# 5 Cable Structures

**Abstract**

Historically, cables have been used as structural components in bridge structures. In this chapter, we first examine how the geometry of a cable is related to the loading that is applied to it. We treat concentrated loadings first and then incorporate distributed loadings leading up to a theory for continuously loaded inclined cables. We also analyze the effect of temperature on the cable geometry. Lastly, we develop an approximate formula for estimating the stiffness of a cable modeled as an equivalent straight member. This modeling strategy is used when analyzing cable-stayed structures.

## 5.1 Introduction

A cable is a flexible structural component that offers no resistance when compressed or bent into a curved shape. Technically, we say a cable has zero bending rigidity. It can support only tensile loading. The first cables were made by twisting vines to form a rope-like member. There are many examples of early cable suspension bridges dating back several thousand years. With the introduction of iron as a structural material, cables were fabricated by connecting wrought iron links. Figure 5.1 shows an example of an iron link suspension bridge, the Clifton Suspension Bridge near Bristol, England built in 1836–1864 and designed by Isambard Brunel.

When high-strength steel wires became available, steel replaced wrought iron as the material of choice for cables. Modern cables are composed of multiple wires (up to 150 wires) clustered in a circular cross-section and arranged in a parallel pattern, as illustrated in Fig. 5.2. This arrangement is used for cable-stayed bridges and suspension bridges. The cable is normally coated with a protective substance such as grease and wrapped or inserted in a plastic sheathing.

One of the most notable early applications of steel cables was the Brooklyn Bridge built in 1870–1883 and designed by John Roebling and Wilhelm Hildebrandt. John Roebling also invented and perfected the manufacture of steel wire cable which was used for the bridge. At the time of completion, the total length of the Brooklyn Bridge was 50 % greater than any existing suspension bridge, an extraordinary advancement in bridge engineering (Fig. 5.3).



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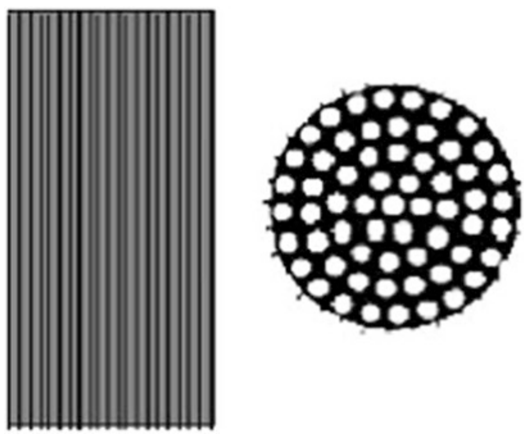
**Fig. 5.1** Clifton Suspension Bridge, England

**Fig. 5.2** Cable–strand arrangements



Cable nets are also used as the primary structural elements for long-span horizontal roof structures. Figure 5.4 shows a single-layer cable net structure with a double-curved saddle-shaped surface designed by Schlaich Bergermann and partners for a stadium in Kuwait.

Cable-stayed structures employ cables fabricated from ultra high-strength steel to allow for the high level of tension required for stiffness. The cable-stayed bridge concept has emerged as the predominant choice for main spans up to about 1000 m, replacing the conventional truss structural system. Figure 5.5 shows the Normandy Bridge, with a main span of 856 m. Built in 1995, it held the record for the largest main span until 1999, when it was exceeded by the Tatara Bridge in Japan.

**Fig. 5.3** Brooklyn Bridge, USA

**Fig. 5.4** Doubly curved single-layer cable net, Kuwait

**Fig. 5.5** Normandy Bridge, France

## 5.2 Cables Subjected to Concentrated Loads

### 5.2.1 Horizontal Cables

Suppose we conduct the following experiment shown in Fig. 5.6. We start with a horizontally aligned cable that is pin connected at A, supported with a roller support at B, and tensioned with a force $H$. We then apply a concentrated load, $P$, at mid-span. The cable adopts the triangular shape shown under the action of $P$. Two questions are of interest. Firstly, why a triangular shape? Secondly, how is the

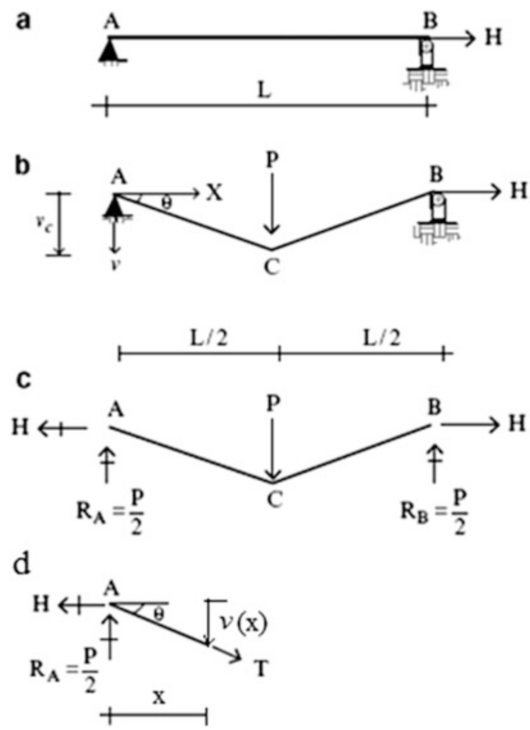



**Fig. 5.6** Transverse loading on pretensioned cable. (**a**) Axial load. (**b**) Transverse load added. (**c**) Free body diagram. (**d**) Free body diagram of cable segment

downward vertical displacement at mid-span related to $P$ and $H$? Historically, the term "sag" is used to describe the vertical motion of a cable.

We answer these questions by noting that the magnitude of the moment at any section along the length of the cable must be zero since a cable has no resistance to bending. Summing moments about B

$$\sum_{\text{at B}} M = R_A L - P\frac{L}{2} = 0 \quad \Rightarrow \quad R_A = \frac{P}{2} \uparrow$$

Next, we consider the free body diagram for the arbitrary segment shown in Fig. 5.6d. Setting the moment at $x$ equal to zero leads to an expression for the sag, $v(x)$.

$$\sum M_{\text{at}\,x} = \frac{P}{2}x - Hv(x) = 0 \tag{5.1}$$

$$v(x) = \frac{P}{2H}x \tag{5.2}$$

Finally, evaluating $v(x)$ at $x = L/2$ results in an equation relating $v_C$ and $P$.

$$v_C = \frac{PL}{4H} \tag{5.3}$$

The relationship between $v_C$ and $H$ is plotted below in Fig. 5.7. Usually, one specifies $H$ and determines $v_C$. However, there are cases where one specifies $v_C$ and determines the required value of $H$. *In general for cable systems, one needs to specify either a force or a sag in order to define the solution.*

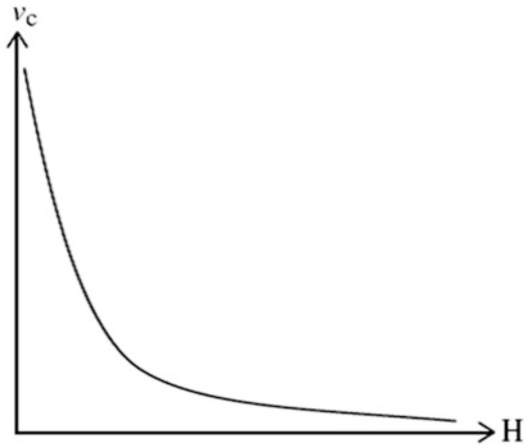
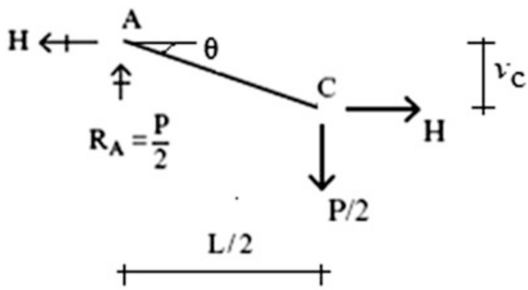



**Fig. 5.7** Relationship between $v_c$ and H

The tension in the cable is given by

$$T = \sqrt{H^2 + \left(\frac{P}{2}\right)^2} = H\sqrt{1 + \left(\frac{P}{2H}\right)^2} \tag{5.4}$$

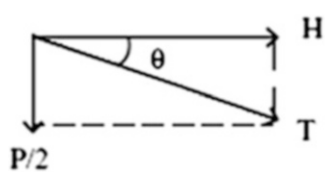



Noting that the angle of inclination of the cable is related to the sag by

$$\tan\theta = \frac{v_C}{L/2} = \frac{P/2}{H} \tag{5.5}$$

leads to an alternative expression for the tension,

$$T = H\sqrt{1 + \left(\frac{P}{2H}\right)^2} = H\sqrt{1 + \tan^2\theta} = \frac{H}{\cos\theta} \tag{5.6}$$




When $\theta$ is small, $T$ is approximately equal to $H$.

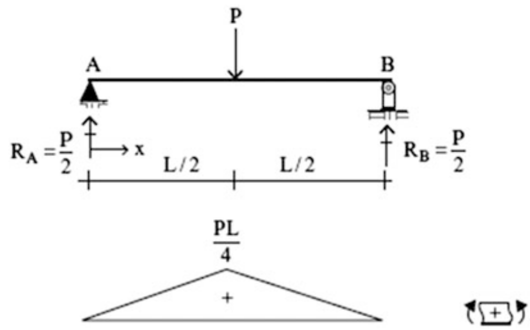



**Fig. 5.8** Moment distribution for simply supported beam $M_0(x)$

Equation (5.1) combines two moment distributions, one due to the transverse loading $P$ and the other due to $H$. The moment due to $P$ can be interpreted as the moment in a simply supported beam spanning between points A and B, the support points for the cable. Figure 5.8 shows this distribution.

We express (5.1) as

$$M_0(x) - v(x)H = 0 \tag{5.7}$$

where $M_0(x)$ is the moment due to the transverse loading acting on the simply supported beam spanning between A and B. Then, the expression for the sag can be written as

$$v(x) = \frac{M_0(x)}{H} \tag{5.8}$$

We interpret this result as follows. *The shape of the vertical sag of the cable from the horizontal chord is a scaled version of the moment diagram for the transverse loading acting on a simply supported beam spanning between the cable supports.*

We extend this reasoning to a cable subjected to multiple concentrated loads. Figure 5.9a illustrates this case. The moment diagram for a set of concentrated loads is piecewise linear, with peak values at the points of application of the concentrated loads. It follows from (5.8) that the shape of the cable is also piecewise linear. One generates $M_0(x)$, the corresponding shear $V_0(x)$, the displacement $v$, and the tension $T$. Details are listed in Fig. 5.9b–d. Note that one has to specify either $H$ or one of the vertical coordinates ($v_C$ or $v_D$) in order to compute the shape.

