Statically Determinate Beams 3

Abstract

Our focus in this chapter is on describing how beams behave under transverse loading, i.e., when the loading acts normal to the longitudinal axes. This problem is called the "beam bending" problem. The first step in the analysis of a statically determinate beam is the determination of the reactions. Given the reactions, one can establish the internal forces using equilibrium-based procedures. These forces generate deformations that cause the beam to displace. We discuss in detail the relationship between the internal forces and the corresponding displacements and describe two quantitative analysis procedures for establishing the displacements due to a particular loading. The last section of the chapter presents some basic analysis strategies employed in the design of beams such as influence lines and global envelopes.

3.1 Definition of a Prismatic Beam

Beams are used extensively in structures, primarily in flooring systems for buildings and bridges. They belong to the line element category, i.e., their longitudinal dimension is large in comparison to their cross-sectional dimensions. Whereas truss members are loaded axially, beams are loaded normal to the longitudinal direction, and carry the loading by bending and twisting action. This mode is illustrated in Fig. 3.1. The transverse loading produces transverse deflection, which results in a nonuniform distribution of stress throughout the body.

Most of the applications of beams in building structures involve straight beams with constant cross-section. We refer to this subgroup as prismatic beams. Figure 3.2 defines the geometrical parameters and notation used for prismatic beams. The longitudinal axis-X passes through the centroid of the cross-section, and the Y, Z axes are taken as the principal inertia directions. The relevant definition equations are

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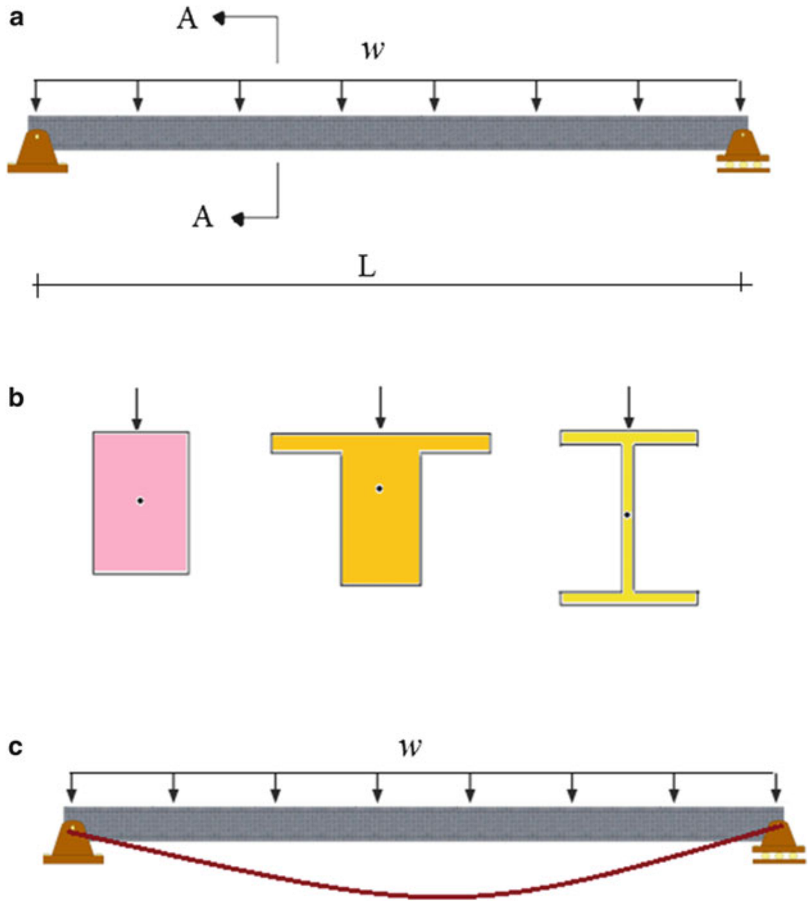


Fig. 3.1 Beam cross-sections and bending mode. (**a**) Simply supported beam. (**b**) Section A-A—cross-section examples. Rectangular, T shape, I shape. (**c**) Bending mode

$$\begin{aligned} &\int_A y\,\mathrm{d}A = \int_A z\,\mathrm{d}A = \int_A yz\,\mathrm{d}A = 0 \\ &I_z = \int_A y^2\mathrm{d}A \\ &I_y = \int_A z^2\mathrm{d}A \end{aligned} \tag{3.1}$$

These conditions ensure that when the applied loads are in the $X - Y$ plane, points on the longitudinal axis will not displace in the Z direction. Figure 3.3 illustrates this mode of behavior, the longitudinal axis-X becomes a curve $\nu(x)$ contained in the $X - Y$ plane. This type of behavior is Stabilitycalled *planar bending*.

There are cases where the line of action of the loading does not pass through the X-axis, such as illustrated in Fig. 3.4. The eccentricity produces a torsional moment about the X-axis, and the cross-section will rotate as well as deflect. This behavior is called "combined bending and torsion." A prismatic member acted upon by just a torsional moment will experience only torsional behavior, i.e., the cross-section will just twist.

Mechanics of Solids texts deal with stresses and strains in beams. *Our objective here is not to redevelop this material but rather to utilize it and formulate a structural theory for beams that will provide the basis for analyzing the behavior of structures composed of beam elements.* Since structural theory is founded on Engineering Mechanics Theory, at least one subject dealing with

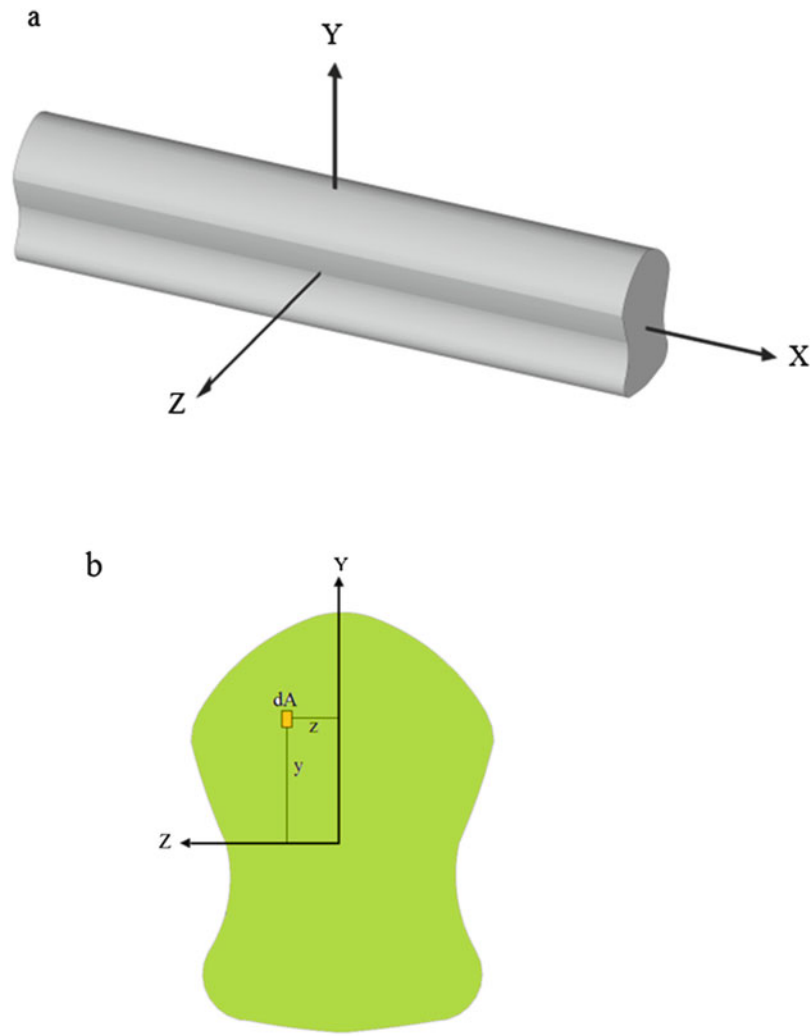


Fig. 3.2 Notations for prismatic beam—symmetrical cross-section

Engineering Mechanics is usually required before studying Structural Theory. We assume that the reader has this level of exposure to Engineering Mechanics.

3.2 Stability and Determinacy of Beams: Planar Bending

We presented the general concept of stability of a rigid body in Chap. 1 and used the general concept to develop stability criteria for truss-type structures in Chap. 2. In what follows, we examine the stability question for beam-type structures and develop similar criteria. For completeness, we first briefly review the basis for stability discussed in Chap. 1.

Consider the rigid body shown in Fig. 3.5. Assume the body can move only in the $X - Y$ plane. There are three types of planar motion for a rigid body: translation in the x direction, u_A, translation in the y direction, v_A, and rotation about an axis normal to the $X - Y$ plane, ω_A. A body is said to be stable when rigid body motion is prevented. Therefore, it follows that one must provide three motion constraints to restrain motion in the $X - Y$ plane.

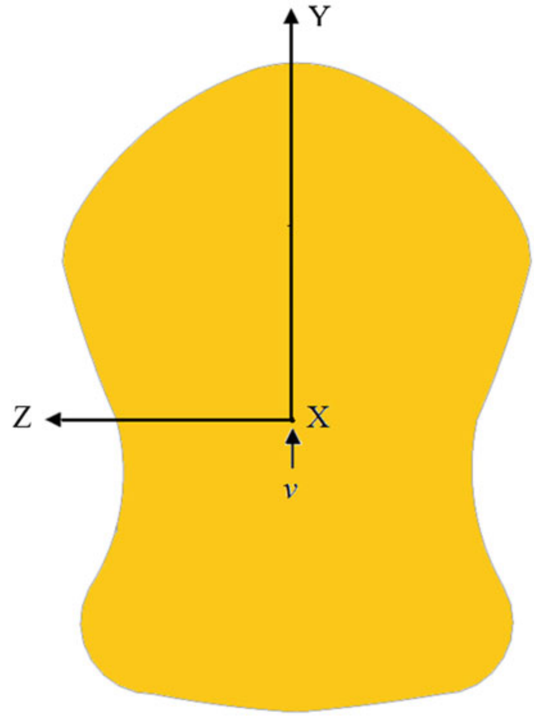


Fig. 3.3 Planar bending mode

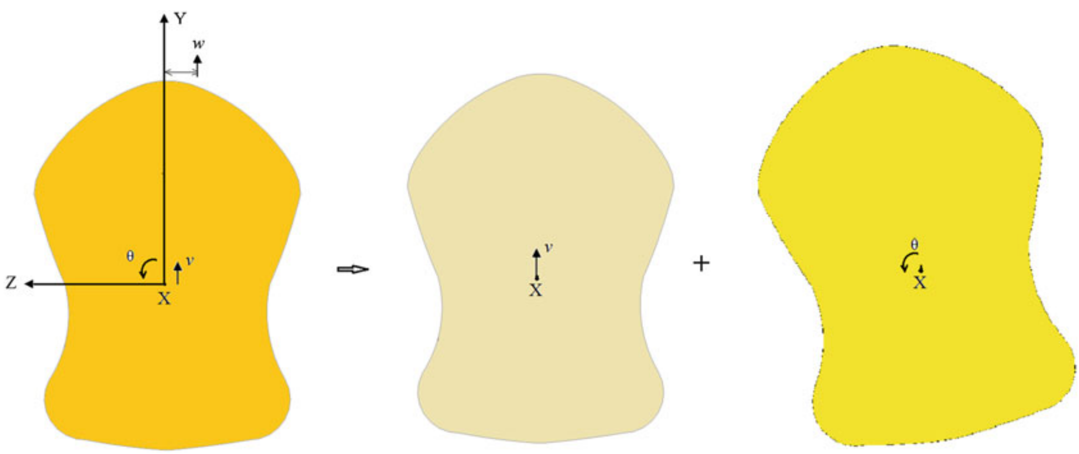


Fig. 3.4 Combined bending and torsion

One needs to be careful in selecting the orientation of the three translation constraints. Consider Fig. 3.6. We first choose two directions, "*a*" and "*b*" in the $X - Y$ plane. They intersect at point o. With these two constraints, the only possible rigid body motion is rotation about point o. If we take the third direction as "*c*," this rotation is not prevented. *Therefore, it follows that the three directions must be nonconcurrent as well as coplanar*, i.e.*, they cannot intersect at a common point*. This implies that they must not be parallel. Any other direction, such as "*d*′" is permissible.

Fig. 3.5 Planar rigid body motions

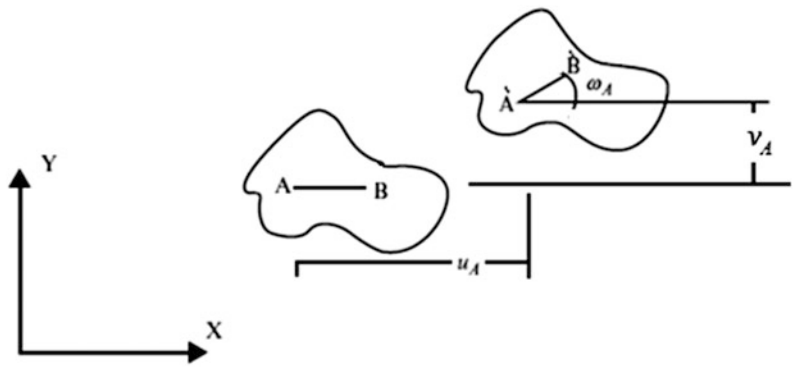
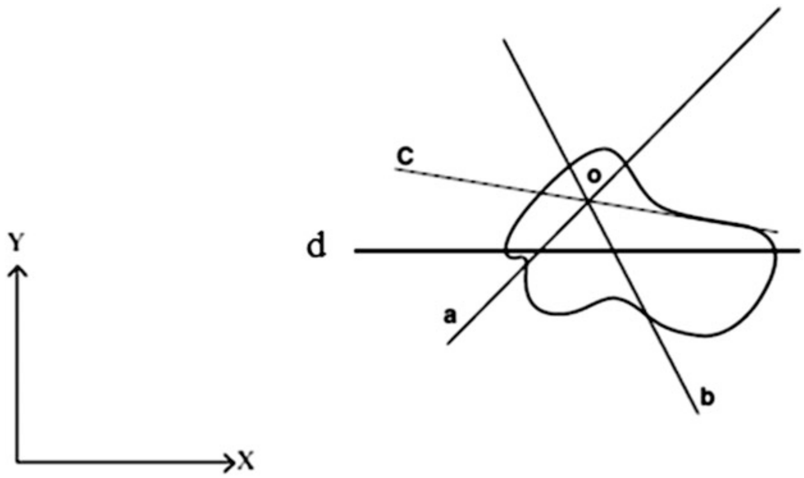
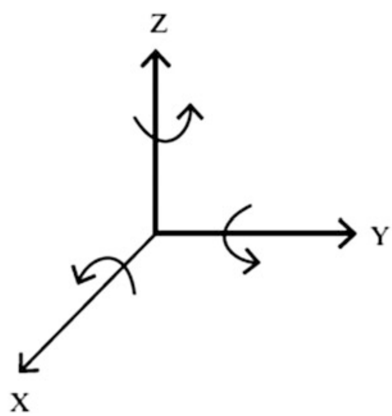


Fig. 3.6 Concurrent displacement constraints



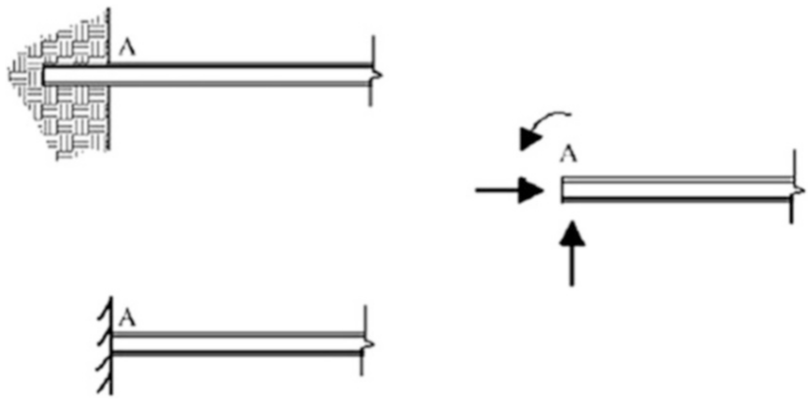
When the loading is arbitrary, the body needs to be constrained against motion in any plane. This requires six constraints, three with respect to translation and three with respect to rotation about the X, Y, and Z direction. The strategy for selecting restraints is similar to the treatment of 3-D truss structures. We point out that for pure rotational loading only one rotational restraint is required.



Motion constraints produce reaction forces when the body is loaded. The nature of the reaction forces depends on the constraints. Various types of supports for beams subjected to planar bending are illustrated below.

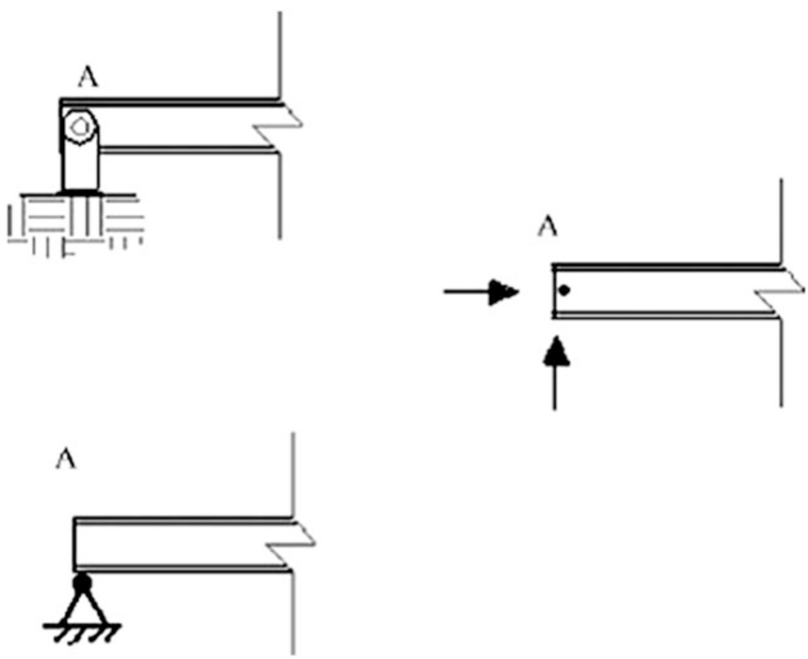
3.2.1 Fixed Support: Planar Loading

The beam is embedded at point A in such a way that the end is prevented from translating or rotating. We say the member is "fixed" at A. The reactions consist of two forces and one moment.



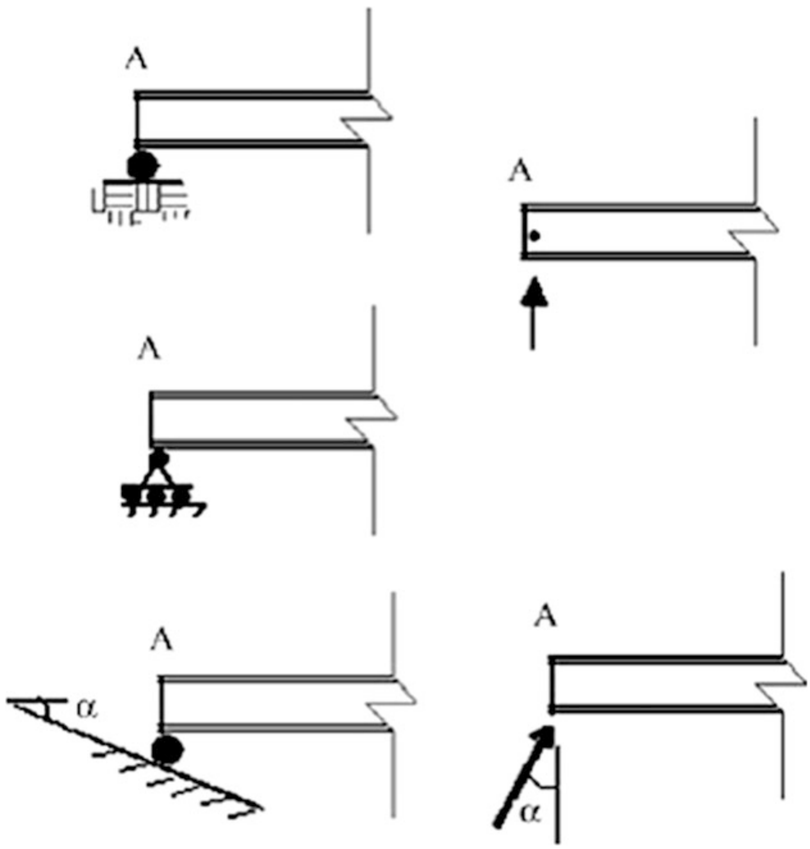
3.2.2 Hinged Support: Planar Loading

Suppose A is to be fully restrained against translation. This can be achieved by pinning the member. Horizontal and vertical reactions are produced.



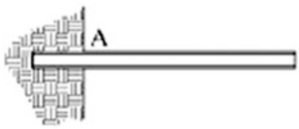
3.2.3 Roller Support: Planar Loading

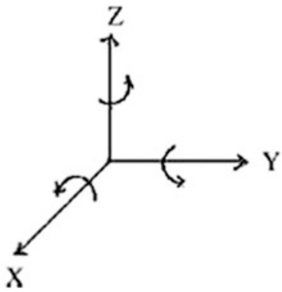
Suppose A is to be restrained against motion perpendicular to the surface of contact. We add a restraint to A by inserting a device that allows motion parallel to the surface of contact but fully restrains motion in the direction perpendicular to the surface. We refer to this device as a roller. This restraint produces a reaction force perpendicular to the surface of contact.

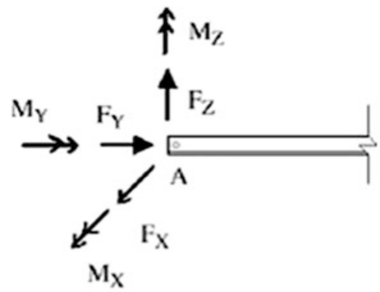


When the loading is three dimensional, additional restraints are required. The supports described above needs to be modified to deal with these additional restraints. Typical schemes are shown below.

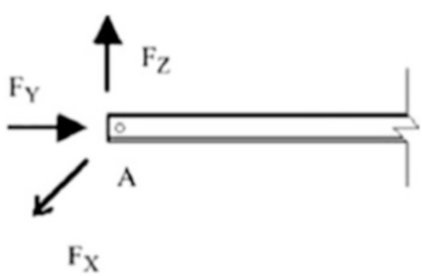
3.2.4 3-D Fixed Support



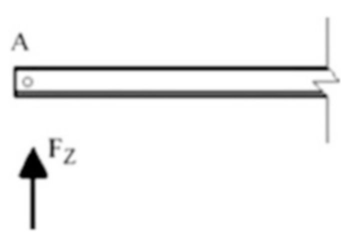




3.2.5 3-D Hinged Support



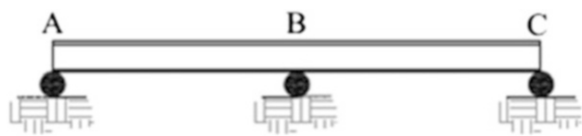
3.2.6 3-D Roller Support: Z Direction



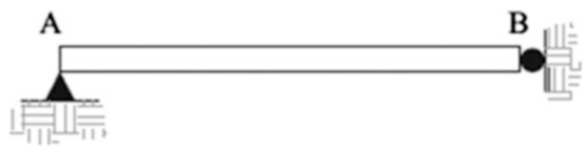
3.2.7 Static Determinacy: Planar Beam Systems

In general, a body restrained with three *nonconcurrent* coplanar displacement constraints is stable for planar loading. When loading is applied, the only motion that occurs is due to deformation of the body resulting from the stresses introduced in the body by the loading. The motion restraints introduce reaction forces. Since there are three equations of force equilibrium for a body, and only three unknown forces, one can determine these force unknowns using only the force equilibrium equations. *In this case, we say that the structure is stable and statically determinate*. *If a body is over restrained, i.e., if there are more than three nonconcurrent displacement restraints, we say that the structure is statically indeterminate*. This terminology follows from the fact that now there are more than three force unknowns and consequently one cannot uniquely determine these unknowns with only the three available force equilibrium equations. Statically indeterminate structures require a more rigorous structural theory and therefore we postpone their treatment to part II of the text. In what follows, we present some examples of statically determinate and statically indeterminate planar beams.

3.2.8 Unstable Support Arrangements



The beam shown above has the proper number of constraints, but they are all vertical. There is no constraint against horizontal motion, and therefore the beam is unstable.



The beam shown above is unstable. The roller support at B constrains motion in the horizontal direction but does not prevent rigid body motion about point A.

3.2.9 Beam with Multiple Supports

There are three vertical restraints and one horizontal restraint. These restraints produce the four reaction forces shown below.

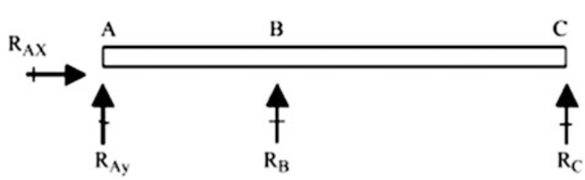


Fig. 3.7 Two-span beam

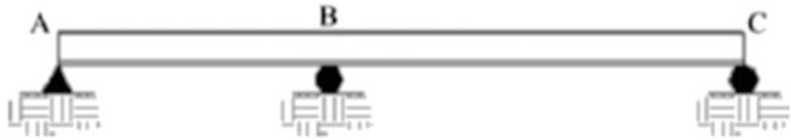


Fig. 3.8 Beam with moment release

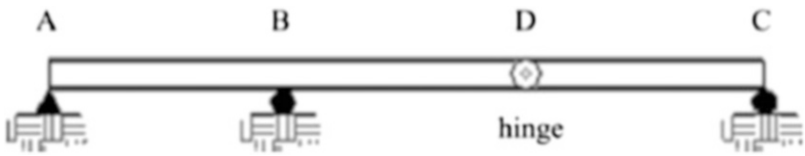
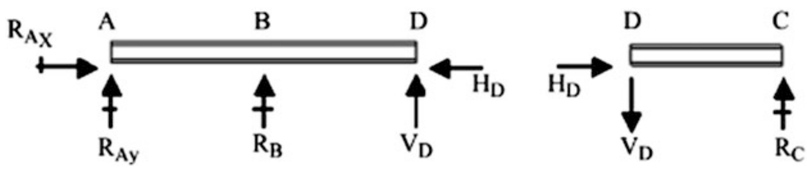
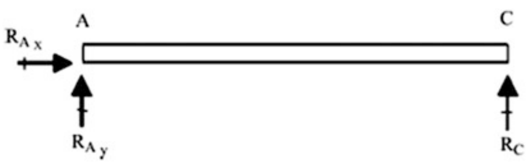


Fig. 3.9 Free body diagram for beam with moment release



One of the vertical restraints is redundant, i.e., is not needed for stability and therefore can be deleted. Deleting the support at B results in the structure shown below.



A beam supported only at its ends in a minimal way is referred to as a simple supported beam.

The beam depicted in Fig. 3.7 is called a two-span continuous beam. This beam is statically indeterminate to the first degree. We will show later that multi-span continuous beams are more structurally efficient than simply supported beams in the sense that they deflect less for a given design loading.

3.2.10 Beam with a Moment Release

Suppose we cut the beam shown in Fig. 3.8 at point D and insert a frictionless hinge. We refer to the hinge as a moment release since the moment is zero. The hinge does not restrain rotation at D, and member DC is free to rotate about D. The beam is now statically determinate. The corresponding reaction forces are listed below on the free body diagrams (Fig. 3.9).

Fig. 3.10 Three-span beam

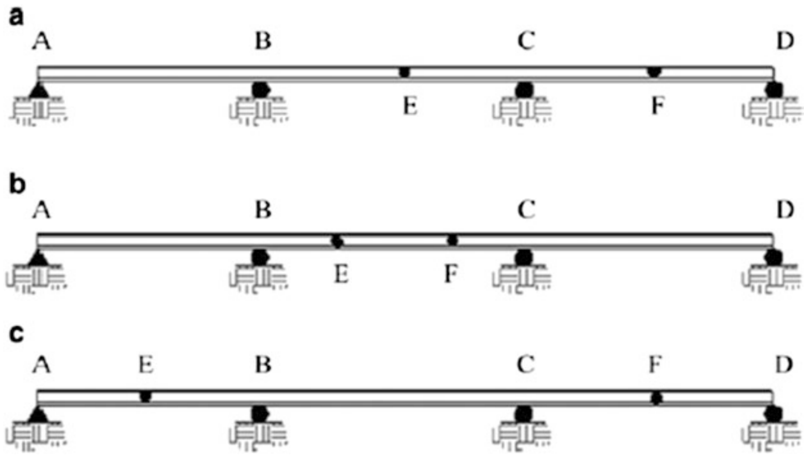


Fig. 3.11 Statically determinate versions of three-span beam with moment releases

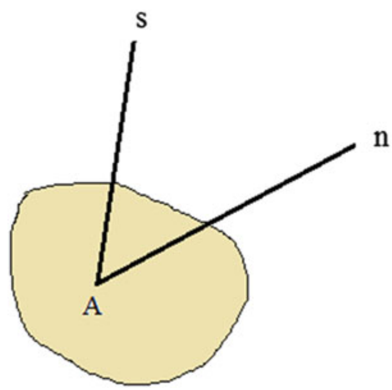
Member DC is statically determinate since there are only three reaction forces. Once the forces at D are known, the remaining reactions for member ABD can be determined. Therefore, it follows that inserting a hinge at D reduces the static indeterminacy by 1°.

We consider next the three-span continuous beam shown in Fig. 3.10. This structure is indeterminate to the second degree since there are two extra vertical supports. One can reduce the structure to a statically determinate structure by inserting two moment releases. Various possibilities are listed in Fig. 3.11. The optimal location of moment releases is illustrated in Examples 3.33 and 3.34.

3.3 Reactions: Planar Loading

When a structure is subjected to external loads, the displacement restraints develop reaction forces to resist the tendency for motion. If the structure is statically determinate, we can determine these forces using the three global force equilibrium equations for planar loading applied to a body. One selects a set of directions *n*–*n* and *s*–*s*, where *s*–*s* is not parallel to *n*–*n*. The steps are

$$\begin{aligned}
&\text{(i)}\ \ \text{Summation of forces in direction}\ \ n - n = 0\\
&\text{(ii)}\ \ \text{Summation of forces in direction}\ \ s - s = 0\\
&\qquad \text{where direction}\ s - s\ \text{is not parallel to direction}\ n - n\\
&\text{(iii)}\ \ \text{Summation of moments about an arbitrary point},\ A = 0
\end{aligned} \tag{3.2}$$



One constructs a free body diagram of the structure and applies these equations in such a way as to obtain a set of uncoupled equations, which can be easily solved.

When a statically indeterminate structure has a sufficient number of releases such that it is reduced to being statically determinate, we proceed in a similar way except that now we need to consider more than one free body. The following series of examples illustrate the strategy for computing the reactions.

Example 3.1 Beam with Two Over Hangs

Given: The beam shown in Fig. E3.1a.
Determine: The reactions.

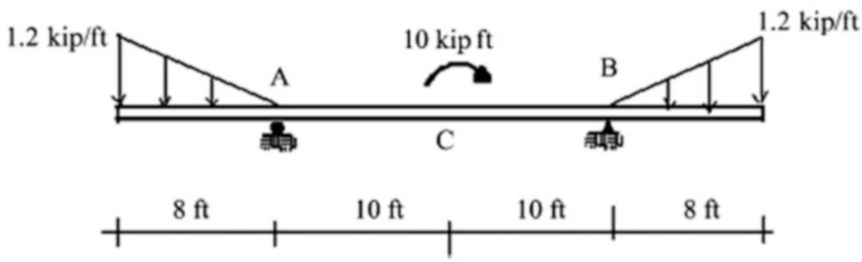


Fig. E3.1a

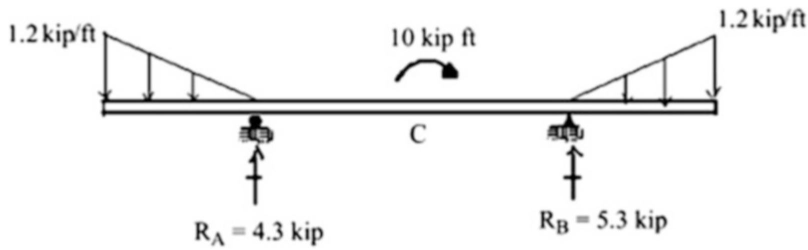
Solution: Summing moments about B leads to the vertical reaction at A.

$$\sum M_{\mathrm{B}} = 0$$
$$R_{\mathrm{A}}(20) + 10 + \frac{1}{2}(1.2)(8)\frac{2}{3}(8) - 1.2\left(\frac{8}{2}\right)\left(20 + \frac{2}{3}8\right) = 0$$
$$\therefore R_{\mathrm{A}} = 4.3 \uparrow$$

Summing the vertical forces,

$$\sum F_Y = 0 \quad R_B + 4.3 - 1.2\left(\frac{8}{2}\right)(2) = 0$$
$$\therefore R_B = 5.3 \uparrow$$

The reactions are listed below.



Example 3.2 Simply Supported Beam

Given: The beam shown in Fig. E3.2a.
Determine: The reactions.

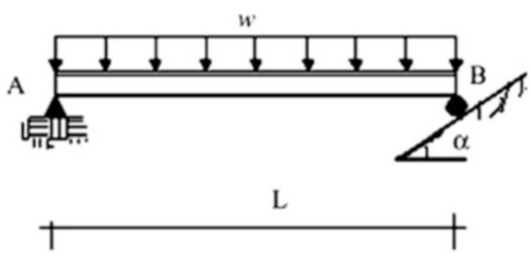


Fig. E3.2a

Solution: As a first step, we construct the free body diagram for the beam. The reaction at B is normal to the inclined surface. We resolve it into horizontal and vertical components using (Fig. E3.2b)

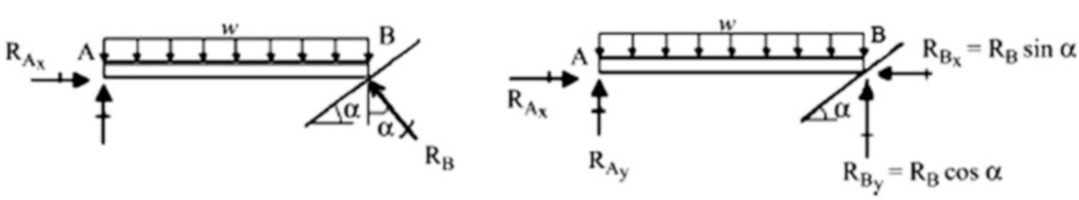


Fig. E3.2b

$$R_{By} = R_B \cos \alpha \quad R_{Bx} = R_B \sin \alpha$$

Summing moments about A leads to the vertical reaction at B.

$$\sum M_{\mathrm{A}} = 0$$
$$wL\left(\frac{L}{2}\right) - LR_{\mathrm{B}y} = 0$$
$$\therefore\ R_{\mathrm{B}y} = \frac{wL}{2}\uparrow$$

Given $R_{\mathrm{B}y}$, we find the reaction R_{B}

$$R_{\mathrm{B}} = \frac{R_{\mathrm{B}y}}{\cos\alpha} = \frac{wL}{2\cos\alpha}$$

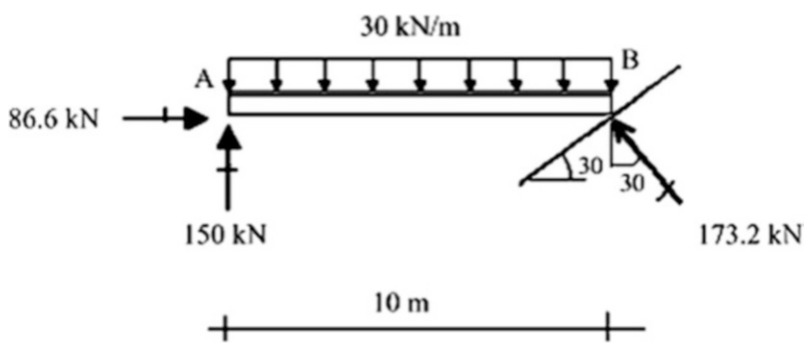
The corresponding horizontal component is

$$R_{\mathrm{B}x} = R_{\mathrm{B}}\sin\alpha = \frac{wL}{2}\tan\alpha\leftarrow$$

We determine the reactions at A using force summations.

$$\sum F_x = 0 \qquad R_{\mathrm{A}x} = -R_{\mathrm{B}x} = \frac{wL}{2}\tan\alpha\rightarrow$$
$$\sum F_y = 0\uparrow^{+} \quad R_{\mathrm{A}y} + R_{\mathrm{B}y} - wL = 0 \quad R_{\mathrm{A}y} = \frac{wL}{2}\uparrow$$

Suppose $w = 30$ kN/m, $\alpha = 30°$ and $L = 10$ m. The reactions are listed below.



Example 3.3 Two-Span Beam with a Moment Release

Given: The beam shown in Fig. E3.3a. There is a moment release at D.

Determine: The reactions.

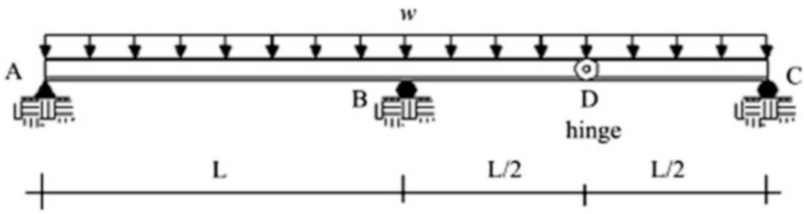
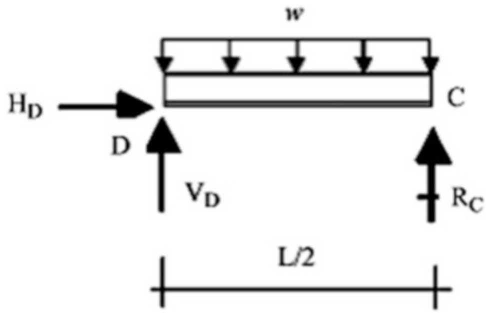


Fig. E3.3a

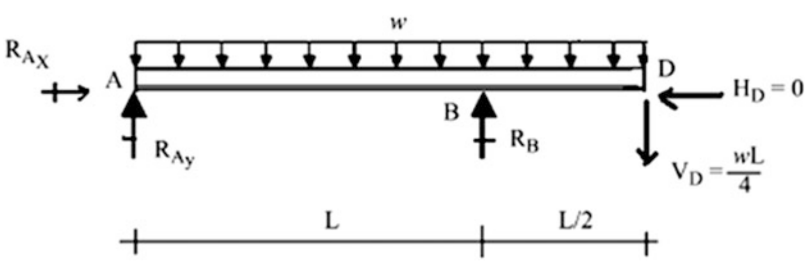
Solution: The most direct way of analyzing this structure is to first work with a free body diagram of beam segment DC.



Applying the equilibrium conditions to this segment results in

$$\sum M_{\mathrm{D}} = 0 \quad R_{\mathrm{C}} = \frac{wL}{4} \uparrow$$
$$\sum F_Y = 0 \quad V_{\mathrm{D}} = \frac{wL}{4} \uparrow$$
$$\sum F_x = 0 \quad H_{\mathrm{D}} = 0$$

With the internal forces at D known, we can now proceed with the analysis of segment ABD.



