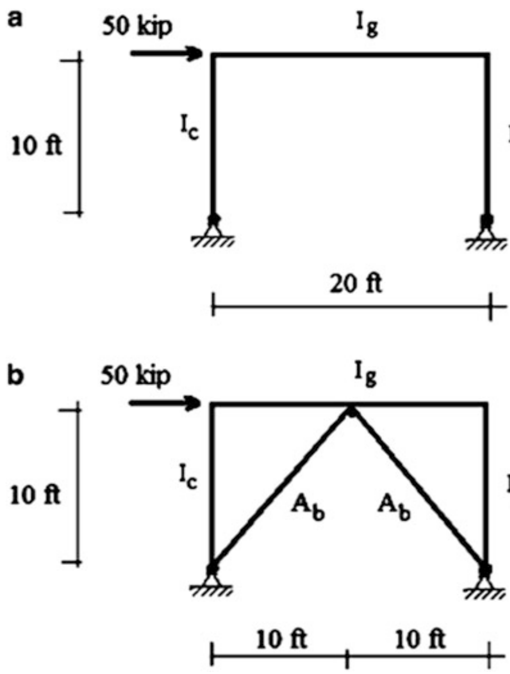




Problem 11.11. Estimate the column axial forces in the bottom story for the distribution of column areas shown.



Problem 11.12. Estimate the column shears for cases (a) and (b). Compare your results with computer-based solutions. Assume $I_g = 300$ in.^{4}, $A_g = 20$ in.^{2}, $I_c = 100$ in.^{4}, $A_c = 10$ in.^{2}, $A_b = 0.5$ in.^{2}, and $E = 29{,}000$ ksi.



Problem 11.13. Consider the rigid steel frame with bracing shown below. Estimate the column shears and brace forces. Compare your results with a computer-based solution. Take $I_g = 120(10)^6$ mm^4, $A_g = 6000$ mm^2, $I_c = 40(10)^6$ mm^4, $A_c = 2000$ mm^2, $A_b = 650$ mm^2, and $E = 2000$ GPa.