$$T = \sqrt{V_0^2 + H^2} = \frac{H}{\cos\theta}$$

*Example 5.1* Cable with Multiple Concentrated Loads

**Given:** The cable and loading shown in Fig. E5.1a.

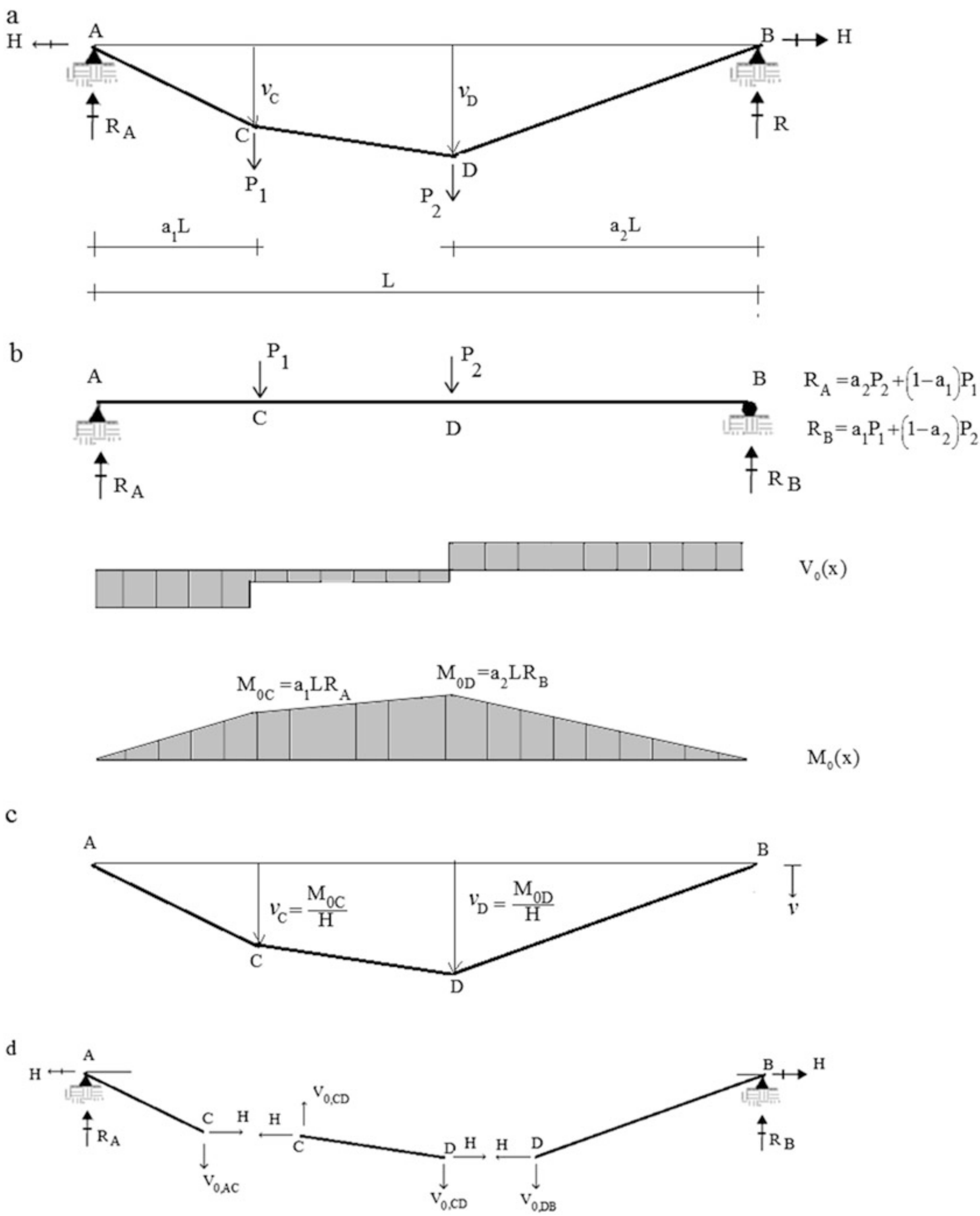



**Fig. 5.9** Cable with two concentrated loads. (**a**) Loading. (**b**) $V_0(x)$, $M_0(x)$ diagrams. (**c**) Cable sag profile. (**d**) Cable tension computation

**Determine:** The shape corresponding to this loading. Assume (a) $v_D = 6$ ft (b) $v_D = 12$ ft.

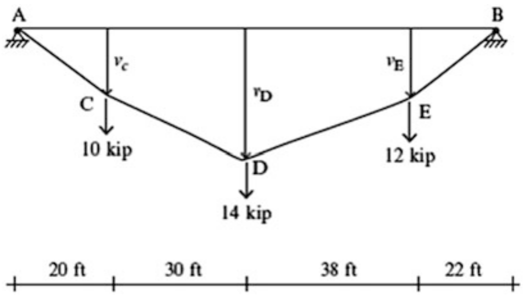



**Fig. E5.1a** Cable geometry and loading

**Solution:** First, we find the vertical reactions and generate the shear diagram $V_0(x)$ and moment diagram, $M_0(x)$, treating chord AB as a simply supported beam acted upon by the three vertical forces (Fig. E5.1b).

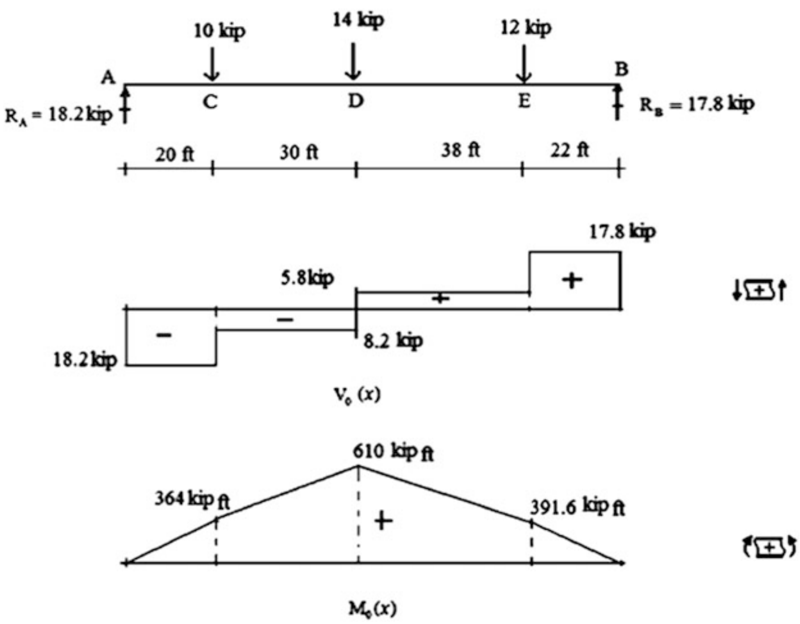



**Fig. E5.1b** Simply supported beam results

The downward vertical sag from the chord AB is determined with (5.8).

$$+\downarrow \quad v(x) = \frac{M_0(x)}{H}$$

In order to compute $v(x)$, we need the horizontal force, $H$.

(a) Taking $v_D = 6$ ft results in

$$6 = \frac{610}{H} \quad \Rightarrow \quad H = 101.67\,\text{kip}$$

The remaining sags are

$$v_C = \frac{364}{101.67} = 3.58\,\text{ft}$$
$$v_E = \frac{391.6}{101.67} = 3.85\,\text{ft}$$

The final results for the shape are plotted below (Fig. E5.1c).

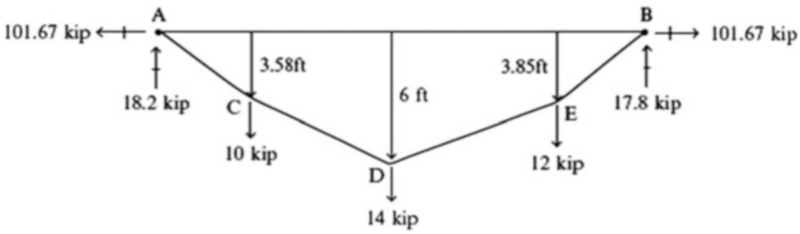



**Fig. E5.1c** Sag profile for $v_D = 6$ ft

Once the shape is known, one can find the tension in the various segments using (Fig. E5.1d)

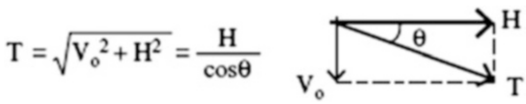



**Fig. E5.1d** Force decomposition

$$T_{AC} = \sqrt{18.2^2 + 101.67^2} = 103.3\,\text{kip}$$
$$T_{CD} = \sqrt{8.2^2 + 101.67^2} = 102\,\text{kip}$$
$$T_{DE} = \sqrt{5.8^2 + 101.67^2} = 101.8\,\text{kip}$$
$$T_{EA} = \sqrt{17.8^2 + 101.67^2} = 103.2\,\text{kip}$$

(b) Taking $v_D = 12$ ft results in

$$H = \frac{610}{12} = 50.83\,\text{kip}$$
$$v_C = \frac{364}{50.83} = 7.16\,\text{ft}$$
$$v_E = \frac{391.6}{50.83} = 7.7\,\text{ft}$$

and

$$T_{AC} = 54\,\text{kip}$$
$$T_{CD} = 51.5\,\text{kip}$$
$$T_{DE} = 51.16\,\text{kip}$$
$$T_{EB} = 53.85\,\text{kip}$$

The sag profile is plotted below (Fig. E5.1e)

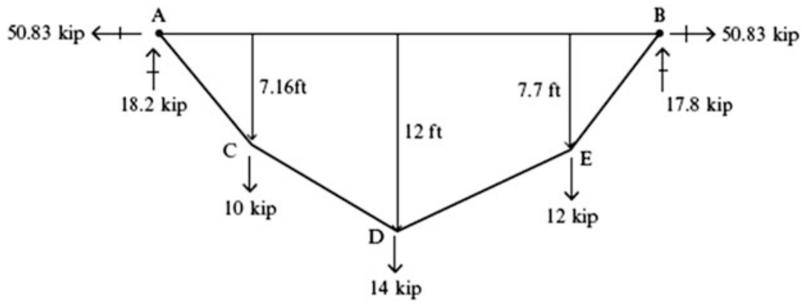
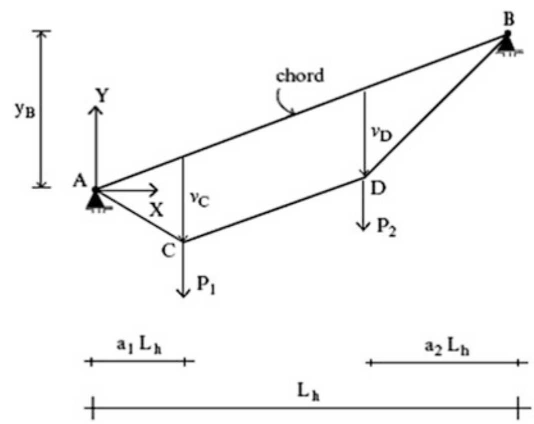



**Fig. E5.1e** Sag profile for $v_D = 12$ ft

Note that increasing the prescribed value of $v_D$ decreases the cable forces.

### 5.2.2 Inclined Cables

**Fig. 5.10** Inclined cable with concentrated loads




When the cable is inclined, we follow the same approach except that now we measure the *cable sag with respect to the inclined chord*. Consider the cable defined in Fig. 5.10. This example differs from the previous examples only with respect to the inclination of the chord AB.

The reactions and corresponding bending moment distribution generated by the vertical loads are shown in Fig. 5.11. Note that these moment results are identical to the results for the case of a horizontal chord orientation. The reactions generated by the horizontal cable force, $H$ are defined in Fig. 5.12.

Setting the total moment equal to zero leads to

$$M_0(x) - H\frac{y_B}{L_h}x + Hy(x) = 0$$
$$\Downarrow$$
$$M_0(x) = H\left(\frac{y_B}{L_h}x - y(x)\right) \equiv Hv(x)$$
$$\Downarrow$$
$$v(x) = \frac{M_0(x)}{H}$$

Note that the solution for $v(x)$ is identical to the results for the horizontal cable except that *now one measures the sag from the inclined chord*.