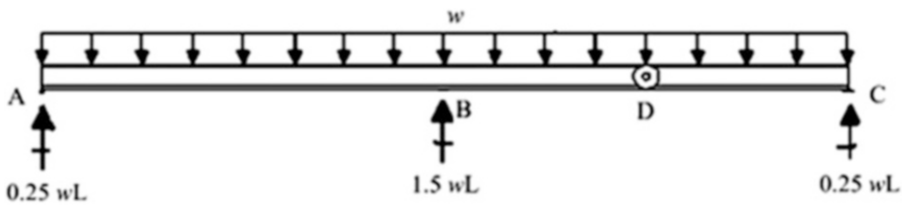
Summing moments about A leads to R_B

$$\sum M_A = 0 \quad (0.75L)(1.5wL) + (1.5L)(0.25wL) - LR_B = 0$$
$$R_B = 1.5wL \uparrow$$

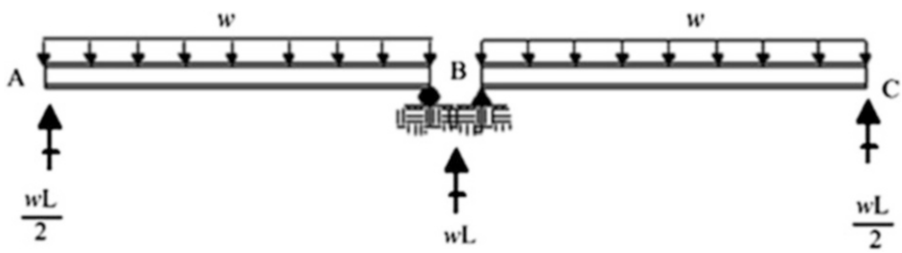
Summing the vertical and horizontal forces,

$$\sum F_Y = 0 \quad R_{Ay} = 1.75wL - R_B = 0.25wL \uparrow$$
$$\sum F_x = 0 \quad R_{Ax} = 0$$

The reactions are listed below.



If the hinge was placed at point B, the structure would act as two simply supported beams, and the reactions would be as shown below.



Example 3.4

Given: The beam shown in Fig. E3.4a.
Determine: The reactions.

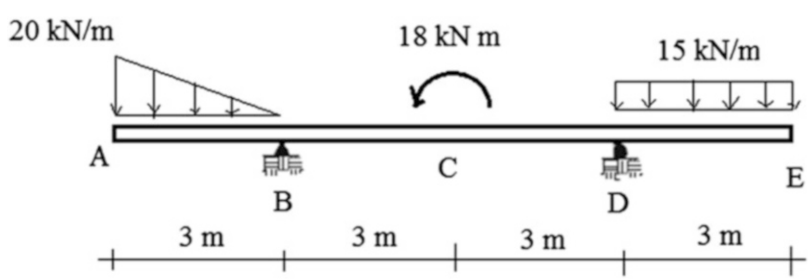


Fig. E3.4a

Solution: Summing moments about B leads to the vertical reaction at D.

$$\sum M_{\mathrm{B}} = 0$$

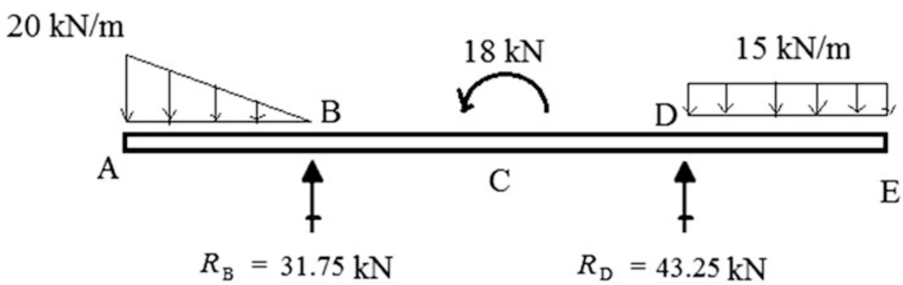
$$R_{\mathrm{D}}(6) + 20\left(\frac{3}{2}\right)(2) + 18 - 15(3)(7.5) = 0 \quad \therefore R_{\mathrm{D}} = 43.25 \uparrow$$

Summing the vertical forces,

$$\sum F_Y = 0 \quad R_{\mathrm{B}} + 43.25 - 15(3) - 20\left(\frac{3}{2}\right) = 0$$

$$\therefore R_{\mathrm{B}} = 31.75 \uparrow$$

The reactions are listed below.



Example 3.5 Three-Span Beam with Two Moment Releases

Given: The beam shown in Fig. E3.5a.
Determine: The reactions.

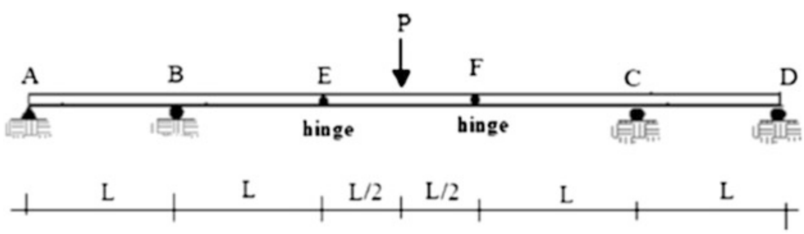
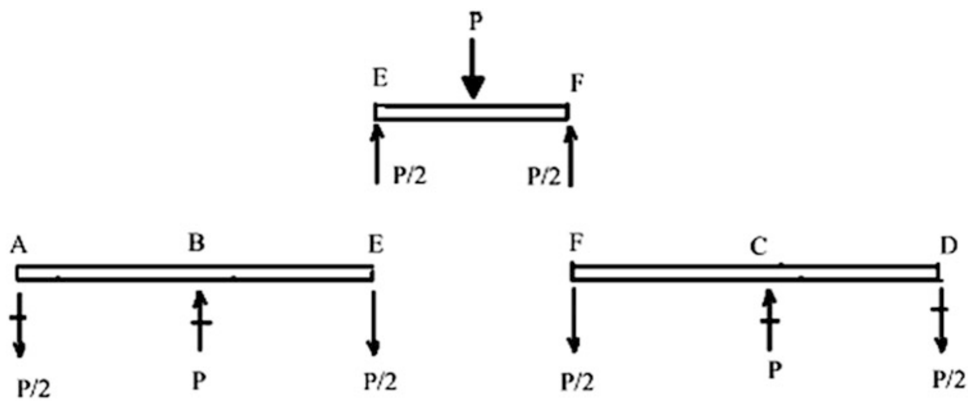
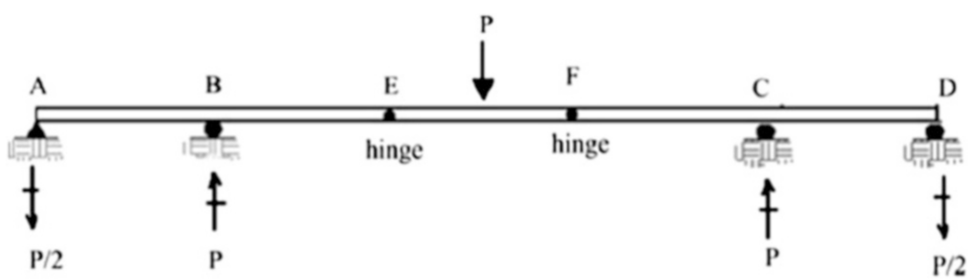


Fig. E3.5a

Solution: We first work with a free body diagram of beam segment EF. Then, with the internal forces at E and F known, we precede with the analysis of segment ABE and FCD.



The reactions are listed below.



Example 3.6 Horizontal Beam Supporting a Vertical Sign

Given: The structure defined in Fig. E3.6a. Member BED is rigidly attached to the beam, ABC. Member FG is simply supported on member BED. Assume member FG has some self-weight, W and is acted upon by a uniform horizontal wind load p. This structure is an idealization of a highway sign supported on a beam.

Determine: The reactions.

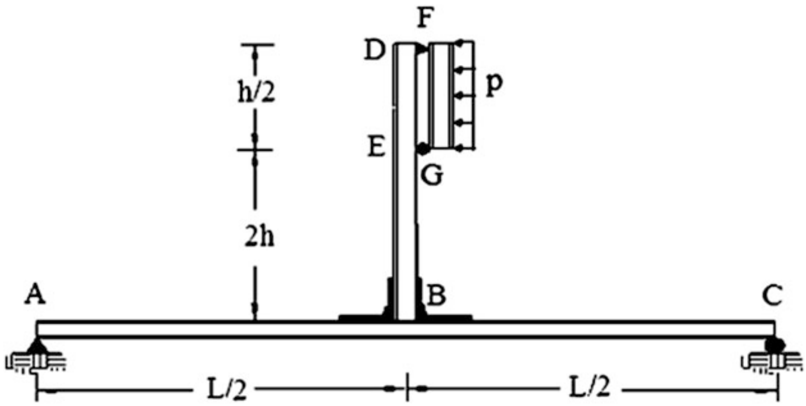
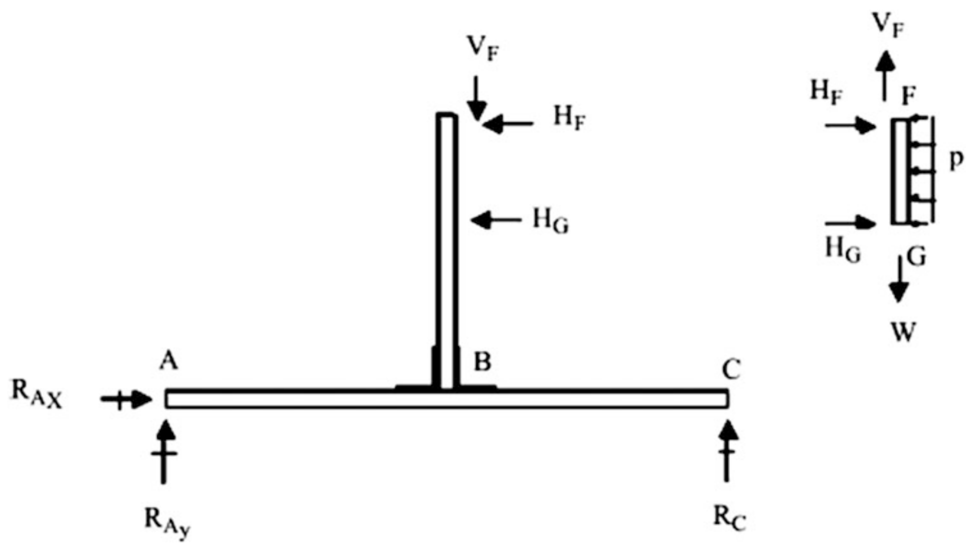


Fig. E3.6a

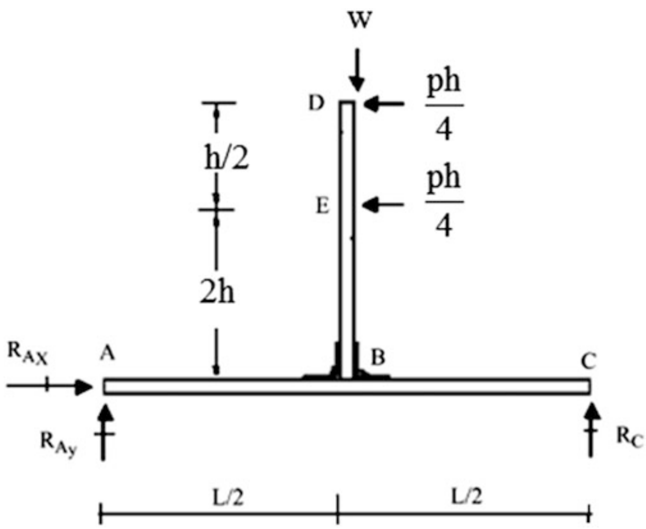
Solution: We work with two free body diagrams, one for member FG and the other for the remaining part of the structure.



Consider first member FG. Enforcing equilibrium leads to:

$$V_{\mathrm{F}} = W$$
$$H_{\mathrm{F}} = H_{\mathrm{G}} = \frac{ph}{4}$$

Next, we apply these forces to the structure composed of member ABC and member BED. The free body diagram is shown below.



Summing moments about A leads to R_C

$$\sum M_A = 0$$

$$W\frac{L}{2} = \frac{ph}{4}(2h) + \frac{ph}{4}(2.5h) + R_C L$$

$$\therefore R_C = \frac{W}{2} - ph\left(1.125\frac{h}{L}\right)$$

The horizontal and vertical reactions at A are

$$R_{Ax} = \frac{ph}{2}$$

$$R_{Ay} = \frac{W}{2} + ph\left(1.125\frac{h}{L}\right)$$

Note that the vertical reaction at C may become negative if ph is large with respect to W and h is of the order of L.

3.4 Internal Forces: Planar Loading

We have shown that external loads produce reaction forces. The next question we need to address is: What is the effect of this combination of external loads and reaction forces on the body? We answer this question by examining the equilibrium of an arbitrary segment of the body.

Consider the uniformly loaded, simply supported beam shown in Fig. 3.12a. We pass a cutting plane a distance x from the left end and consider either the left or right segment.

The external loads create a force unbalance. To maintain equilibrium, a vertical force, $V(x)$, and a moment, $M(x)$, are required at the section. We refer to these quantities as the *internal shear force and bending moment*. The magnitudes of $V(x)$ and $M(x)$ for this section are

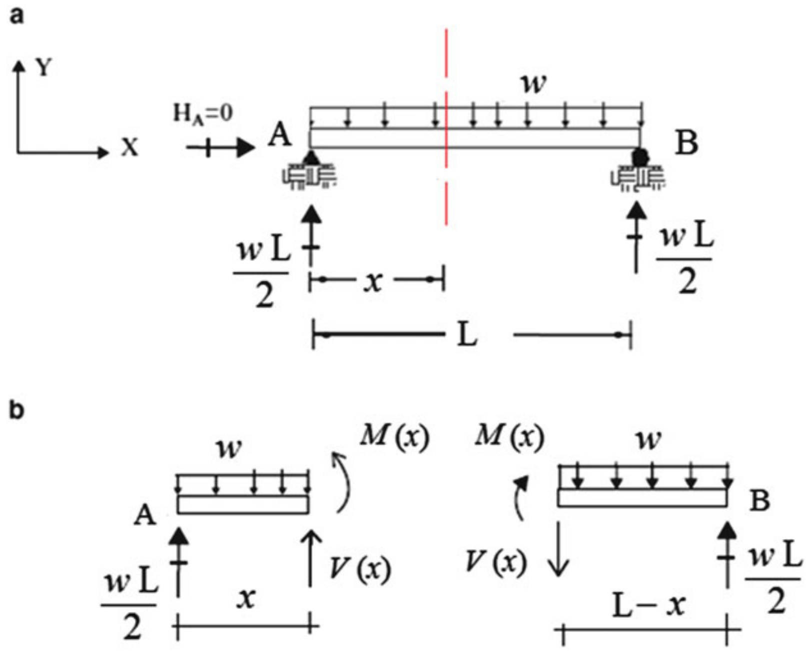


Fig. 3.12 Internal shear and moment. (**a**) beam. (**b**) Segmented beam

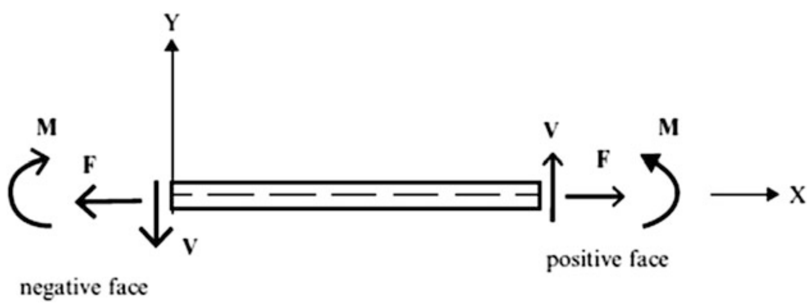


Fig. 3.13 Sign convention for internal forces

$$V(x) = -\frac{wL}{2} + wx$$
$$M(x) = \frac{wL}{2}x - \frac{wx^2}{2} \tag{3.3}$$

We need to first define a sign convention for the positive directions of the internal force quantities. This notation is shown in Fig. 3.13 for a positive face, i.e., a face whose outward normal points in the $+X$ direction. The shear force is positive when it points in the $+Y$ direction, and the positive sense for moment is from X to Y. Depending on the external loading, there may also be an axial force. The positive sense for the axial force is taken as the $+X$ direction. These directions are reversed for a negative face.

This sign convention is also used in the matrix formulation of the beam bending problem which is the basis for computer-based analysis software. Historically, some authors use a sign convention for shear which is opposite to this choice. We prefer to employ the above convention since it is consistent with the output of structural software systems and therefore allows the reader to transition easily from analytical to digital computation schemes.

Fig. 3.14 Shear and moment diagrams

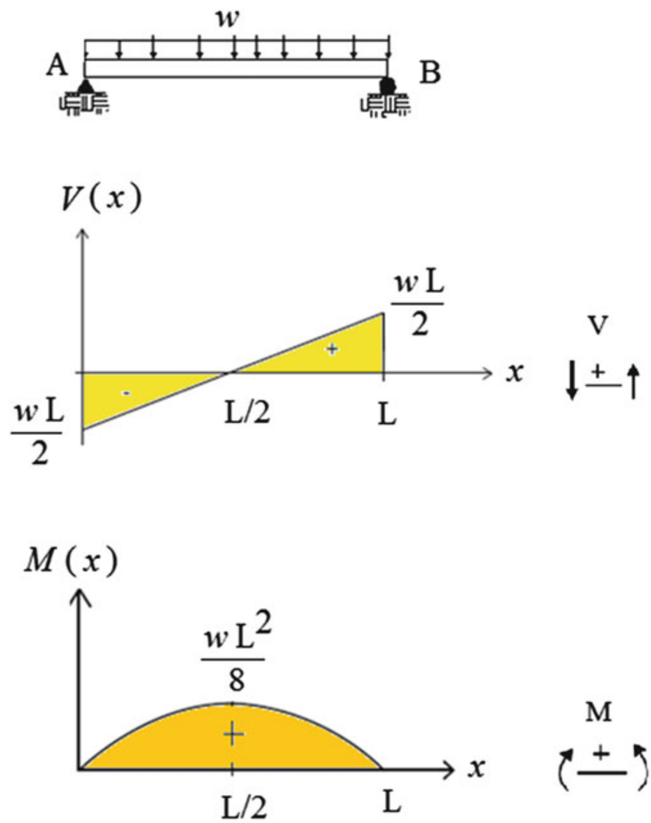


Figure 3.14 shows the variation of these quantities for the beam defined in Fig. 3.12. The shear varies linearly, with maximum values at the supports. The moment varies parabolically, and the maximum value occurs at mid-span. These plots are called “shear” and “moment” diagrams. Positive moment is plotted on the top face in the USA. In the UK, positive moment is plotted on the bottom face. Again, it is a question of what convention one is most comfortable with.

The maximum bending moment and shear force are used to determine the dimensions of the cross-section. The specific design procedure depends on the material selected, such as wood, steel, or concrete, and the design code adopted.

One constructs the internal force distributions by selecting various cutting planes, evaluating the corresponding values, and then extrapolating between the sections. With some experience, one can become very proficient at this operation. We illustrate the process with the following examples.

Example 3.7 Cantilever Beam with Multiple Concentrated Loads

Given: The cantilever beam with two concentrated loads shown in Fig. E3.7a.

Determine: The shear and moment diagrams.

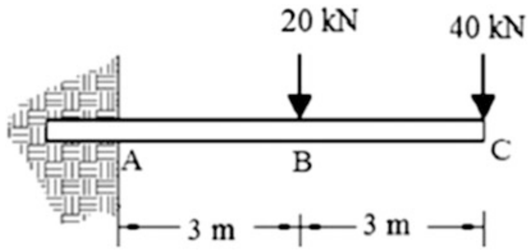
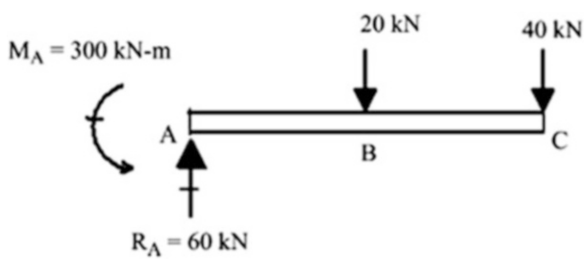


Fig. E3.7a

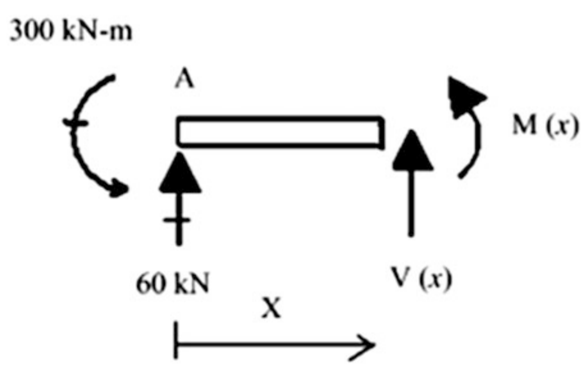
Solution: We first determine the reactions at A by enforcing the equilibrium equations.

$$\sum F_y = 0 \quad R_A - 20 - 40 = 0 \Rightarrow R_A = 60\,\text{kN}$$
$$\sum M_A = 0 \quad M_A - 20(3) - 40(6) = 0 \Rightarrow M_A = 300\,\text{kN m}$$



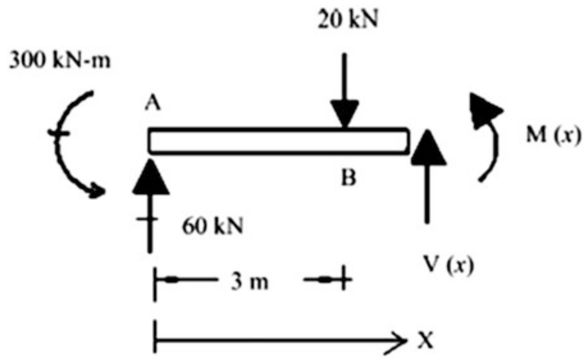
Then, we pass a cutting plane between points A and B

$$0 \le x < 3 \quad \begin{aligned} V(x) &= -60 \\ M(x) &= -300 + 60x \end{aligned}$$

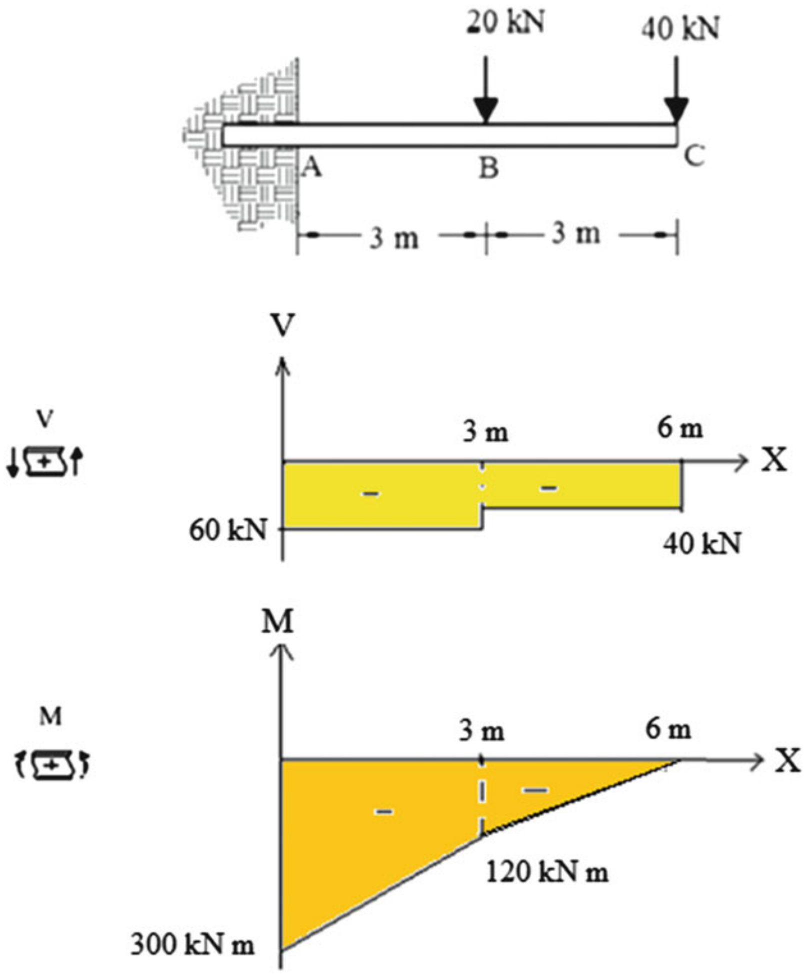


Lastly, we cut between B and C.

$$3 < x \le 6 \quad V(x) = -40$$
$$M(x) = -300 + 60x - 20(x-3) = 40x - 240$$



The distributions are plotted below.



There are some features that we want to point out. Firstly, a concentrated load produces a discontinuity in the form of a "jump" in the shear force, such as at points B and C. Secondly, when the loading consists only of concentrated loads, the shear diagram consists of segments having constant values, and the moment diagram is composed of a set of straight-line segments. We have demonstrated these features here. Later in the next section, we will establish a proof based on equilibrium considerations. A thought question: When would the moment diagram have a jump in the moment value? Hint: Consider Example 3.15.

Example 3.8 Cantilever Beam with Uniform Loading

Given: The uniformly loaded cantilever beam shown in Fig. E3.8a.
Determine: The shear and moment distributions.

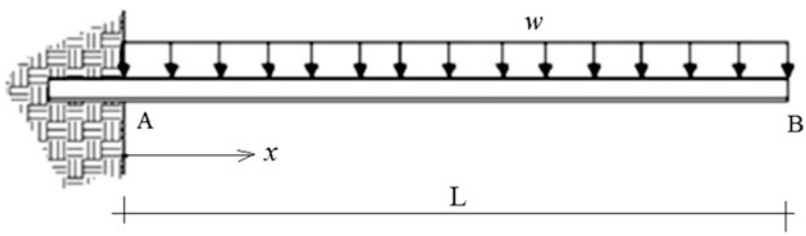
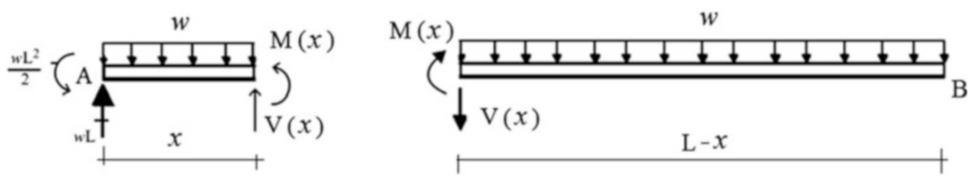


Fig. E3.8a

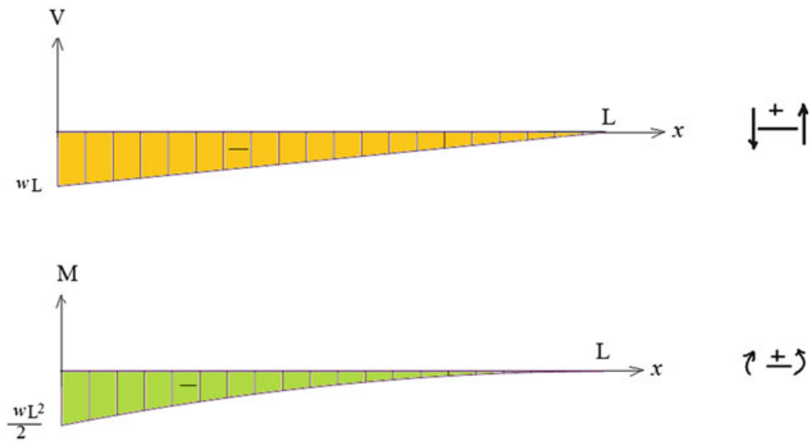
Solution: We pass a cutting plane between points A and B. Then, we can consider either segment shown below.



The shear and moment required for equilibrium are

$$0 \le x \le L \quad V(x) = -w(L-x)$$
$$M(x) = -\frac{w}{2}(L-x)^2$$

These functions are plotted below. Note that the maximum moment varies as L^2.



Example 3.9 Beam with an Eccentric Lateral Load

Given: The structure defined in Fig. E3.9a. Member BC is rigidly attached to member AB at B.
Determine: The axial, shear, and moment diagrams.

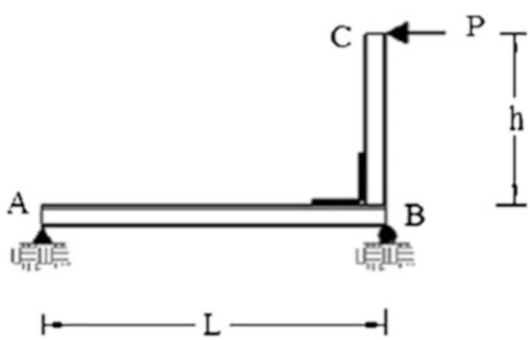


Fig. E3.9a

Solution: Member BC is rigidly attached to the beam, AB, and has a horizontal load applied at its end. The effect of this force is to apply a bending moment at B, which causes beam AB to bend. Figure E3.9b illustrates the deflected shape.

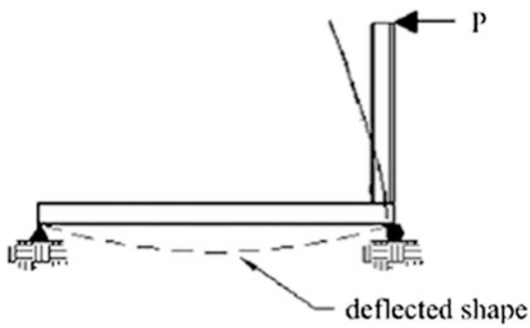
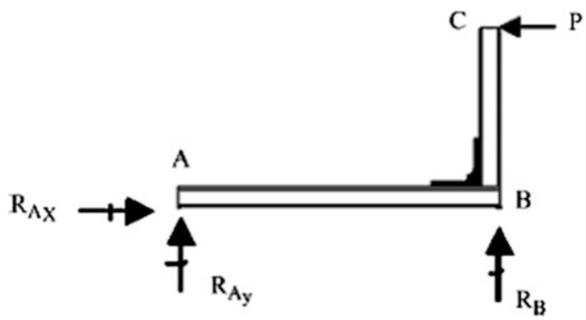


Fig. E3.9b

We determine the reactions first. The free body diagram is shown below.



Moment summation about A leads to

$$\sum M_A = 0$$

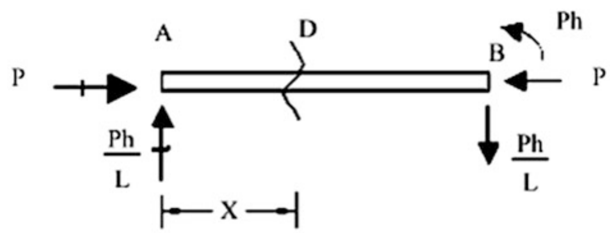
$$R_B L + Ph = 0 \quad \Rightarrow \quad R_B = -\frac{h}{L}P \quad \Rightarrow \quad R_B = \frac{h}{L}P \downarrow$$

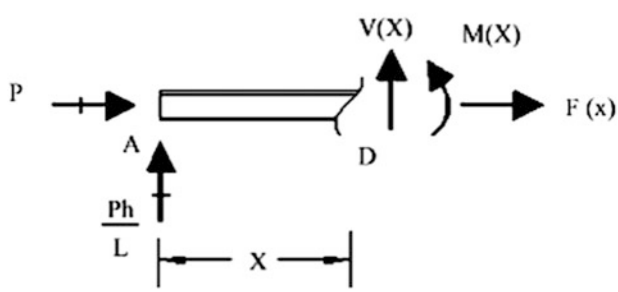
The reactions at A required for equilibrium are

$$\sum F_y = 0 \quad R_{Ay} = -V_B = \frac{h}{L}P \quad \Rightarrow \quad R_{Ay} = \frac{h}{L}P \uparrow$$

$$\sum F_x = 0 \quad \Rightarrow \quad R_{Ax} = P \rightarrow$$

Next, we pass a cutting plane at D, isolate the left segment, and enforce equilibrium.

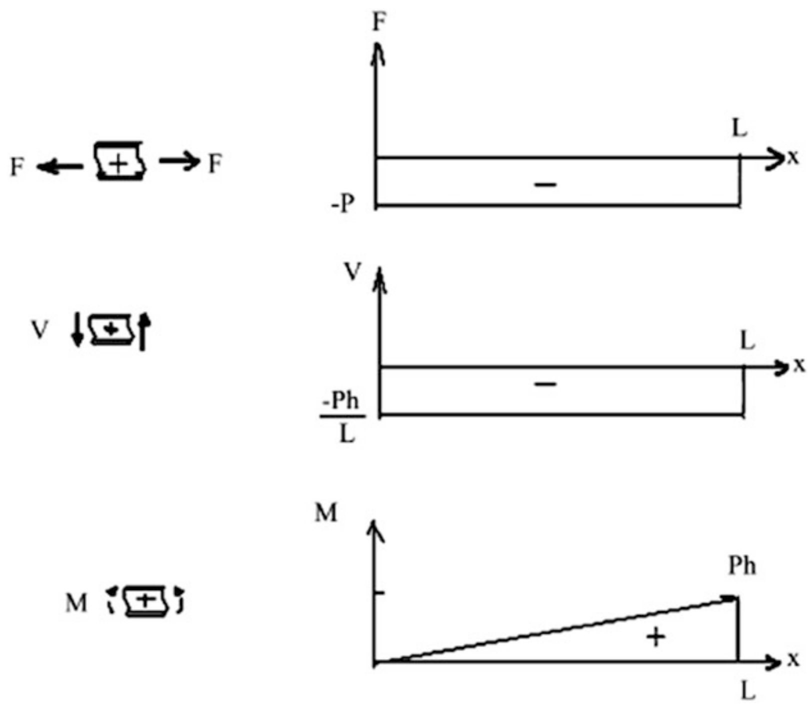




The results are

$$V(x) = -Ph/L$$
$$M(x) = (Ph/L)x$$
$$F(x) = -P$$

The beam is subjected to combined compression and bending: The maximum moment is equal to Ph and occurs at B. This is the critical section for design. Plots of F, V, and M for member AB are shown below.



3.5 Differential Equations of Equilibrium: Planar Loading

The strategy described in the previous section was based on working with a free body diagram of a large segment of the beam and determining the shear and moment by applying the equilibrium equations. We generate the distributions of these quantities by selecting various free body diagrams. This approach is convenient when the loading is fairly simple, i.e., it consists of a combination of concentrated forces and uniformly distributed loadings. For complex distributed loadings expressed as analytic functions, one needs a more systematic approach for enforcing the equilibrium conditions. In what follows, we describe an approach based on applying the equilibrium conditions to a differential element of the beam, resulting in a set of differential equations relating the shear force and moment to the applied distributed loading.

Consider the beam and the differential element shown in Fig. 3.15. We use the same sign convention for V and M as defined in Sect. 3.4. We take the positive sense of the distributed loading to be "downward" since these loadings are generally associated with gravity. Considering V and M to be functions of x, expanding these variables in terms of their differentials, and retaining up to first-order terms results in the forces shown in Fig. 3.15b.

Summing forces in the Y direction,

$$V + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{2} - \left(V - \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{2}\right) - w\,\mathrm{d}x = 0$$

and combining terms leads to

$$\left(\frac{\mathrm{d}V}{\mathrm{d}x} - w\right)\mathrm{d}x = 0$$

Lastly, since this equation must be satisfied for arbitrary dx, it follows that

$$\frac{\mathrm{d}V}{\mathrm{d}x} = w \tag{3.4}$$

In words, "*the rate of change of the shear force is equal to the applied distributed loading*."

Repeating this analysis for moment summation about point o, the steps are

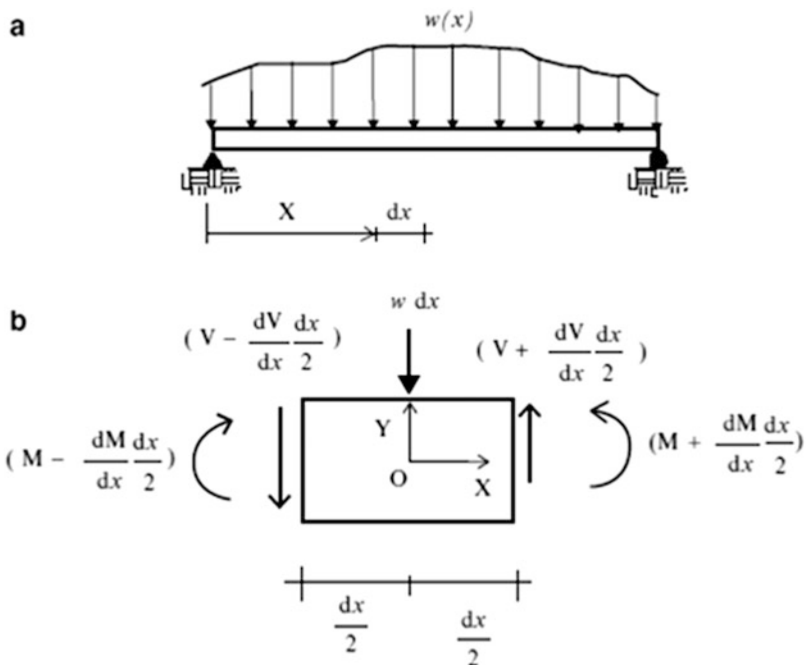


Fig. 3.15 Beam with arbitrary distributed loading. (**a**) Beam. (**b**) Differential beam element

$$M + \frac{\mathrm{d}M}{\mathrm{d}x}\frac{\mathrm{d}x}{2} - \left(M - \frac{\mathrm{d}M}{\mathrm{d}x}\frac{\mathrm{d}x}{2}\right) + \left(V + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{2}\right)\frac{\mathrm{d}x}{2} + \left(V - \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{2}\right)\frac{\mathrm{d}x}{2} = 0$$

$$\Downarrow$$

$$\left(\frac{\mathrm{d}M}{\mathrm{d}x} + V\right)\mathrm{d}x = 0$$

$$\Downarrow \tag{3.5}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} + V = 0$$

$$\Downarrow$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -V$$

Equation (3.5) states that "*the rate of change of the bending moment is equal to minus the shear force*."

These two relations are very useful for checking the consistency of the shear and moment diagrams. One can reason qualitatively about the shape of these diagrams using only information about the loading on a segment of the beam. For example, if $w = 0$, the shear is constant and the moment varies linearly. If $w =$ constant, the shear varies linearly and the moment varies quadratically.

One can establish a set of integral equations by integrating the derivative terms. Consider two points, x_1 and x_2, on the longitudinal X-axis. Integrating (3.4) and (3.5) between these points leads to

$$V_2 - V_1 = \int_{x_1}^{x_2} w\,\mathrm{d}x \tag{3.6}$$

$$M_2 - M_1 = -\int_{x_1}^{x_2} V\,\mathrm{d}x \tag{3.7}$$

Equation (3.6) can be interpreted as: "*The difference in shear between two points is equal to the area under the distributed loading diagram included between these points*." Equation (3.7) relates the *change in moment to the area under the shear diagram between these points*. Figure 3.16 illustrates these interpretations.

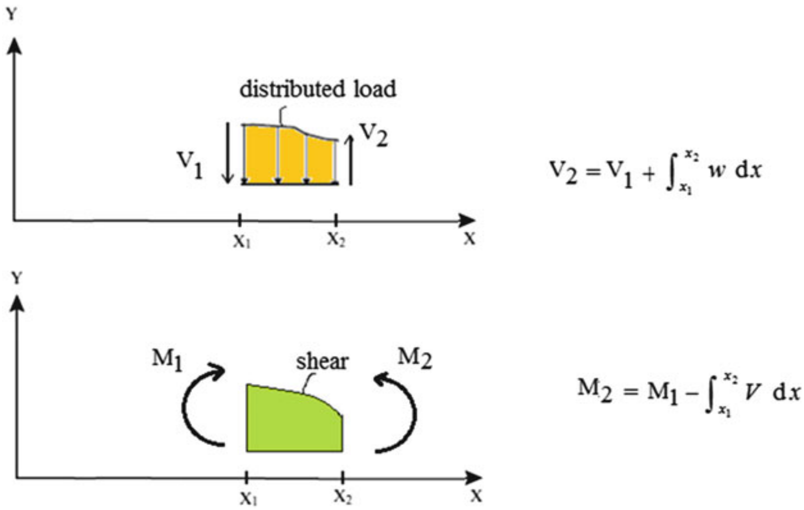
The integral forms are useful if one wants to either compute values at discrete points or determine analytical solutions. The differential forms are more convenient for qualitatively reasoning about the shape of the diagrams. We generally use both approaches to construct shear and moment diagrams.