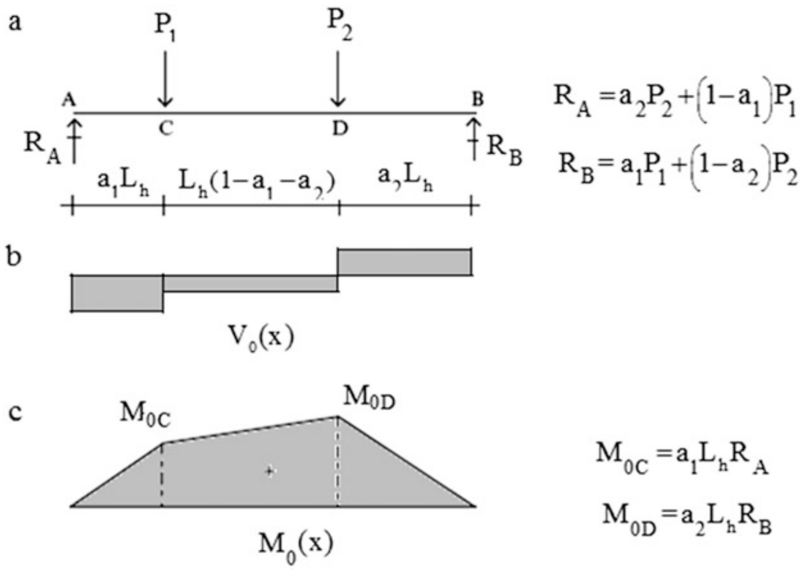



**Fig. 5.11** Simply supported beam results. (**a**) Vertical loading. (**b**) $V_0(x)$ diagram. (**c**) $M_0(x)$ diagram

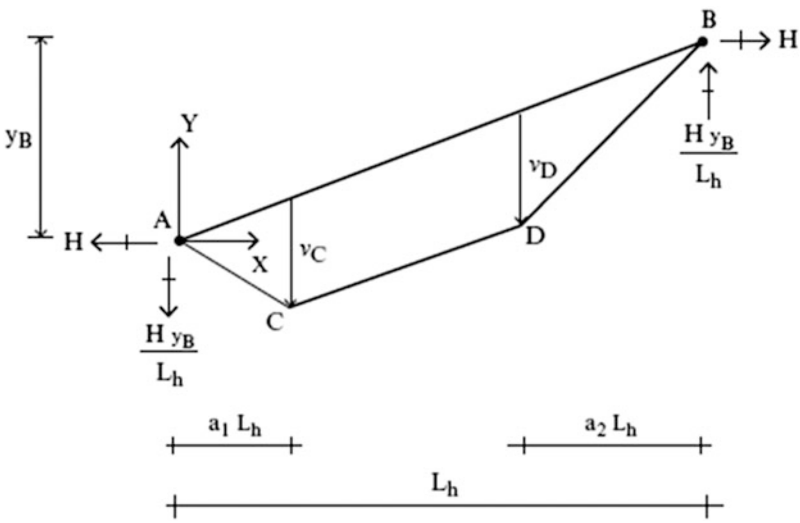



**Fig. 5.12** Reactions due to horizontal force, $H$

*Example 5.2* Analysis of an Inclined Cable

**Given:** The inclined cable and loading shown in Fig. E5.2a.

**Determine:** The sag of the cable. Assume $v_D = 6$ ft.

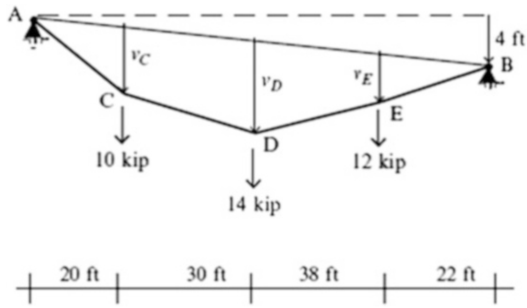



**Fig. E5.2a** Inclined geometry

**Solution:** According to the theory presented above, the sag with respect to the inclined chord is given by

$$+\downarrow \quad v(x) = \frac{M_0(x)}{H}$$

where $M_0(x)$ is the simply supported beam moment (Fig. E5.2b).

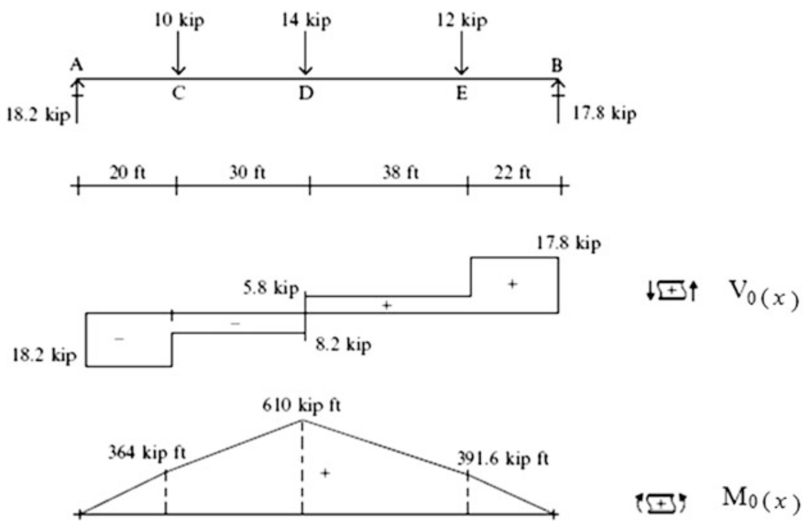



**Fig. E5.2b** Simply supported beam results

Then,

$$v_C = \frac{364}{H} \quad v_D = \frac{610}{H} \quad v_E = \frac{391.6}{H}$$

For $v_D = 6$ ft, the value of $H$ follows from

$$H = \frac{M_{0D}}{v_D} = \frac{610}{6} = 101.67\,\text{kip}$$

Finally, the values of sag at C and E are

$$v_C = \frac{364}{101.67} = 3.58\,\text{ft}$$

$$v_E = \frac{391.6}{101.67} = 3.85\,\text{ft}$$

To determine the tension, we need to compute the vertical shear in each panel. The vertical reactions due to $H$ (Fig. E5.2c) are

$$\frac{H y_B}{L} = \frac{101.67(4)}{110} = 3.7\,\text{kip}$$

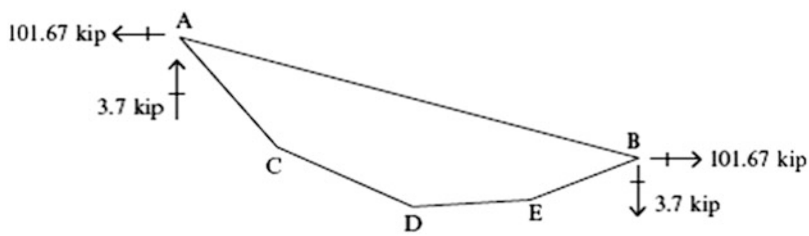



**Fig. E5.2c** Vertical reactions due to $H$

The net results for vertical shear are shown in Fig. E5.2d.

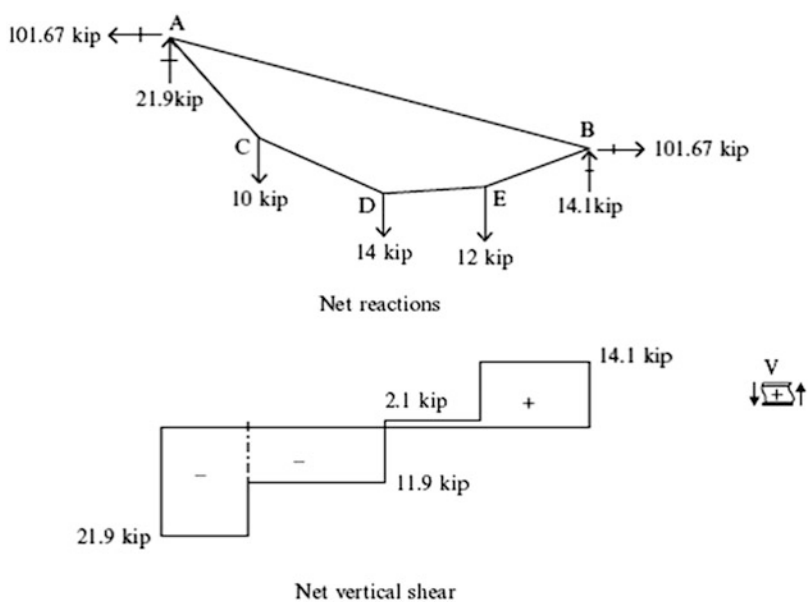



**Fig. E5.2d** Vertical shear

Lastly, the tension in each segment is computed using these values for $V$ and $H$. The maximum tension is in segment AC.

$$T_{AC} = \sqrt{21.9^2 + 101.67^2} = 104\,\text{kip}$$
$$T_{CD} = \sqrt{11.9^2 + 101.67^2} = 102.4\,\text{kip}$$
$$T_{DE} = \sqrt{2.1^2 + 101.67^2} = 101.7\,\text{kip}$$
$$T_{EA} = \sqrt{14.09^2 + 101.67^2} = 102.6\,\text{kip}$$

## 5.3 Cables Subjected to Distributed Loading

### 5.3.1 Horizontal Cable: Uniform Loading per Horizontal Projection

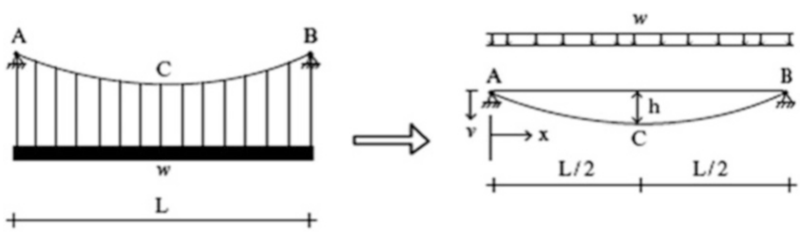



**Fig. 5.13** Cable with a uniformly distributed loading

We consider next the cable system shown in Fig. 5.13. The cable supports a horizontal platform, which in turn, supports a uniform vertical loading. We represent the action of the closely spaced vertical hangers on the cable *as a uniform downward loading per unit horizontal projection*. The self weight of the cable, which is usually small in comparison to the applied loading, is neglected. Following the procedure described in the previous section, we determine the moment diagram for a simply supported beam spanning between the end supports. *The sag of the cable with respect to the horizontal chord AB is an inverted scaled version of the moment diagram*. The details are shown in Fig. 5.14.

The sag, tan $\theta$, and $T$ are given by

$$\begin{aligned} v(x) &= \frac{M_0(x)}{H} = \frac{(wL/2)x - (wx^2/2)}{H} = \frac{w}{2H}(Lx - x^2) \\ \tan\theta &= \frac{dv}{dx} = \frac{1}{H}\frac{dM_0(x)}{dx} = \frac{w}{2H}(L - 2x) \\ T &= \frac{H}{\cos\theta} \end{aligned} \tag{5.9}$$

It follows that the shape due to a uniform load is parabolic and the maximum sag occurs at mid-span, point c.

$$v_C = h = \frac{w}{2H}\left(L^2/2 - L^2/4\right) = \frac{wL^2}{8H} \tag{5.10}$$

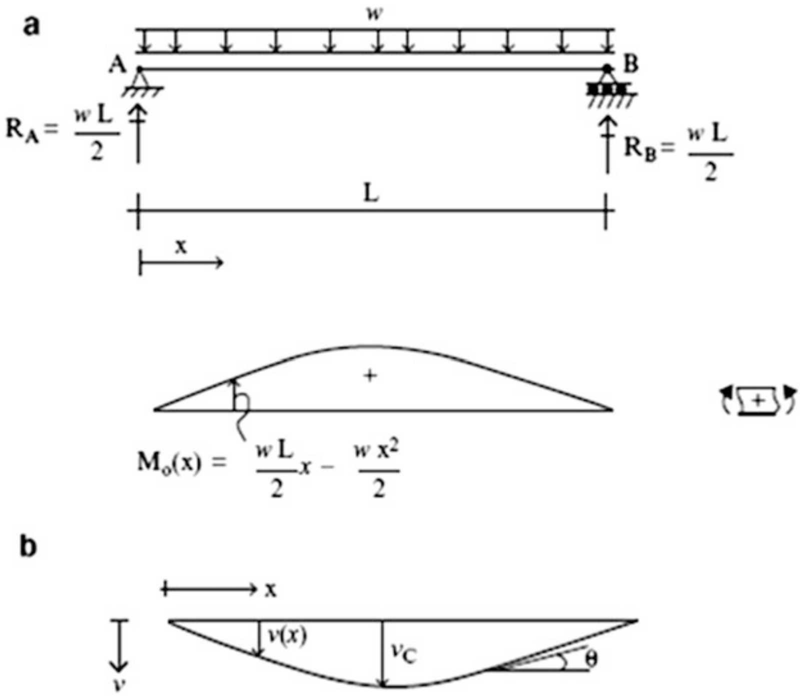



**Fig. 5.14** Horizontal cable. (**a**) Simply supported beam results. (**b**) Cable sag

*Example 5.3*

**Given:** The cable shown in Fig. E5.3a. The loading and desired cable geometry is specified.
**Determine:** The value of the horizontal tension force, *H* and the peak value of cable tension, which produces this geometry under the given loading.