Another useful property that can be established from (3.5) relates to the maximum values of the moment. We know from calculus that extreme values of a continuous function are located at points where the first derivative is zero. Applying this theorem to the moment function, $M(x)$, the location x^*, of an extreme value (either maximum or minimum) of moment is found by solving:

$$\left.\frac{\mathrm{d}M}{\mathrm{d}x}\right|_{x=x^*} = 0 \tag{3.8}$$

Noting (3.5), it follows that *extreme values of moment occur at points where the shear force is zero*. One first generates the shear diagrams from the applied loading. This process identifies the points of zero shear. If only peak values of moment are of interest, one selects free body diagrams by passing cutting planes through these locations and applies the equilibrium conditions. This approach is the most direct procedure.

Fig. 3.16 Interpretation of shear and moment in terms of segmental loadings



When the loading consists of concentrated forces, the shear diagram has a discontinuity at the point of application of each concentrated force. By considering the equilibrium of a differential element at the point (see Fig. 3.17), one can establish that *the "jump" in shear is equal to the applied load. Similarly, the jump in moment is equal to the applied external moment*, M.

$$\mathbf{V}_{+\text{face}} - \mathbf{V}_{-\text{face}} = \mathbf{P} \tag{3.9}$$

$$\mathbf{M}_{+\text{face}} - \mathbf{M}_{-\text{face}} = -\mathbf{M} \tag{3.10}$$

One applies (3.6) and (3.7) to generate solutions for the segments adjacent to the discontinuities and uses (3.9) and (3.10) to connect the solutions.

In what follows, we illustrate the application of the differential/integral equation representation to generate shear and moment diagrams. This material overlaps slightly with the material presented in the previous section. Some repetition is useful for reinforcing basic concepts.

Example 3.10 Cantilever Beam—Triangular Loading

Given: A cantilever beam with a triangular distributed loading (Fig. E3.10a).
Determine: $V(x)$ and $M(x)$.

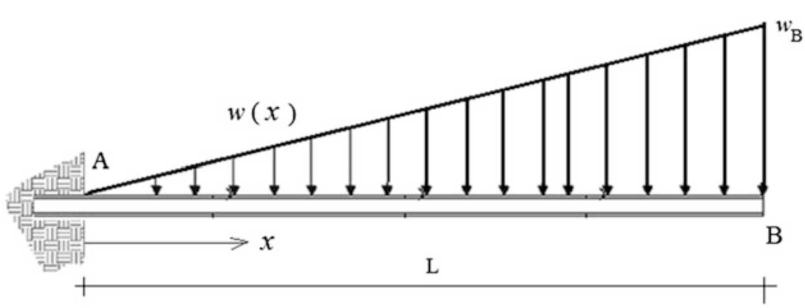


Fig. E3.10a

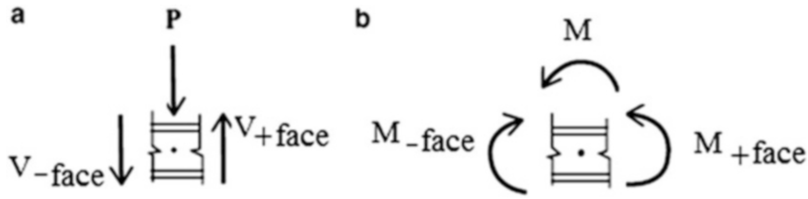


Fig. 3.17 Jump conditions. (**a**) Shear. (**b**) Moment

Solution: First, we determine the reactions at A (Fig. E3.10b)

$$\sum F_Y = 0 \quad R_A = \frac{w_B L}{2} \uparrow$$

$$\sum M_A = 0 \quad M_A = \frac{w_B L^2}{3}$$

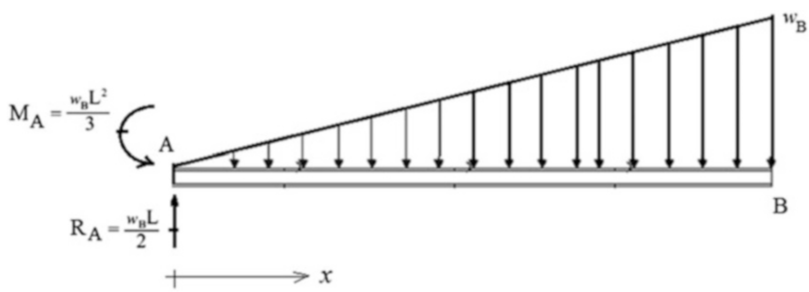


Fig. E3.10b Reactions

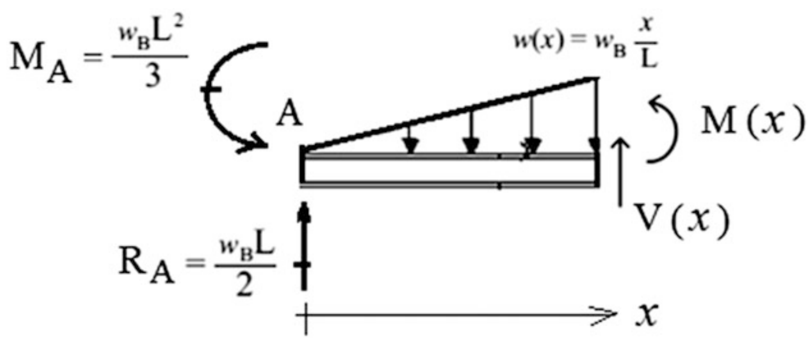


Fig. E3.10c Internal shear and moment

Next, we determine the shear, $V(x)$, with (3.6) (Fig. E3.10c). Integrating between points A and x

$$V(x) - V_A = \int_0^x w(x)\mathrm{d}x = \int_0^x \frac{w_B x}{L}\mathrm{d}x = \left. \frac{w_B x^2}{2L} \right|_0^x = \frac{w_B x^2}{2L}$$

Noting that $V_A = -R_A = -\dfrac{w_B L}{2}$, the solution for $V(x)$ reduces to

$$V(x) = \frac{w_B}{2}\left(-L + \frac{x^2}{L}\right)$$

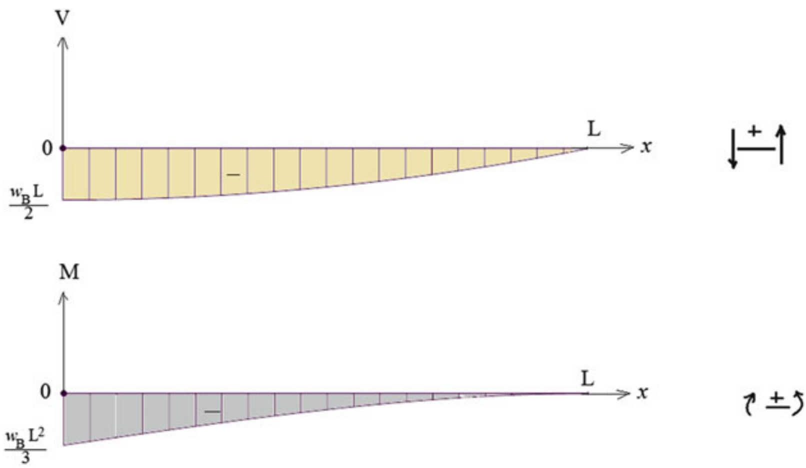
We determine the moment, $M(x)$, with (3.7)

$$M(x) - M_A = -\int_0^x V(x)\mathrm{d}x = \int_0^x \frac{w_B}{2}\left(L - \frac{x^2}{L}\right)\mathrm{d}x = \frac{w_B}{2}\left(Lx - \frac{x^3}{3L}\right)$$

Noting that $M_A = -\frac{w_B L^2}{3}$, one obtains

$$M(x) = w_B\left(-\frac{x^3}{6L} + \frac{Lx}{2} - \frac{L^2}{3}\right)$$

The shear and moment distribution are plotted below. Note that the peak values of shear and moment occur at $x = 0$. Also note that the boundary conditions at B are $V_B = M_B = 0$ since this cross-section is free, i.e., unrestrained and unloaded.



Example 3.11 Distributed and Concentrated Loads

Given: The beam and loading defined in Fig. E3.11a.

Determine: $V(x)$ and $M(x)$.

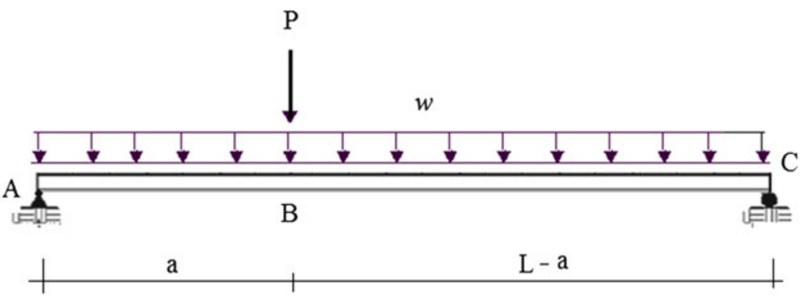


Fig. E3.11a

Solution: This example illustrates how to deal with a combination of distributed and concentrated loads. We separate the distributed and the concentrated loads and then superimpose the results (Fig. E3.11b).

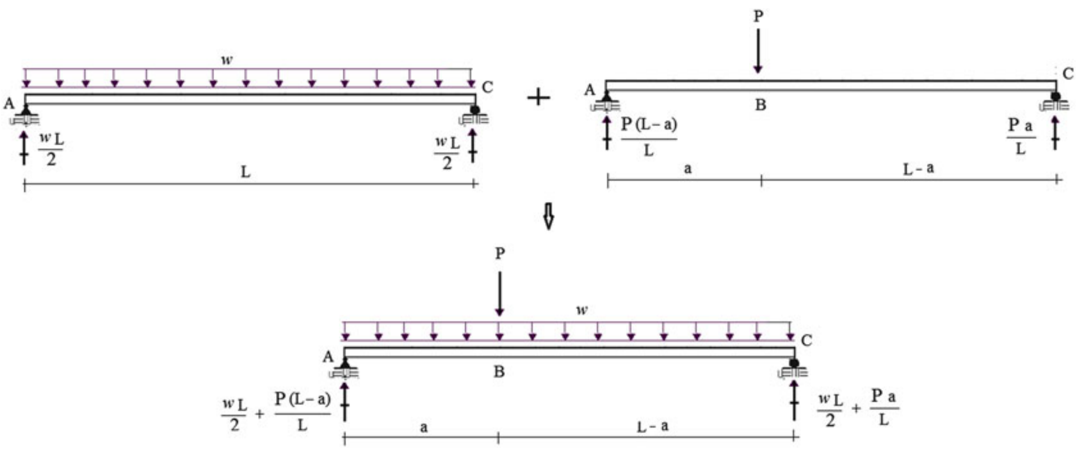


Fig. E3.11b Reactions

We consider first the segment AB. Applying (3.6) and (3.7), and noting the boundary conditions at $x = 0$, the distributions for $0 \le x < a$ are

$$V(x) = -\frac{wL}{2} - \frac{(L-a)P}{L} + wx$$

$$M(x) = \frac{wL}{2}x + \frac{P(L-a)}{L}x - \frac{1}{2}wx^2$$

The values of V and M just to the left of point B are

$$V_{B-} = -\frac{wL}{2} + wa - \frac{P(L-a)}{L}$$

$$M_{B-} = \frac{wL}{2}a - \frac{1}{2}wa^2 + \frac{P(L-a)}{L}a$$

Applying (3.9) and (3.10) for the jump conditions at B, and noting signs, the quantities just to the right of B are

$$V_{B+} = P + V_{B-} = -\frac{wL}{2} + wa - \frac{Pa}{L}$$

$$M_{B+} = M_{B-} = \frac{wL}{2}a - \frac{1}{2}wa^2 + \frac{P(L-a)}{L}a$$

Note that there is no jump in moment for this example.

Applying (3.6) and (3.7), these expressions for $a < x \leq L$ expand to

$$V(x) = \frac{Pa}{L} - \frac{wL}{2} + wx$$

$$M(x) = Pa - \frac{Pax}{L} + \frac{wLx}{2} - \frac{wx^2}{2}$$

The approach we followed here is general and applies for all loadings. It is fairly straightforward to establish the expressions for the regions $0 \leq x < a$ and $a < x \leq L$. An easier way to obtain the shear and moment diagrams for this example would be to generate separate diagrams for the two types of loadings and then superimpose the results. The individual shear and moment diagrams are plotted below (Figs. E3.11c and E3.11d).

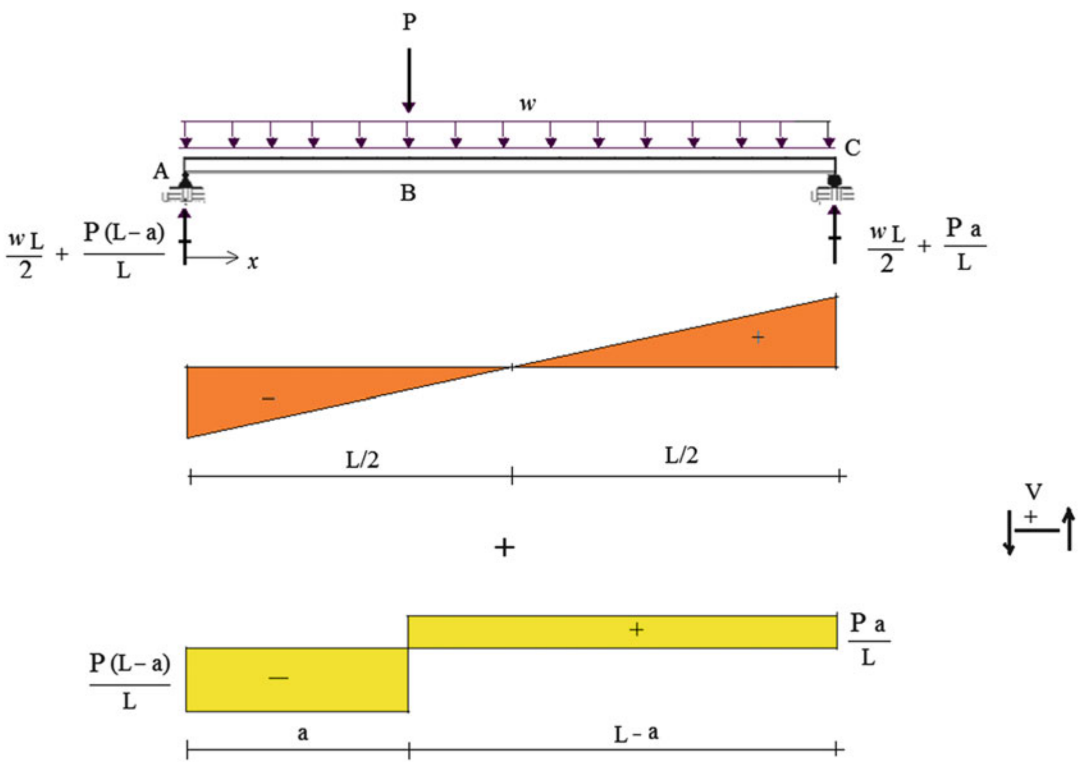


Fig. E3.11c Shear diagrams

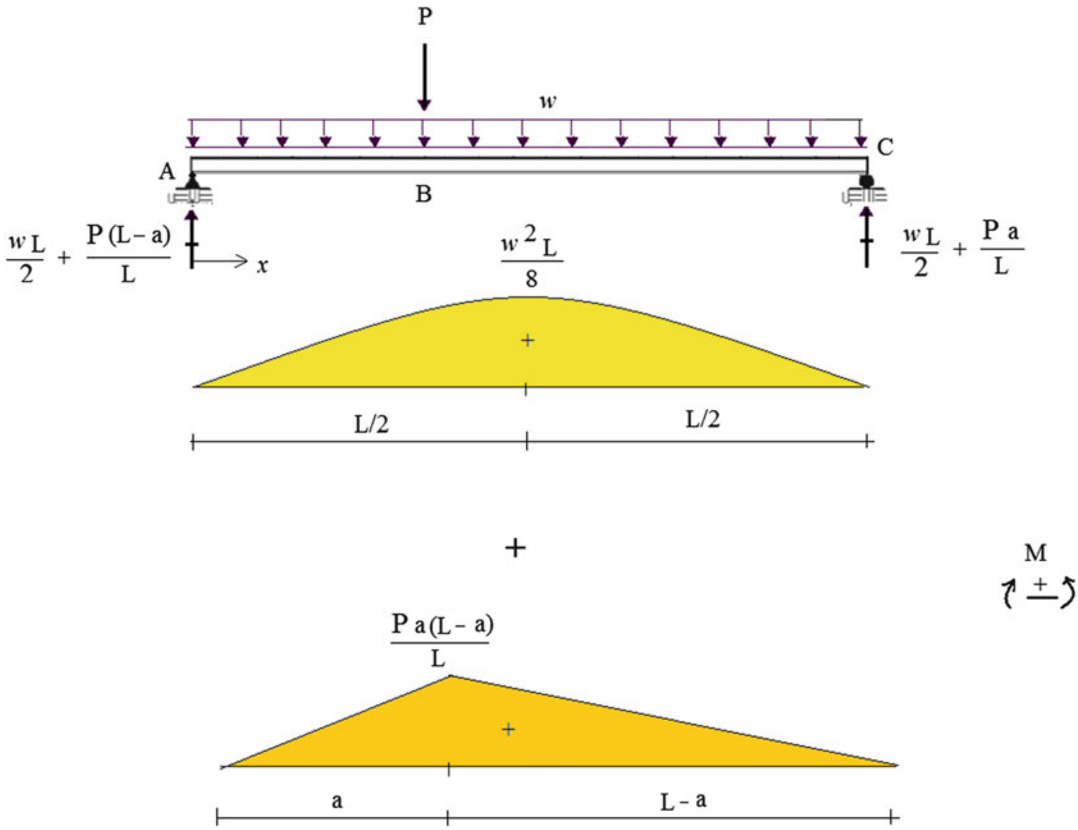
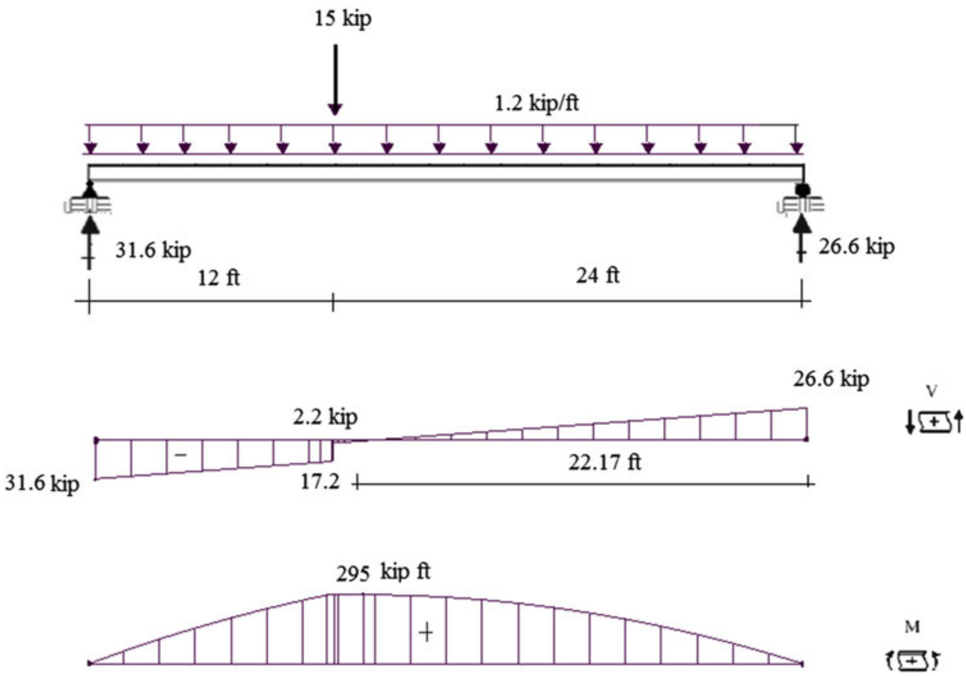


Fig. E3.11d Moment diagrams Suppose $P = 15$ kip, $a = 12$ ft, $L = 36$ ft, and $w = 1.2$ kip/ft. The combined shear and moment diagrams are plotted below.



Example 3.12 Uniform Loading Combined with End Moments

Given: A simply supported beam subjected to a uniform loading and bending moments at the ends. This is a typical case for a floor beam in a rigid building frame, i.e., where the beam-column connections apply moment to the ends of the beam (Fig. E3.12a).

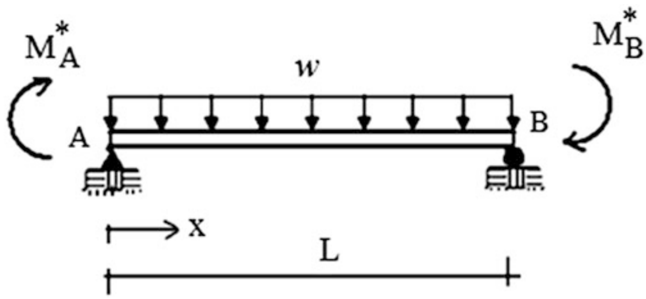


Fig. E3.12a

Determine: The location and magnitude of the maximum moment.

Solution: We consider separate loadings and then superimpose the results. The solution due to the end moment is (Fig. E3.12b)

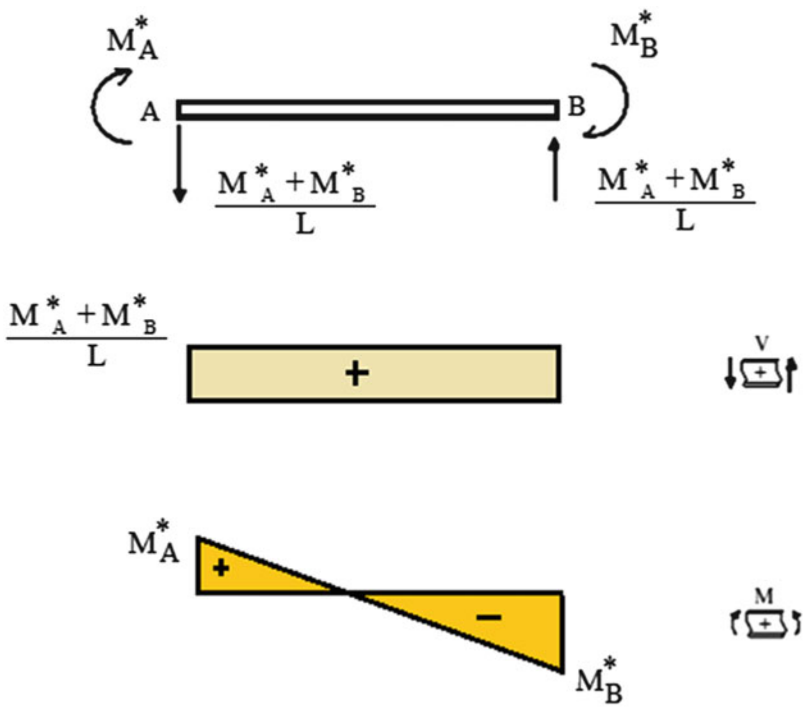


Fig. E3.12b

The uniform loading provides the following distribution (Fig. E3.12c):

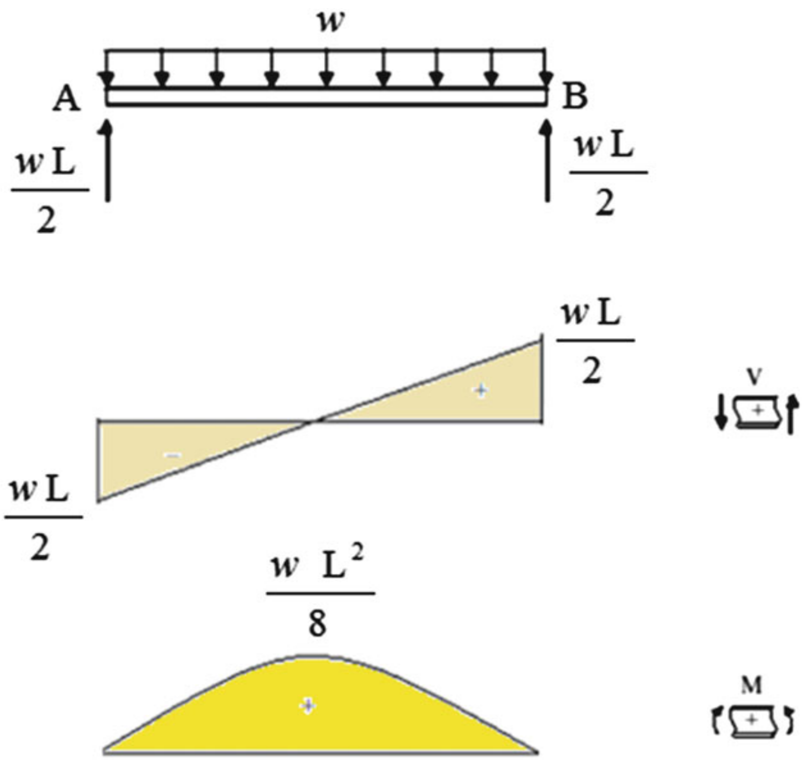


Fig. E3.12c

Combining these solutions leads to the analytical solution

$$V(x) = -\frac{wL}{2} + wx + \frac{M_{\mathrm{A}}^{*} + M_{\mathrm{B}}^{*}}{L}$$
$$M(x) = M_{\mathrm{A}}^{*} - \frac{M_{\mathrm{A}}^{*} + M_{\mathrm{B}}^{*}}{L}x + \frac{wL}{2}x - \frac{wx^2}{2}$$

These functions are plotted below (Figs. E3.12d and E3.12e).

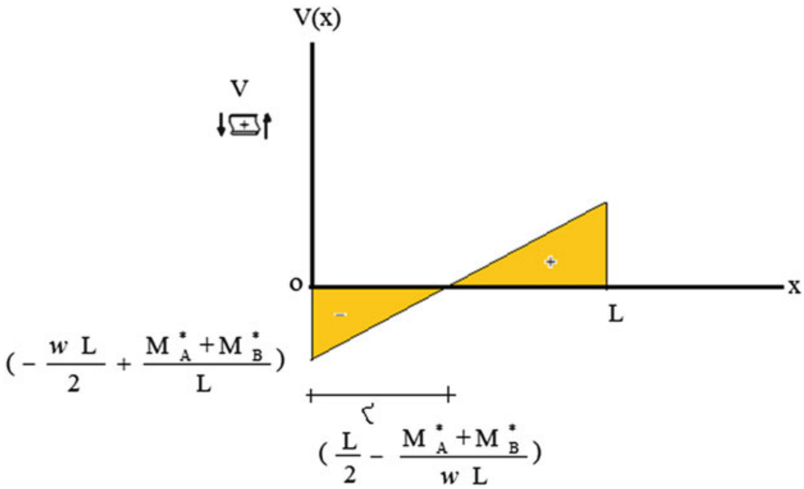


Fig. E3.12d

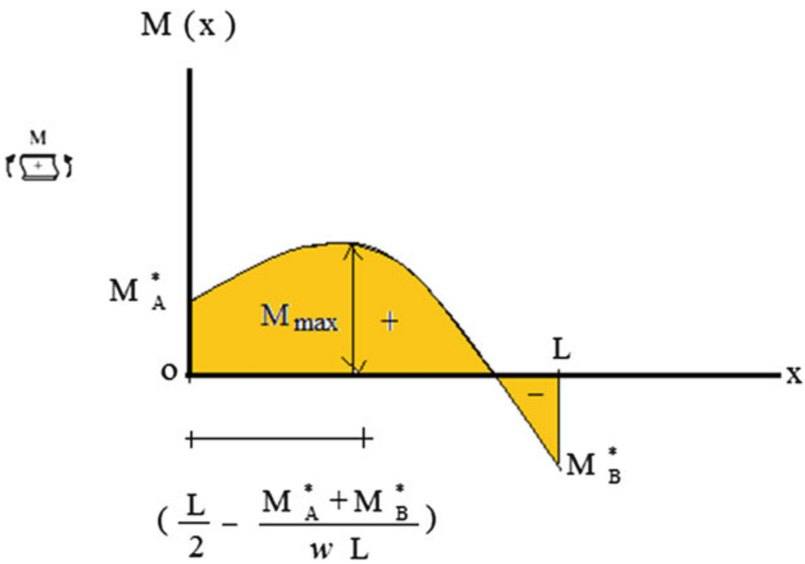


Fig. E3.12e

The peak moment occurs where the shear is zero. Noting the plot of $V(x)$, the shear is zero at x_{max}.

$$V = 0$$
$$\Downarrow$$
$$x_{max} = \frac{L}{2} - \frac{M_A^* + M_B^*}{wL}$$

The form of the solution suggests that we express the sum of the end moments as

$$M_A^* + M_B^* = \frac{\alpha}{2} wL^2$$

where α is a dimensionless parameter. Substituting for this term, the equation simplifies to

$$x_{\max} = \frac{L}{2}(1 - \alpha)$$

Lastly, we determine $M_{\max}$ using this value for x.

$$M_{\max} = M_{\mathrm{A}}^{*} + \frac{wL^2}{8}(1 - \alpha)^2$$

Given w and the end moments, one evaluates α,

$$\alpha = \frac{M_{\mathrm{A}}^{*} + M_{\mathrm{B}}^{*}}{\frac{wL^2}{2}}$$

and then $M_{\max}$. When $\alpha = \pm 1$, the peak moment occurs at an end point and equals the applied end moment.

The case where the end moments are equal in magnitude but opposite in sense is of considerable interest. One sets $M_{\mathrm{A}}^{*} = -M_{\mathrm{B}}^{*} = -M^{*}$, and it follows that $\alpha = 0$. The moment diagram is symmetrical with respect to the centerline. The peak negative values of moment occur at the end points; the peak positive moment occurs at the center point of the span (Fig. E3.12f).

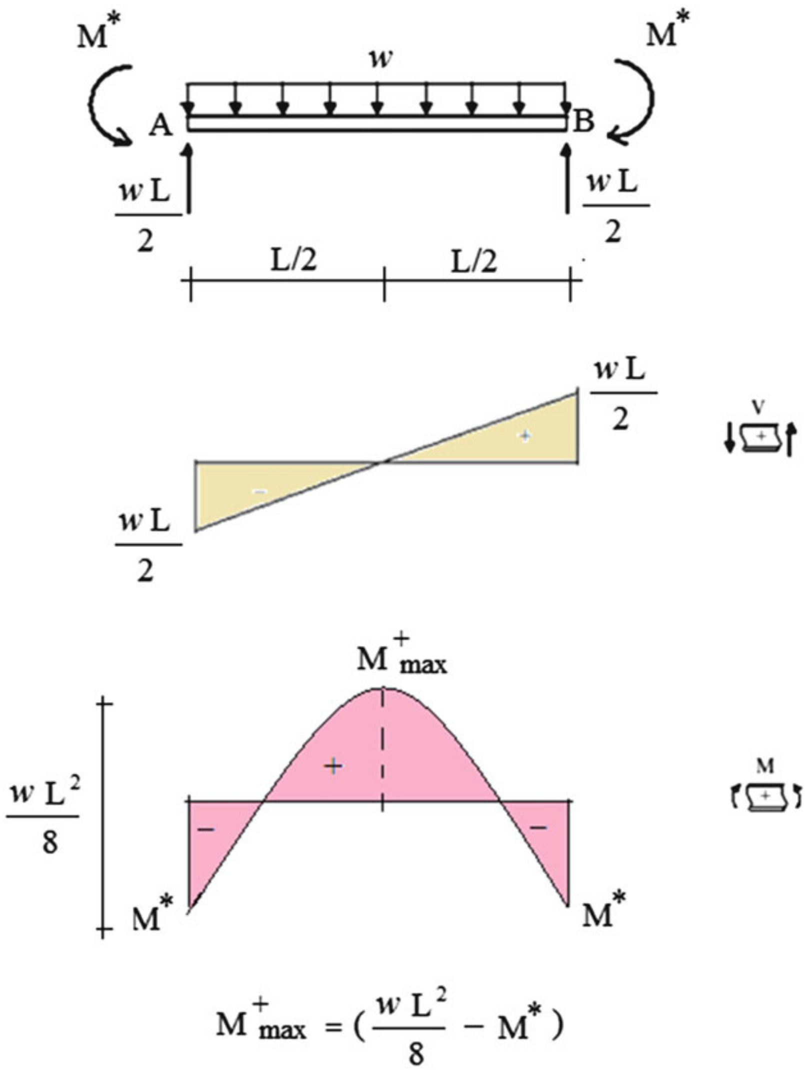


Fig. E3.12f

When there is no end restraint, $M^* = 0$. Then, $M^{+}_{\text{max}} = wL^2/8$. The effect of end restraint is to *reduce* the positive moment and introduce a negative moment at the ends. This behavior is typical for rigid frames such as building frames subjected to gravity loading. We examine this behavior in more detail in Chap. 15.

Example 3.13

Given: The beam shown in Fig. E3.13a

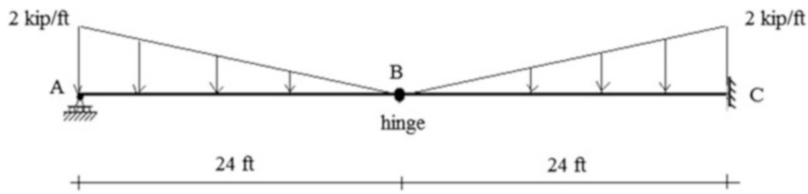


Fig. E3.13a

Determine: The reactions, shear, and bending moment distributions.
Solution: We draw the free body diagram of beam segment AB. Applying the equilibrium conditions to this segment results in (Fig. E3.13b)

$$\sum M_{\rm A}=0 \quad V_{\rm B}(24)-(2)\frac{(24)}{2}\frac{(24)}{3}=0 \quad V_{\rm B}=8\,\text{kip}$$
$$\sum F_y=0 \quad R_{\rm A}-(2)\frac{(24)}{2}+8=0 \qquad R_{\rm A}=16\,\text{kip}$$

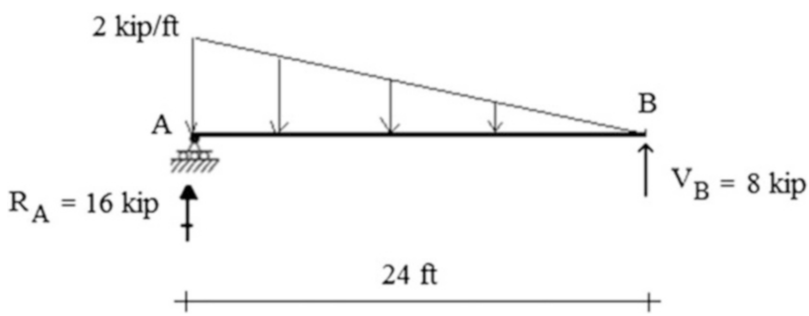


Fig. E3.13b Segment AB

With the internal force at B known, one can now proceed with the analysis of segment BC (Fig. E3.13c).

$$\sum F_Y=0 \;-(2)\frac{(24)}{2}-8+R_{\rm C}=0 \quad \Rightarrow \quad R_{\rm C}=32\,\text{kip}\uparrow$$
$$\sum M_{\rm C}=0 \;-M_{\rm C}+8(24)+(2)\frac{(24)}{2}\frac{(24)}{3}=0 \quad \Rightarrow \quad M_{\rm C}=384\,\text{kip ft clockwise}$$

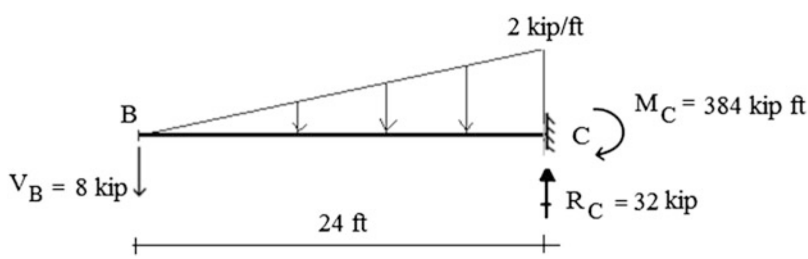
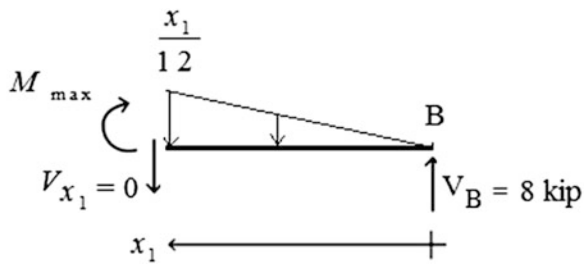


Fig. E3.13c Segment BC

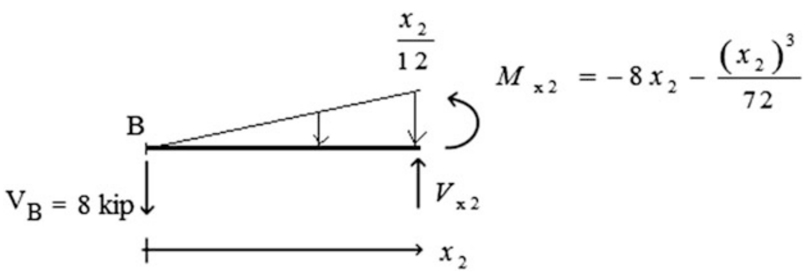
The peak moment occurs where the shear is zero.
Segment AB:



$$\sum F_y = 0 \quad 8 - \frac{1}{2}\left(\frac{x_1}{12}\right)x_1 = 0 \quad \rightarrow \quad x_1 = 13.85\,\text{ft}$$

$$\therefore M_{\max} = 8(13.85) - (13.85)\left(\frac{13.85}{12}\right)\frac{1}{2}\left(\frac{13.85}{3}\right) = +73.9\,\text{kip ft}$$

Segment BC:



$$\sum F_y = 0 \quad V_{x_2} = 8 + \frac{1}{2}\left(\frac{x_2}{12}\right)x_2 \neq 0$$

Therefore, there is no peak moment between B and C.

The shear and bending moment diagrams are listed below (Fig. E3.13d).

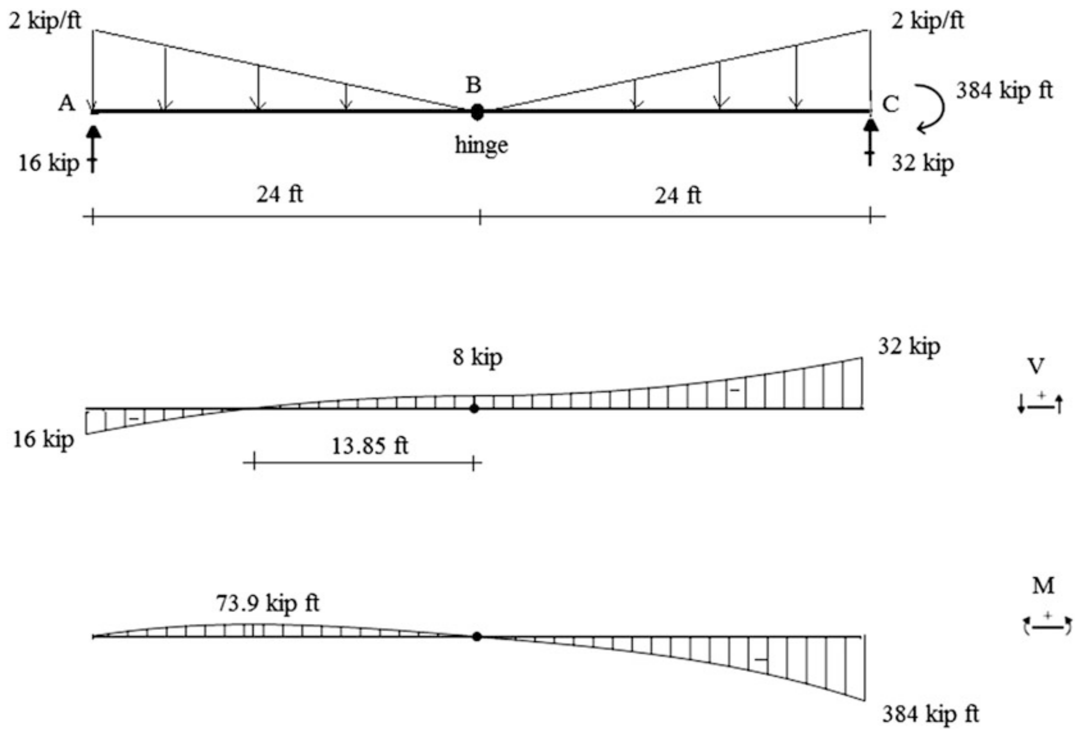


Fig. E3.13d

Example 3.14

Given: The beam shown in Fig. E3.14a

Determine: The reactions, and the shear and bending moment distributions.