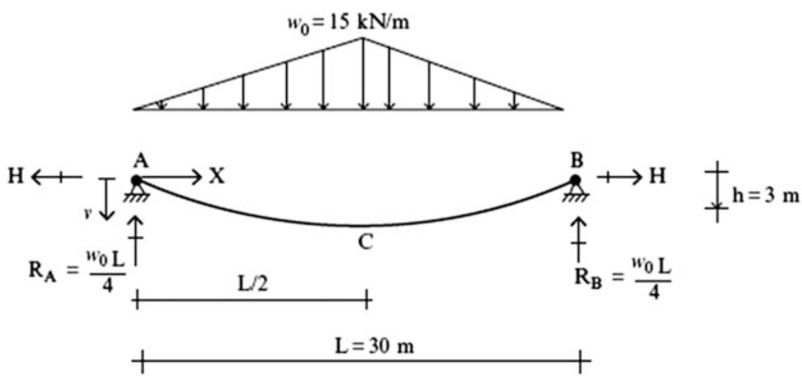
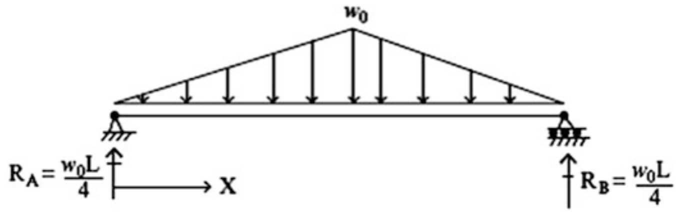



**Fig. E5.3a**

**Solution:** We note that the maximum value of $v$ occurs at $x = L/2$. Then, specializing (5.9) for this value of $x$ leads to the value of $H$:




$$M_0(x) = \frac{w_0 L}{4}x - \frac{w_0}{3L}x^3 \quad 0 \le x \le \frac{L}{2}$$

$$\tan\theta = \frac{1}{H}\frac{dM_0(x)}{dx} = \frac{1}{H}\left(\frac{w_0 L}{4} - \frac{w_0}{L}x^2\right)$$

$$H = \frac{M_0(x = L/2)}{v_C} = \frac{w_0 L^2}{12}\frac{1}{v_C} = \frac{(15)(30)^2}{12(3)} = 375\,\text{kN}$$

The tension is related to $H$ by:

$$T = \frac{H}{\cos\theta}$$

The peak values of $\theta$ occur at $x = 0$ and $x = L$.

$$\tan\theta_{\text{at}\,x=0} = \frac{1}{H}\left(\frac{w_0 L}{4}\right) = \frac{(15)(30)}{(375)(4)} = 0.3$$

$$\theta_{\text{at}\,x=0} = 16.7°$$

It follows that

$$\theta_{\max} = \pm 16.7°$$

$$T_{\max} = \frac{H}{\cos\theta} = 391.5\,\text{kN}$$

### 5.3.2 Inclined Cables

Suppose the cable is inclined and subjected to an arbitrary loading. We define the shape by the function $y(x)$. Figure 5.15 defines this notation.

Since the cable has no bending rigidity, the shape of the cable must adjust itself so that the resultant moment due to the vertical load and $H$ vanishes at all points along the cable. Then, setting the total moment at $x$ equal to zero leads to

$$\sum M_{\text{at}\,x} = M_0(x) + Hy(x) - \frac{Hy_B}{L_h}x = 0$$

$$\Downarrow \tag{5.11}$$

$$y(x) = \frac{y_B}{L_h}x - \frac{M_0(x)}{H}$$

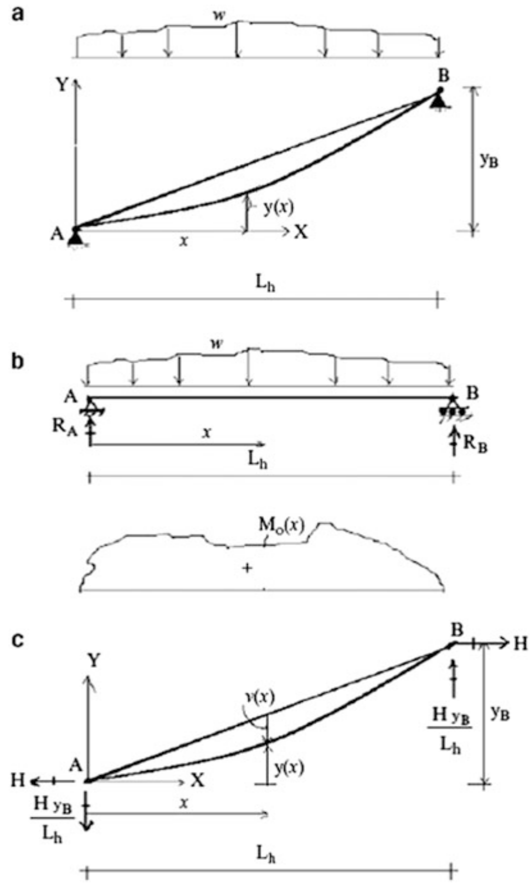



**Fig. 5.15** Inclined cable geometry—arbitrary loading. (**a**) Geometry-arbitrary loading. (**b**) Simply supported beam results. (**c**) Reactions due to horizontal force, *H*

We note from Fig. 5.15 that

$$\frac{y(x) + v(x)}{x} = \frac{y_B}{L_h}$$
$$\Downarrow \tag{5.12}$$
$$y(x) = \frac{y_B}{L_h}x - v(x)$$

Finally, equating (5.11) and (5.12) leads to the expression for the sag,

$$v(x) = \frac{M_0(x)}{H} \tag{5.13}$$

We observe that the solution for the sag is identical to the result that we obtained for the horizontal chord orientation *except now one measures the sag from the inclined chord*. The solution is also similar to the case of a set of concentrated loads.

The lowest point on the cable (point C in Fig. 5.16) is determined by setting the slope equal to zero.

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_C} = 0 \tag{5.14}$$

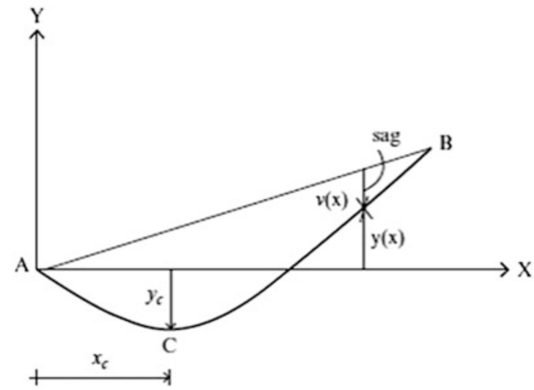



**Fig. 5.16** Cable geometry—lowest point

Noting (5.11),

$$\frac{y_B}{L_h} - \frac{1}{H}\frac{dM_0(x)}{dx} = 0 \tag{5.15}$$

For the case where the distributed load is uniform, $M_0(x)$ is parabolic, and (5.15) expands to

$$\frac{y_B}{L_h} - \frac{1}{H}\left(-wx_C + \frac{wL_h}{2}\right) = 0 \tag{5.16}$$

Solving for $x$ leads to

$$x_C = \frac{L_h}{2} - \frac{y_B}{L_h}\frac{H}{w} \tag{5.17}$$

For an arbitrary loading, we need to use (5.15).

*Example 5.4*

**Given:** The inclined cable is defined in Fig. E5.4a. Point C is the lowest point of the cable.

**Determine:** The coordinates of point C and the peak values of cable tension.

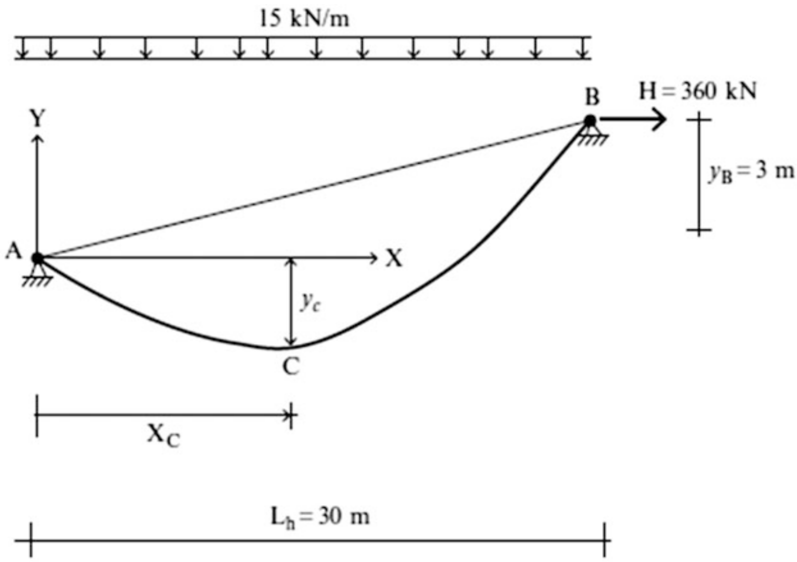



**Fig. E5.4a**

**Solution**: Noting (5.17),

$$x_C = \frac{L_h}{2} - \frac{y_B}{L_h}\frac{H}{w} = \frac{30}{2} - \frac{3}{30}\left(\frac{360}{15}\right) = 12.6\,\text{m}$$

Applying (5.11) for point C,

$$y_C = x_C\frac{y_B}{L_h} - \frac{w}{2H}\left\{L_h x_C - (x_C)^2\right\} = 12.6\left(\frac{3}{30}\right) - \frac{15}{2(360)}\left(30(12.6) - (12.6)^2\right)$$
$$= -3.3\,\text{m}$$

Given $H$, we can find the cable tension at any point with:

$$T = \frac{H}{\cos\theta}$$

where

$$\tan\theta = \frac{dy}{dx} = \frac{y_B}{L_h} - \frac{wL_h}{2H} + \frac{wx}{H}$$

The critical locationsare at the support points A and B.

$$\tan\theta_A = \frac{3}{30} - \frac{15(30)}{2(360)} = -0.525 \quad \theta_A = -27.7^\circ$$
$$\tan\theta_B = \frac{3}{30} - \frac{15(30)}{2(360)} + \frac{15(30)}{360} = +0.725 \quad \theta_B = +35.9^\circ$$
$$T_A = \frac{H}{\cos\theta_A} = 406.6\,\text{kN}$$
$$T_{max} = T_B = \frac{H}{\cos\theta_B} = 444.6\,\text{kN}$$

## 5.4 Advanced Topics

This section deals with the calculation of arch length, the axial stiffness, and the effect of temperature. We also discuss a modeling strategy for cable-stayed structures such as guyed towers and cable-stayed bridges.