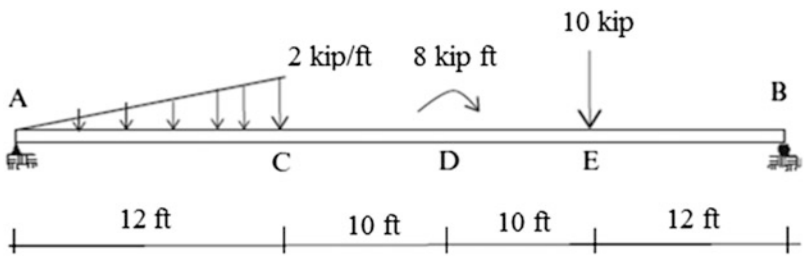


Fig. E3.14a

Solution: We first determine the reaction at B using $\sum M_A = 0$. We then compute the reaction at A by summing forces in the Y direction (Fig. E3.14b).

$$\sum M_A = 0 \quad -R_B(44) + (2)\left(\frac{12}{2}\right)\frac{2}{3}(12) + 8 + 10(32) = 0 \quad R_B = 9.64 \uparrow$$

$$\sum F_Y = 0 \quad R_A - 2\left(\frac{12}{2}\right) - 10 + 9.64 = 0 \qquad R_A = 12.36 \uparrow$$

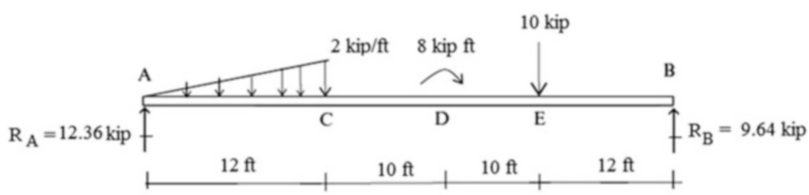


Fig. E3.14b

We determine the shear, $V(x)$, with (3.6) (Fig. E3.14c). Integrating between points A and x

$$V(x) - V_A = \int_0^x w(x)\mathrm{d}x = \int_0^x \frac{x}{6}\mathrm{d}x = \frac{x^2}{12}\bigg|_0^x = \frac{x^2}{12} \quad 0 \le x \le 12$$

Noting that $V_A = -12.36$, the solution for $V(x)$ reduces to

$$V(x) = \frac{x^2}{12} - 12.36$$

We determine the *moment* $M(x)$, with (3.7) (Fig. E3.14c). Integrating between points A and x

$$M(x) - M_A = -\int_0^x V(x)\mathrm{d}x = -\int_0^x \left(\frac{x^2}{12} - 12.36\right)\mathrm{d}x = -\frac{x^3}{36} + 12.36x \quad 0 \le x \le 12$$

Noting that $M_A = 0$, one obtains

$$M(x) = -\frac{x^3}{36} + 12.36x$$

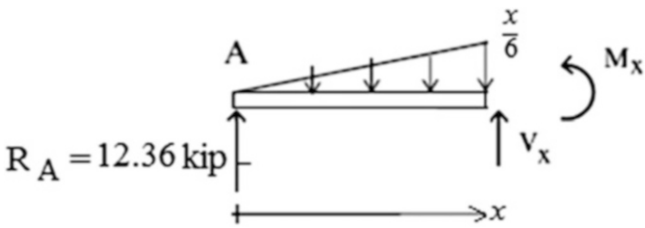
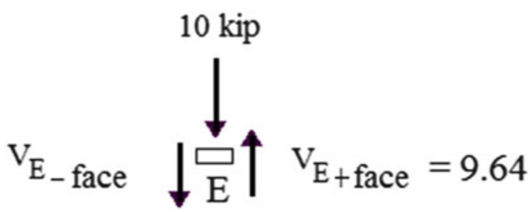


Fig. E3.14c

Note that there is a jump in the shear at E.



$$\sum F_y = 0 \quad V_{\text{E-face}} - 9.64 + 10 = 0 \quad V_{\text{E-face}} = -0.36 \quad V_{\text{E-face}} = 0.36 \uparrow \text{kip}$$

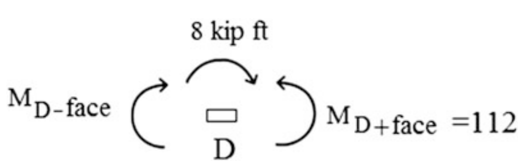
Applying (3.7) to the different segments results in:
Segment EB

$$M_{\text{B}} - M_{\text{E}} = -\int_{\text{E}\to\text{B}} V\,\mathrm{d}x = -9.64(12) = -115.68$$
$$M_{\text{B}} = 0 \quad \therefore M_{\text{E}} = 115.68\,\text{kip ft}$$

Segment DE

$$M_{\text{E}} - M_{\text{D}} = -\int_{\text{E}\to\text{B}} V\,\mathrm{d}x = -(-0.36)(10) = 3.6$$
$$M_{\text{E}} = 115.68 \quad \therefore M_{\text{D}} = 112\,\text{kip ft}$$

Note that there is a jump in the bending moment at D.



$$\sum M = 0 \quad M_{\text{D-face}} + 8 - 112 = 0 \quad M_{\text{D-face}} = 104\,\text{kip ft}$$

Segment CD

$$M_{\text{D}} - M_{\text{C}} = -\int_{\text{E}\to\text{B}} V\,dx = -(-0.36)(10) = 3.6$$
$$M_{\text{D}} = 104. \quad \therefore M_{\text{C}} = 100.4\,\text{kip ft}$$

The reactions, shear, and bending moment distributions are listed below (Fig. E3.14d).

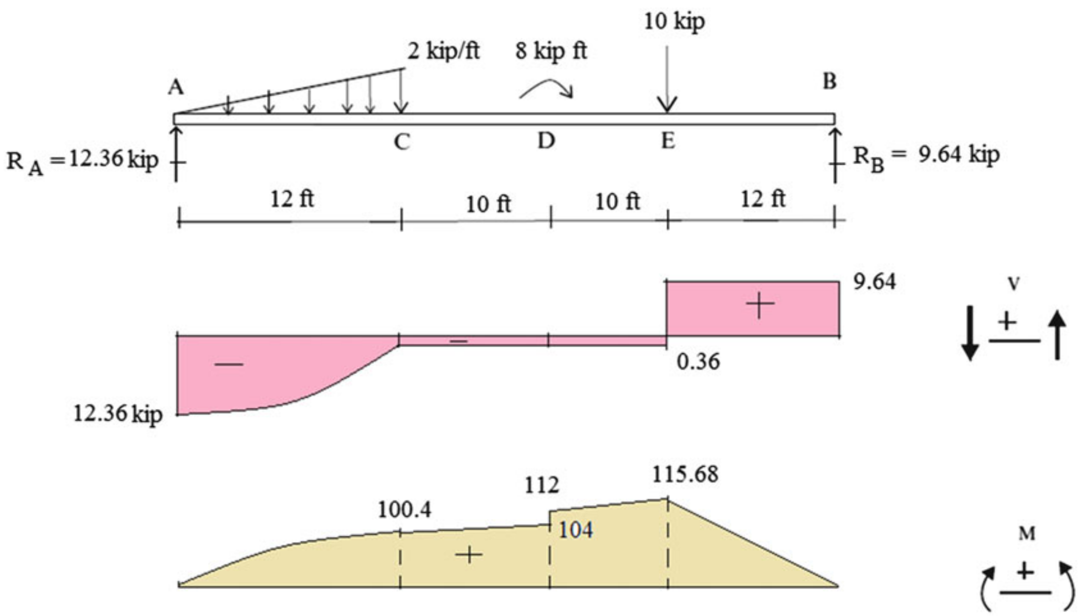


Fig. E3.14d

3.6 Displacement and Deformation of Slender Beams: Planar Loading

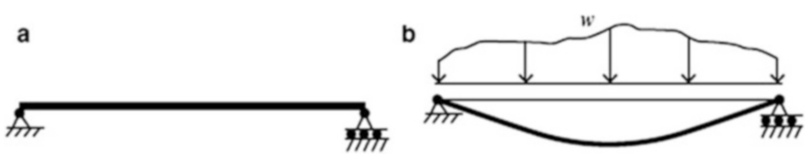


Fig. 3.18 Slender beam. (**a**) Initial. (**b**) Deformed

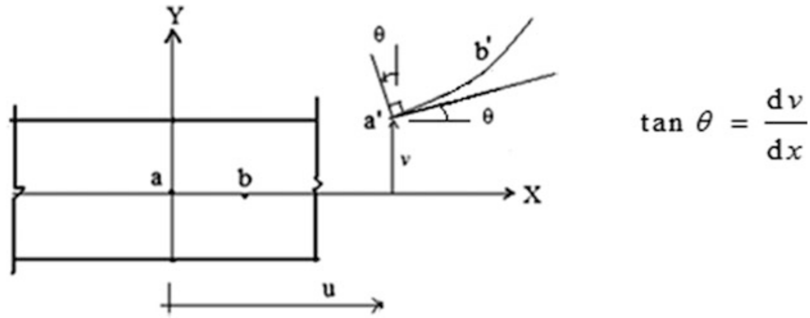


Fig. 3.19 Definition of displacement components

Figure 3.18 shows how a slender beam responds to a transverse planar loading. The geometric quantities that define the movement of the beam from its unloaded position due to an applied loading are defined as the displacements. Displacements are also referred to as deflections. Consider the segment of a homogeneous beam shown in Fig. 3.19. We take the X-axis to coincide with the initial position of the centroidal axis and the Y-axis to be 90° counterclockwise from the X-axis. When the loading is applied in the $X - Y$ plane, points on the centroidal axis move horizontally and vertically. We assume the cross-section, which is initially normal to the centroidal axis, remains normal to the curve connecting the displaced points. This is a standard assumption for beams known as "Kirchoff's" hypothesis and implies that the cross-section rotates through the same angle as the tangent to the centroidal axis. Kirchoff's hypothesis is valid for slender beams, i.e., beams having a depth to span ratio less than about 0.1. With this assumption, the independent geometric measures are the two displacement components, $u(x)$ and $v(x)$, which are functions of x for static loading. Given $v(x)$, we find the cross-sectional rotation, $\theta(x)$, with the geometric relation.

The next assumption that we introduce concerns the magnitude of θ. We assume here that θ^2 is small in comparison to unity, which implies that the tangent is essentially equal to the angle in radians:

$$\tan\theta \approx \theta \tag{3.11}$$

Then, the expression for θ reduces to

$$\theta \approx +\frac{\mathrm{d}v}{\mathrm{d}x} \tag{3.12}$$

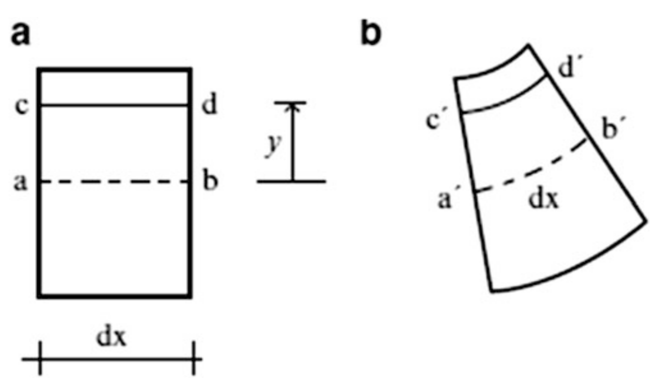


Fig. 3.20 Differential beam elements. (**a**) Initial. (**b**) Deformed

Deformations are dimensionless strain measures resulting from displacements. Consider the differential elements shown in Fig. 3.20. The initial rectangular shape is transformed to a quadrilateral shape with curved upper and lower edges. Adjacent cross-sections experience a relative rotation equal to $(\mathrm{d}\theta/\mathrm{d}x)\mathrm{d}x$, which causes line elements parallel to the centroidal axis to either elongate or contract. These changes in length produce extensional strains. A line element located y units above the centroidal axis experiences a strain $\varepsilon(y)$ equal to

$$\varepsilon(y) = -y\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{3.13}$$

According to this model, the strain varies linearly over the cross-section and the peak strain values occur at the upper and lower surfaces; the centroidal axis is not strained.

At this point, we introduce some standard notation for the derivative of the cross-section rotation angle, θ.

$$\begin{aligned} \chi &= \text{curvature} \equiv \frac{\mathrm{d}\theta}{\mathrm{d}x} \approx \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}(\text{units of radians/length}) \\ \rho &= \text{radius of curvature} = \frac{1}{\chi}(\text{units of length}) \end{aligned} \tag{3.14}$$

We prefer to work with the curvature and express the extensional strain as

$$\varepsilon = -y\chi \tag{3.15}$$

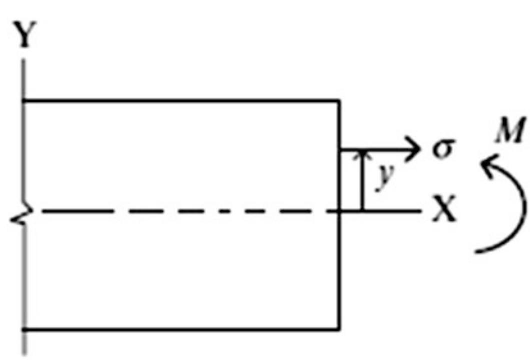
Given χ, one can establish qualitatively the shape of the curve defining the displaced centroidal axis. An analytical solution for the displacement, v, can also be determined by integrating (3.14). We will illustrate both procedures in later sections.

3.6.1 Moment: Curvature Relationship

We have demonstrated how to establish the bending moment distribution corresponding to a given loading. We have also showed how the displacement field can be generated once the curvature is known. To find the displacements due to a given loading, we need to relate the moment and the corresponding curvature along the centroidal axis. Given this relationship, it is a fairly straightforward process to move from prescribed loading to the resulting displacement.

The positive sense of the bending moment on a positive cross-section is defined as counterclockwise. Then noting Fig. 3.21, the moment and normal stress are related by

Fig. 3.21 Definition of normal stress and moment



$$M = \int_A - y\sigma \, \mathrm{d}A \tag{3.16}$$

We determine the stress using the stress–strain relation. In what follows, we assume the material behavior is linear elastic. The stress is a linear function of the strain in this case.

$$\sigma = E\varepsilon = -yE\chi \tag{3.17}$$

where E is Young's modulus for the material. Substituting for σ in (3.16) leads to

$$M = EI\chi \tag{3.18}$$

where $I = \int y^2 \mathrm{d}A$. Given M and EI, one finds the curvature (χ) with

$$\chi = \frac{M}{EI} \tag{3.19}$$

and then the displacement v by integrating

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \chi = \frac{M}{EI} \tag{3.20}$$

The complete solution of (3.20) consists of a homogeneous term and a particular term,

$$v = c_0 + c_1 x + v_\mathrm{p} \tag{3.21}$$

where v_p is the particular solution corresponding to the function, M/EI, and c_0, c_1 are constants. Two boundary conditions on v are required to determine c_0 and c_1.

3.6.2 Qualitative Reasoning About Deflected Shapes

Noting (3.18) and the fact that EI is always positive, it follows that the sense of curvature χ is the same as the sense of M. The deflected shapes corresponding to positive and negative curvature are shown in Fig. 3.22. It is more convenient to interpret these deflected shapes as the result of applying positive and negative moments. Figure 3.23 illustrates this interpretation.

We divide the moment diagrams into positive and negative moment zones and identify, using Fig. 3.23, the appropriate shape for each zone. Points where the moment changes sign are called inflection points. The curvature is zero at an inflection point, which implies that the curve is locally straight. We deal with inflection points by adjusting the orientation of adjoint shapes such that their tangents coincide at the inflection point. Figure 3.24 illustrates this process.

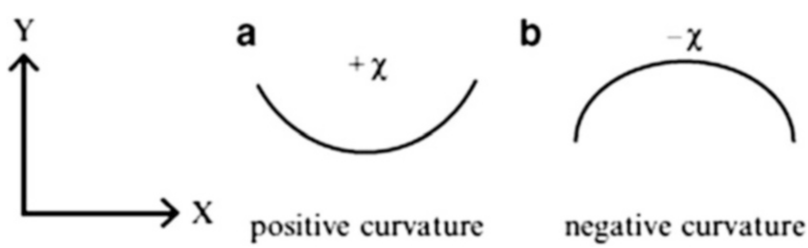


Fig. 3.22 Deflected shapes for positive and negative curvature

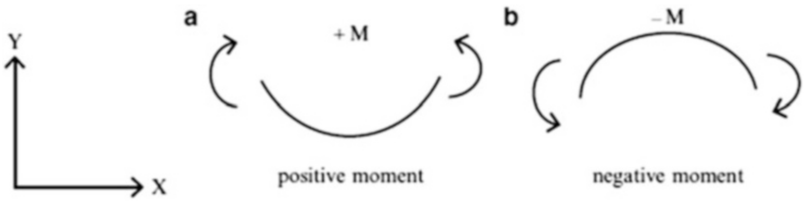


Fig. 3.23 Deflected shape for positive and negative moments

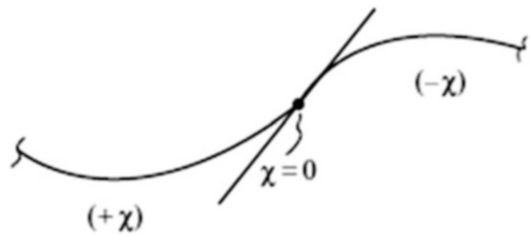


Fig. 3.24 Shape transition at an inflection point

The last step involves enforcing the displacement boundary conditions associated with end conditions. Figure 3.25 shows four types of end conditions (full fixity, hinge, roller, and free) with their corresponding displacement measures that are constrained by these conditions.

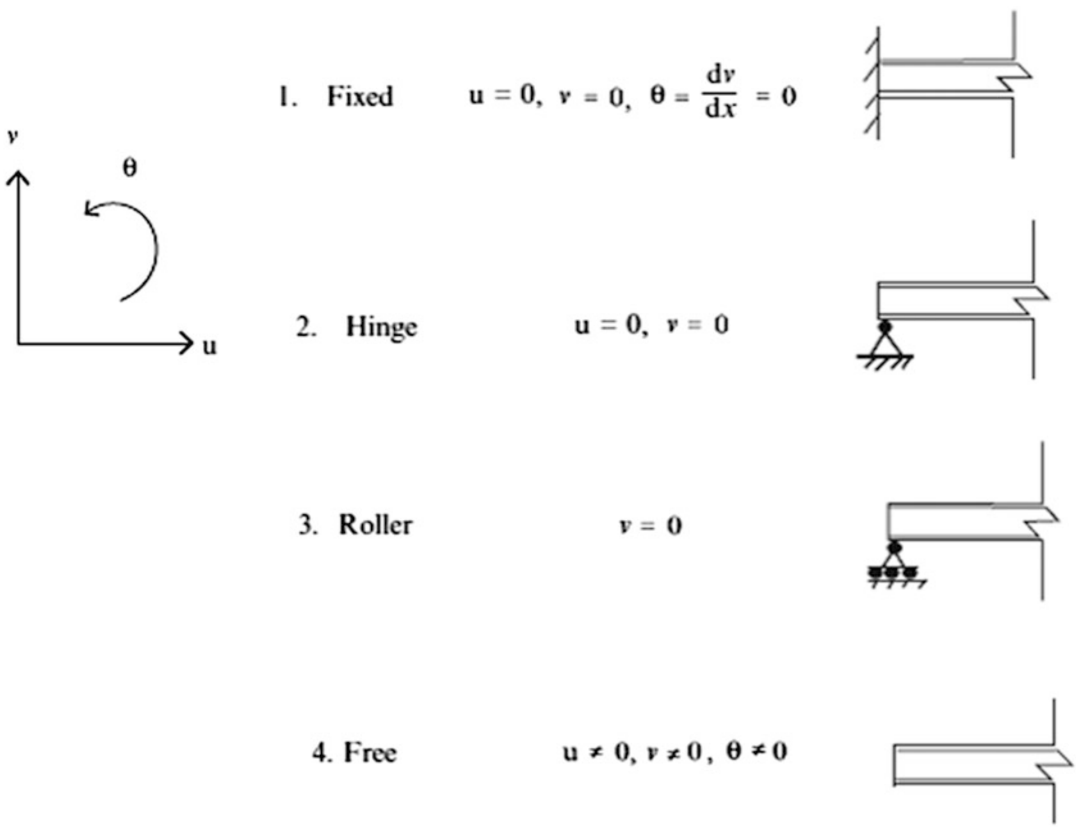


Fig. 3.25 Types of end conditions—displacement measures

The deflected shape must pass through a support. If an end is fixed, the cross-section cannot rotate at that point. We need to orient the deflected shape such that the tangent coincides with the initial centroidal axis. In what follows, we present a series of examples which illustrate the process of developing qualitative estimates of deflected shapes given the bending moment distribution.

Example 3.15 Deflected Shape—Uniformly Loaded, Simply Supported Beam

Given: The uniformly loaded, simply supported beam shown in Fig. E3.15a.
Determine: The deflected shape.

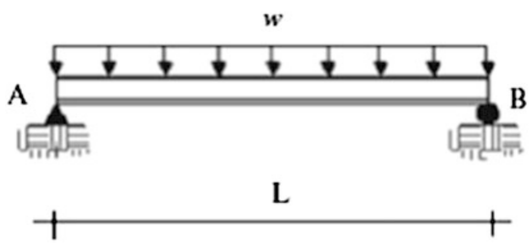


Fig. E3.15a

Solution: The moment is positive throughout the span so Fig. 3.23a applies. The displacement boundary conditions require

$$v(0) = v(L) = 0$$

One starts at the left end, sketches a curve with increasing positive curvature up to mid-span and then reverses the process. The deflected shape is symmetrical with respect to mid-span since the moment diagrams and support locations are symmetrical (Figs. E3.15b and E3.15c).

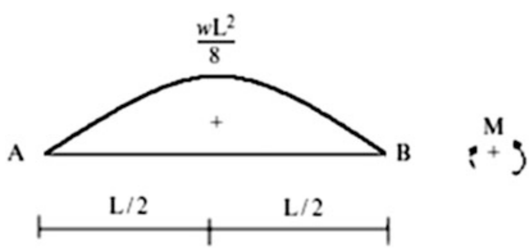


Fig. E3.15b Moment diagram

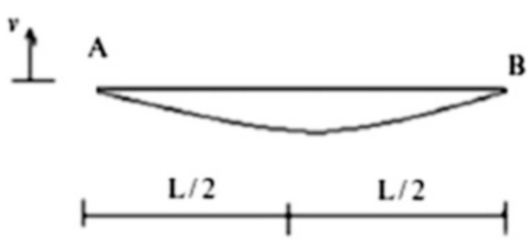


Fig. E3.15c Deflected shape

Example 3.16 Deflected Shape—Cantilever Beam

Given: The cantilever beam defined in Fig. E3.16a.
Determine: The deflected shape.

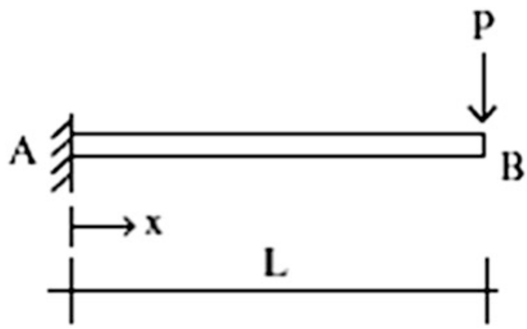


Fig. E3.16a

Solution: We note that the moment is negative throughout the span. Point A is fixed and therefore the tangent must be horizontal at this point. The displacement boundary conditions require

$$v(0) = \theta(0) = 0$$

We start at point A and sketch a curve with decreasing negative curvature up to $x = L$ (Figs. E3.16b and E3.16c).

Fig. E3.16b Moment diagram

Fig. E3.16c Deflected shape

Example 3.17 Deflected Shape of a Beam with an Overhang

Given: The beam with overhang shown in Fig. E3.17a.
Determine: The deflected shape.

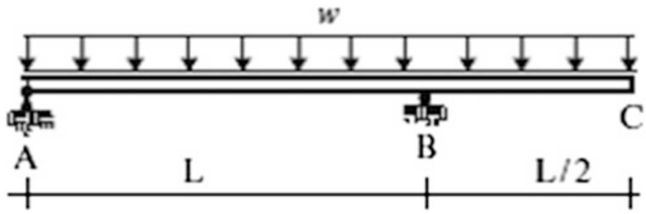


Fig. E3.17a

Solution: We note that Fig. E3.17b shows that the moment diagram has both positive and negative regions with an inflection point at $x = 0.75L$. Therefore, it follows that the left segment has positive curvature and the right segment has negative curvature. We need to join these shapes such that the tangent is continuous at point D and the deflections are zero at points A and B (Fig. E3.17c).

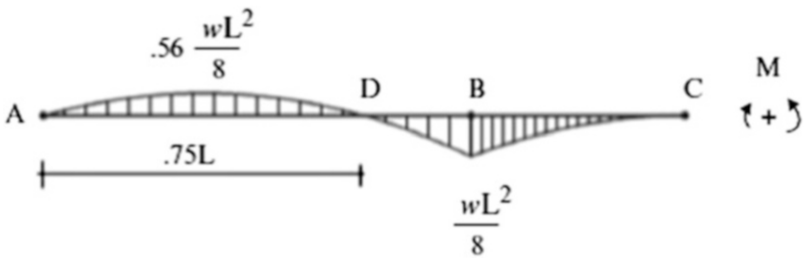


Fig. E3.17b Moment diagram

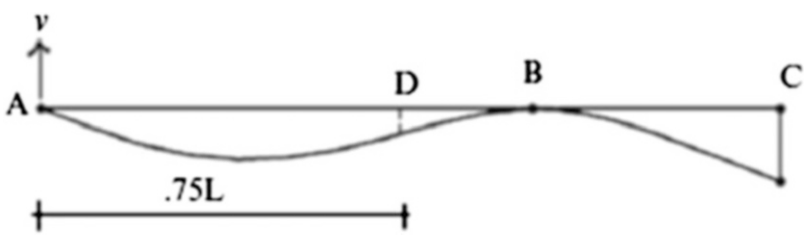


Fig. E3.17c Deflected shape

Example 3.18 Deflected Shape—Beam with a Moment Release

Given: The beam shown in Fig. E3.18a.
Determine: The deflected shape.

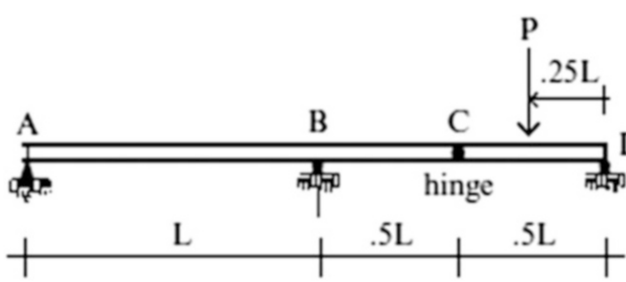


Fig. E3.18a

Solution: Member CD is connected to member ABC with a hinge at point C. A hinge is a physical artifact that allows the members connected to it to rotate freely, i.e., no moment is introduced. A hinge point is different from an inflection point. Although the moment is zero for both hinge and inflection points, the cross-sectional rotation is discontinuous at a hinge, whereas it is continuous at an inflection point. This feature is illustrated in the displacement sketch shown below. The left segment (ABC) has negative curvature. The right segment (CD) has positive curvature (Figs. E3.18b and E3.18c).

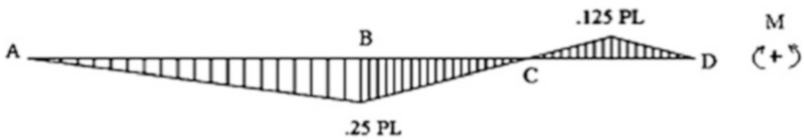


Fig. E3.18b Moment diagram

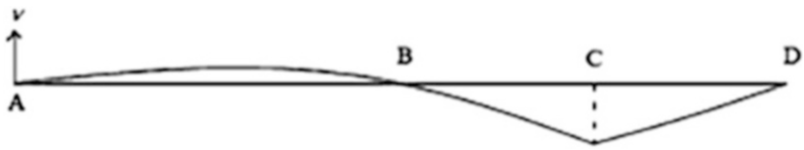


Fig. E3.18c Deflected shape

3.6.3 Moment Area Theorems

The starting point for quantitative analysis is the set of differential equations relating the moment, the cross-sectional rotation, and the deflection.

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}x} &= \frac{M(x)}{EI}\\ \frac{\mathrm{d}v}{\mathrm{d}x} &= \theta(x)\end{aligned} \tag{3.22}$$

Given $M(x)/EI$, we integrate $\mathrm{d}\theta/\mathrm{d}x$ between two points x_1 and x_2 on the x-axis and write the result as

$$\theta(x_2) - \theta(x_1) = \int_{x_1}^{x_2} \frac{M(x)}{EI}\mathrm{d}x \tag{3.23}$$

We interpret (3.23) as "*The difference in rotation between 2 points is equal to the area of the M/EI diagram included between these points*." This statement is referred to as the "First Moment Area" theorem. Taking x_2 as x in (3.23), we can express $\theta(x)$ as

$$\theta(x) = \theta(x_1) + \int_{x_1}^{x} \frac{M(x)}{EI}\mathrm{d}x \tag{3.24}$$

Given $\theta(x)$, we solve for $v(x_2)$.

$$v(x_2) = v(x_1) + \int_{x_1}^{x_2} \theta(x)\mathrm{d}x \tag{3.25}$$

Evaluating (3.24) first, and then substituting for $\theta(x)$ in (3.25) leads to

$$v(x_2) - v(x_1) = (x_2 - x_1)\theta(x_1) + \int_{x_1}^{x_2} \left\{ \int_{x_1}^{x} \frac{M(x)}{EI} \mathrm{d}x \right\} \mathrm{d}x \tag{3.26}$$

The double integral in (3.26) can be evaluated using integration by parts. First, we note the following identity,

$$\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u \tag{3.27}$$

Integrating between x_1 and x_2,

$$\int_{x_1}^{x_2} \mathrm{d}(uv) = \int_{x_1}^{x_2} (u\,\mathrm{d}v + v\,\mathrm{d}u) \tag{3.28}$$

and rearranging terms leads to

$$\int_{x_1}^{x_2} u\,\mathrm{d}v = uv|_{x_1}^{x_2} - \int_{x_1}^{x_2} v\,\mathrm{d}u \tag{3.29}$$

We take

$$\begin{aligned} u &= \int_{x_1}^{x} \frac{M}{EI} \mathrm{d}x \\ \mathrm{d}v &= \mathrm{d}x \end{aligned} \tag{3.30}$$

in (3.26). Using (3.29), the double integral can be expressed as

$$\begin{aligned} \int_{x_1}^{x_2} \left\{ \int_{x_1}^{x} \frac{M(x)}{EI} \mathrm{d}x \right\} \mathrm{d}x &= \left[x \int_{x_1}^{x} \frac{M(x)}{EI} \mathrm{d}x \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} x \frac{M(x)}{EI} \mathrm{d}x \\ = \int_{x_1}^{x_2} (x_2 - x) \frac{M(x)}{EI} \mathrm{d}x \end{aligned} \tag{3.31}$$

Finally, an alternate form of (3.26) is

$$v(x_2) - v(x_1) = (x_2 - x_1)\theta(x_1) + \int_{x_1}^{x_2} (x_2 - x) \frac{M(x)}{EI} \mathrm{d}x \tag{3.32}$$

This form is referred to as the "Second Moment Area Theorem." Figure 3.26 shows that the last term can be interpreted as the moment of the M/EI diagram with respect to x_2. It represents the deflection from the tangent at point 1, as indicated in Fig. 3.27.

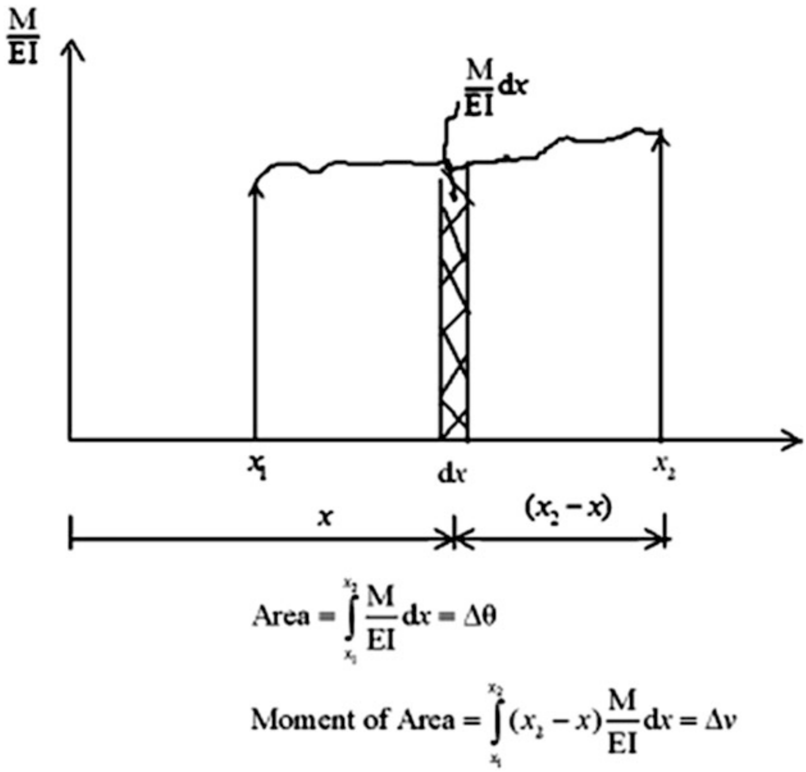


Fig. 3.26 Area and moment of area

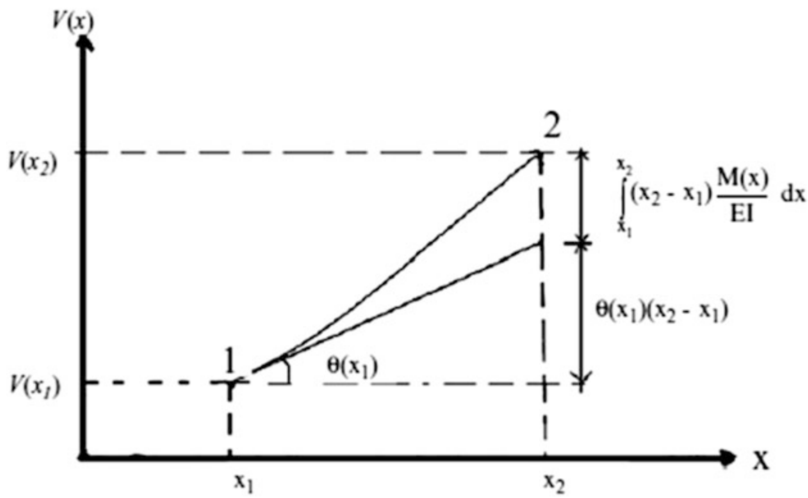


Fig. 3.27 Graphic interpolation of (3.26)

Using the Moment Area theorems, one has to evaluate only two integrals,

$$
\begin{aligned}
J(x) &= \int_{x_1}^{x} \frac{M(x)}{EI} \mathrm{d}x \\
H(x) &= \int_{x_1}^{x} x \frac{M(x)}{EI} \mathrm{d}x
\end{aligned}
\tag{3.33}
$$

When I is a complicated function of x, these integrals can be evaluated using a symbolic integration scheme or the numerical integration scheme described in Sect. 3.6.5. The final expressions for $v(x)$ and $\theta(x)$ in terms of $v(x_1)$, $\theta(x_1)$ and these integrals are (we take $x_2 = x$ in (3.23) and (3.32))

$$\begin{aligned} \theta(x) &= \theta(x_1) + J(x) \\ v(x) &= v(x_1) + (x - x_1)\theta(x_1) + xJ(x) - H(x) \end{aligned} \tag{3.34}$$

Example 3.19 Deflected Shape—Cantilever Beam

Given: The cantilever beam shown in Fig. E3.19a. Consider EI is constant.

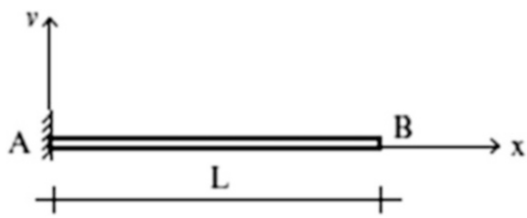


Fig. E3.19a

Determine: The deflected shapes for various loadings: concentrated moment, concentrated force, and uniform load.

Solution: We measure x from the left support. The displacement boundary conditions are

$$v_A = v(0) = 0$$

$$\theta_A = \theta(0) = 0$$

Taking $x_1 = 0$ and noting the boundary conditions at $x = 0$, (3.34) reduces to

$$0 \le x \le L$$

$$\theta(x) = \int_0^x \frac{M(x)}{EI} \mathrm{d}x$$

$$v(x) = x\int_0^x \frac{M(x)}{EI} \mathrm{d}x - \int_0^x x\frac{M(x)}{EI} \mathrm{d}x$$

Solutions for various loadings are listed below.

1. *Concentrated moment* (Fig. E3.19b)

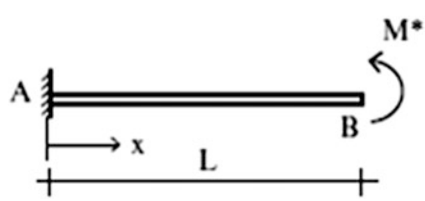
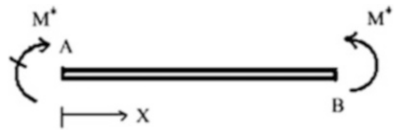
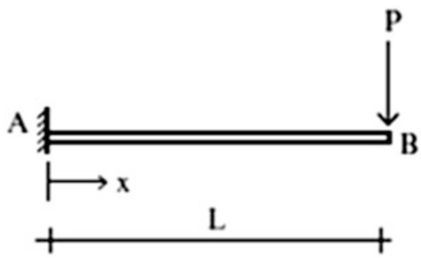


Fig. E3.19b

The expressions for $M(x)$, $\theta(x)$, and $v(x)$ for a concentrated moment are as follows:





$$M(x) = M^* \quad 0 \le x \le L$$

$$\theta(x) = \frac{1}{EI}\int_0^x M(x)\mathrm{d}x = \frac{M^*}{EI}x$$

$$v(x) = \frac{x}{EI}\int_0^x M(x)\mathrm{d}x - \frac{1}{EI}\int_0^x xM(x)\mathrm{d}x = \frac{M^* x^2}{EI} - \frac{M^* x^2}{2EI} = +\frac{M^* x^2}{2EI}$$

Specific values are

$$\theta_\mathrm{B} = \frac{M^* L}{EI}$$

$$v_\mathrm{B} = \frac{M^* L^2}{2EI}$$

2. *Concentrated Force* (Fig. E3.19c)

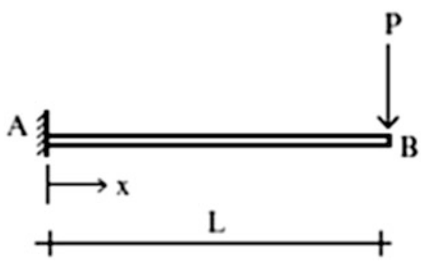
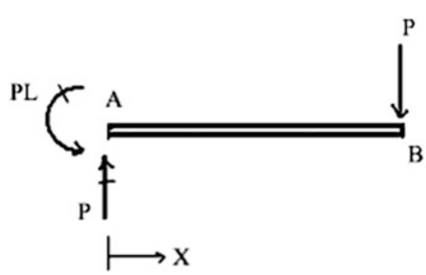


Fig. E3.19c

The expressions for $M(x)$, $\theta(x)$, and $v(x)$ for a concentrated load are as follows:



$$M(x) = +P(x - L) \quad 0 \le x \le L$$
$$\theta(x) = \frac{1}{EI}\int_0^x M(x)\mathrm{d}x = \frac{P}{EI}\left(\frac{x^2}{2} - Lx\right)$$
$$v(x) = \frac{x}{EI}\int_0^x M(x)\mathrm{d}x - \frac{1}{EI}\int_0^x xM(x)\mathrm{d}x = \frac{P}{EI}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right)$$

Specific values are

$$\theta_{\mathrm{B}} = -\frac{PL^2}{2EI}$$
$$v_{\mathrm{B}} = -\frac{PL^3}{3EI}$$

3. *Uniform Loading* (Fig. E3.19d)

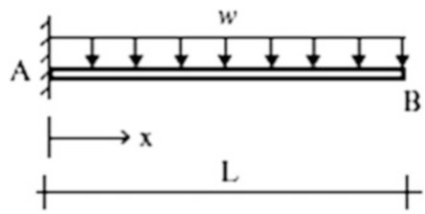
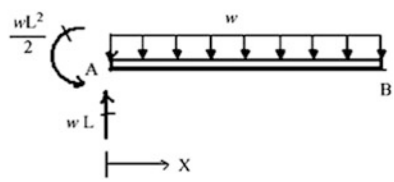


Fig. E3.19d

The expressions for $M(x)$, $\theta(x)$, and $v(x)$ for a uniform load are as follows:



$$M(x) = -\frac{wx^2}{2} + wLx - \frac{wL^2}{2} \quad 0 \le x \le L$$
$$\theta(x) = \frac{1}{EI}\int_0^x M(x)\mathrm{d}x = \frac{w}{6EI}\left(-x^3 + 3Lx^2 - 3L^2x\right)$$
$$v(x) = \frac{x}{EI}\int_0^x M(x)\mathrm{d}x - \frac{1}{EI}\int_0^x xM(x)\mathrm{d}x = \frac{w}{24EI}\left(-x^4 + 4Lx^3 - 6L^2x^2\right)$$

Specific values are

$$\theta_{\mathrm{B}} = -\frac{wL^3}{6EI}$$
$$v_{\mathrm{B}} = -\frac{wL^4}{8EI}$$

Example 3.20 Deflected Shape—Simply Supported Beam

Given: The simply supported beam shown in Fig. E3.20a. Consider EI is constant.

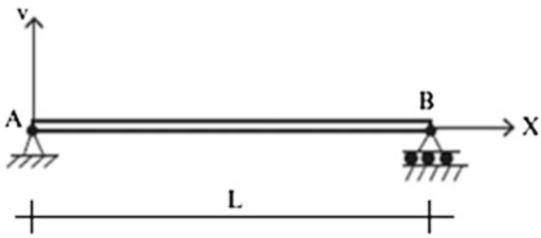


Fig. E3.20a

Determine: The deflected shape under different load conditions.
Solution: We measure x from the left support. The displacement boundary conditions are

$$v_{\text{A}} = v(0) = 0$$
$$v_{\text{B}} = v(L) = 0$$
$$\theta_{\text{A}} = \theta(0) \neq 0$$
$$\theta_{\text{B}} = \theta(L) \neq 0$$

Noting the boundary conditions at $x = 0$, the general solution (3.34) for constant EI is given by

$$0 \leq x \leq L$$
$$\theta(x) = \theta(0) + \int_0^x \frac{M(x)}{EI} \mathrm{d}x$$
$$v(x) = x\theta(0) + x\int_0^x \frac{M(x)}{EI} \mathrm{d}x - \int_0^x x\frac{M(x)}{EI} \mathrm{d}x$$

We determine $\theta(0)$ using the remaining boundary condition, $v(L) = 0$. Evaluating $v(x)$ at $x = L$ and equating the result to 0 leads to

$$\theta(0) = -\int_0^L \frac{M(x)}{EI} \mathrm{d}x + \frac{1}{L}\int_0^L x\frac{M(x)}{EI} \mathrm{d}x$$

Various loading cases are considered below. We omit the integral details and just present the final solutions.

1. *Concentrated Moment* (Fig. E3.20b)

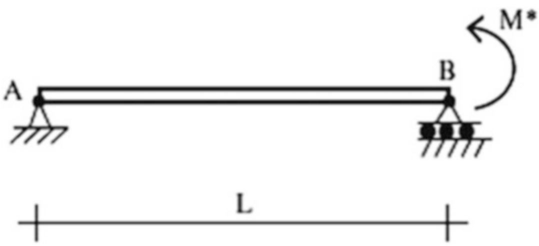
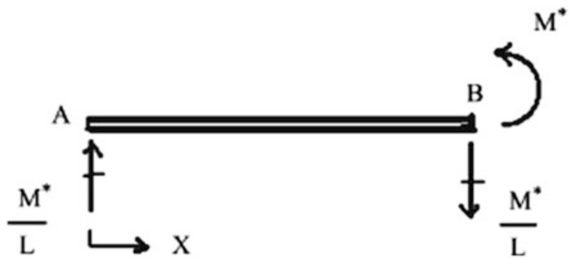


Fig. E3.20b

The expressions for $M(x)$, $\theta(x)$, and $v(x)$ for a concentrated moment are as follows:



$$M(x) = \frac{M^*}{L}x \quad 0 \le x \le L$$

$$\theta(x) = -\int_0^L \frac{M(x)}{EI}\mathrm{d}x + \frac{1}{L}\int_0^L x\frac{M(x)}{EI}\mathrm{d}x + \int_0^x \frac{M(x)}{EI}\mathrm{d}x$$

$$= \frac{M^* L}{EI}\left(\frac{x^2}{2L^2} - \frac{1}{6}\right)$$

$$v(x) = -x\int_0^L \frac{M(x)}{EI}\mathrm{d}x + \frac{x}{L}\int_0^L x\frac{M(x)}{EI}\mathrm{d}x + x\int_0^x \frac{M(x)}{EI}\mathrm{d}x - \int_0^x x\frac{M(x)}{EI}\mathrm{d}x$$

$$= \frac{M^* L^2}{6EI}\left(\frac{x^3}{L^3} - \frac{x}{L}\right)$$

Specific values are

$$\theta_A = -\frac{M^* L}{6EI}$$

$$\theta_B = \frac{M^* L}{3EI} \quad \frac{M^* L^2}{EI} \quad \text{at} \quad x = \frac{L}{\sqrt{3}} \approx 0.58L$$

$$v_{max} = -\frac{\sqrt{3}}{27}$$

2. *Concentrated Force* (Fig. E3.20c)

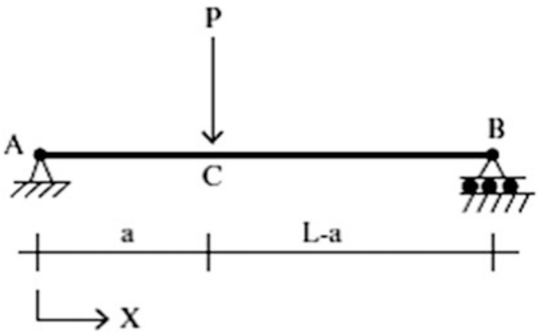
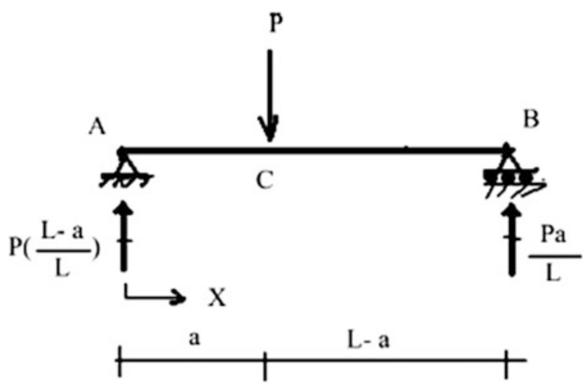


Fig. E3.20c

The expressions for $M(x)$, $\theta(x)$, and $v(x)$ for a concentrated load are as follows:



$$M(x) = P\left(\frac{L-a}{L}\right)x \quad 0 \le x \le a$$

$$M(x) = \frac{Pa}{L}(L-x) \quad a \le x \le L$$

Segment AC $0 \le x \le a$

$$\begin{aligned}\theta(x) &= -\int_0^L \frac{M(x)}{EI}\mathrm{d}x + \frac{1}{L}\int_0^L x\frac{M(x)}{EI}\mathrm{d}x + \int_0^x \frac{M(x)}{EI}\mathrm{d}x \\ &= -\frac{PL^2}{EI}\left(1-\frac{a}{L}\right)\left[-\frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{1a}{6L}\left(2-\frac{a}{L}\right)\right]\end{aligned}$$

$$v(x) = -x\int_0^L \frac{M(x)}{EI}\mathrm{d}x + \frac{x}{L}\int_0^L x\frac{M(x)}{EI}\mathrm{d}x + x\int_0^x \frac{M(x)}{EI}\mathrm{d}x - \int_0^x x\frac{M(x)}{EI}\mathrm{d}x$$
$$= -\frac{PL^3}{EI}\left(1-\frac{a}{L}\right)\left[-\frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1ax}{6LL}\left(2-\frac{a}{L}\right)\right]$$

Segment CB $a \le x \le L$

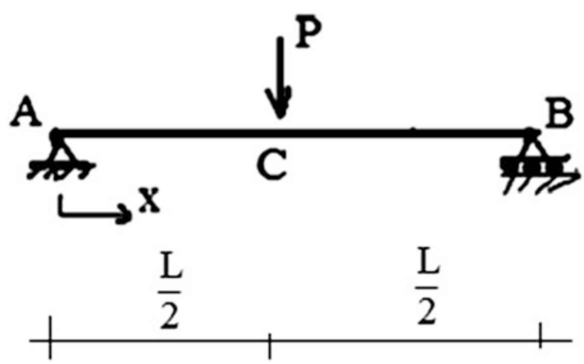
$$\theta(x) = -\frac{PL^2}{EI}\left(\frac{1}{3}+\frac{1}{6}\left(\frac{a}{L}\right)^2 - \frac{x}{L} + \frac{1}{2}\left(\frac{x}{L}\right)^2\right)\left(\frac{a}{L}\right)$$
$$v(x) = -\frac{PL^3}{EI}\left(\frac{1a}{6L}\right)\left[-\left(\frac{a}{L}\right)^2 + \frac{x}{L}\left(2+\left(\frac{a}{L}\right)^2\right) - 3\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right]$$

Specific values are

$$\theta_A = -\frac{Pa\left(2L^2 - 3aL + a^2\right)}{6EIL}$$
$$\theta_B = \frac{Pa}{6EIL}\left(L^2 - a^2\right)$$
$$v_C = -\frac{Pa^2(L-a)^2}{3EIL}$$

The maximum deflection occurs at the point, where $\theta(x) = 0$. This location depends on a. When $a < L/2$ the peak displacement occurs in segment CB. The location reverses when $a > L/2$.

Special case: $a = L/2$



$$\theta_{max} = \theta_B = -\theta_A = \frac{PL^2}{16EI}$$
$$\theta_C = 0$$
$$v_{max} = -\frac{PL^3}{48EI} \quad \text{at} \quad x = \frac{L}{2}$$

3. *Uniform Loading* (Fig. E3.20d)

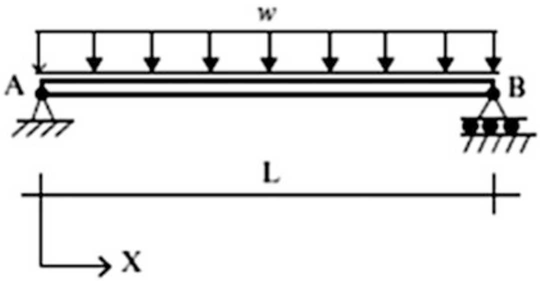
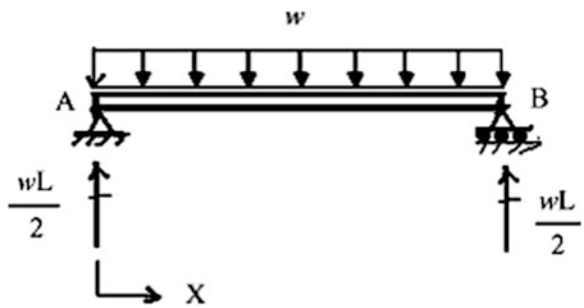


Fig. E3.20d

The expressions for $M(x)$, $\theta(x)$, and $v(x)$ for a uniform load are as follows:



$$M(x) = -\frac{wx^2}{2} + \frac{wL}{2}x \quad 0 \le x \le L$$

$$\theta(x) = -\int_0^L \frac{M(x)}{EI}\mathrm{d}x + \frac{1}{L}\int_0^L x\frac{M(x)}{EI}\mathrm{d}x + \int_0^x \frac{M(x)}{EI}\mathrm{d}x$$

$$= \frac{wL^3}{24EI}\left(-4\frac{x^3}{L^3} + 6\frac{x^2}{L^2} - 1\right)$$

$$v(x) = -x\int_0^L \frac{M(x)}{EI}\mathrm{d}x + \frac{x}{L}\int_0^L x\frac{M(x)}{EI}\mathrm{d}x + x\int_0^x \frac{M(x)}{EI}\mathrm{d}x - \int_0^x x\frac{M(x)}{EI}\mathrm{d}x$$

$$= \frac{wL^4}{24EI}\left(-\frac{x^4}{L^4} + 2\frac{x^3}{L^3} - \frac{x}{L}\right)$$

Specific values are

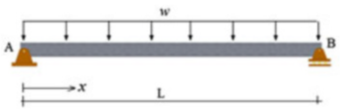
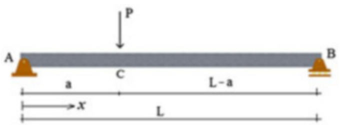
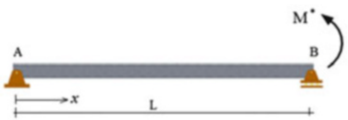
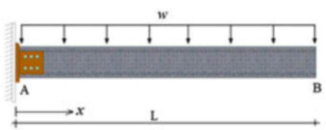
$$\theta_A = -\theta_B = -\frac{wL^3}{24EI}$$

$$v_{\max} = -\frac{5wL^4}{384EI} \quad \text{at} \quad x = \frac{L}{2}$$

Note that the rotation is zero at mid-span since the loading and the structure are symmetrical.

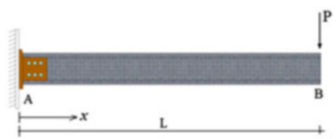
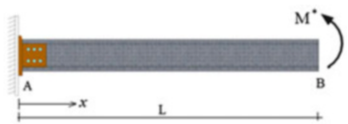
For future reference, the end displacements corresponding to typical loading condition are summarized in Table 3.1. We utilize these results in formulating the Force method to be presented in Chap. 9.

Table 3.1 Catalogue of displacements for various loading condition cases

Loading	Rotation θ^{+} counter clockwise	Translation $v^{+}\uparrow$
	$\theta(x) = -\frac{wL^3}{24EI}\left(\frac{4x^3}{L^3} - \frac{6x^2}{L^2} + 1\right)$ $\theta_B = -\theta_A = \frac{wL^3}{24EI}$	$v(x) = -\frac{wL^4}{24EI}\left(\frac{x^4}{L^4} - \frac{2x^3}{L^3} + \frac{x}{L}\right)$ $v_{max} = \frac{-5wL^4}{384EI}$ at $x = \frac{L}{2}$
	$\theta(x) = \frac{PL^2}{6EI}\left(1 - \frac{a}{L}\right)\left[3\left(\frac{x}{L}\right)^2 - \frac{2a}{L} + \frac{a^2}{L^2}\right] \quad 0 \le x \le a$ $\theta_A = -\frac{Pa\left(2L^2 - 3aL + a^2\right)}{6EIL}$ $\theta_B = \frac{Pa}{6EIL}(L^2 - a^2)$	$v(x) = \frac{PL^3}{6EI}\left(\frac{L-a}{L}\right)\left[\frac{x^3}{L^3} - \frac{2ax}{L^2} + \frac{a^2x}{L^3}\right] \quad 0 \le x \le a$ $v_C = -\frac{Pa^2(L-a)^2}{3EIL}$
	$\theta(x) = \frac{M^*L}{EI}\left(\frac{x^2}{2L^2} - \frac{1}{6}\right)$ $\theta_A = \frac{-M^*L}{6EI}$ $\theta_B = \frac{M^*L}{3EI}$	$v(x) = \frac{M^*L^2}{6EI}\left(\frac{x^3}{L^3} - \frac{x}{L}\right)$ $v_{max} = \frac{-\sqrt{3}}{27}\frac{M^*L^2}{EI}$ at $x = \frac{L}{\sqrt{3}} \approx 0.58L$
	$\theta(x) = -\frac{wL^3}{6EI}\left(\frac{x^3}{L^3} - \frac{3x^2}{L^2} + \frac{3x}{L}\right)$ $\theta_B = \frac{-wL^3}{6EI}$	$v(x) = -\frac{wL^4}{24EI}\left(\frac{x^4}{L^4} - \frac{4x^3}{L^3} + \frac{6x^2}{L^2}\right)$ $v_B = \frac{-wL^4}{8EI}$

(continued)

Table 3.1 (continued)

Loading	Rotation θ^{+} counter clockwise	Translation $v^{+} \uparrow$
	$\theta(x) = \frac{P}{EI}\left(\frac{x^2}{2} - Lx\right)$ $\theta_{\mathrm{B}} = \frac{-PL^2}{2EI}$	$v(x) = \frac{P}{6EI}(x^3 - 3Lx^2)$ $v_{\mathrm{B}} = \frac{-PL^3}{3EI}$
	$\theta(x) = \frac{M^*}{EI} x \theta_{\mathrm{B}} = \frac{M^* L}{EI}$	$v(x) = \frac{M^* x^2}{2EI}$ $v_{\mathrm{B}} = \frac{M^* L^2}{2EI}$

3.6.4 Computing Displacements with the Method of Virtual Forces

The procedures described in the previous section are intended to generate analytical solutions for the displacement and rotation. In many cases, one is interested only in the motion measures for a particular point. Rather than generate the complete analytical solution and then evaluate it at the point of interest, one can apply the Method of Virtual Forces. The Method of Virtual Forces specialized for bending of slender beams is defined in [1]. We express the principle as

$$d \cdot \delta P = \int_L (\text{bending deformation})(\delta M(x))\mathrm{d}x \tag{3.35}$$

where d is the desired displacement measure, δP is the virtual force in the direction of d, and $\delta M(x)$ is the virtual moment due to δP. The deformation due to transverse shear is not included since it is negligible for slender beams. When the behavior is linear elastic, the bending deformation is related to the moment by

$$\text{bending deformation} \equiv \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{M(x)}{EI}$$

and (3.35) takes the form

$$d \cdot \delta P = \int_L \frac{M(x)}{EI} \delta M(x)\mathrm{d}x \tag{3.36}$$

The steps involved in applying the principle are as follows. We use as an example, the beam shown in Fig. 3.28. To determine a desired vertical displacement or rotation such as v_{A} or θ_{B}, one applies the corresponding virtual force or virtual moment in the direction of the desired displacement or rotation, determines the virtual moment $\delta M_v(x)$ or $\delta M_\theta(x)$, and then evaluates the following integrals.

$$v_{\mathrm{A}} = \int_L \frac{M(x)}{EI} \delta M_v(x)\mathrm{d}x$$

$$\theta_{\mathrm{B}} = \int_L \frac{M(x)}{EI} \delta M_\theta(x)\mathrm{d}x$$

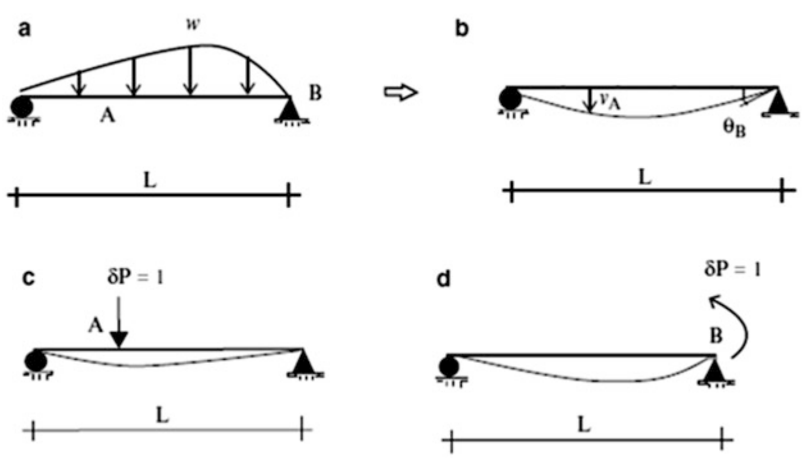


Fig. 3.28 Actual and virtual loads moments. (**a**) Actual load $M(x)$. (**b**) Deflected shape. (**c**) Virtual load δM_v for v_{A}. (**d**) Virtual load $\delta M_\theta(x)$ for θ_{B}

Just as we did for truss structures in Chap. 2, one takes δP to be a unit value. We illustrate the application of this procedure with the following examples.

Example 3.21 Deflection Computation—Method of Virtual Forces

Given: A uniformly loaded cantilever beam shown in Fig. E3.21a.

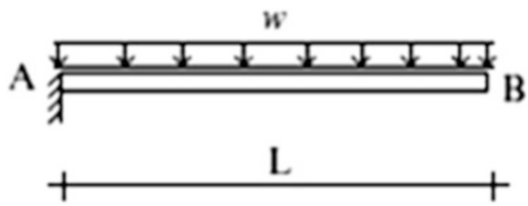


Fig. E3.21a

Determine: The vertical displacement and rotation at B. Take EI as constant.
Solution: We start by evaluating the moment distribution corresponding to the applied loading. This is defined in Fig. E3.21b. The virtual moment distributions corresponding to v_B, θ_B are defined in Figs. E3.21c and E3.21d. Note that we take δP to be either a unit force (for displacement) or a unit moment (for rotation).

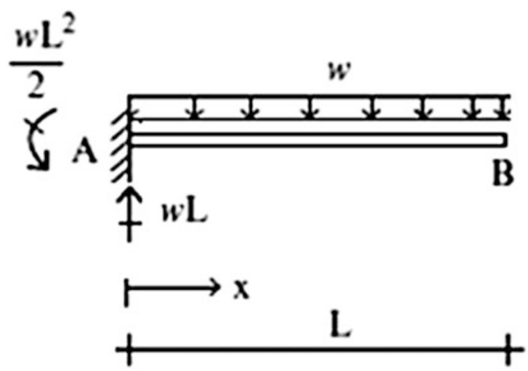


Fig. E3.21b $M(x)$

The actual moment $M(x)$ is

$$0 \le x \le L \quad M(x) = wLx - w\frac{x^2}{2} - \frac{wL^2}{2} = -\frac{w}{2}(x - L)^2$$

Vertical deflection at B: We apply the virtual vertical force, $\delta P = 1$ at point B and compute the corresponding virtual moment.

$$0 \le x \le L \quad \delta M_{v_B}(x) = x - L$$

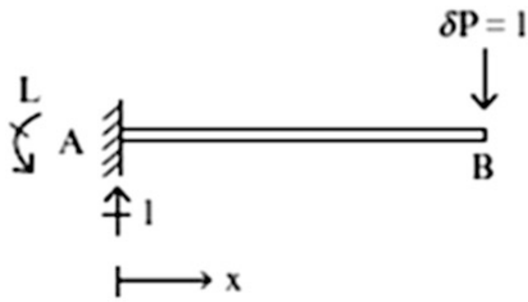


Fig. E3.21c $\delta M_{v_B}(x)$

Then, noting (3.36)

$$v_B = \int_L \frac{M(x)}{EI} \delta M_{v_B}(x) dx = \frac{1}{EI} \int_0^L -\frac{w}{2}(x-L)^2 (x-L) dx$$
$$= \frac{1}{EI} \int_0^L -\frac{w}{2}(x-L)^3 dx$$

Integrating leads to

$$v_B = \frac{wL^4}{8EI} \downarrow$$

Rotation at B: We apply the virtual moment, $\delta P = 1$ at point B and determine $\delta M(x)$. This loading produces a constant bending moment,

$$0 \le x \le L \quad \delta M_{\theta_B}(x) = -1$$

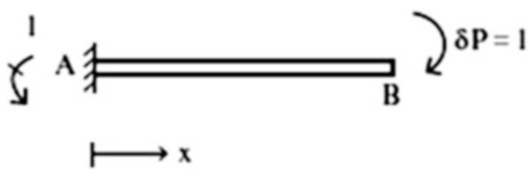


Fig. E3.21d $\delta M_{\theta_B}(x)$

Then, noting (3.36)

$$\theta_B = \int_L \frac{M(x)}{EI} \delta M_{\theta_B}(x) dx = \frac{1}{EI} \int_0^L -\frac{w}{2}(x-L)^2(-1) dx$$

Finally, one obtains

$$\theta_B = \frac{wL^3}{6EI} \text{clockwise}$$

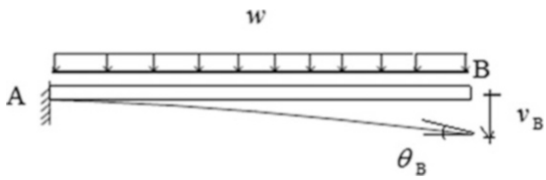


Fig. E3.21e Deflected shape

Example 3.22 Deflection Computation—Method of Virtual Forces

Given: The simply supported beam shown in Fig. E3.22a.

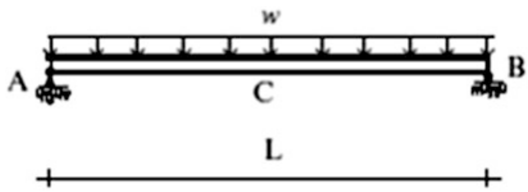


Fig. E3.22a

Determine: The vertical deflection and rotation at point C located at mid-span. Take EI is constant.
Solution: We start by evaluating the moment distribution corresponding to the applied loading. This is defined in Fig. E3.22b. The virtual moment distributions corresponding to v_C, θ_C are defined in Figs. E3.22c and E3.22d. Note that we take δP to be either a u*n*it force (for displacement) or a unit moment (for rotation).

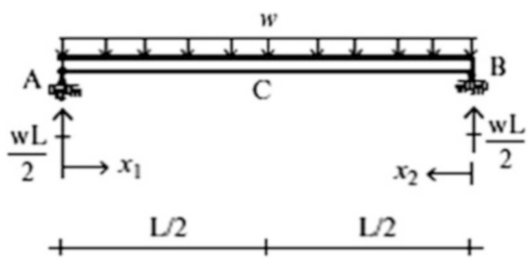


Fig. E3.22b $M(x)$

The actual moment is

$$0 < x_1 < L \quad M(x_1) = \frac{wL}{2}x_1 - \frac{wx_1^2}{2}$$

$$0 < x_2 < L \quad M(x_2) = \frac{wL}{2}x_2 - \frac{wx_2^2}{2}$$

Vertical displacement at C: We apply a unit virtual load at point C and determine $\delta M(x)$.

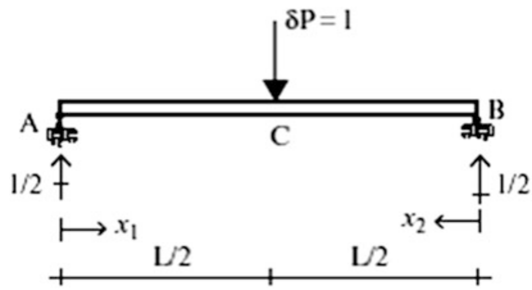


Fig. E3.22c $\delta M_{\nu C}(x)$

$$0 < x_1 < \frac{L}{2} \quad \delta M_{v_C}(x_1) = \frac{1}{2}x_1$$

$$0 < x_2 < \frac{L}{2} \quad \delta M_{v_C}(x_2) = \frac{1}{2}x_2$$

Then, evaluating the integral in (3.36), we obtain

$$\begin{aligned} v_C &= \int_{AC}\left(\frac{M(x_1)}{EI}\delta M_{v_C}(x_1)\right)\mathrm{d}x_1 + \int_{BC}\left(\frac{M(x_2)}{EI}\delta M_{v_C}(x_2)\right)\mathrm{d}x_2 \\ &= \frac{1}{EI}\left[\int_0^{L/2}\left(\frac{1}{2}x_1\right)\left(\frac{wLx_1}{2} - \frac{wx_1^2}{2}\right)\mathrm{d}x_1 + \int_0^{L/2}\left(\frac{1}{2}x_2\right)\left(\frac{wLx_2}{2} - \frac{wx_2^2}{2}\right)\mathrm{d}x_2\right] \\ &= \frac{5wL^4}{384EI}\downarrow \end{aligned}$$

Rotation at C: We apply a unit virtual moment at point C and determine $\delta M(x)$.

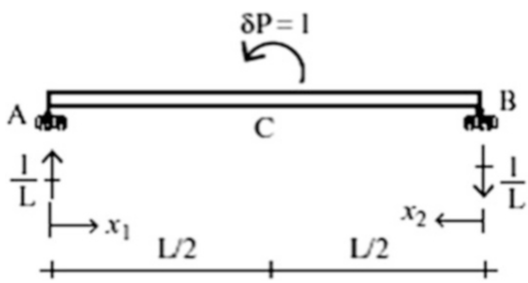


Fig. E3.22d $\delta M_{\theta C}(x)$

$$0 < x_1 < \frac{L}{2} \quad \delta M_{\theta_C}(x_1) = \frac{x_1}{L}$$

$$0 < x_2 < \frac{L}{2} \quad \delta M_{\theta_C}(x_2) = -\frac{x_2}{L}$$

Then, evaluating the integral in (3.36), we obtain

$$\theta_{\mathrm{C}} = \int_{\mathrm{AC}} \left(\frac{M(x_1)}{EI} \delta M_{\theta_{\mathrm{C}}}(x_1) \right) \mathrm{d}x_1 + \int_{\mathrm{BC}} \left(\frac{M(x_2)}{EI} \delta M_{\theta_{\mathrm{C}}}(x_2) \right) \mathrm{d}x_2$$
$$= \frac{1}{EI} \left\{ \int_0^{L/2} \left(\frac{x_1}{L} \right) \left(\frac{wLx_1}{2} - \frac{wx_1^2}{2} \right) \mathrm{d}x_1 + \int_0^{L/2} \left(\frac{-x_2}{L} \right) \left(\frac{wLx_2}{2} - \frac{wx_2^2}{2} \right) \mathrm{d}x_2 \right\} = 0$$

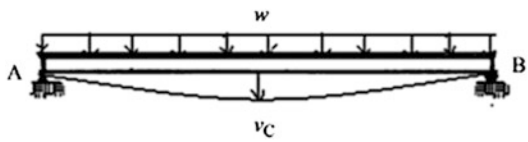


Fig. E3.22e Deflected shape

Example 3.23 Deflection Computation—Method of Virtual Forces

Given: The beam shown in Fig. E3.23a.

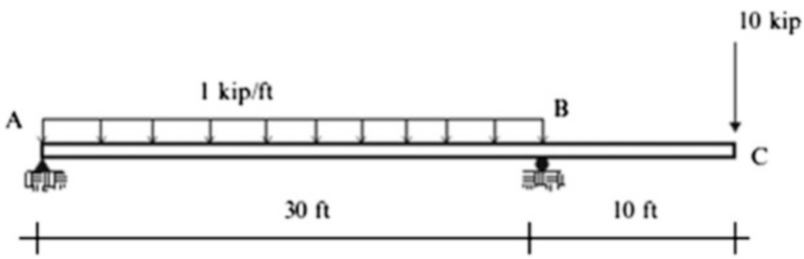


Fig. E3.23a

Determine: Use the virtual force method to determine the vertical deflection and rotation at C. $E = 29{,}000$ ksi and $I = 300$ in.^{4}

Solution: We start by evaluating the moment distribution corresponding to the applied loading. We divide up the structure into two segments AB and CB.

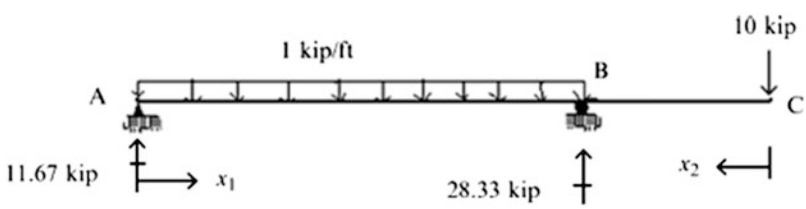


Fig. E3.23b $M(x)$

$$0 < x_1 < 30 \quad M(x_1) = 11.67x_1 - \frac{x_1^2}{2}$$
$$0 < x_2 < 10 \quad M(x_2) = -10x_2$$

Vertical deflection at C: We apply a unit virtual load at point C and determine $\delta M(x)$.

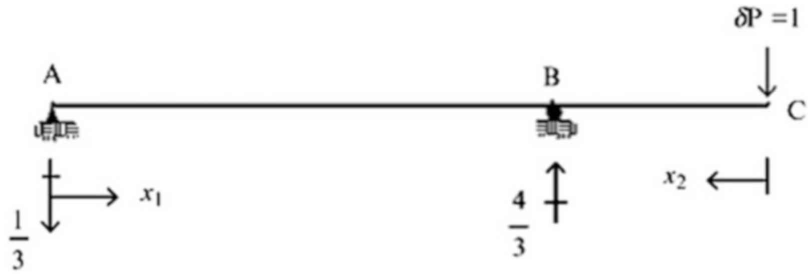


Fig. E3.23c $\delta M_{\nu C}(x)$

$$0 < x_1 < 30 \quad \delta M_{v_C}(x_1) = -\frac{x_1}{3}$$

$$0 < x_2 < 10 \quad \delta M_{v_C}(x_2) = -x_2$$

Then, noting (3.36), we divide up the structure into two segments AB and CB and integrate over each segment. The total integral is given by

$$\begin{aligned} v_C &= \int_{AB}\left(\frac{M(x_1)}{EI}\delta M_{v_C}(x_1)\right)dx_1 + \int_{CB}\left(\frac{M(x_2)}{EI}\delta M_{v_C}(x_2)\right)dx_2 \\ &= \frac{1}{EI}\int_0^{30}\left(11.67x_1 - \frac{x_1{}^2}{2}\right)\left(-\frac{x_1}{3}\right)dx_1 + \frac{1}{EI}\int_0^{10}(-10x_2)(-x_2)dx_2 \\ &= +\frac{2073.33\,\text{kip ft}^3}{EI} = \frac{2073.33(12)^3}{29{,}000(300)} = +0.41\,\text{in.} \end{aligned}$$

The positive sign indicates that the vertical displacement is in the direction of the unit load.

$$\therefore v_C = 0.41\,\text{in.} \downarrow$$

Rotation at C: We apply a unit virtual moment at point C and determine $\delta M(x)$.

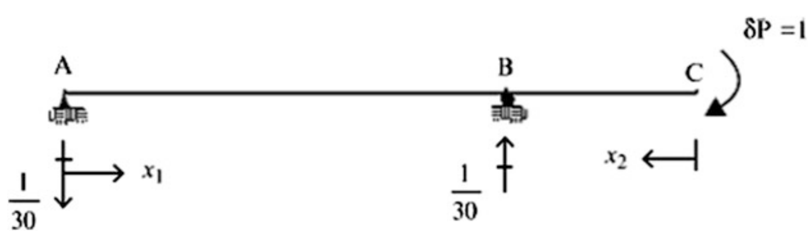


Fig. E3.23d $\delta M_{\theta C}(x)$

$$0 < x_1 < 30 \quad \delta M_{\theta_C}(x) = -\frac{x_1}{30}$$

$$0 < x_2 < 10 \quad \delta M_{\theta_C}(x) = -1$$

Then, noting (3.36)

$$\theta_C = \int_{AB} \left(\frac{M(x)}{EI} \delta M_{\theta_C}(x) \right) dx_1 + \int_{CB} \left(\frac{M(x)}{EI} \delta M_{\theta_C}(x) \right) dx_2$$

$$= \frac{1}{EI} \int_0^{30} \left(11.67 x_1 - \frac{x_1^2}{2} \right) \left(-\frac{x_1}{30} \right) dx_1 + \frac{1}{EI} \int_0^{10} (-10 x_2)(-1) dx_2$$

$$= \frac{374 \, \text{kip ft}^2}{EI} = \frac{374(12)^2}{29,000(300)} = +0.0063 \, \text{rad}$$

The positive sign indicates that the rotation is in the direction of the unit moment.

$$\therefore \theta_C = 0.0063 \, \text{rad clockwise}$$

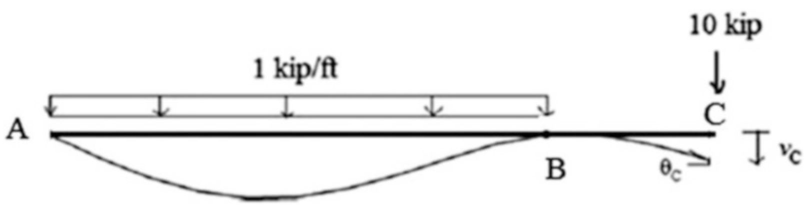


Fig. E3.23e Deflected shape

Example 3.24 Deflection Computation—Method of Virtual Forces

Given: The beam shown in Fig. E3.24a.

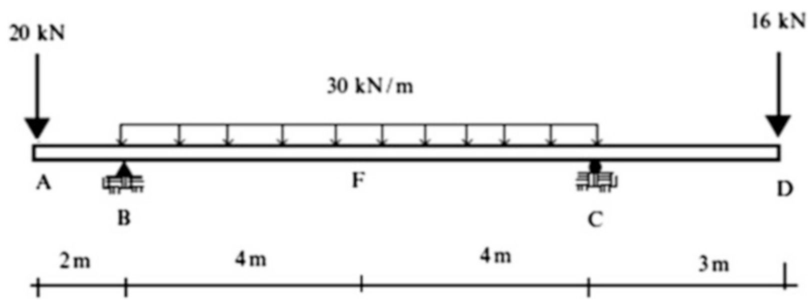


Fig. E3.24a

Determine: Use the virtual force method to determine the vertical deflection at F, rotation at B, and rotation at D. Assume $E = 200$ GPa and $I = 120(10)^6$ mm^4.

Solution: We start by evaluating the moment distribution corresponding to the applied loading.