### 5.4.1 Arc Length

We consider first the uniformly loaded horizontal cable shown in Fig. 5.17. We have shown that the sag profile due to a uniform load is parabolic,

$$v(x) = \frac{w\mathrm{L}}{2H}x - \frac{wx^2}{2H}$$

and the maximum sag occurs at mid-span,

$$v_{\max} \equiv h = \frac{wL^2}{8H}$$

Given $H$ and $L$, of interest is the total arc length of the cable. We need this quantity in order to determine the effect on the cable geometry of a temperature increase in the cable. Figure 5.17 shows the initial and loaded shapes of the cable. Note that the deformed length is greater than $L$. We denote this quantity as $L + \Delta$.

The differential arc length, d$s$, is related to its horizontal and vertical projections by

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{5.18}$$

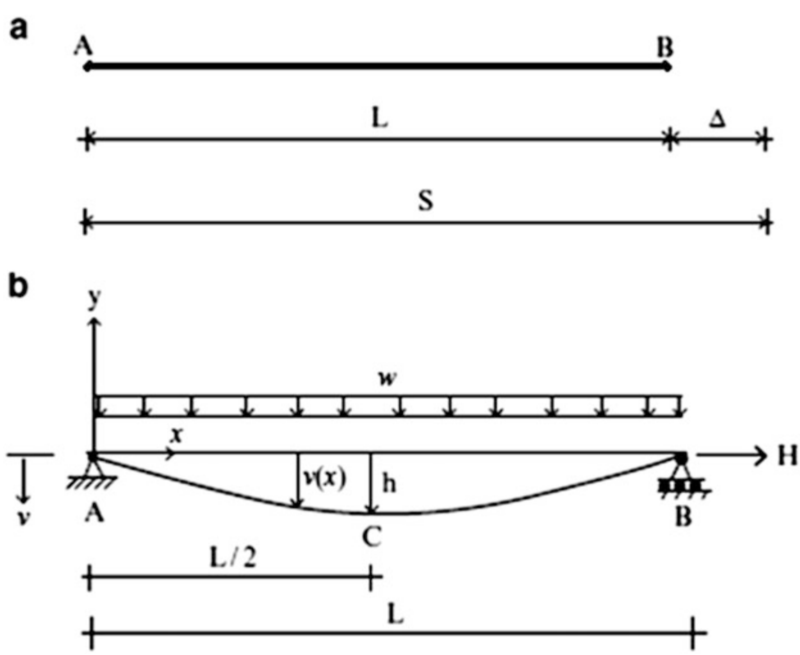



**Fig. 5.17** Cable geometry. (**a**) Initial unloaded. (**b**) Loaded shape

Integrating between 0 and $L$ leads to an expression for the total arc length

$$S = \int_0^L \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx \tag{5.19}$$

Given $y(x)$, one evaluates the integral using either symbolic or numerical integration. When the cable is horizontal, $y(x) = -v(x)$.

$$y(x) = \frac{wL}{2H}\left(-x + \frac{x^2}{L}\right) = \frac{4h}{L}\left(-x + \frac{x^2}{L}\right)$$

When the maximum sag $h$ is small with respect to $L$, we can assume that $dy/dx$ is small with respect to 1 and simplify the integral in (5.19) using the following binominal series expression,

$$\begin{aligned} &(1 + f)^{\frac{1}{2}} = 1 + \frac{1}{2}f - \frac{1}{8}f^2 + \cdots \\ &|f| < 1 \end{aligned} \tag{5.20}$$

Taking $f = (dy/dx)^2$ and retaining only the first three terms, we obtain the following approximation for $S$:

$$S \approx \int_0^L \left\{1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 - \frac{1}{8}\left(\frac{dy}{dx}\right)^4\right\} dx \tag{5.21}$$

Noting Fig. 5.17a, we see that $\Delta \approx \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx$ for a small sag ratio.

Lastly, we evaluate $S$ for the case when the loading is uniform. Retaining the first three terms in (5.21) leads to

$$S \approx L\left\{1 + \frac{8}{3}\left(\frac{h}{L}\right)^2 - \frac{32}{5}\left(\frac{h}{L}\right)^4\right\} \tag{5.22}$$

We refer to $h/L$ as the sag ratio. Equation (5.22) shows that the effect of decreasing the sag ratio is to transform the "curved" cable to essentially a straight segment connecting the two end points. *The cables used for guyed towers and cable-stayed bridges have small sag ratios and are approximated as equivalent straight axial elements. We will discuss this topic in a later section.*

*Example 5.5*

**Given:** The cable defined in Fig. E5.5a.

**Determine:** The length of the cable corresponding to this geometry. Also determine the change in geometry due to a temperature increase of 150 °F. Take $\alpha = 6.6 \times 10^{-6}$/°F.

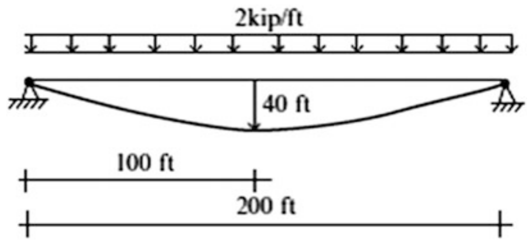



**Fig. E5.5a**

**Solution:** The horizontal reaction due to the loading shown is

$$H = \frac{wL^2}{8h} = 250\,\text{kip}$$

We evaluate $S$ using (5.22),

$$S = 200\left\{1 + \frac{8}{3}\left(\frac{40}{200}\right)^2 - \frac{32}{5}\left(\frac{40}{200}\right)^4\right\} = 200\{1 + 0.107 - 0.01\}$$

$$S = 219.4\,\text{ft}$$

The change in cable length due to a temperature increase is

$$\Delta S = S(\alpha \Delta T) \approx 219.4\left(6.6 \times 10^{-6}\right)(150) \approx 0.217\,\text{ft}$$

This length change produces a change in the sag. We differentiate (5.22) with respect to $h$,

$$\frac{\mathrm{d}S}{\mathrm{d}h} \approx \frac{16}{3}\frac{h}{L} - \frac{128}{5}\left(\frac{h}{L}\right)^3$$

and solve for d$h$.

$$\mathrm{d}h \approx \frac{\mathrm{d}S}{(16/3)(h/L)\left\{1 - 4.8(h/L)^2\right\}}$$

Substituting for d$S$ leads to

$$\mathrm{d}h \approx \frac{0.217}{(16/3)(40/200)\left\{1 - 4.8(40/200)^2\right\}} = 0.25\,\text{ft}$$

Finally, we update $H$ using the new values for $h = 40 + 0.25 = 40.25$ ft

$$H = \frac{wL^2}{8h} = \frac{2(200)^2}{8(40.25)} = 248.5\,\text{kip}$$

The effect of temperature increase on $H$ is small for this geometry.

*Example 5.6*

**Given:** The uniformly loaded inclined cable is shown in Fig. E5.6a.

**Determine:** The sag profile and total arc length.

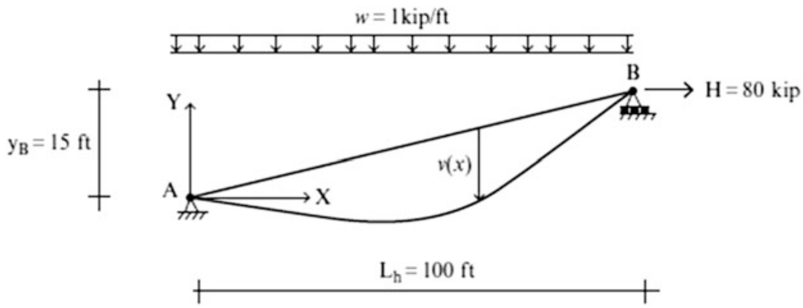



**Fig. E5.6a**

**Solution:** The profile defined in terms of $y(x)$ is given by (5.11). For the given dimensions, it expands to

$$y(x) = \frac{y_B}{L_h}x - \frac{M_0(x)}{H}$$
$$= \frac{15}{100}x - \left(50x - \frac{x^2}{2}\right)\frac{1}{80}$$

Then, the sag profile is given by

$$v(x) = +\left(50x - \frac{x^2}{2}\right)\frac{1}{80} = \frac{5}{8}x - \frac{x^2}{160}$$

We determine the total arc length using (5.19).

$$S = \int_0^{L_h} \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} dx$$

Substituting for $y(x)$, $S$ expands to

$$S = \int_0^{100} \left\{1 + \left[\frac{15}{100} - \frac{1}{80}(50 - x)\right]^2\right\}^{1/2} dx$$

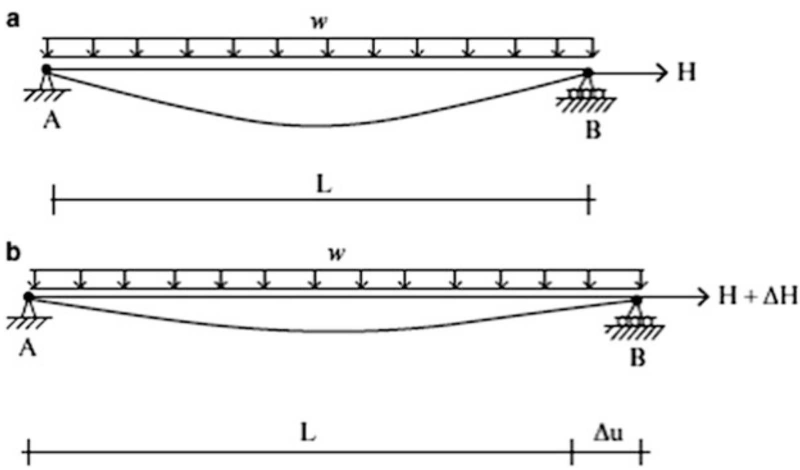
We evaluate the integral using numerical integration. The result is

$$S = 107.16\,\text{ft}$$

### 5.4.2 Equivalent Axial Stiffness

In what follows, we establish a procedure for modeling a shallow horizontal cable as an equivalent straight axial member. Consider the cable shown in Fig. 5.18. Suppose the horizontal force, $H$, is increased by a small amount, say $\Delta H$. This action causes the support at B to displace horizontally, an amount $\Delta u$. The ratio $\Delta H/\Delta u$ is a measure of the axial stiffness for the cable. We interpret it as the tangent stiffness since we perturbed the system from a "loaded" state.

**Fig. 5.18** Actual and perturbed configurations




We generate an expression for the tangent stiffness in the following way. We start with the straight unloaded cable shown in Fig. 5.19 and apply a horizontal force. The cable stretches an amount $u_1$. Next, we apply the uniform downward load, holding $H$ constant. Point B moves to the left, an amount $u_2$. We estimate $u_2$ using (5.21) specified for a parabolic shape and *small* sag ratio,

$$u_2 \approx \int_0^L \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{w^2L^3}{24H^2}$$

The net motion of B is $u_\mathrm{B}$.

$$u_\mathrm{B} = u_1 - u_2 = \frac{HL}{AE} - \frac{w^2L^3}{24H^2} \tag{5.23}$$

Equation (5.23) is plotted in Fig. 5.20. For large $H$, the first term dominates and the behavior approaches the behavior of an axial member. We want to determine $\mathrm{d}H/\mathrm{d}u$. Since $u_\mathrm{B}$ is a nonlinear function of $H$, we first find the derivative $\mathrm{d}u/\mathrm{d}H$, and then invert.

$$\begin{gathered}\frac{\mathrm{d}u_\mathrm{B}}{\mathrm{d}H} = \frac{L}{AE} + \frac{w^2L^3}{12H^3} = \frac{L}{AE}\left\{1 + \frac{1}{12}\frac{AE}{H}\left(\frac{wL}{H}\right)^2\right\} \\ \downarrow \\ \frac{\mathrm{d}H}{\mathrm{d}u_\mathrm{B}} = k_\mathrm{t} = \left(\frac{1}{1 + (1/12)(AE/H)(wL/H)^2}\right)\frac{AE}{L}\end{gathered} \tag{5.24}$$

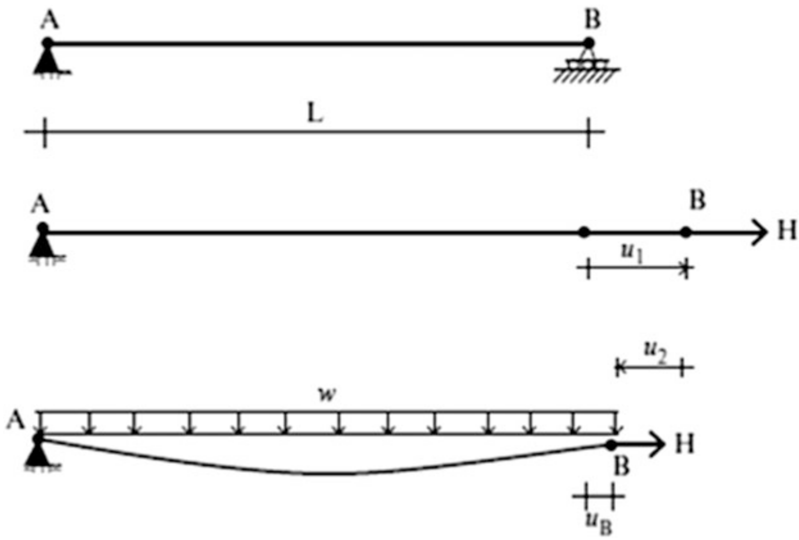



**Fig. 5.19** Deflection patterns

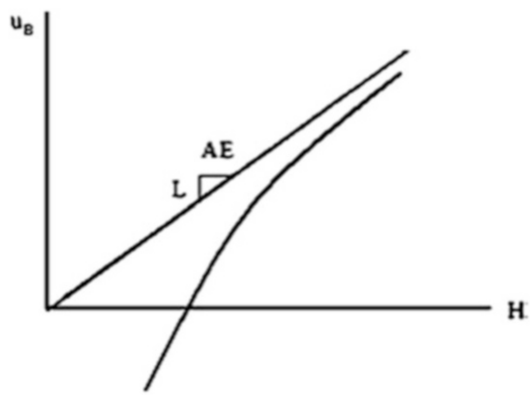



**Fig. 5.20** $u_B$ vs. $H$ relationship

Note that $AE/L$ is the axial stiffness of a straight member. Equation (5.24) shows that the tangent stiffness for the horizontal cable approaches $AE/L$ as the tension $H$ is increased.

The tangent stiffness $k_t$ can also be expressed in terms of a modified elastic modulus $E_{eq}$.

We write (5.24) as $k_t = (A/L)E_{eq}$. Then, the definition equation for $E_{eq}$ follows:

$$E_{eq} = \frac{E}{1 + (1/12)(AE/H)(wL/H)^2} \tag{5.25}$$

In general, $E_{eq} < E$. Substituting the terms,

$$\frac{A}{H} = \frac{1}{\sigma}$$
$$\frac{wL}{H} = 8\left(\frac{h}{L}\right)$$

transforms (5.25) to

$$E_{eq} = \frac{E}{1 + (16/3)(E/\sigma)(h/L)^2} \tag{5.26}$$

where $\sigma$ is the stress in the cable. It follows that the equivalent modulus depends on the initial stress in the cable and the sag ratio. A typical value of initial stress is on the order of 50–100 ksi (344,700–1,034,100 kN/m$^2$). Values of sag ratio range from 0.005 to 0.02. The corresponding variation in $E_{eq}$ for a steel cable with $\sigma = 50$ ksi (344,700 kN/m$^2$) is tabulated below.

| $E/\sigma$ | $h/L$ | $E_{eq}/E$ |
|---|---|---|
| 580 | 0.005 | 0.928 |
| | 0.01 | 0.764 |
| | 0.02 | 0.447 |

Note that a typical sag ratio of 0.01 results in a 25 % reduction in $E$. One uses high-strength steel strands, on the order of 150 ksi (1,034,100 kN/m$^2$) yield stress, for cable-stayed structures in order to minimize their loss of stiffness due to cable sag.

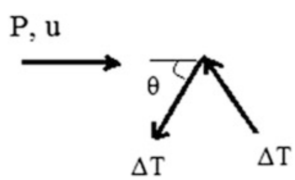
### 5.4.3 Equivalent Axial Stiffness for an Inclined Cable

In this section, we extend the modeling strategy to deal with shallow inclined cables. Inclined cables with *small sag* ratios are used in cable-stayed bridges and also as supports for guyed towers. Figure 5.21 shows the Millau Viaduct Bridge in France. Figure 5.22 illustrates a two-cable scheme for a guyed tower subjected to wind loading.

We model each cable as a straight axial member with a modulus of elasticity, $E_{eq}$ which depends on the initial tension and geometry of the cable. This approach is reasonable when the changes in geometry and tension due to the applied load are small in comparison to the initial properties.

Equilibrium of the tower requires

$$2\Delta T \cos\theta = P \tag{5.27}$$




The corresponding extension of the "equivalent" straight member due to $\Delta T$ is:

$$\Delta e = \frac{\Delta T L}{A E_{eq}} \tag{5.28}$$

Lastly, we relate $\Delta e$ to the horizontal displacement $u$.

$$\Delta e = u \cos\theta$$

Combining these equations leads to an expression relating $P$ and $u$.

$$P = \left[\frac{2AE_{eq}}{L}(\cos\theta)^2\right] u \tag{5.29}$$

The bracketed term represents the lateral stiffness of the tower for a lateral load applied at the top of the tower. Given $E_{eq}$, one can evaluate the lateral response of the tower with (5.29).

**Fig. 5.21** Millau Viaduct Bridge in France

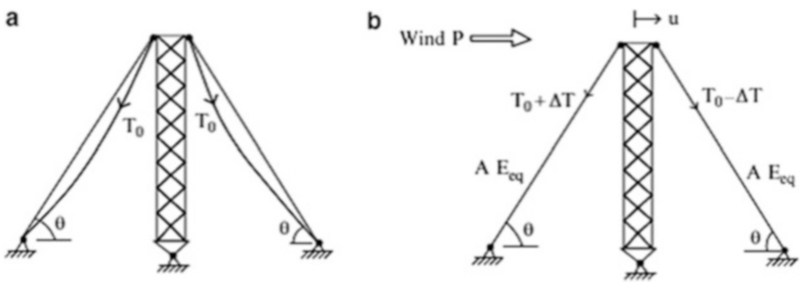



**Fig. 5.22** Guyed tower modeling scheme. (**a**) Initial position. (**b**) Loaded position

We develop an expression for $E_{eq}$ by modifying (5.25). Figure 5.23 shows a typical inclined cable and the notation introduced here. The loading acting on the cable is assumed to be the self weight, $w_g$. Also when the cable is rotated from the horizontal position up to the inclined position, $H$ is now the cable tension, $T$; the normal distributed load $w$ becomes $w_g \cos \theta$; and the loading term becomes

$$wL = \left(w_g \cos \theta\right)L = w_g L_h \tag{5.30}$$

Substituting for these terms in (5.25) leads to

$$E_{eq} \approx \frac{E}{1 + (1/12)(AE/T)\left(w_g L_h/T\right)^2} \tag{5.31}$$

Lastly, we introduce the following definitions involving the initial stress and weight density,

$$\begin{gathered} \frac{A}{T} = \frac{1}{\sigma} \\ w_g = \gamma_g A \end{gathered} \tag{5.32}$$

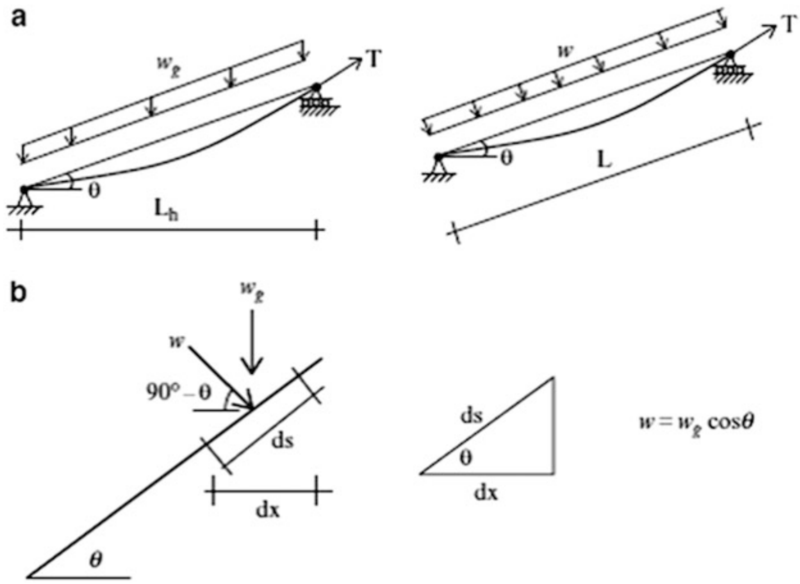



**Fig. 5.23** Inclined cable geometry. (**a**) Vertical versus normal loading. (**b**) Loading components

The final form of (5.31) for an individual cable is

$$E_{eq} = \frac{E}{1 + (1/12)(E/\sigma)\left(\gamma_g L_h/\sigma\right)^2} \tag{5.33}$$

Equation (5.33) is known as Ernst's Formula. This expression is used when modeling the cables in a cable-stayed scheme with equivalent axial member properties.

*Example 5.7*

**Given:** The steel cable shown in Fig. E5.7a. Take the initial stress as 700 MPa.

**Determine:** The equivalent modulus, $E_{eq}$.

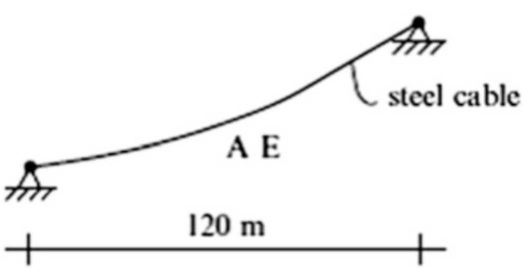
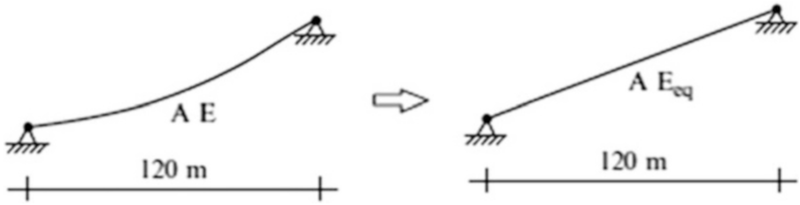



**Fig. E5.7a**

**Solution:** The properties of steel are $E = 200$ GPa and $\gamma_g = 77$ kN/m$^3$. Substituting these values in (5.33) leads to

$$\frac{E_{eq}}{E} = \frac{1}{1 + (1/12)\left(200\left(10^3\right)/700\right)(77(120)/700{,}000)^2} = 0.996$$

One uses $E_{eq}$ when specifying the properties of the "equivalent" straight axial member.




## 5.4.4 Cable Shape Under Self Weight: Catenary

There are cases where the loading on a cable is due only to self weight. Electrical transmission lines are one example. The previous analyses have assumed the loading is defined in terms of the horizontal projection (d$x$). This assumption is reasonable when the slope of the cable is small. In order to investigate the case when the slope is not small, we need to work with the exact equilibrium equation.