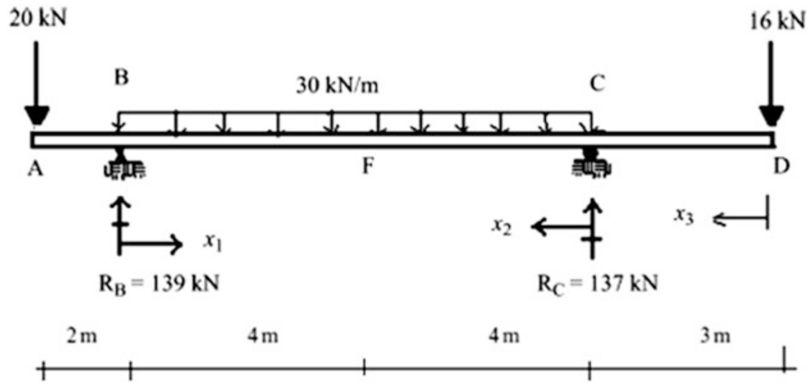


Fig. E3.24b $M(x)$

$$0 < x_1 < 8 \quad M(x_1) = -15x_1^2 + 139x_1 - 20(x_1 + 2) = -15x_1^2 + 119x_1 - 40$$
$$0 < x_2 < 8 \quad M(x_2) = -15x_2^2 + 137x_2 - 16(x_2 + 3) = -15x_2^2 + 121x_2 - 48$$
$$0 < x_3 < 3 \quad M(x_3) = -16x_3$$

Vertical deflection at F: We apply a unit virtual load at point F and determine δM.

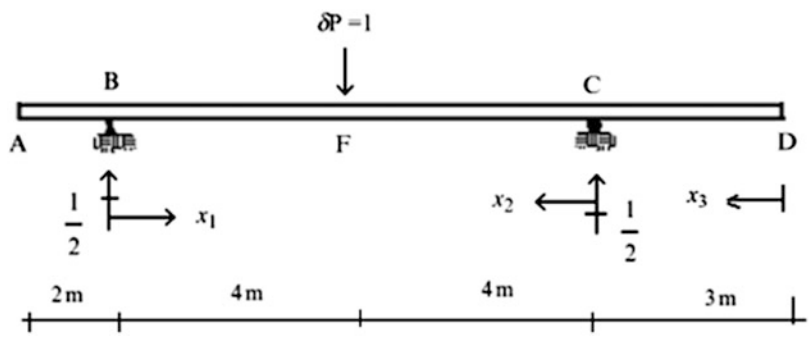


Fig. E3.24c $\delta M_{v_F}(x)$

$$0 < x_1 < 4 \quad \delta M_{v_F}(x_1) = \frac{1}{2}x_1$$
$$0 < x_2 < 4 \quad \delta M_{v_F}(x_2) = \frac{1}{2}x_2$$

Then, noting (3.36)

$$\begin{aligned} v_F &= \int_{BF} \left(\frac{M(x_1)}{EI} \delta M_{v_F}(x_1) \right) dx_1 + \int_{CF} \left(\frac{M(x_2)}{EI} \delta M_{v_F}(x_2) \right) dx_2 \\ &= \frac{1}{EI} \left\{ \int_0^4 \left(-15x_1^2 + 119x_1 - 40\right) \left(\frac{x_1}{2}\right) dx_1 + \int_0^4 \left(-15x_2^2 + 121x_2 - 48\right) \left(\frac{x_2}{2}\right) dx_2 \right\} \\ &= \frac{1248\,\text{kN}\,\text{m}^3}{EI} = \frac{1248(10)^9}{200(120)(10)^6} = 52\,\text{mm} \end{aligned}$$

The positive sign indicates that the vertical displacement is in the direction of the unit load.

$$\therefore \quad v_F = 52\,\text{mm} \downarrow$$

Rotation at B: We apply a unit moment at point B and determine $\delta M(x)$.

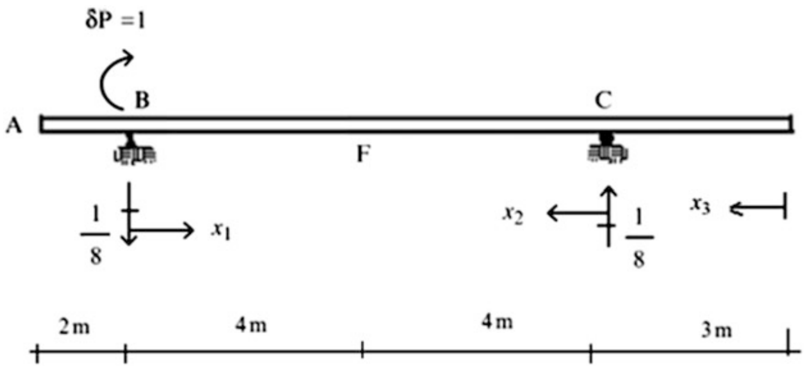


Fig. E3.24d $\delta M_{\theta_B}(x)$

$$0 < x_2 < 8 \quad \delta M_{\theta_B}(x_2) = \frac{1}{8}x_2$$

Then, noting (3.36)

$$\theta_B = \int_{CB} \frac{M(x_2)}{EI} \delta M_{\theta_B}(x_2) dx_2 = \frac{1}{EI}\int_0^8 \left(-15x_2^2 + 121x_2 - 48\right)\left(\frac{x_2}{8}\right) dx_2$$

$$= \frac{469.3\,\text{kN}\,\text{m}^2}{EI} = \frac{469.3(10)^6}{200(120)(10)^6} = +0.0195\,\text{rad}$$

The positive sign indicates that the rotation is in the direction of the unit moment.

$$\therefore \theta_B = 0.0195\,\text{rad}\,\text{clockwise}$$

Rotation at D: We apply a unit moment at point D and determine $\delta M(x)$.

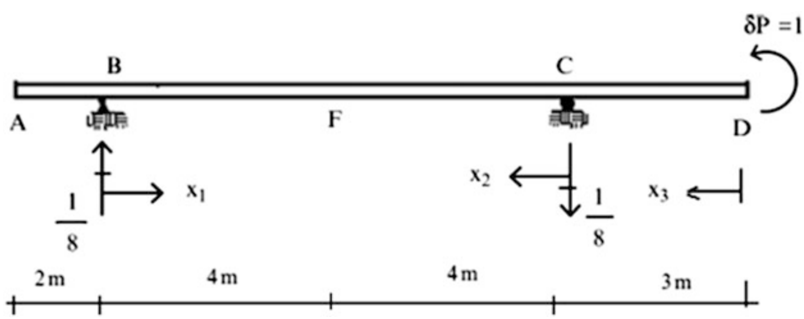


Fig. E3.24e $\delta M_{\theta_D}(x)$

$$0 < x_1 < 8 \quad \delta M_{\theta_D}(x_1) = \frac{1}{8}x_1$$

$$0 < x_3 < 3 \quad \delta M_{\theta_D}(x_3) = 1$$

Then, noting (3.36)

$$\begin{aligned}\theta_D &= \int_{BC} \frac{M(x_1)}{EI}\delta M_{\theta_D}(x_1)\mathrm{d}x_1 + \int_{DC} \frac{M(x_3)}{EI}\delta M_{\theta_D}(x_3)\mathrm{d}x_3 \\ &= \frac{1}{EI}\int_0^8 \left(-15x_1^2 + 119x_1 - 40\right)\left(\frac{x_1}{8}\right)\mathrm{d}x_1 + \frac{1}{EI}\int_0^3 (-16x_3)(1)\mathrm{d}x_3 \\ &= +\frac{386.7\,\mathrm{kN\,m^2}}{EI} = \frac{386.7(10)^6}{200(120)(10)^6} = 0.016\,\mathrm{rad}\end{aligned}$$

The positive sign indicates that the rotation is in the direction of the unit moment.

$$\therefore \theta_D = 0.016\,\mathrm{rad\,counterclockwise}$$

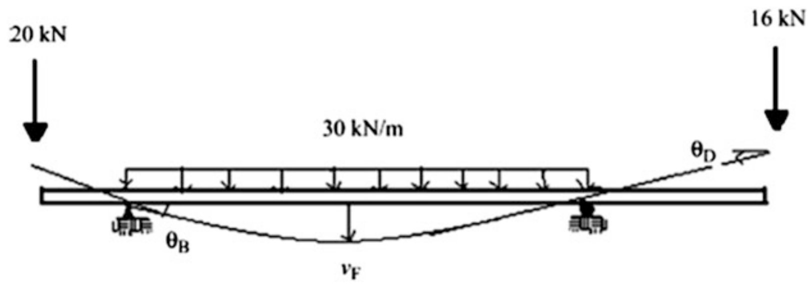


Fig. E3.24f Deflected shape

3.6.5 Computing Displacements for Non-prismatic Members

When the member is non-prismatic, I is a function of x and it may be difficult to obtain a closed form solution for the integral involving $1/I$. In this case, one can employ a numerical integration scheme. In what follows, we describe a numerical integration procedure which can be easily programmed.

Consider the problem of evaluating the following integral

$$J = \int_{x_A}^{x_B} f(x)\mathrm{d}x \tag{3.37}$$

We divide the total interval into n equal segments of length h

$$h = \frac{x_B - x_A}{n} \tag{3.38}$$

and denote the values of x and f at the equally spaced points as

$$\begin{gathered} x_1, x_2, x_3, \ldots, x_{n+1} \\ f_1, f_2, f_3, \ldots, f_{n+1} \end{gathered}$$

This notation is illustrated in Fig. 3.29.

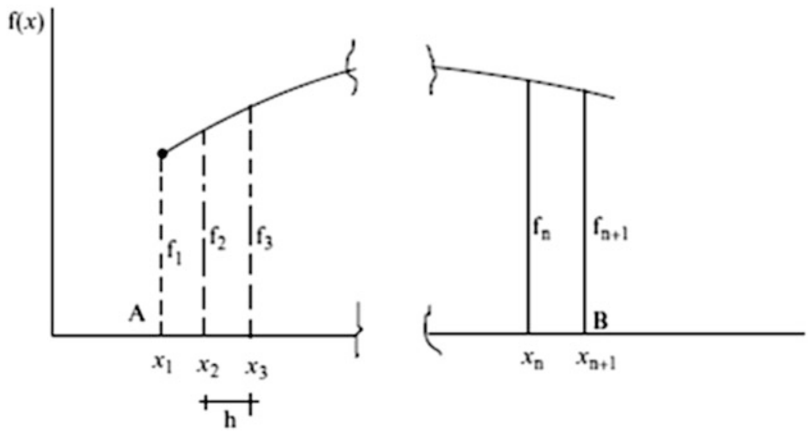


Fig. 3.29 Piecewise function approximation

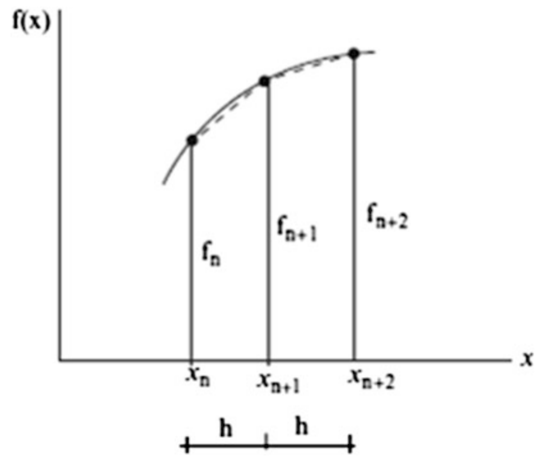


Fig. 3.30 Notation

The simplest approach is based on approximating the actual curve of $f(x)$ with a set of straight lines connecting (f_n, f_{n+1}), (f_{n+1}, f_{n+2}), etc. as shown in Fig. 3.30.

The incremental area between x_n and x_{n+1} is approximated as

$$\Delta J_{n,n+1} = \int_{x_n}^{x_{n+1}} f(x)\mathrm{d}x \approx \frac{h}{2}\left(f_n + f_{n+1}\right) \tag{3.39}$$

Also, the area between x_1 and x_n is expressed as

$$J_n = \int_{x_1}^{x_n} f(x)\mathrm{d}x \tag{3.40}$$

Starting with $J_1 = 0$, one generates successive areas with

$$\begin{aligned} J_2 &= J_1 + \Delta J_{1,2} = \Delta J_{1,2} \\ J_3 &= J_2 + \Delta J_{2,3} \\ &\vdots \\ J_n &= J_{n-1} + \Delta J_{n-1,n} \\ J_{n+1} &= J_n + \Delta J_{n,n+1} \end{aligned} \tag{3.41}$$

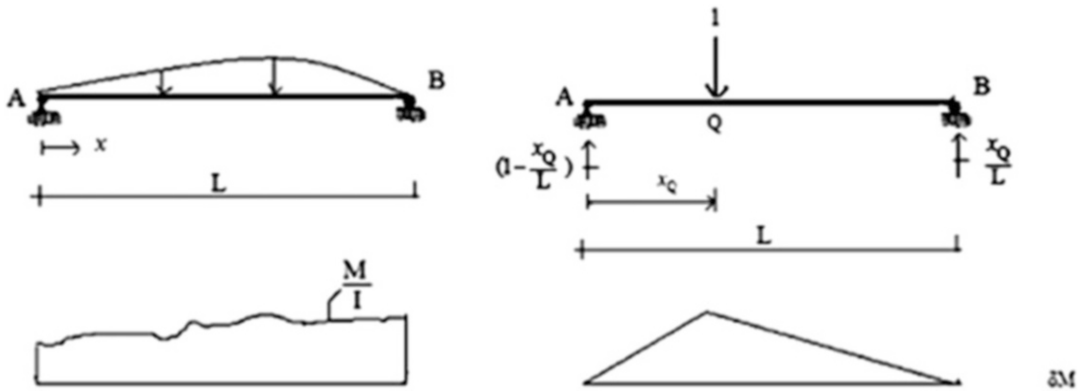


Fig. 3.31 Moment distribution

The total integral, J_{n+1} expands to

$$J_{n+1} = \int_{x_1}^{x_{n+1}} f(x)\mathrm{d}x \approx h\left\{\frac{1}{2}(f_1 + f_{n+1}) + \sum\nolimits_{j=2}^{n} f_j\right\} \tag{3.42}$$

Equation (3.42) is known as the "Trapezoidal" Rule. One uses (3.41) to evaluate the intermediate integrals when applying the Moment Area Theorems such as (3.33) and (3.34). Equation (3.42) is also used with the Virtual Force Method.

We illustrate the application of this approach defined by (3.36) to the beam defined in Fig. 3.31. Suppose the vertical displacement at point Q is desired. Given $M(x)$ and $I(x)$, we subdivide the X-axis into n equal intervals and evaluate M/I and $\delta M(x)$ at each point.

$$h = \frac{L}{n}$$
$$x_k = (k-1)h \quad k = 1, 2, \ldots, n+1$$
$$\delta M(x_k) = \left(1 - \frac{x_Q}{L}\right)x_k \quad x < x_Q$$
$$\delta M(x_k) = (L - x_k)\frac{x_Q}{L} \quad x > x_Q$$

Lastly, we take $f = \left(\frac{M}{I}\right)\delta M$ in (3.42) and evaluate the summation. The choice of h depends on the "smoothness" of the function M/I; a typical value is $L/20$. One can assess the accuracy by refining the initial choice for h and comparing the corresponding values of the integral.

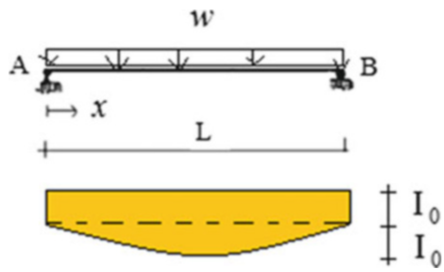
Suppose the deflection at $x = L/2$ is desired. The virtual moment for this case is

$$\begin{cases} \delta M(x) = \left(1 - \frac{1}{2}\right)x_k = \frac{1}{2}x_k & \text{for} \quad x_k < \frac{L}{2} \\ \delta M(x) = (L - x_k)\frac{1}{2} & \text{for} \quad x_k > \frac{L}{2}. \end{cases}$$

We also suppose the loading is uniform and the variation of I is given by

$$I = I_0\left\{1 + 4\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]\right\}$$

where I_0 is constant.



The corresponding moment is

$$M = \frac{wL}{2}x - \frac{wx^2}{2} = \frac{wL^2}{8}\left\{4\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]\right\}$$

Substituting for M, δM, and I, the virtual force expression for the displacement takes the form

$$v\left(x = \frac{L}{2}\right) = \frac{wL^4}{8EI_0}\int_0^1 \left\{\frac{4\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]\frac{\delta M}{L}}{1 + 4\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]}\right\} \mathrm{d}\left(\frac{x}{L}\right) = \frac{wL^4}{8EI_0}\alpha$$

where α is a dimensionless coefficient that depends on the interval size.

We subdivide the interval 0–1 into n such intervals. Applying Equation (3.42) and taking a range of values for n leads to

$$\begin{aligned} n &= 10 \quad \alpha = 0.065 \\ n &= 20 \quad \alpha = 0.0559 \\ n &= 30 \quad \alpha = 0.0559 \end{aligned}$$

We note that taking $n = 20$ is sufficiently accurate. We used MATLAB [2] to program the computation associated with Equation (3.42).

3.7 Deformation–Displacement Relations for Deep Beams: Planar Loading

When the depth to span ratio is greater than 0.1, the theory presented in Sect. 3.6, which is based on Kirchhoff's hypothesis, needs to be modified to include the transverse shear deformation. Figure 3.32 illustrates this case: the cross-section remains a plane but is no longer normal to the centroidal axis. *Defining β as the rotation of the cross-section, and γ as the transverse shear strain*, it follows that

$$\gamma = \theta - \beta \approx \frac{\mathrm{d}v}{\mathrm{d}x} - \beta \tag{3.43}$$

The extensional strain now involves β rather than θ.

$$\varepsilon(y) = -y\frac{\mathrm{d}\beta}{\mathrm{d}x} \tag{3.44}$$

Expressions for the internal force variables, V and M, in terms of the deformation measures are derived in a similar way as followed in Sect. 3.6.1. We express them as:

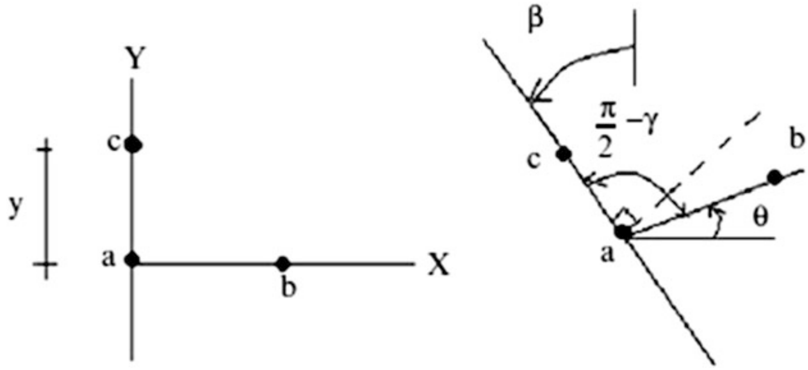


Fig. 3.32 Deformation with transverse shear deformation

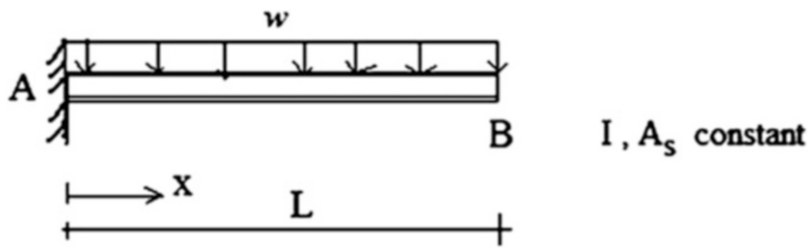


Fig. 3.33 Cantilever beam

$$
\begin{aligned}
M &= EI\frac{d\beta}{dx} \\
V &= GA_s\gamma
\end{aligned}
\tag{3.45}
$$

where G is the material shear modulus and A_s is the effective shear area, i.e., the cross-sectional area over which the shear stress is essentially uniformly distributed. For an I shape steel section, A_s is taken as the web area.

Given M and V, one first determines β by integrating between two points, x_A and x

$$
\beta(x) - \beta(x_A) = \int_{x_A}^{x} \frac{M}{EI} dx \tag{3.46}
$$

If A is a fixed support, $\beta(x_A) = 0$. Once β is known, we find v by integrating

$$
\frac{dv}{dx} = \beta + \frac{V}{GA_s}
$$

This leads to

$$
v(x) - v(x_A) = \int_{x_A}^{x} \left(\beta + \frac{V}{GA_s}\right) dx \tag{3.47}
$$

In general, two boundary conditions are required to specify the two integration constants.

For example, consider the structure and loading defined in Fig. 3.33.

The transverse shear force and moment expressions are

$$V(x) = -w(L - x)$$
$$M(x) = -\frac{w}{2}(L - x)^2$$

Point A is a fixed support. Then, $\beta(x_A) = v(x_A) = 0$. Noting (3.46),

$$\beta = \int_0^x -\frac{w}{2EI}(L - x)^2 \mathrm{d}x$$
$$\Downarrow$$
$$\beta = \frac{w}{6EI}(L - x)^3 - \frac{w}{6EI}L^3$$

Substituting for β in (3.47) leads to

$$\begin{aligned} v(x) &= \left[\tfrac{w}{2GA_s}(L - x)^2\right]_0^x + \left[-\tfrac{w}{24EI}(L - x)^4 - \tfrac{w}{6EI}L^3 x\right]_0^x \\ &= \frac{w}{2GA_s}\left((L - x)^2 - L^2\right) + \frac{w}{6EI}\left(-\frac{1}{4}(L - x)^4 - L^3 x + \frac{L^4}{4}\right) \end{aligned} \tag{3.48}$$

Specializing for $x = L$, the end displacement is equal to

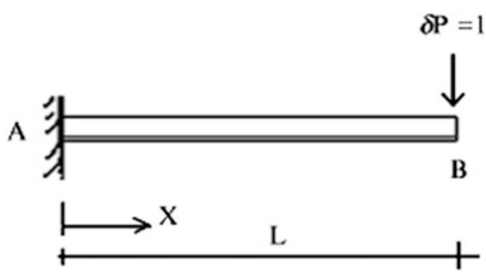
$$+\uparrow v(L) = -\frac{wL^4}{8EI}\left(1 + \frac{4EI}{GA_s L^2}\right)$$

The effect of shear deformation is to "increase" the displacement by a dimensionless factor which is proportioned to the ratio EI/GA_sL^2. This factor is usually small with respect to 1 for a homogeneous cross-section. It may be large for composite beams that have a "soft" core, i.e., where $G \ll E$.

Rather than work with the deformation–displacement results, one can apply an extended form of the Principle of Virtual Forces. We add the shear deformation term to the integral and also introduce the virtual shear force δV. Then, (3.36) expands to

$$d \cdot \delta P = \int_L \left(\frac{M(x)}{EI}\delta M(x) + \frac{V(x)}{GA_s}\delta V(x)\right)\mathrm{d}x \tag{3.49}$$

The steps involved are the same as for slender beams. One now has to determine δV as well as δM for a given δP. Revisiting the previous example defined in Fig. 3.33, we compute $v(L)$. The details are as follows.



$$\delta V(x) = -1$$
$$\delta M(x) = -(L - x)$$

$$\begin{aligned}
+\downarrow v(L) &= \int_0^L \frac{w}{2EI}(L-x)^2(L-x)\mathrm{d}x + \int_0^L \frac{-w}{GA_s}(L-x)(-1)\mathrm{d}x \\
&= \left[\tfrac{-w}{8EI}(L-x)^4\right]_0^L + \left[-\tfrac{w}{2GA_s}(L-x)^2\right]_0^L \\
&= \frac{w}{8EI}L^4 + \frac{w}{2GA_s}L^2
\end{aligned}$$

Applying the Principle of Virtual Forces for this example involves less algebra than required for integration.

3.8 Torsion of Prismatic Members

Consider the prismatic member shown in Fig. 3.34. Up to this point, we have assumed the line of action of the external loading passes through the centroidal axis, and consequently the member just bends in the $X - Y$ plane. This assumption is not always true, and there are cases where the loading may have some eccentricity with respect to the X-axis. When this occurs, the member twists about the X-axis as well as bends in the $X - Y$ plane.

We deal with an eccentric load by translating its line of action to pass through the X-axis. This process produces a torsional moment about X as illustrated in Fig. 3.34.

The torsional moment is resisted by shearing stresses acting in the plane of the cross-section, resulting in shear strain and ultimately rotation of the cross-section about the X-axis. Mechanics of Solids texts such as [1] present a detailed theory of torsion of prismatic members so we just list the resultant equations here. First, we introduce the following notation listed in Fig. 3.35

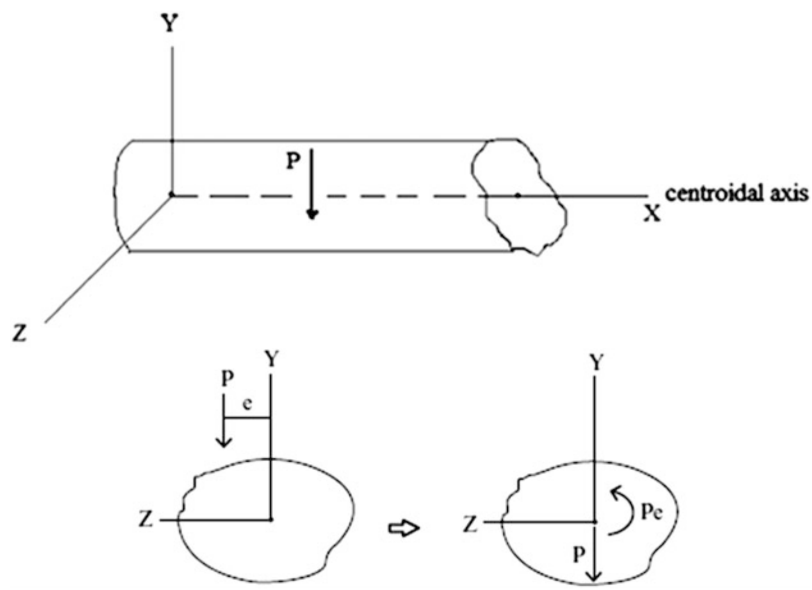
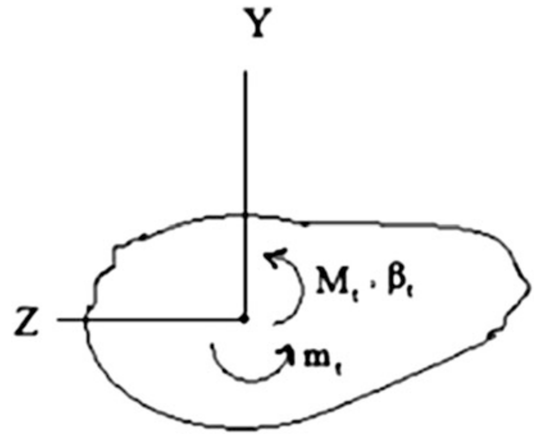


Fig. 3.34 Prismatic member—eccentric load

Fig. 3.35 Notation



M_t = moment vector about the X-axis (positive sense from Y toward Z)
β_t = rotational vector about the X-axis
m_t = distributed external torsional moment loading
J = torsional cross-sectional property (similar to I for plane bending)

The differential equation of equilibrium for torsion has the form

$$\frac{\mathrm{d}M_t}{\mathrm{d}x} + m_t = 0 \tag{3.50}$$

One needs to restrain the member at one point for stability. A free end has $M_t = 0$. Given M_t, one determines the rotation with

$$M_t = GJ\frac{\mathrm{d}\beta_t}{\mathrm{d}x} \tag{3.51}$$

Note the similarity between the expression for bending and twisting. We find β_t by integrating (3.51).

$$\beta_t(x) - \beta_t(x_\mathrm{A}) = \int_{x_\mathrm{A}}^{x} \frac{M_t}{GJ}\mathrm{d}x \tag{3.52}$$

A boundary condition on β_t is required to determine $\beta_t(x)$. Typical boundary conditions are illustrated below.

The principle of Virtual Forces can be extended to deal with combined bending and twisting by adding the twist deformation term to the integration. The general expression which includes all deformation terms is

$$d\,\delta P_\mathrm{A} = \int_L \left(\frac{M}{EI}\delta M + \frac{V}{GA_\mathrm{s}}\delta V + \frac{M_t}{GJ}\delta M_t\right)\mathrm{d}x \tag{3.53}$$

where δM_t is the virtual torsional moment.

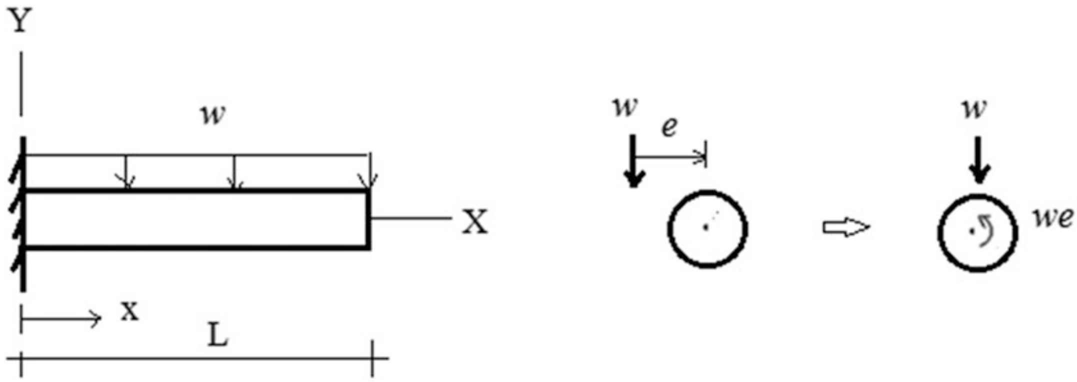


Fig. 3.36 Eccentrically loaded member

When bending and twisting are coupled because of eccentric loading, it is convenient to solve the bending and twisting problems separately, and then combine the solutions.

In what follows, we illustrate this approach.

The eccentric load shown in Fig. 3.36 produces the distributed torsional loading equal to *we*, and the planar loading *w*. Noting (3.50), the torsional moment is

$$M_t = we(L - x)$$

We determine the twist with (3.52). The left end is fixed, so $\beta_t(0) = 0$. Then,

$$\beta_t(x) = \int_0^x \frac{1}{GJ}[we(L - x)] = -\frac{we}{2GJ}(L - x)^2 \Big|_0^x$$
$$\beta_t(x) = \frac{we}{2GJ}(2Lx - x^2)$$

The solution for plane bending is generated with (3.22)

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\frac{w}{2EI}(L - x)^2$$
$$\theta(x) = \left[\tfrac{w}{6EI}(L - x)^3\right]_0^x = \frac{w}{6EI}(L - x)^3 - \frac{wL^3}{6EI}$$
$$v(x) = \left[-\tfrac{w}{24EI}(L - x)^4 - \tfrac{wL^3 x}{6EI}\right]_0^x = \frac{w}{6EI}\left\{-\frac{1}{4}(L - x)^4 - L^3 x + \frac{L^4}{4}\right\}$$

The solution for a cantilever beam subjected to a concentrated torsional moment at the free end is needed later when we deal with plane grids.

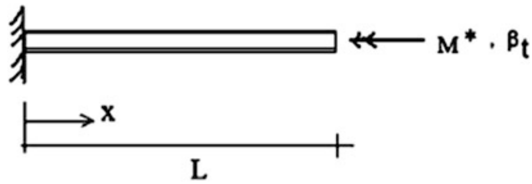
Noting Fig. 3.37, the torsional moment is constant,

$$M_t = M^*$$

and the twist angle varies linearly with x

$$\beta_t = \frac{M^*}{GJ}x \tag{3.54}$$

Fig. 3.37 Pure torsion



3.9 Symmetry and Anti-symmetry

3.9.1 Symmetry and Anti-symmetry: Shear and Moment Diagrams

This section discusses the relationship between certain properties of the shear and moment diagrams and the nature of the loading distribution and support locations. We first introduce some background material on symmetrical and anti-symmetrical functions.

Consider the function $f(x)$ shown in Fig. 3.38. We say the function is symmetrical with respect to $x = 0$ when $f(-x) = f(x)$ and anti-symmetrical when $f(-x) = -f(x)$. Symmetrical functions have $\mathrm{d}f/\mathrm{d}x = 0$ at $x = 0$. Anti-symmetrical functions have $f = 0$ at $x = 0$. One can establish that the derivative of a symmetrical function is an anti-symmetrical function. Similarly, the derivative of an anti-symmetrical function is a symmetrical function. If we know that a function is either symmetrical or anti-symmetrical, then we have to generate only one-half the distribution. The shape of the other half follows by definition of the symmetry properties.

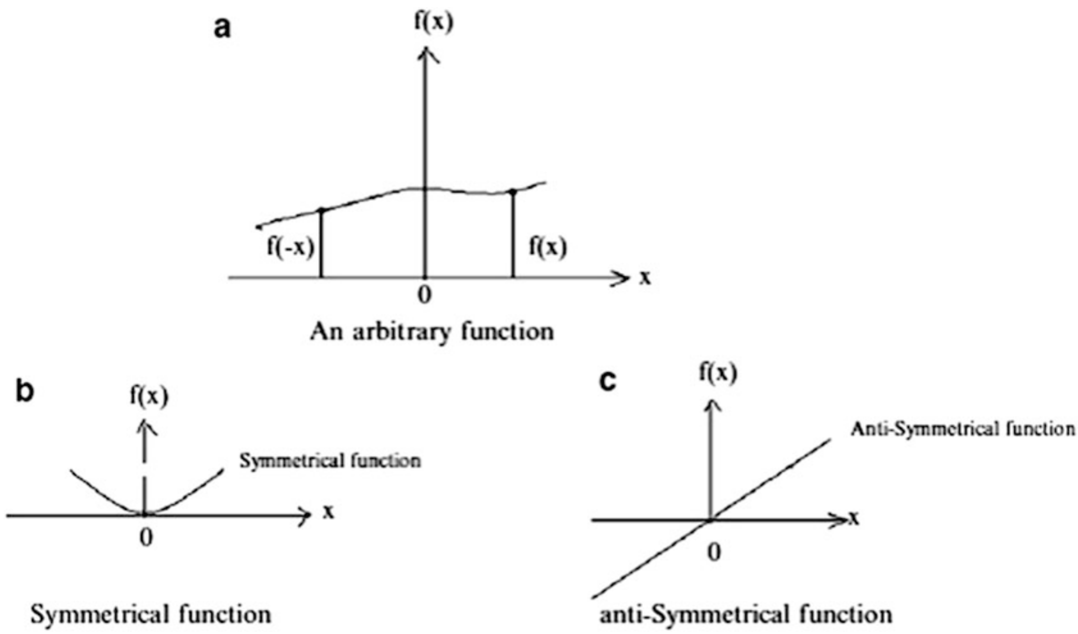


Fig. 3.38 Symmetry and anti-symmetry properties

Starting with the basic differential equations relating the shear, moment, and applied distributed loadings:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = w$$
$$\frac{\mathrm{d}M}{\mathrm{d}x} = -V$$
$$\frac{\mathrm{d}M_t}{\mathrm{d}x} + m_t = 0$$

we can deduce the following properties for V, M, and M_t, given the nature of the loading

1. w is a symmetrical function
   - V is anti-symmetrical
   - M is symmetrical
2. w is an anti-symmetrical function
   - V is symmetrical
   - M is anti-symmetrical
3. m_t is a symmetrical function
   - M_t is anti-symmetrical
4. m_{t} is an anti-symmetrical function
   - M_t is symmetrical

(3.55)

The following cases illustrate these rules.

Symmetrical—planar loading:

Case (*a*) (Fig. 3.39):

Case (*b*) (Fig. 3.40):

Note that the center section is in pure bending, i.e., the shear force is zero. This loading scheme is used to test beams in bending

Anti-symmetry—planar loading:

Case (*a*) (Fig. 3.41):

Case (*b*) (Fig. 3.42):

We use the concept of symmetry to represent an arbitrary loading as a superposition of symmetrical and anti-symmetrical loadings. Then, we generate the individual shear and moment diagrams and combine them. As an illustration, consider a simply supported beam with a single concentrated force shown in Fig. 3.43a. We replace it with two sets of forces, one symmetrical and the other anti-symmetrical, as shown in Fig. 3.43b. Then, we use the results shown in Figs. 3.40 and 3.42 to construct the shear and moment diagrams.