Consider the segment shown in Fig. 5.24b. Enforcing equilibrium and noting that the loading is vertical leads to following equations:

$$\begin{aligned} &\sum F_y = 0 \quad \frac{\mathrm{d}}{\mathrm{d}x}(T\sin\theta)\mathrm{d}x = w_g \mathrm{d}s \\ &\sum F_x = 0 \quad \frac{\mathrm{d}}{\mathrm{d}x}(T\cos\theta) = 0 \Rightarrow T\cos\theta = \text{Constant} = H \end{aligned} \tag{5.34}$$

Substituting for $T$

$$T = \frac{H}{\cos\theta} \Rightarrow T\sin\theta = H\tan\theta = H\frac{\mathrm{d}y}{\mathrm{d}x}$$

in the first equation in (5.34) leads to

$$H\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = w_g\frac{\mathrm{d}s}{\mathrm{d}x} = w_g\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{\frac{1}{2}} \tag{5.35}$$

The general solution of (5.35) is

$$y = \frac{H}{w_g}\cosh\left(\frac{w_g}{H}x + c_1\right) + c_2 \tag{5.36}$$

where $c_1$ and $c_2$ are integration constants which are determined using the coordinates of the support points. For the unsymmetrical case, we locate the origin at the left support (Fig. 5.24a). When the cable is symmetrical, it is more convenient to locate the origin at the lowest point.

We consider the symmetrical case shown in Fig. 5.25. We locate the origin at the lowest point. Then for this choice,

$$\begin{aligned} c_1 &= 0 \\ c_2 &= -\frac{H}{w_g} \end{aligned}$$

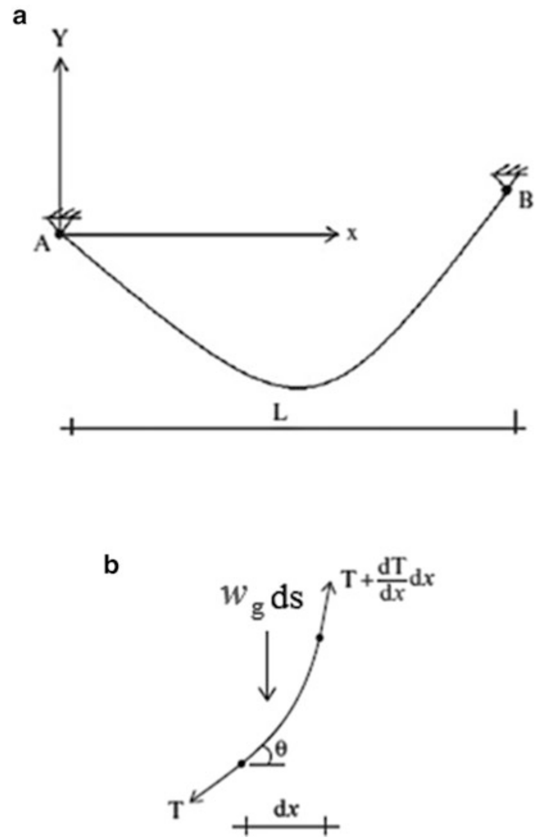



**Fig. 5.24** (**a**) Cable shape under self weight—catenary. (**b**) Differential segment

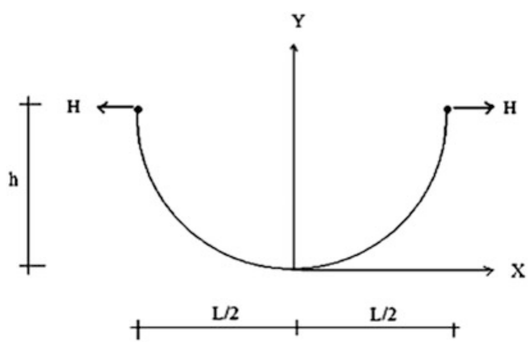



**Fig. 5.25** Catenary–symmetrical

and

$$y = \frac{H}{w_g}\left\{\cosh\left(\frac{w_g}{H}x\right) - 1\right\}$$

The force $H$ is determined from the condition $y(L/2) = h$

$$h = \frac{H}{w_g}\left\{\cosh\left(\frac{w_g L}{2H}\right) - 1\right\} \tag{5.37}$$

We need to solve (5.37) using iteration since it is a transcendental equation.
Expanding the cosh term,

$$\begin{aligned}\cosh x &= 1+\frac{x^2}{2}+\frac{x^4}{24}+\cdots+\frac{x^n}{n!}\\ &= 1+\frac{x^2}{2}\left\{1+\frac{x^2}{12}+\cdots+2\frac{x^{(n-2)}}{n!}\right\}\end{aligned} \tag{5.38}$$

and noticing that when $x^2$ is small with respect to 1, the expression can be approximated as

$$\cosh x \approx 1+\frac{x^2}{2}\left\{1+\frac{x^2}{12}\right\}$$

and taking $x=\frac{w_g L}{2H}$ leads to

$$h \approx \frac{w_g L^2}{8H}\left\{1+\frac{1}{12}\left(\frac{w_g L}{2H}\right)^2\right\} \tag{5.39}$$

When the loading is assumed to be per unit projected length, the corresponding expression for $h$ is $h = wL^2/8H$. For a given $H$, $h$ is larger for the self weight case. Also for a given $h$, $H$ is larger for the self weight case. The difference increases with the sag ratio, $h/L$.

We find the arc length using (5.35).

$$H\frac{d^2y}{dx^2}dx = w_g ds$$

Integrating,

$$\begin{aligned}S &= 2\int_0^{\frac{L}{2}}\left(\frac{1}{w_g}\right)H\frac{d^2y}{dx^2}dx = \frac{2}{w_g}H\frac{dy}{dx}\bigg|_0^{\frac{L}{2}}\\ S &= \frac{2H}{w_g}\sinh\left(\frac{w_g L}{2H}\right)\end{aligned} \tag{5.40}$$

The maximum tension, which occurs at $x = \pm(L/2)$, is determined using

$$T_{max} = H\cosh\left(\frac{w_g L}{2H}\right) \tag{5.41}$$

*Example 5.8*

**Given:** The cable shown in Fig. E5.8a has a self weight of 1.2 kip/ft.

**Determine:** The arc length, $h$ the maximum tension in the cable using the catenary equations, and the percent of error in the maximum tension value when using parabolic equations. Consider the following values for $H$: $H = 75$, 100, and 250 kip.

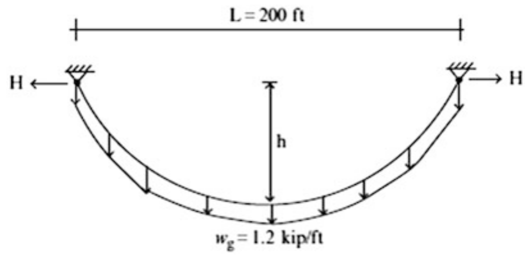



**Fig. E5.8a**

**Solution:** The relevant equations are listed below.

$$h = \frac{H}{w_g}\left\{\cosh\left(\frac{w_g L}{2H}\right) - 1\right\}$$

$$h_{ap} \approx \frac{w_g L^2}{8H}\left\{1 + \frac{1}{12}\left(\frac{w_g L}{2H}\right)^2\right\}$$

$$S = \frac{2H}{w_g}\sinh\left(\frac{w_g L}{2H}\right)$$

$$T_{max} = H\cosh\left(\frac{w_g L}{2H}\right)$$

These equations are evaluated using a digital computer. The results are summarized in the table below. Note that when $h/L$ is large, the error introduced by the parabolic approximation is significant.

| | Catenary | | | | Parabola | | |
|---|---|---|---|---|---|---|---|
| $H$ | $S$ | $h$ | $h_{ap.}$ | $T_{max}$ | $h$ | $T_{max}$ | % difference $T_{max}$ |
| 75 | 296.9 | 98.6 | 97 | 193 | 80 | 141.5 | 27 % |
| 100 | 251.6 | 67.5 | 67.2 | 181 | 60 | 156.2 | 14 % |
| 250 | 207.7 | 24.5 | 24.5 | 279 | 24 | 277.3 | 1 % |

## 5.5 Summary

### 5.5.1 Objectives

- To describe how a cable adjusts its geometry when subjected to a single vertical concentrated load.
- To extend the analysis to a cable subjected to multi-concentrated vertical loads.
- To derive an expression for the deflected shape of the cable when subjected to an arbitrary vertical loading.
- To present a series of examples which illustrate the computational procedure for finding the deflected shape of a cable.
- To derive an approximate expression for the equivalent axial stiffness of a cable modeled as a straight member.

### 5.5.2 Key Concepts

- Given a cable supported at two points, A and B, and subjected to a vertical loading. The vertical deflection from the chord connecting points A and B is proportional to the bending moment $M$ in a simply supported beam spanning between A and B. One finds the bending moment diagram using a simple equilibrium analysis. The deflection of the cable with respect to the chord AB is an inverted scaled version of the moment diagram.
- Under vertical loading, the horizontal component of the cable force is constant.
- The length of the cable is determined by integrating

$$S = \int_0^L \mathrm{d}s = \int_0^L \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

where

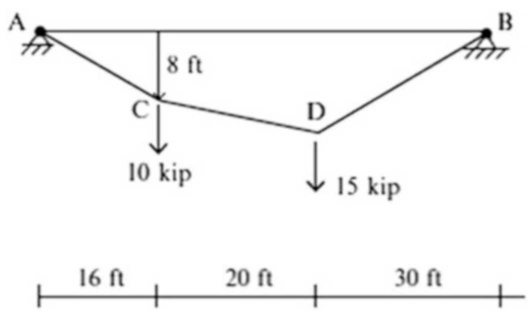
$$y = -\frac{M_0(x)}{H} + \frac{y_B}{L_h}x$$

One usually approximates the integrand with $\mathrm{d}s \approx 1 + (1/2)\ (\mathrm{d}y/\mathrm{d}x)^2$ when $(\mathrm{d}y/\mathrm{d}x)^2$ is small in comparison to 1.

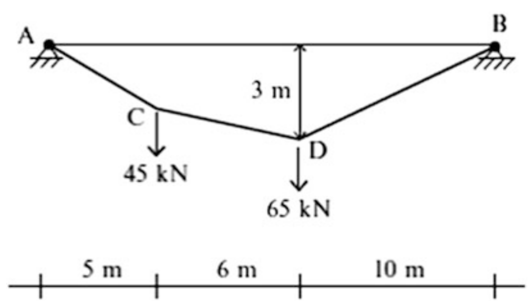
## 5.6 Problems

For Problems 5.1–5.8, determine the reactions at the supports, and the tension in each segment of the cable.

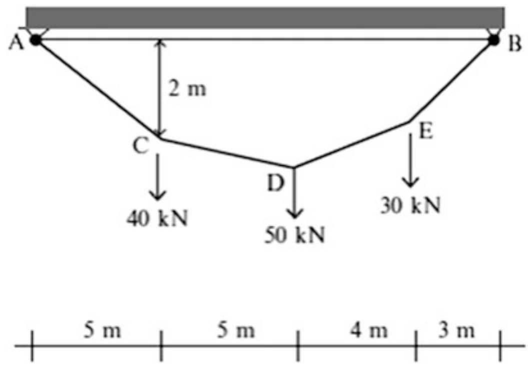
**Problem 5.1**



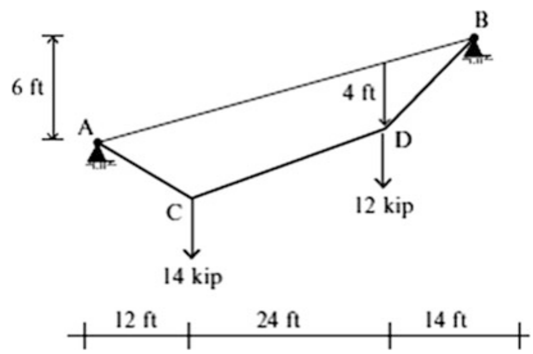

**Problem 5.2**



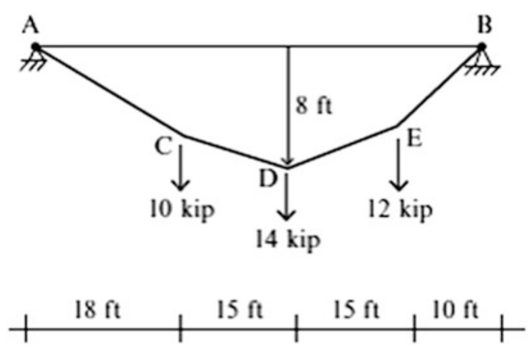
**Problem 5.3**



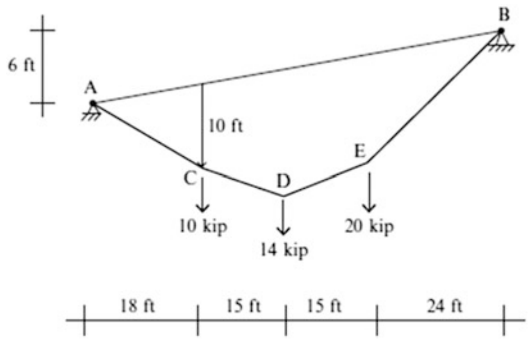

**Problem 5.4**



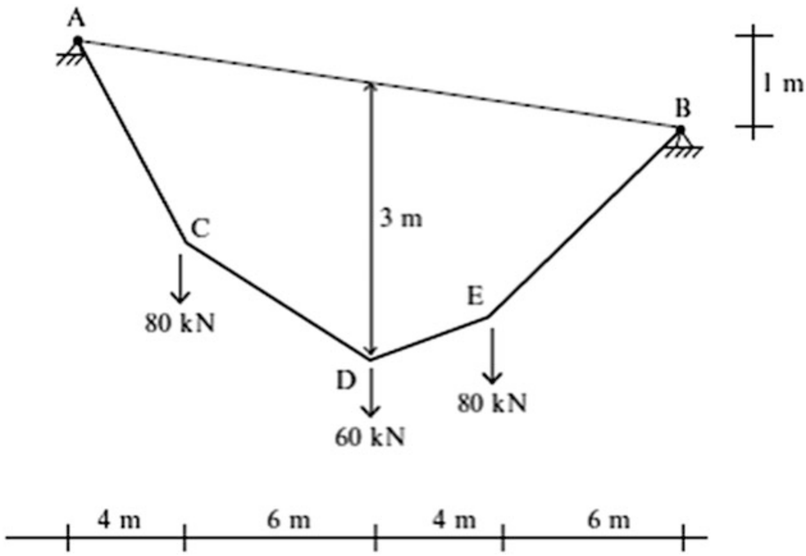

**Problem 5.5**



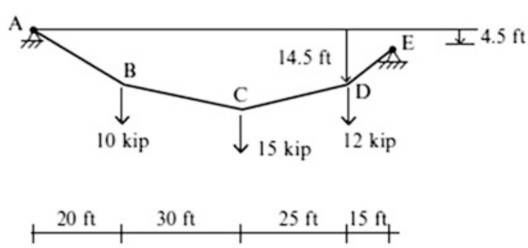
**Problem 5.6**




**Problem 5.7**

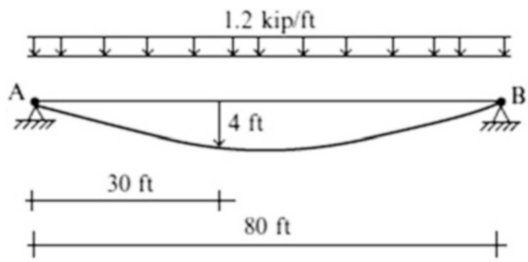



**Problem 5.8**

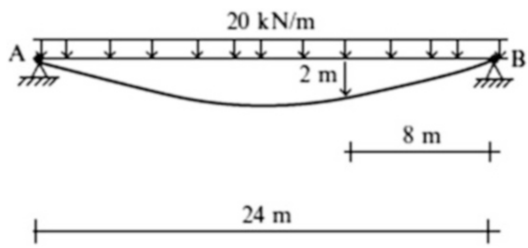



For Problems 5.9–5.14, determine the maximum tension.

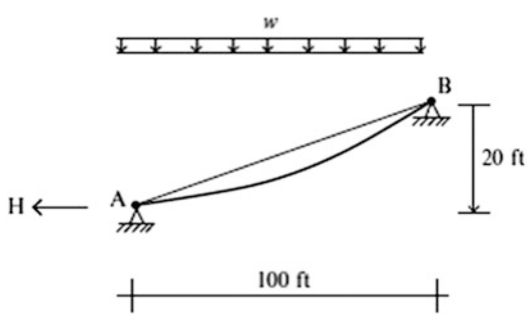
**Problem 5.9**



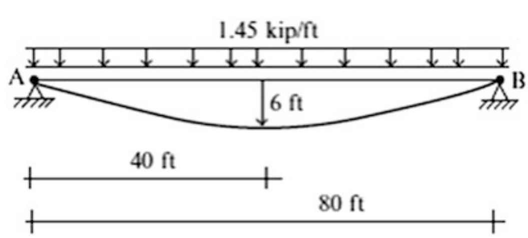

**Problem 5.10**



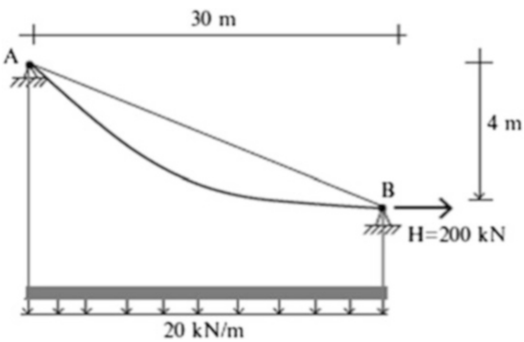

**Problem 5.11**
Assume $w = 1.7$ kip/ft and $H = 40$ kip.




**Problem 5.12**

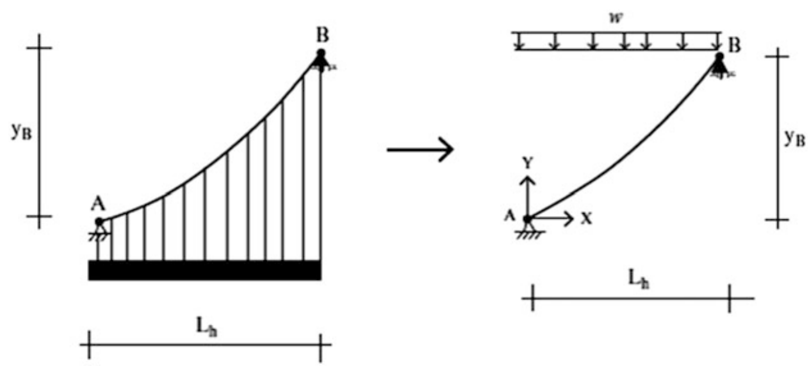


**Problem 5.13**



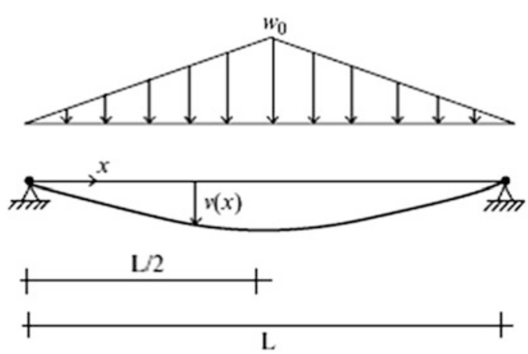

**Problem 5.14**

Assume $w = 1.4$ kip/ft, $y_B = 10$ ft, $H = 100$ kip, and $L_h = 40$ ft.

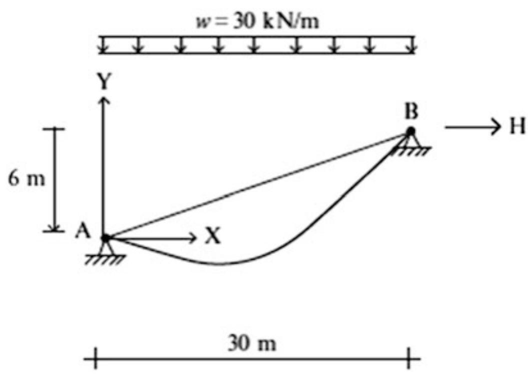



**Problem 5.15**

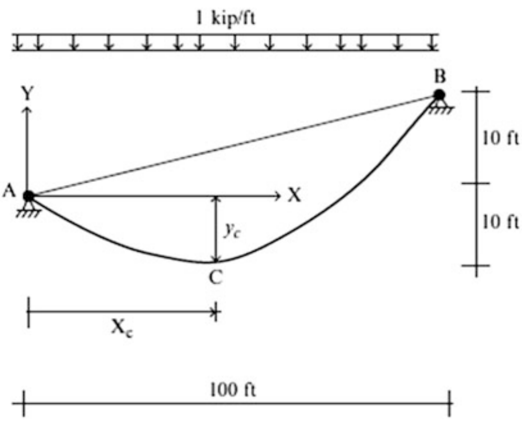
Assume $w_0 = 1.8$ kip/ft, $v_{\text{at } x = 20 \text{ ft}} = 2$ ft and $L = 80$ ft. Determine the deflected shape.



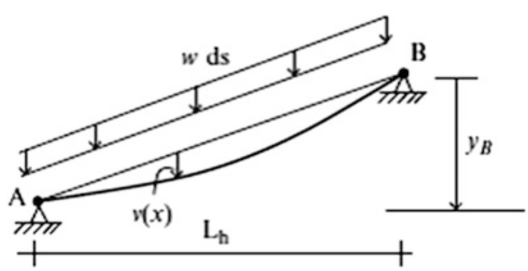
**Problem 5.16**
Determine the coordinates of the lowest point on the cable for $H = 650$ kN




**Problem 5.17**
Determine the peak values of cable tension.

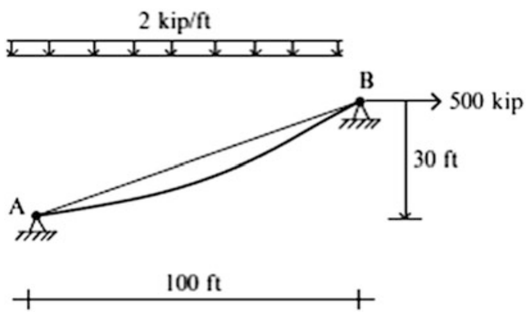



**Problem 5.18**
Consider the case where the loading is defined in terms of per unit arc length. Derive the expression for the deflected shape, $\nu(x)$.



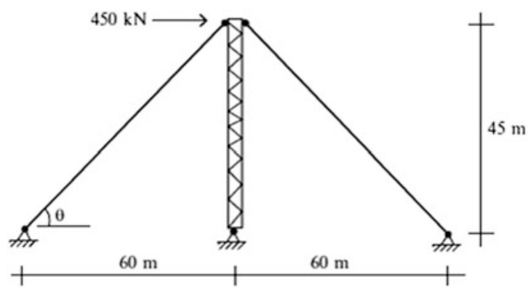
**Problem 5.19**

(a) Determine the total arc length for this geometry.
(b) Determine the effect of a temperature increase of 100 °F. Assume the cable material is steel.




**Problem 5.20**

Consider the guyed tower scheme shown in the sketch below. Assume the guys are steel cables that are stressed initially to 520 MPa. Determine the cable cross-sectional area required to limit the lateral motion at the top of the tower to 10 mm.




**Problem 5.21**

The cable shown below carries its own weight. Determine the arc length and $y_B$. Point C is the lowest point. Assume $w = 0.8$ kip per foot of cable, $L_1 = 60$ ft, $L_2 = 80$ ft, and $H = 150$ kip.