Fig. 3.39 Symmetrical uniform loading

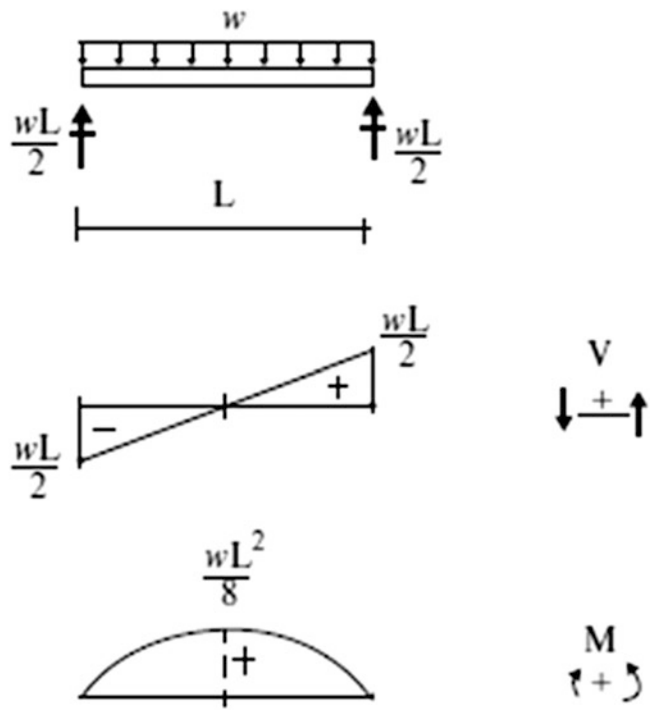
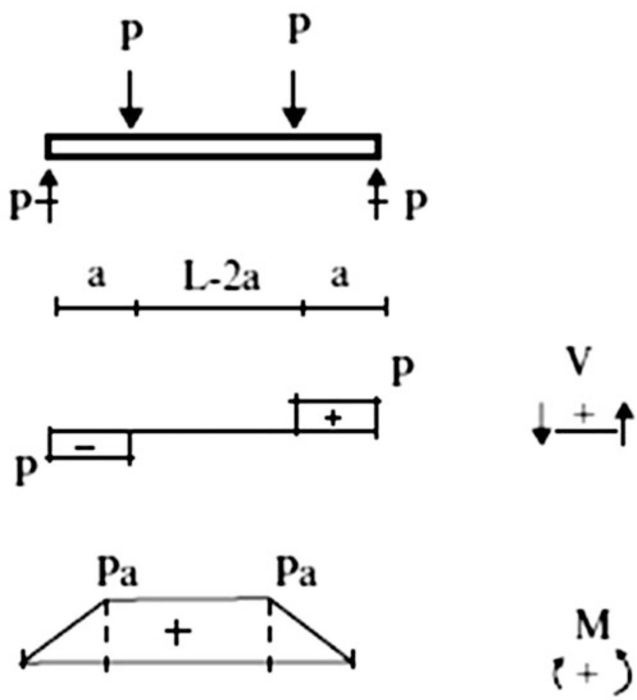


Fig. 3.40 Symmetrical 2-point loading



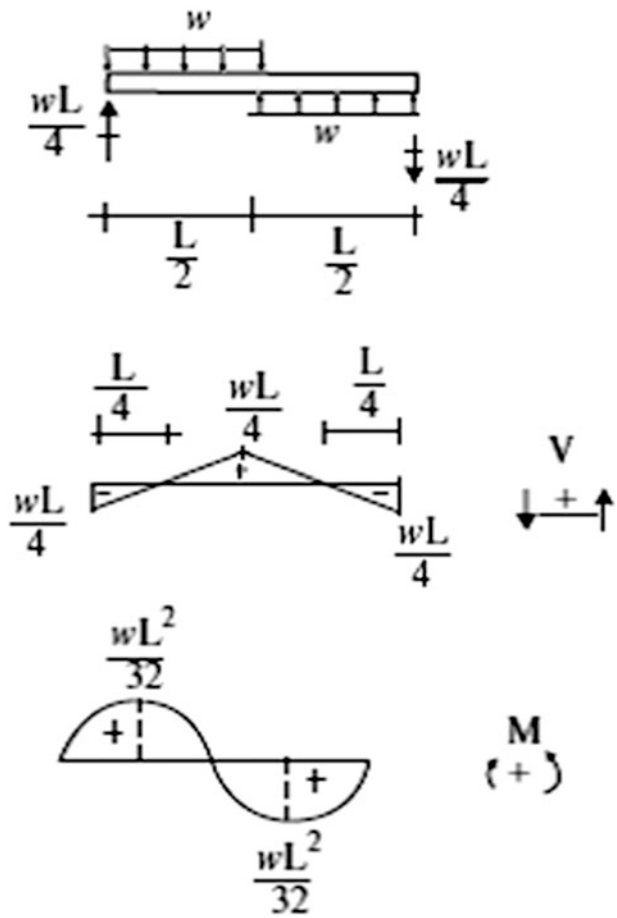


Fig. 3.41 Anti-symmetrical uniform loading

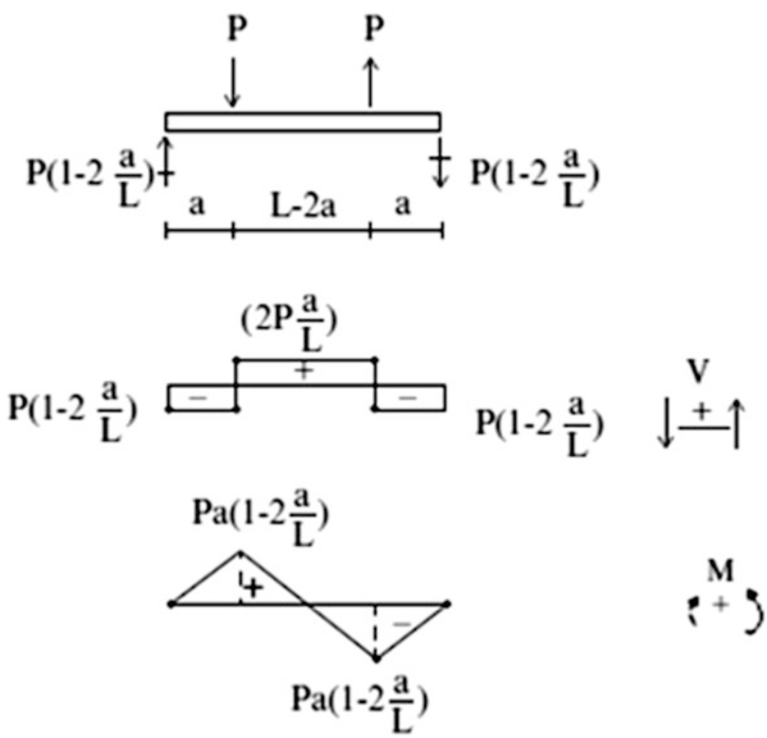


Fig. 3.42 Anti-symmetrical 2-point loading

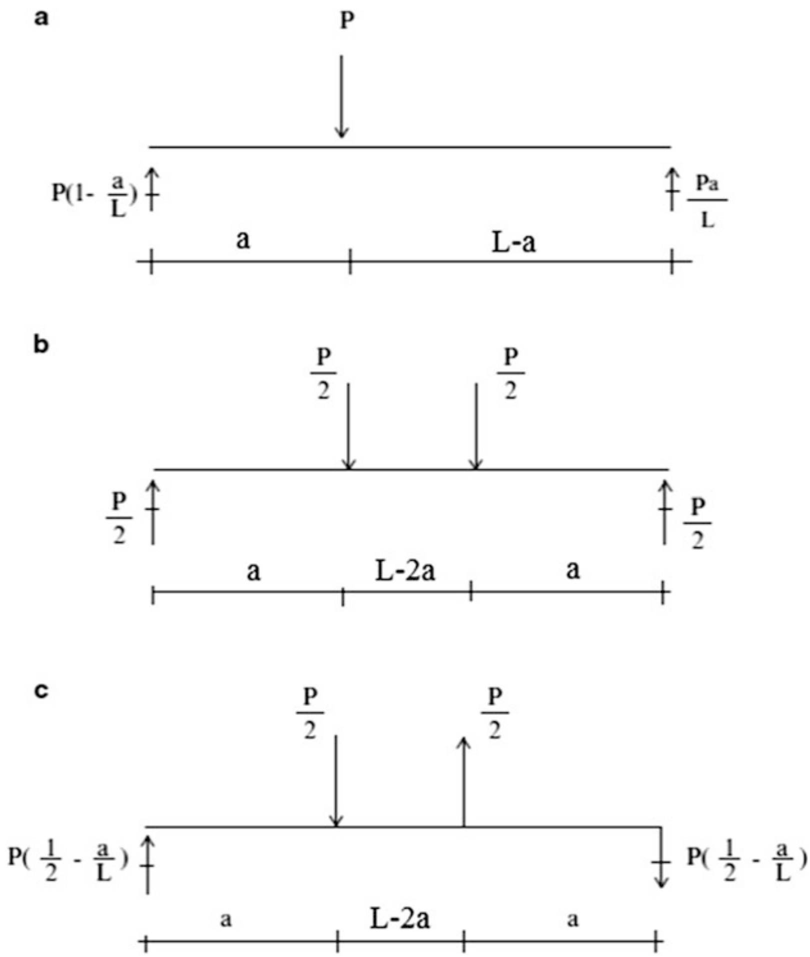


Fig. 3.43 Representation of an arbitrary loading by superposition. (**a**) Single concentrated load. (**b**) Set of symmetrical loads. (**c**) Set of anti-symmetrical loads

3.9.2 Symmetry and Anti-symmetry: Deflected Shapes

A structure is said to be geometrically symmetrical with respect to a particular axis when, if one rotates the portion either to the right or to the left of the axis through 180°, it coincides identically with the other portion. Figure 3.44 illustrates this definition. If we rotate A-B about axis 1-1, it ends up exactly on A-C. A mathematical definition of geometric symmetry can be stated as follows: for every point having coordinates X, Y, there exists a corresponding point with coordinates $-X$, Y.

In addition to geometric symmetry, we also introduce the concept of support symmetry. The supports must be located symmetrically with respect to the axis of geometric symmetry and be of the same nature, e.g., vertical, horizontal, and rotational constraints. Consider Fig. 3.45. There are four vertical restraints at points A, B, C, and D. The geometric symmetry axis, 1-1, passes through mid-span. For complete symmetry, the pin support at point D needs to be shifted to the end of the span. Another example is shown in Fig. 3.46.

Fig. 3.44 Geometric symmetry

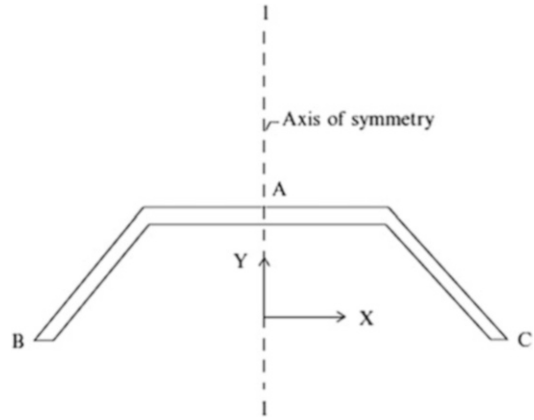


Fig. 3.45 Unsymmetrical support

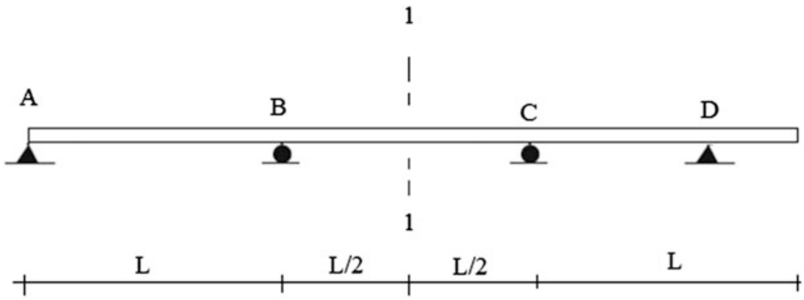
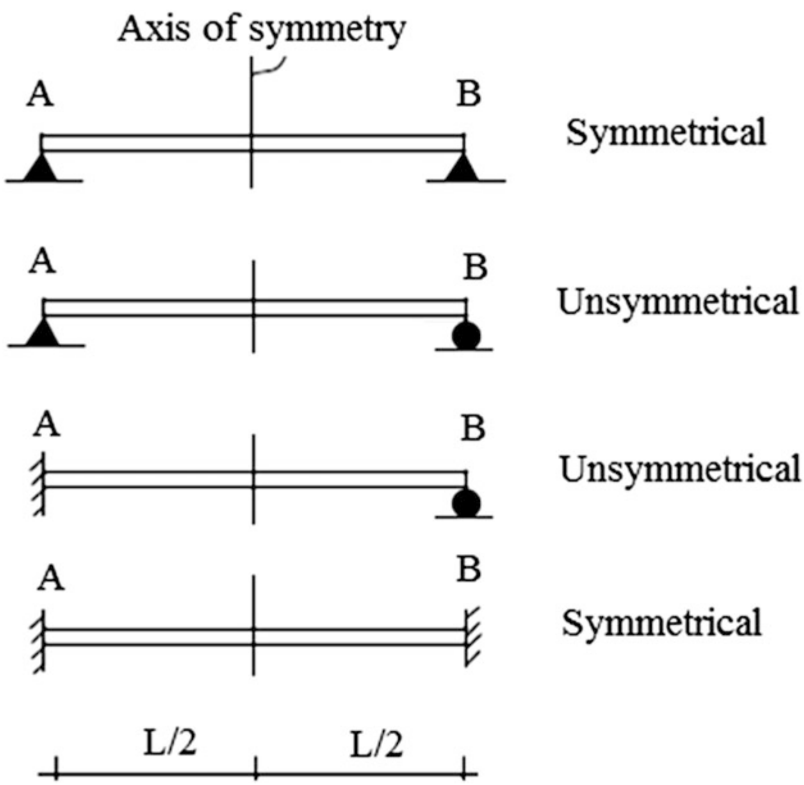


Fig. 3.46 Support symmetry examples



We say a structure is symmetrical when it has both geometric and support symmetry. The symmetry property is very useful since it leads to the following conclusions:

When a symmetrical structure is loaded symmetrically, the resulting deflected shape is also symmetrical. Similarly, a symmetrical structure loaded anti-symmetrically has an anti-symmetric deflected shape.

These conclusions follow from the differential equations listed below and the properties of symmetrical and anti-symmetrical functions:

$$\begin{aligned} &\frac{\mathrm{d}V}{\mathrm{d}x} = w \\ &\frac{\mathrm{d}M}{\mathrm{d}x} = -V \\ &\frac{\mathrm{d}M_t}{\mathrm{d}x} + m_t = 0 \\ &\frac{\mathrm{d}\beta}{\mathrm{d}x} = \frac{M}{EI} \\ &\frac{\mathrm{d}V}{\mathrm{d}x} = \beta + \frac{V}{GJ} \\ &\frac{\mathrm{d}\beta_t}{\mathrm{d}x} = \frac{M_t}{GJ} \end{aligned} \tag{3.56}$$

If $f(x)$ is symmetrical, $\mathrm{d}f/\mathrm{d}x$ is anti-symmetrical; if $f(x)$ is anti-symmetrical, $\mathrm{d}f/\mathrm{d}x$ is symmetrical.

Using these properties, we construct the following table relating the response variables to the loading for a symmetrical structure (Table 3.2).

Table 3.2 Loading response relationships—symmetrical structure

Loading	Response variables
w symmetrical	V anti-symmetrical
	M symmetrical
	β anti-symmetrical
	v symmetrical
w anti-symmetrical	V symmetric
	M anti-symmetrical
	β symmetrical
	v anti-symmetrical
m_t symmetrical	M_t anti-symmetrical
	β_t symmetrical
m_t anti-symmetrical	M_t symmetrical
	β_t anti-symmetrical

We have placed a lot of emphasis here on symmetry because it is useful for qualitative reasoning. It also allows us to work with only one-half the structure provided that we introduce appropriate boundary conditions on the axis of symmetry. The boundary conditions for the symmetrical case follow from the fact that V, β, and M_t are anti-symmetric functions and therefore vanish at the

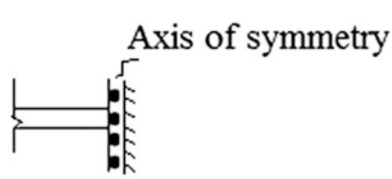


Fig. 3.47 Symmetrical boundary conditions on a symmetry axis

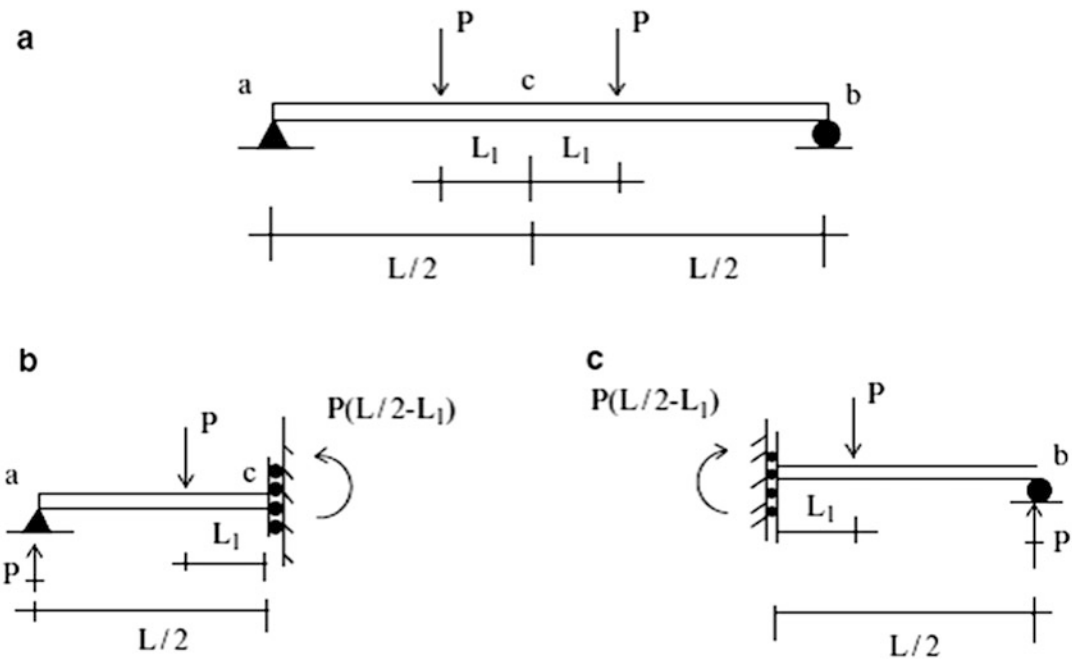


Fig. 3.48 Boundary conditions on symmetry axis—symmetrical planar loading. (**a**) Symmetrical load. (**b**) Left segment. (**c**) Right segment

symmetry axis. We introduce a new support symbol shown in Fig. 3.47, which represents these conditions. The roller support releases V and M_t; the rigid end plate eliminates β.

For example, consider the symmetrically loaded simply supported beam shown in Fig. 3.48a. We can work with either the left or right segment. We choose to work with the left segment, with an appropriate support at c on the axis of symmetry. The displacement boundary conditions for this segment are

$$\begin{aligned} v_a &= 0, \quad \theta_a \neq 0 \\ v_c &\neq 0, \quad \theta_c = 0 \end{aligned}$$

The solution generated with this segment also applies for the other segment (the right portion).

When the loading is anti-symmetrical, the bending moment and displacement are also anti-symmetric functions which vanish at the symmetry axis. For this case, the appropriate support on the axis of symmetry is a roller support. We replace the full beam with the segments shown in Fig. 3.49b, c. We analyze the left segment, and then reverse the sense of the response variables for the other segment.

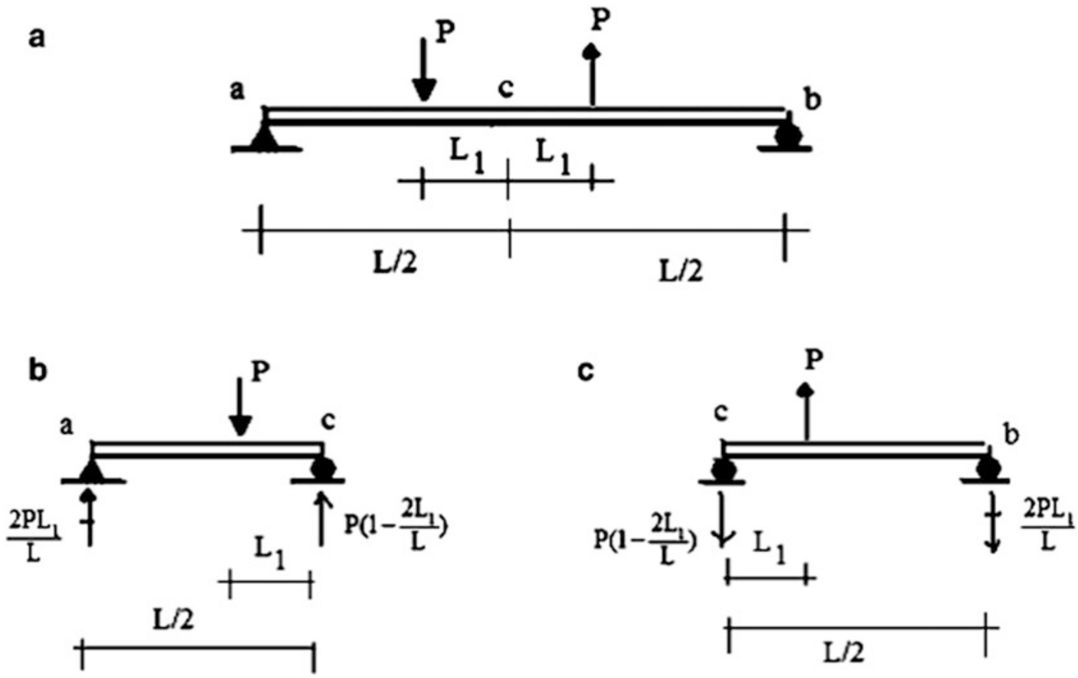


Fig. 3.49 Bending conditions on symmetry axis—anti-symmetrical planar loading. (**a**) Anti-symmetrical planar load. (**b**) Left segment. (**c**) Right segment

3.10 Influence Lines and Force Envelopes for Statically Determinate Beams

3.10.1 The Engineering Process

The Force envelopesInfluence linesobjective of the engineering process is to define the physical makeup of the beam, i.e., the material, the shape of the cross-section, and special cross-section features such as steel reinforcement in the case of a reinforced concrete beam. Cross-sectional properties are governed by the strength of the material and constraints associated with the specific design codes recommended for the different structural materials such as concrete, steel, and wood. Given the maximum values of shear and moment at a particular location, the choice of material, and the general shape of the cross-section, the determination of the specific cross-sectional dimensions at that location involves applying numerical procedures specific to the associated design code. This computational aspect of the engineering process is called design detailing. There are an extensive set of computer-aided design tools available for design detailing. *Therefore, we focus here mainly on that aspect of the engineering process associated with the determination of the "maximum" values of shear and moment for statically determinate beams*. Parts II and III extend the discussion to statically indeterminate structures.

Shear and bending moment result when an external loading is applied to a beam. We described in Sect. 3.4 how one can establish the shear and moment distributions corresponding to a given loading. *For statically determinate beams, the internal forces depend only on the external loading and geometry; they are independent of the cross-sectional properties*. Now, the loading consists of two contributions: dead and live. The dead loading is fixed, i.e., its magnitude and spatial distribution are constant over time. Live loading is, by definition, time varying over the life of the structure. This variability creates a problem when one is trying to establish the maximum values of shear and moment needed to dimension the cross-section. If the cross-section is allowed to vary, one needs the absolute maximum positive and negative values at discrete points along the axis. This information is essential for reinforced concrete beams in order to specify the steel reinforcement.

3.10.2 Influence Lines and Force Envelopes

In what follows, we describe two approaches for treating live loadings. In the first approach, we select a particular location on the longitudinal axis, and determine analytically how the internal forces (shear and moment) vary as the live load is positioned at different points along the span. The analysis is usually carried out for a single concentrated load, and the force magnitude is plotted vs. the load location along the span. This plot is referred to as an influence line, and allows one to easily identify the position of the live load which produces the maximum value of the force quantity at the particular section on the span. By scanning the influence line plot, one can establish the absolute maximum and minimum value for this particular section. This information is sufficient for detailed design of the cross-section at that location.

However, in order to dimension the complete beam, one also needs similar information at other locations along the beam. This data is generated by repeating the influence line process at discrete points along the span, and determining the absolute max/min values from the corresponding influence lines. The results (positive and negative values) are plotted at each discrete point. Plots of this type are called force envelopes. Given the force envelope, one can readily establish the design force requirement at an arbitrary discrete point.

It is important to distinguish between influence lines and force envelopes. An influence line provides information about forces at a particular section due to live loading passing along the span. A force envelope presents information about the extreme force values at discrete points along the span due to live loading passing along the span. Constructing a force envelope based on n discrete points along the span requires n separate analyses. Most commercial civil structural software has the ability to generate force envelope for various live load configurations.

Consider the simply supported beam shown in Fig. 3.50a. Suppose the influence line for the positive moment at A is desired. We apply force P at location x, and evaluate the moment at A. This quantity is a function of x.

$$
\begin{aligned}
M_{\mathrm{A}} &= PL\left(1-\frac{x_{\mathrm{A}}}{L}\right)\frac{x}{L} \quad \text{for} \quad x < x_{\mathrm{A}} \\
M_{\mathrm{A}} &= PL\left(1-\frac{x}{L}\right)\frac{x_{\mathrm{A}}}{L} \quad \text{for} \quad x > x_{\mathrm{A}}
\end{aligned} \tag{3.57}
$$

Letting x range from 0 to L leads to the plot shown in Fig. 3.50c. The maximum value of M_{A} occurs when the load is acting at point A.

$$
\left|\frac{M_{\mathrm{A}}}{PL}\right|_{\max} = \left(1-\frac{x_{\mathrm{A}}}{L}\right)\frac{x_{\mathrm{A}}}{L}
$$

This value provides input for the moment envelope. We repeat the computation taking different points such as x_{A}, x_{B}, and x_{c}. The conventional way of representing this data is to show the discrete points along the span and list the corresponding absolute values at each point. Figure 3.50d illustrates this approach.

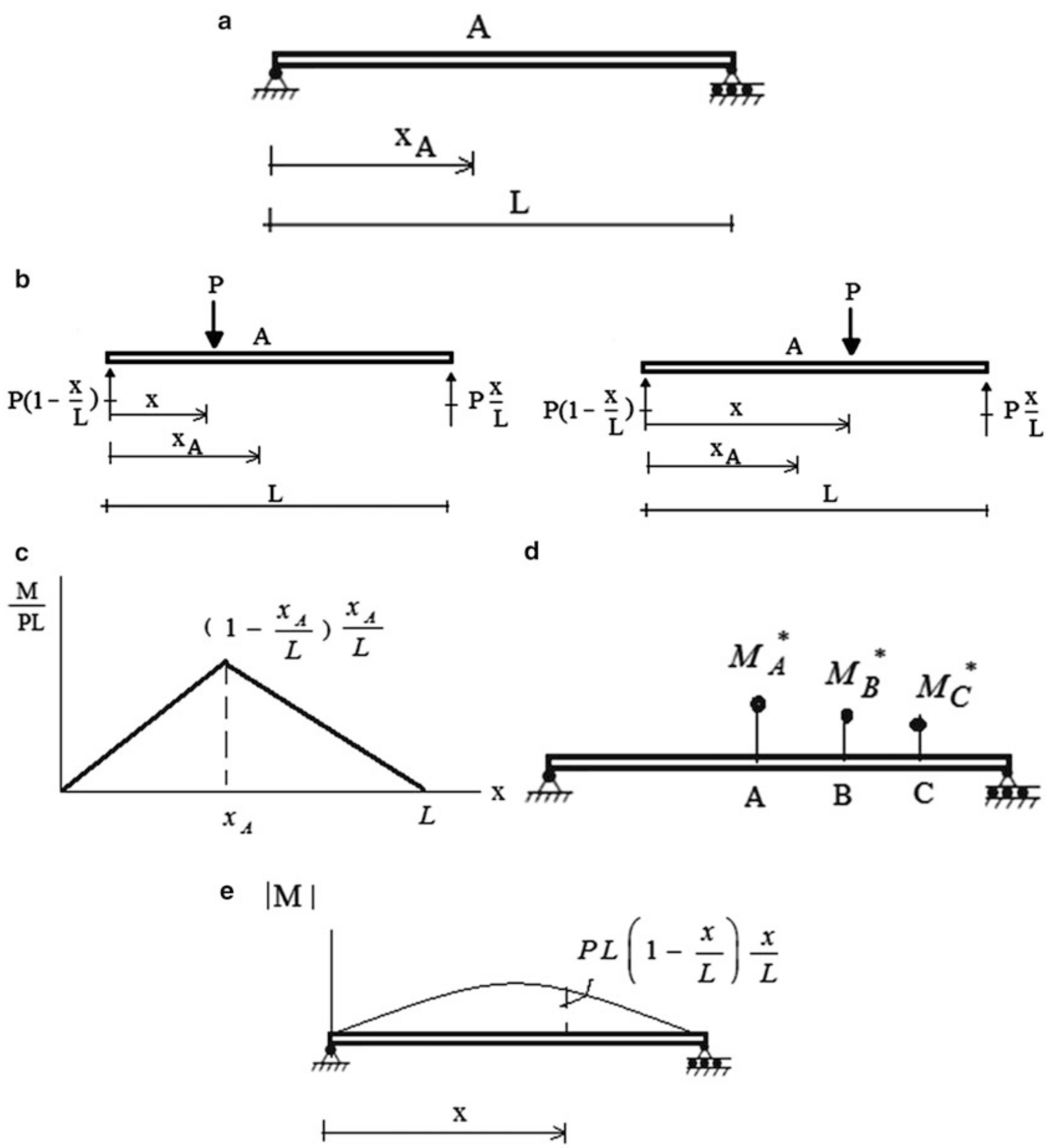


Fig. 3.50 (**a**) Beam. (**b**) Loading patterns – Concentrated load P at $x < x_A$ and $x > x_A$. (**c**) Moment diagram. (**d**) Different load patterns. (**e**) Moment diagram. (**f**) Shear diagrams for concentrated load P at $x < x_A$ and $x > x_A$. (**g**) Influence line for shear at location x_A. (**h**) Maximum and minimum shear. (**i**) Shear force envelope

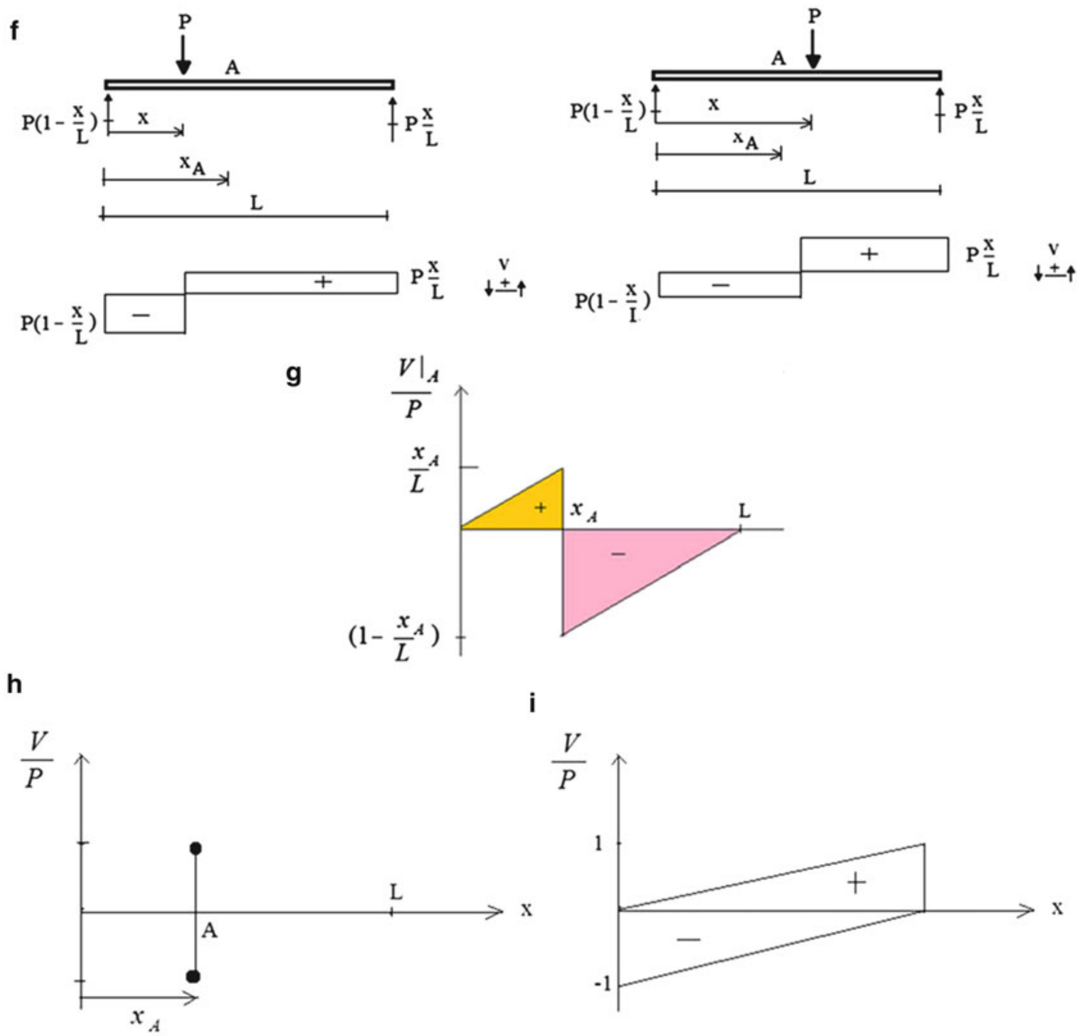


Fig. 3.50 (continued)

Location	Maximum positive moment
x_A	$PL\left(1-\frac{x_A}{L}\right)\frac{x_A}{L}=M_A^*$
:	:
x_B	$PL\left(1-\frac{x_B}{L}\right)\frac{x_B}{L}=M_B^*$
:	:
x_c	$PL\left(1-\frac{x_C}{L}\right)\frac{x_C}{L}=M_C^*$

One selects a sufficient number of points so that the local extremities are identified. The limiting form of the force envelope based on many points is a parabola.

We proceed in a similar manner to establish the influence line and force envelope for the shear force. The shear diagram for a single concentrated force applied at x is shown in Fig. 3.50f.

Suppose we want the influence line for the shear at location x_A. Noting Fig. 3.50f, the shear force at x_A for the different positions of the load is

$$\begin{aligned} x < x_A \quad V|_A &= +\frac{Px}{L} \\ x > x_A \quad V|_A &= -P\left(1-\frac{x}{L}\right) \end{aligned} \tag{3.58}$$

These functions are plotted in Fig. 3.50g. At point x_A, there is a discontinuity in the magnitude of V equal to P and a reversal in the sense. This behavior is characteristic of concentrated forces.

To construct the force envelope, we note that maximum and minimum values of shear at point A are

$$\left(\frac{V_A}{P}\right)_{\max} = +\frac{x_A}{L}$$
$$\left(\frac{V_A}{P}\right)_{\min} = -\left(1-\frac{x_A}{L}\right)$$

These values are plotted on the span at point A (Fig. 3.50h).

Repeating the process for different points, one obtains the force envelope shown in Fig. 3.50i.

Example 3.25 Construction of Influence Lines

Given: The beam shown in Fig. E3.25a.

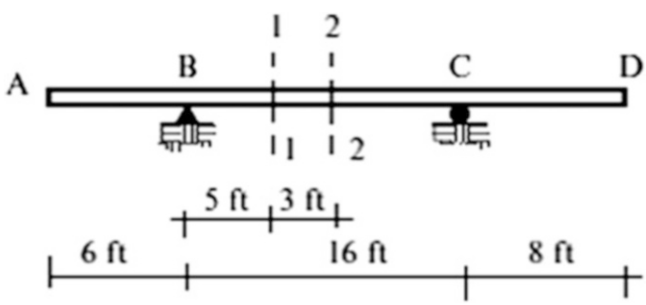
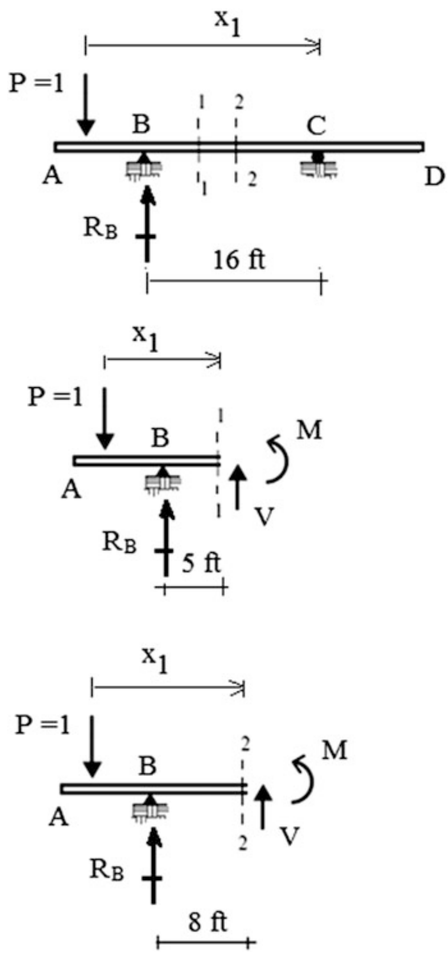


Fig. E3.25a

Determine: The influence lines for the vertical reactions at B and C, moment at section 2-2, and the moment and shear forces at section 1-1. Suppose a uniformly distributed live load of $w_L = 1.2$ kip/ft and uniformly distributed dead load of $w_D = 0.75$ kip/ft are placed on the beam. Using these results, determine the maximum value of the vertical reaction at B and the maximum and minimum values of moment at section 2-2.

Solution: Note that the influence lines are linear because the equilibrium equations are linear in the position variable (see Fig. E3.25b).



$$R_B = \frac{x_1}{16}P$$

$$M = 5R_B - x_1 P$$
$$V = -R_B + P$$

$$M = 8R_B - x_1 P$$
$$V = -R_B + P$$

Fig. E3.25b

The influence lines corresponding to the force quantities of interest are plotted in Fig. E3.25c.

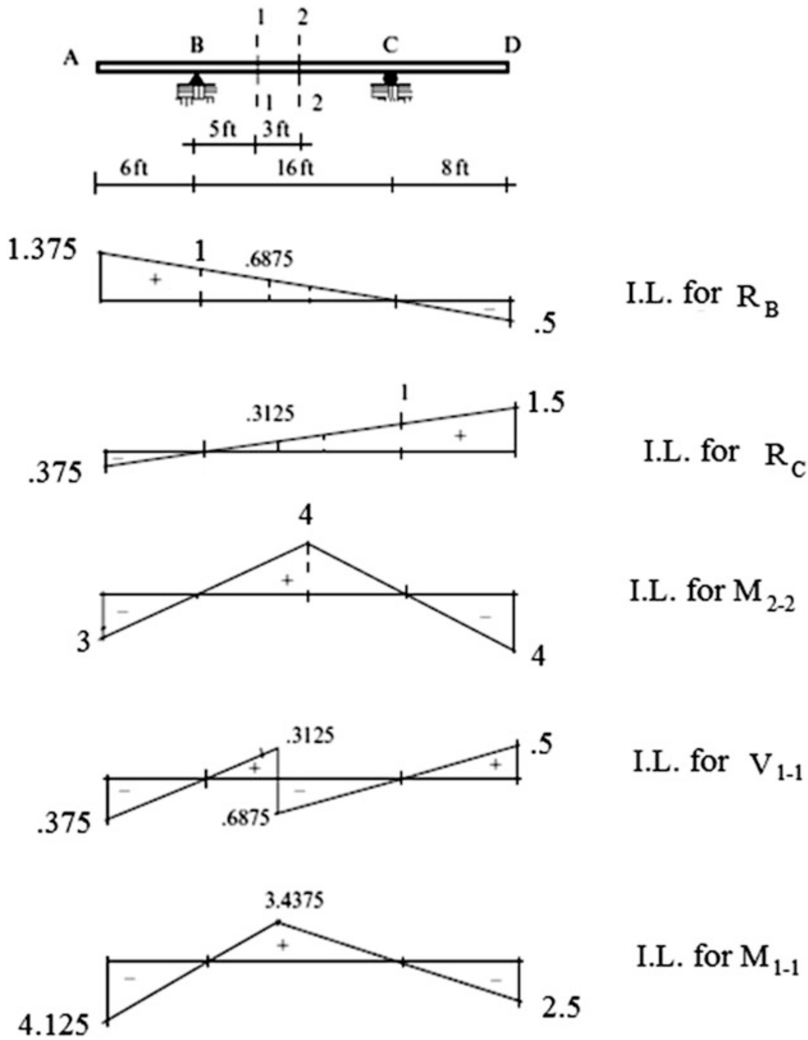


Fig. E3.25c Influence lines for R_B, R_C, $V_{1\text{-}1}$, $M_{2\text{-}2}$, and $M_{1\text{-}1}$

Then, the peak value of R_B is determined using data shown in Fig. E3.25d.

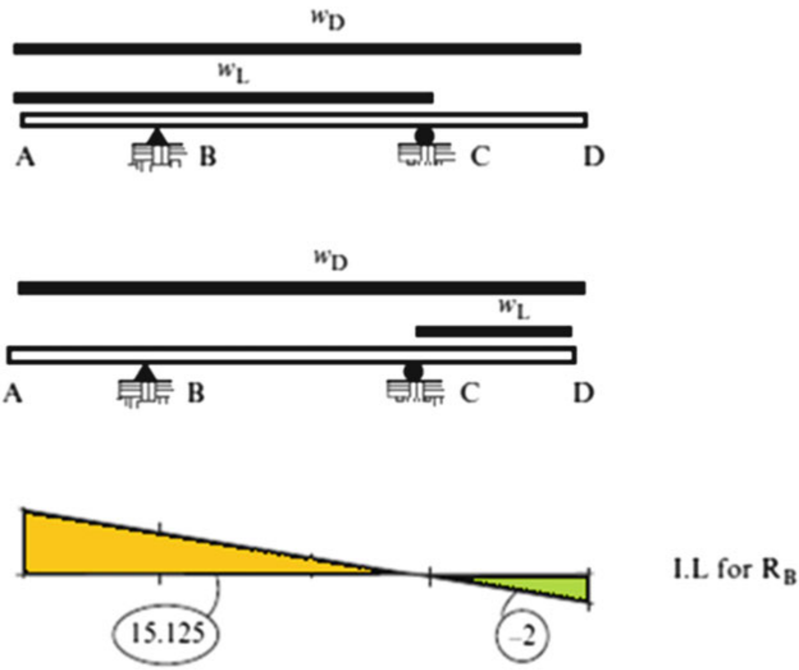


Fig. E3.25d Maximum and minimum values of R_B

$$R_{B_{max}} = 1.2(15.125) + 0.75(15.125 - 2) = 28\,\text{kip}$$

Similarly, the peak values of moment at section 2-2 are generated using the data shown in Fig. E3.25e.

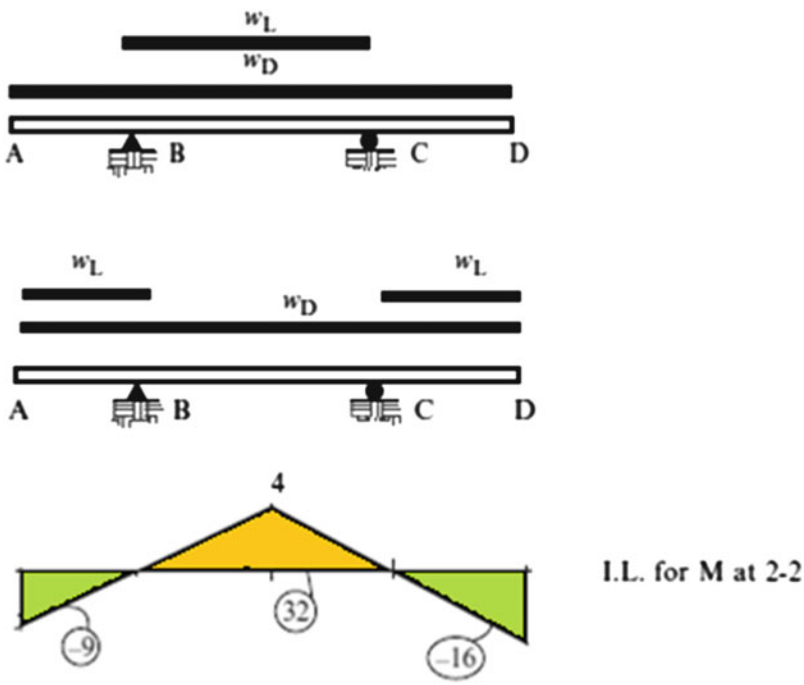


Fig. E3.25e

$$\begin{cases} M_{max} \text{ at } 2\text{-}2 = 1.2(32) + 0.75(32 - 9 - 16) = 43.65\,\text{kip ft} \\ M_{min} \text{ at } 2\text{-}2 = 1.2(-9 - 16) + 0.75(32 - 9 - 16) = -24.75\,\text{kip ft} \end{cases}$$

Example 3.26

Given: The two-span beam shown in Fig. E3.26a. There is a hinge (moment release) at the midpoint of the second span.

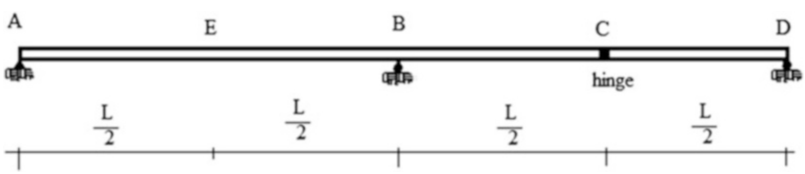


Fig. E3.26a

Determine: The influence line for the bending moment at E and the moment force envelope.
Solution: We consider a unit vertical load moving across the span and use the free body diagrams to determine the moment diagrams. Figure E3.26b shows that the reaction at D equals zero when the load is acting on member ABC.

$$\left.\begin{aligned} R_B &= \frac{x}{L} \\ R_A &= 1 - \frac{x}{L} \end{aligned}\right. \quad \text{for} \quad 0 < x < 1.5L$$

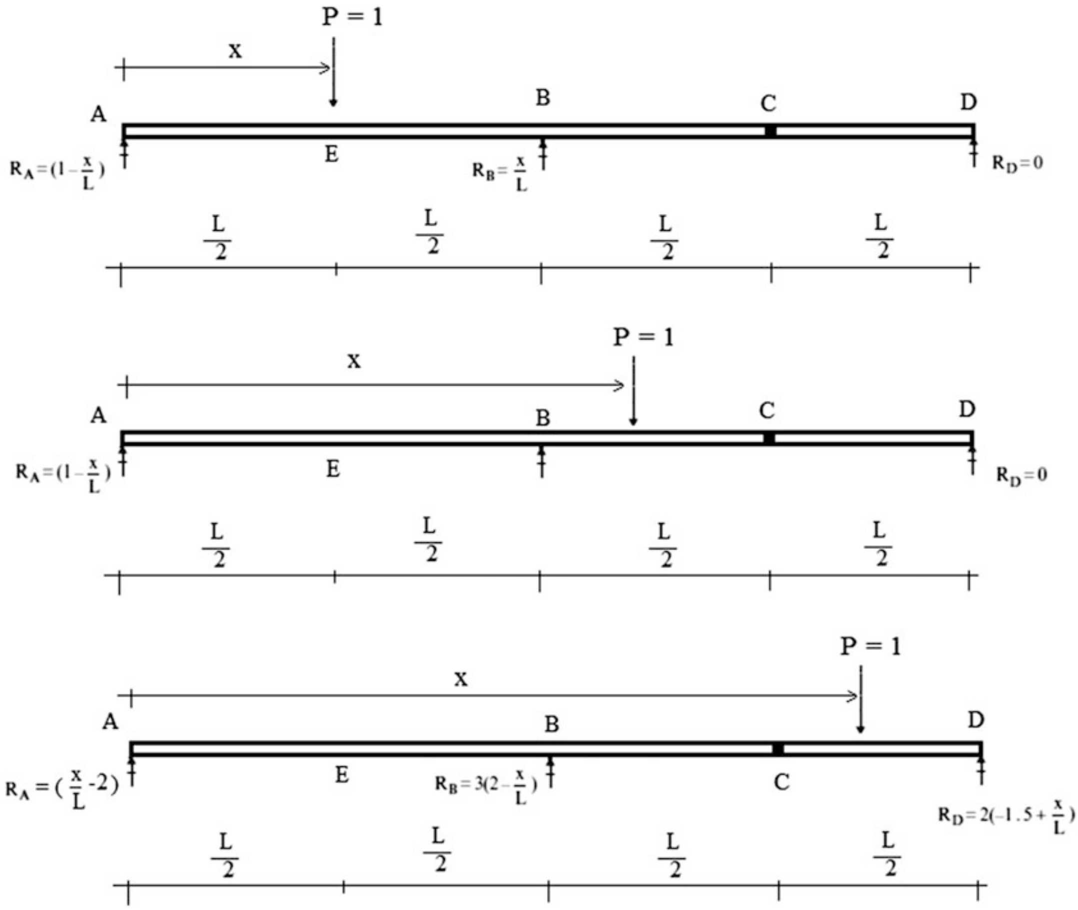


Fig. E3.26b

The behavior changes when the loading passes to member CD. Now there is a reaction at D which releases some of the load on member ABC.

$$\left.\begin{aligned} R_{\mathrm{D}} &= 2\left(-1.5+\frac{x}{L}\right) \\ R_{\mathrm{B}} &= 3\left(2-\frac{x}{L}\right) \qquad \text{for} \quad 1.5L < x < 2L \\ R_{\mathrm{A}} &= \left(\frac{x}{L}-2\right) \end{aligned}\right.$$

The moment distribution corresponding to these loading cases are plotted in Fig. E3.26c.

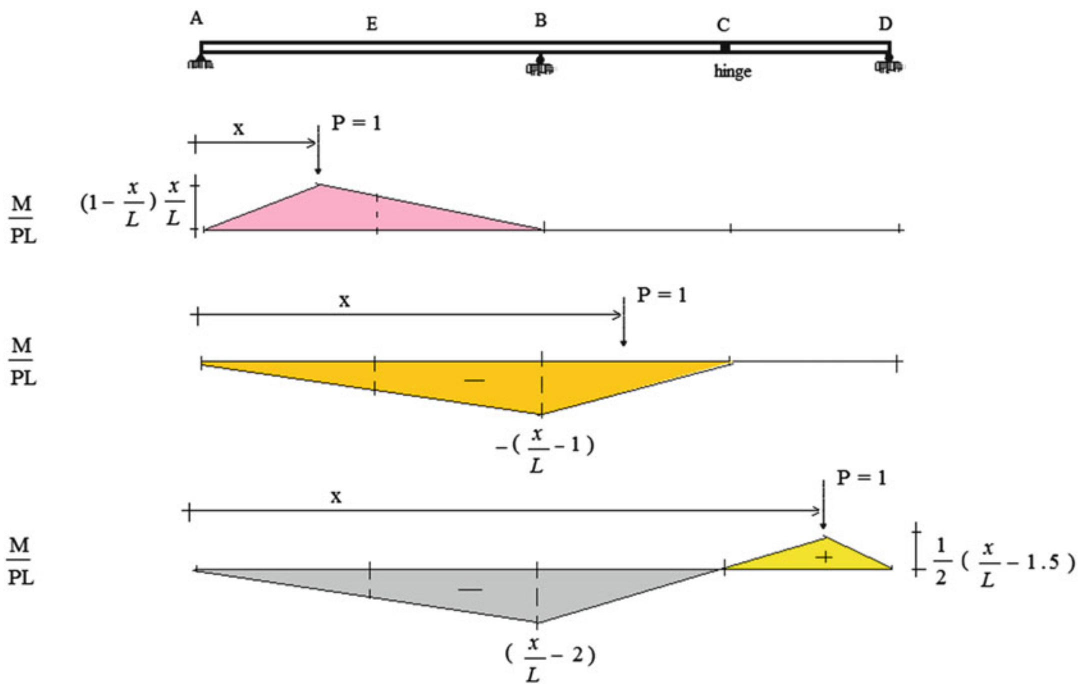


Fig. E3.26c

We note that the moment at E is positive when the load is on span AB, and switches to a negative value when the load moves on to span BCD. The influence line for the bending moment at E is plotted in Fig. E3.26d.

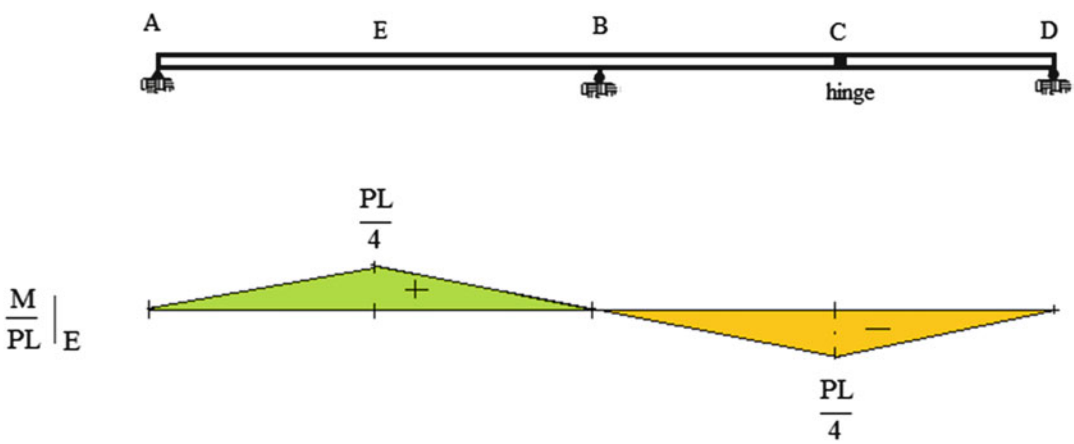


Fig. E3.26d

The moment force envelope is constructed using Fig. E3.26c. Span AB has both positive and negative components; span BC has a negative component; and span CD has a positive component. These segments are plotted in Fig. E3.26e.

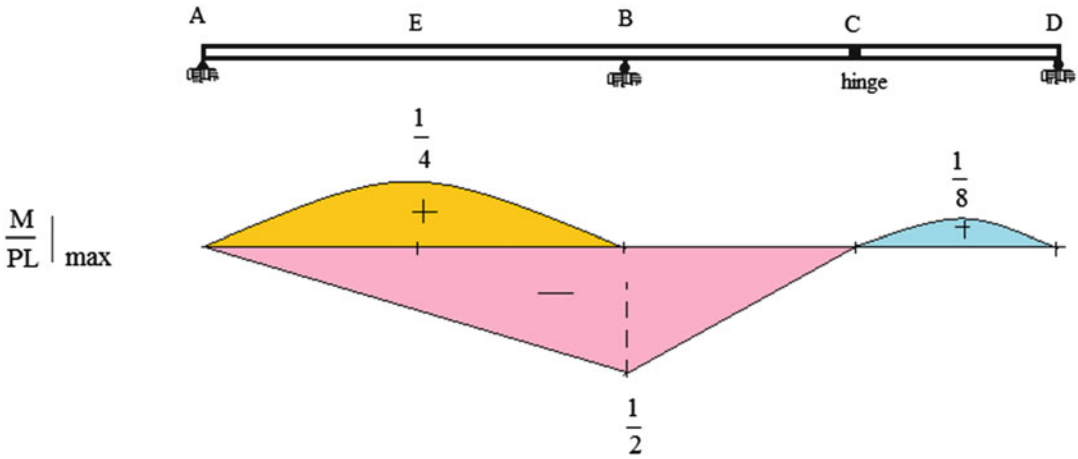


Fig. E3.26e

Example 3.27 Cantilever Construction-Concentrated Loading

Given: The three-span symmetrical scheme shown in Fig. E3.27a. There are two moment releases located symmetrically with respect to the centerline of the center span. This structure is statically determinate: Member cd functions as a simply supported member; segments bc and de act as cantilevers in providing support for member cd. The structural arrangement is called cantilever construction and is used for spanning distances which are too large for a single span or a combination of two spans.

Determine: A method for selecting L_1 and the location of the moment releases corresponding to a concentrated live loading P for a given length, given L_T.

Solution: The optimal geometric arrangement is determined by equating the maximum moments in the different spans. Given the total crossing length, L_T, one generates a conceptual design by selecting L_1, and α which defines the location of the hinges. The remaining steps are straightforward. One applies the design loading, determines the maximum moments for each beam segment, and designs the corresponding cross-sections. The local topography may control where the interior supports may be located. We assume here that we are not constrained in choosing L_1 and describe below how one can utilize moment diagrams to arrive at an optimal choice for L_1 and α.

We consider the design load to be a single concentrated force that can act on any span. The approach that we follow is to move the load across the total span and generate a sequence of moment diagrams. This calculation provides information on the location of the load that generates the maximum moment for each span.

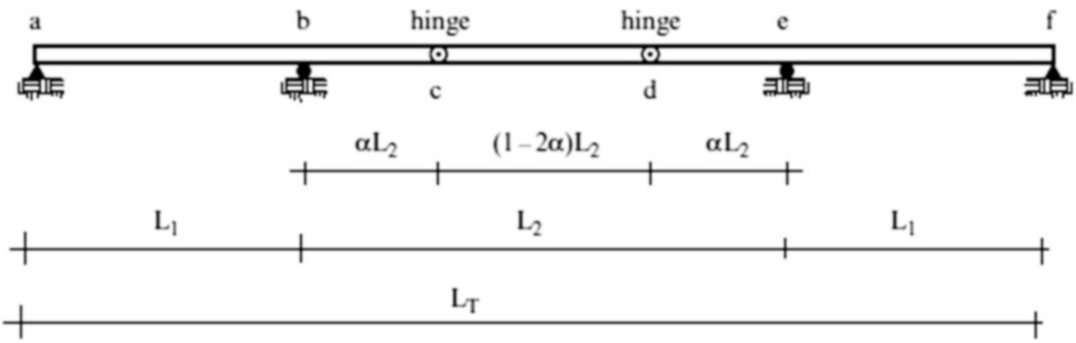


Fig. E3.27a

When the load is on ab, member ab functions as a simply supported beam, and we know from the previous example that the critical location is at mid-span. As the load moves from b to c, bc acts like a cantilever, and the critical location is point c. Lastly, applying the load at the midpoint of c, d produces the maximum moment for cd. Since the structure is symmetrical, we need to move the load over only one-half the span. Moment diagrams for these cases are shown in Figs. E3.327b, E3.327c, and E3.327d.

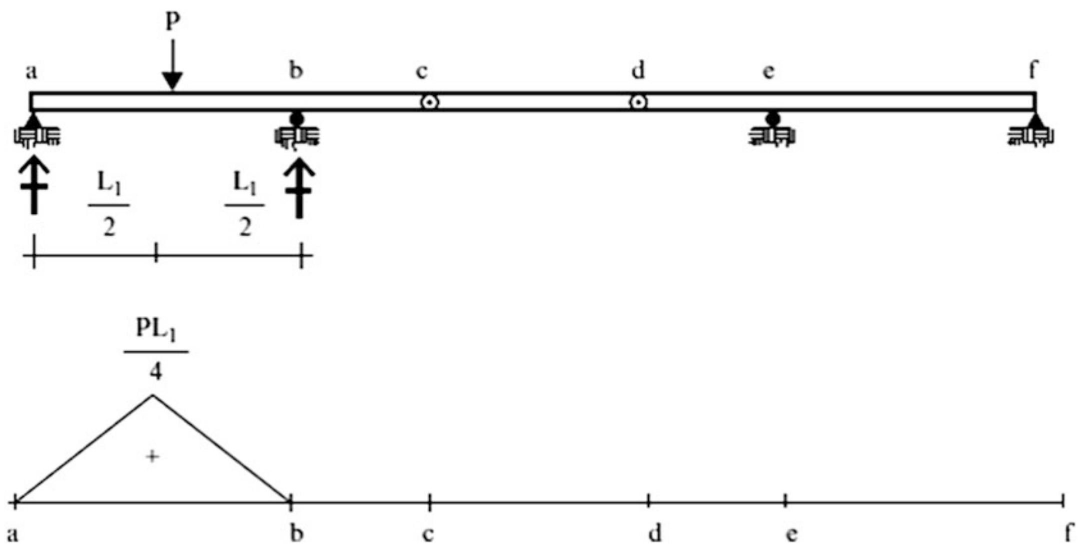


Fig. E3.27b Moment diagram—load on member AB

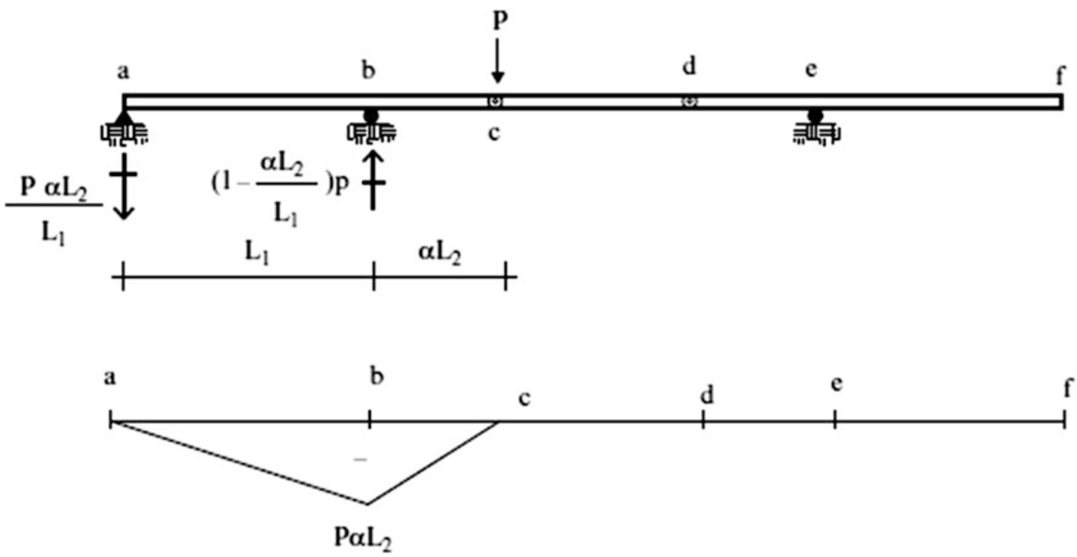


Fig. E3.27c Moment diagram—load on member BC

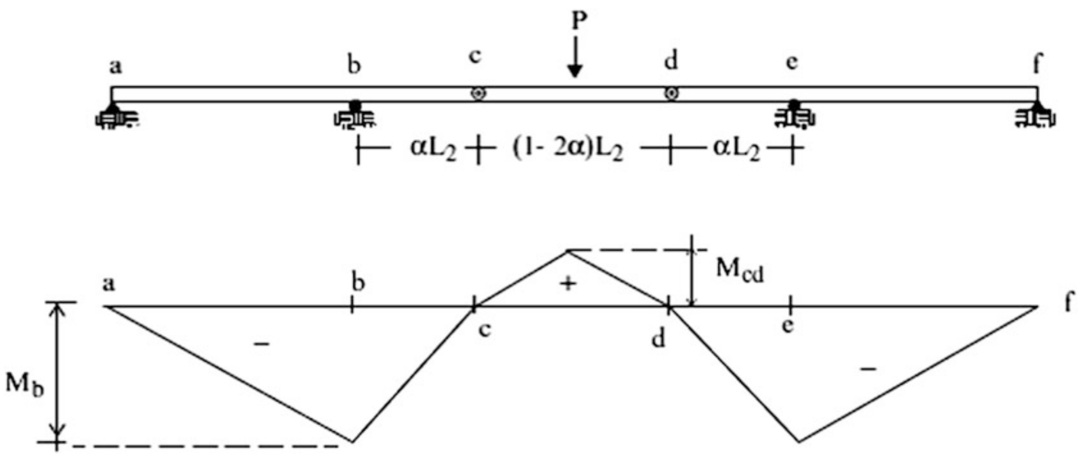


Fig. E3.27d Moment diagram—load on member CD

Based on these analyses, the design moments for the individual spans are

$$M|_{\mathrm{ab}} = \frac{PL_1}{4} \quad M|_{\mathrm{fe}} = M|_{\mathrm{ab}}$$

$$M|_{\mathrm{bc}} = P\alpha L_2 \quad M|_{\mathrm{ed}} = M|_{\mathrm{bc}}$$

$$M|_{\mathrm{cd}} = \frac{PL_2(1-2\alpha)}{4}$$

$$M_{\mathrm{b}} = \frac{P}{2}\alpha L_2 \quad M_{\mathrm{e}} = M_{\mathrm{b}}$$

From a constructability perspective, a constant cross-section throughout the total span is desirable. This goal is achieved by equating the design moments and leads to values for L_1 and α. Starting with $M|_{\mathrm{bc}} = M|_{\mathrm{cd}}$, one obtains

$$P\alpha L_2 = \frac{PL_2(1-2\alpha)}{4}$$
$$\Downarrow \qquad \Downarrow \alpha = \frac{1}{6}$$
$$\alpha = \frac{(1-2\alpha)}{4}$$

Next, we equate $M|_{ab}$ and $M|_{bc}$, resulting in

$$\frac{PL_1}{4} = P\alpha L_2$$
$$\Downarrow$$
$$L_1 = \frac{2}{3}L_2$$

The "optimal" center span is

$$2L_1 + L_2 = L_T$$
$$\Downarrow$$
$$L_2 = \frac{3}{7}L_T = 0.429L_T$$

If the interior supports can be located such that these span lengths can be realized, the design is *optimal for this particular design loading*. We want to emphasize here that analysis is useful for gaining insight about behavior, which provides the basis for rational design. One could have solved this problem by iterating through various geometries, i.e., assuming values for α and L_1, but the strategy described above is a better structural engineering approach.

Example 3.28 Cantilever Construction—Uniform Design Loading

Given: The three-span symmetrical structure shown in Fig. E3.28a.

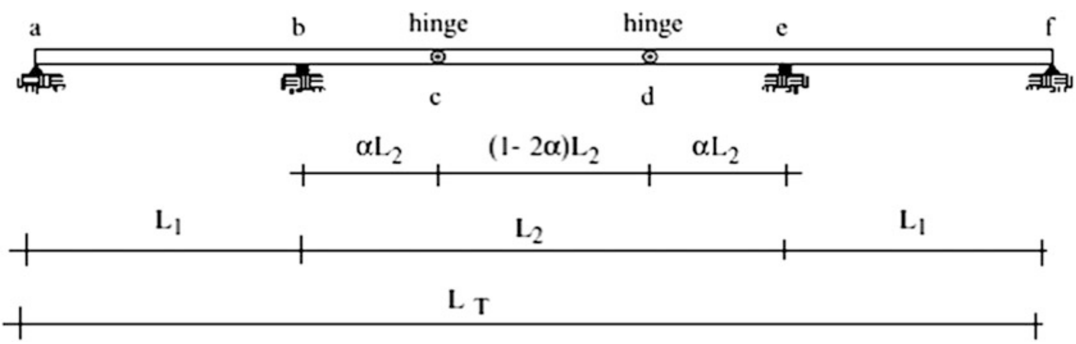


Fig. E3.28a

Determine: The optimal values of L_1 and α corresponding to a uniform live loading w.

Solution: Using the results of the previous example, first, we establish the influence lines for the moment at mid-span of ab ($M_{1\text{-}1}$), at point b (M_b), and at mid-span of member cd ($M_{2\text{-}2}$). They are plotted in Fig. E3.28b.

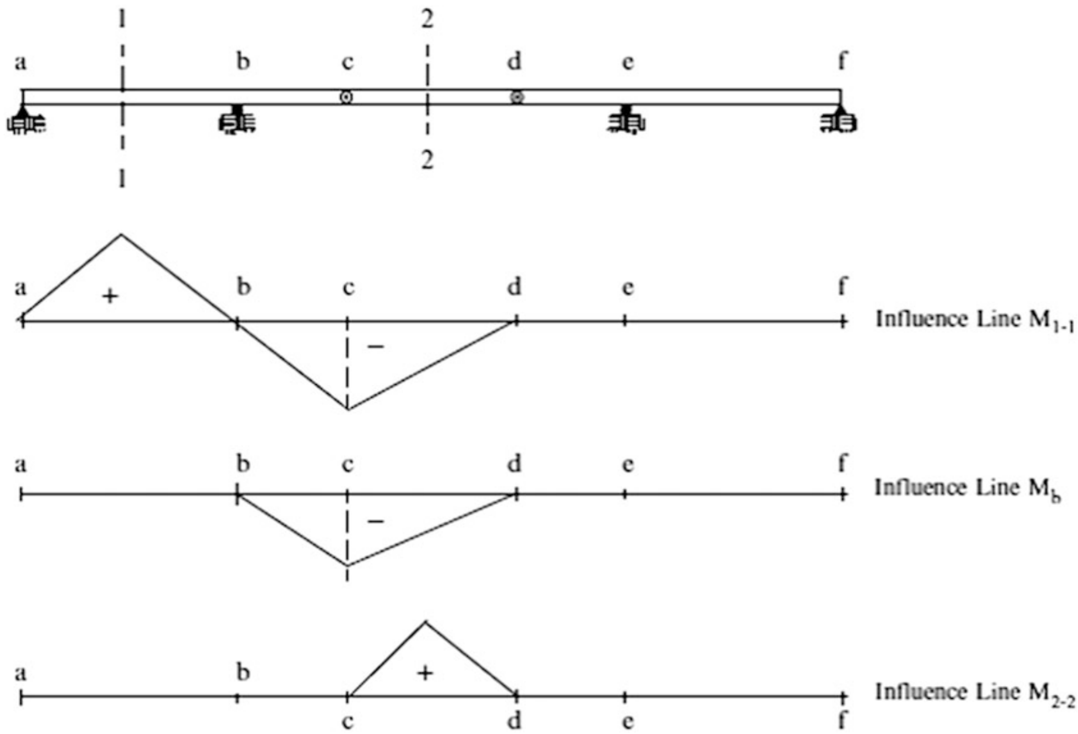


Fig. E3.28b Influence lines

We suppose that the uniformly distributed loading can be applied on an arbitrary segment of a span. We start with the side span, ab. Based on the influence line, we load span ab (Fig. E3.28c).

Next, we load the center span. Loading the segment bcd produces the maximum values for M_b and M_{cd} (Fig. E3.28d). The third option is to load the center span (Fig. E3.28e).

The peak values for these loading schemes are

$$M_{ab} = \frac{wL_1^2}{8}$$

$$M_{cd} = \frac{wL_2^2}{8}(1-2\alpha)^2$$

$$M_{bc} = \frac{wL_2^2}{2}\left[\alpha^2 + \alpha(1-2\alpha)\right]$$

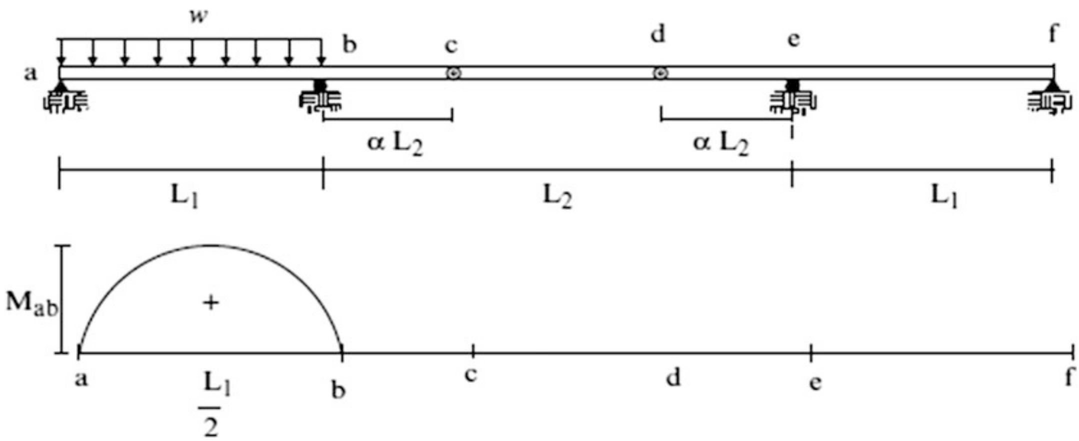


Fig. E3.28c Moment diagram

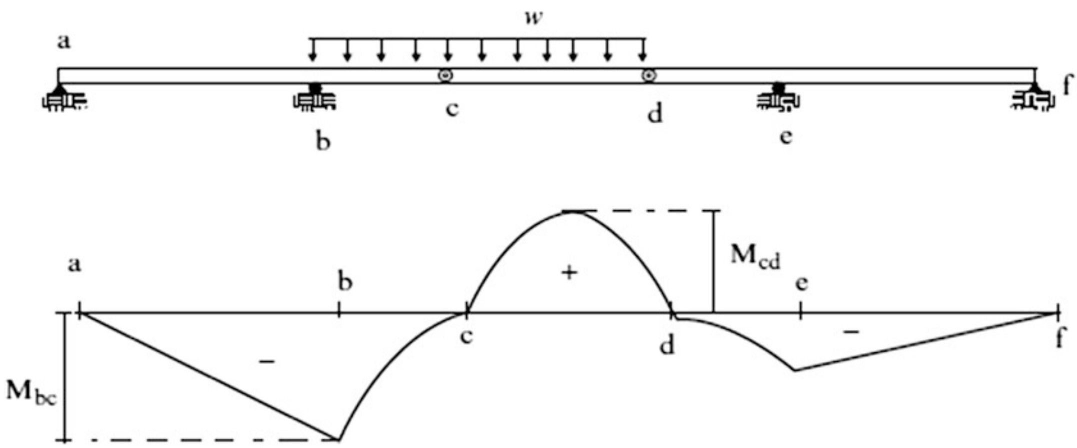


Fig. E3.28d Moment diagram

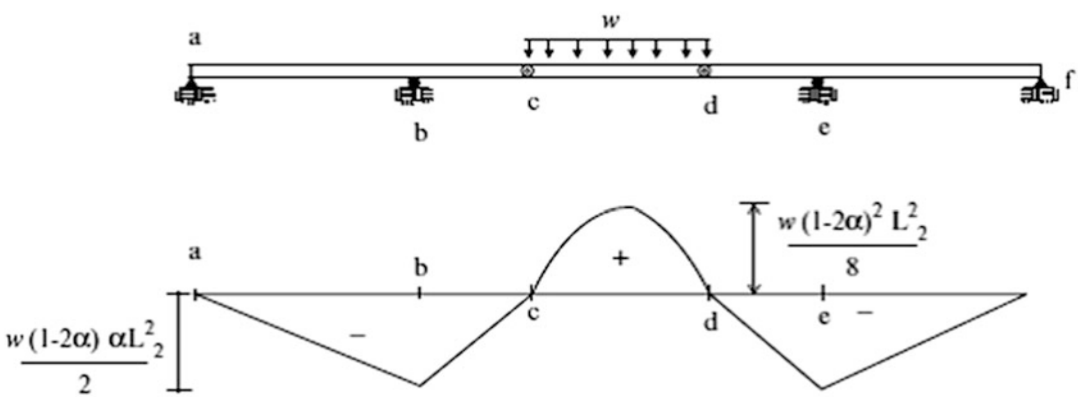


Fig. E3.28e Moment diagram

The remaining steps are the same as for the previous example. We want to use a constant cross-section for the total span and therefore equate the design moments. This step results in values for α and L_1.

Setting $M_{\text{bc}} = M_{\text{cd}}$ results in

$$\frac{1}{8}(1-2\alpha)^2 = \left[\alpha^2 + \alpha(1-2\alpha)\right] 8\alpha^2 - 8\alpha + 1 = 0 \Downarrow \alpha = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right) = 0.147$$

Setting $M_{\text{ab}} = M_{\text{cd}}$ leads to

$$\frac{wL_1^2}{8} = \frac{wL_2^2}{8}(1-2a)^2 = \frac{wL_2^2}{8}\left(\frac{\sqrt{2}}{2}\right)^2 \quad L_2 = 0.707L_2$$

$$\therefore L_1 = \frac{\sqrt{2}}{2}$$

Lastly, L_2 is related to the total span by

$$2L_1 + L_2 = L_{\text{T}}$$
$$\Downarrow$$
$$(1+\sqrt{2})L_2 = L_{\text{T}}$$
$$\Downarrow$$
$$L_2 = 0.414L_{\text{T}}$$

These results are close to the values based on using a single concentrated load.

Examples 3.27 and 3.28 illustrate an extremely important feature of statically determinate structures. The reactions and internal forces produced by a specific loading depend only on the *geometry* of the structure. They are independent of the properties of the components that comprise the structure. This fact allows one to obtain a more favorable internal force distribution by adjusting the geometry as we did here.

These examples also illustrate the use of cantilever construction combined with internal moment releases. In Part II of the text, we rework those problems using beams which are continuous over all three spans, i.e., we remove the moment releases. The resulting structures are statically indeterminate.

3.10.2.1 Multiple Concentrated Loads

We consider next the case where there are two concentrated forces. This loading can simulate the load corresponding to a two axle vehicle. The notation is defined in Fig. 3.51.

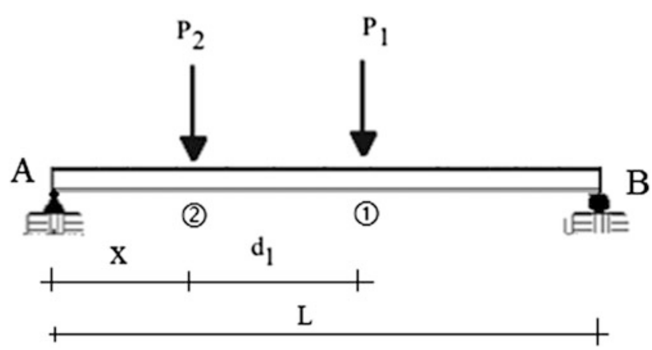
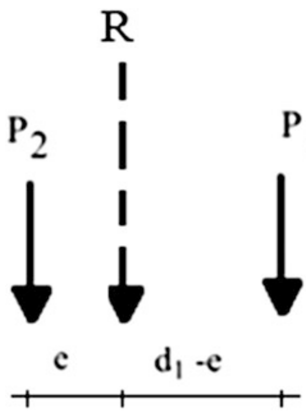


Fig. 3.51 Two concentrated forces

The resultant force $R = P_1 + P_2$ is located e units from the line of action of P_2.



where

$$e = \frac{P_1}{P_1 + P_2} d_1$$

The moment diagram for a set of concentrated forces is piecewise linear with peak values at the points of application of the forces. Figure 3.52 shows the result for this loading case. Analytical expressions for the reactions and the moments at points ② and ② are

$$\begin{aligned} R_A &= (P_1 + P_2)\frac{1}{L}(L - x) - P_1\frac{d_1}{L} \\ R_B &= (P_1 + P_2)\frac{x}{L} + P_1\frac{d_1}{L} \\ M_1 &= (P_1 + P_2)\frac{x}{L}(L - x - d_1) + P_1\frac{d_1}{L}(L - x - d_1) \\ M_2 &= (P_1 + P_2)\frac{x}{L}(L - x) - P_1\frac{d_1}{L}x \end{aligned} \tag{3.59}$$

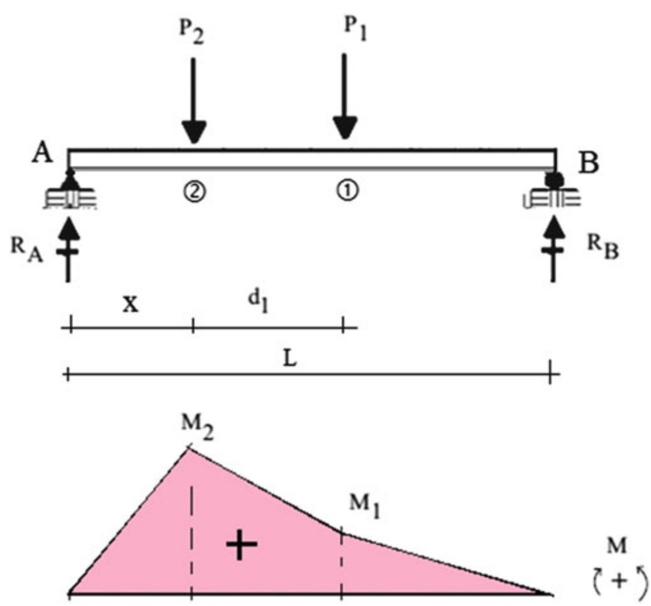


Fig. 3.52 Moment diagram—arbitrary position of loading

These moments are quadratic functions of x. One can compute M_1 and M_2 for a range of x values and determine the values of x corresponding to the peak values. Alternatively, one can determine the value of x corresponding to a maximum value of a particular moment by differentiating the corresponding moment expression with respect to x and setting the result equal to zero.

Maximum value of M_2

$$\frac{\partial M_2}{\partial x} = 0$$

$$(P_1 + P_2)\left(1 - 2\frac{x}{L}\right) - P_1\frac{d_1}{L} = 0 \tag{3.60}$$

$$x\big|_{M_{2\max}} = \frac{L}{2} - \frac{d_1}{2}\left(\frac{P_1}{P_1 + P_2}\right) = \frac{L}{2} - \frac{e}{2}$$

Maximum value of M_1

$$\frac{\partial M_1}{\partial x} = 0$$

$$(P_1 + P_2)(L - 2x - d_1) - P_1 d_1 = 0 \tag{3.61}$$

$$x\big|_{M_{1\max}} = \frac{L}{2} - \frac{d_1}{2} - \frac{e}{2}$$

We can interpret the critical location for the maximum value of M_2 from the sketch shown in Fig. 3.53a. The force P_2 is located $e/2$ units to the left of mid-span and the line of action of the resultant is $e/2$ units to the right of mid-span. A similar result applies for M_1. P_1 is positioned such that P_1 and R are equidistant from mid-span as shown in Fig. 3.53b.

The absolute maximum live load moment is found by evaluating M_1 and M_2 using the corresponding values of $x\big|_{M_{1\max}}$ and $x\big|_{M_{2\max}}$. In most cases, the absolute maximum moment occurs at the point of application of the *largest force* positioned according to (3.60) and (3.61).

Example 3.29 Illustration of Computation of Maximum Moments for Two-Force Loading

Given: The beam shown in Fig. E3.29a and the following data

$$R = W \quad P_1 = 0.2W \quad P_2 = 0.8W \quad d_1 = 14\,\text{ft} \quad L = 40\,\text{ft}$$

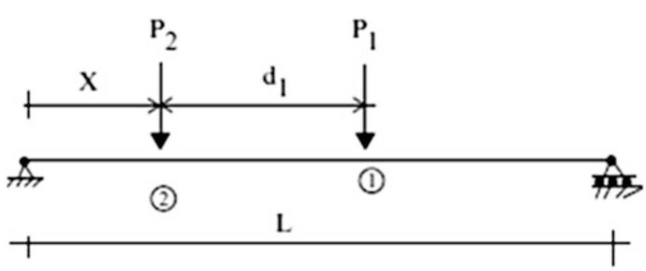


Fig. E3.29a

Determine: The maximum possible moment in the beam as the two-force loading system moves across the span.

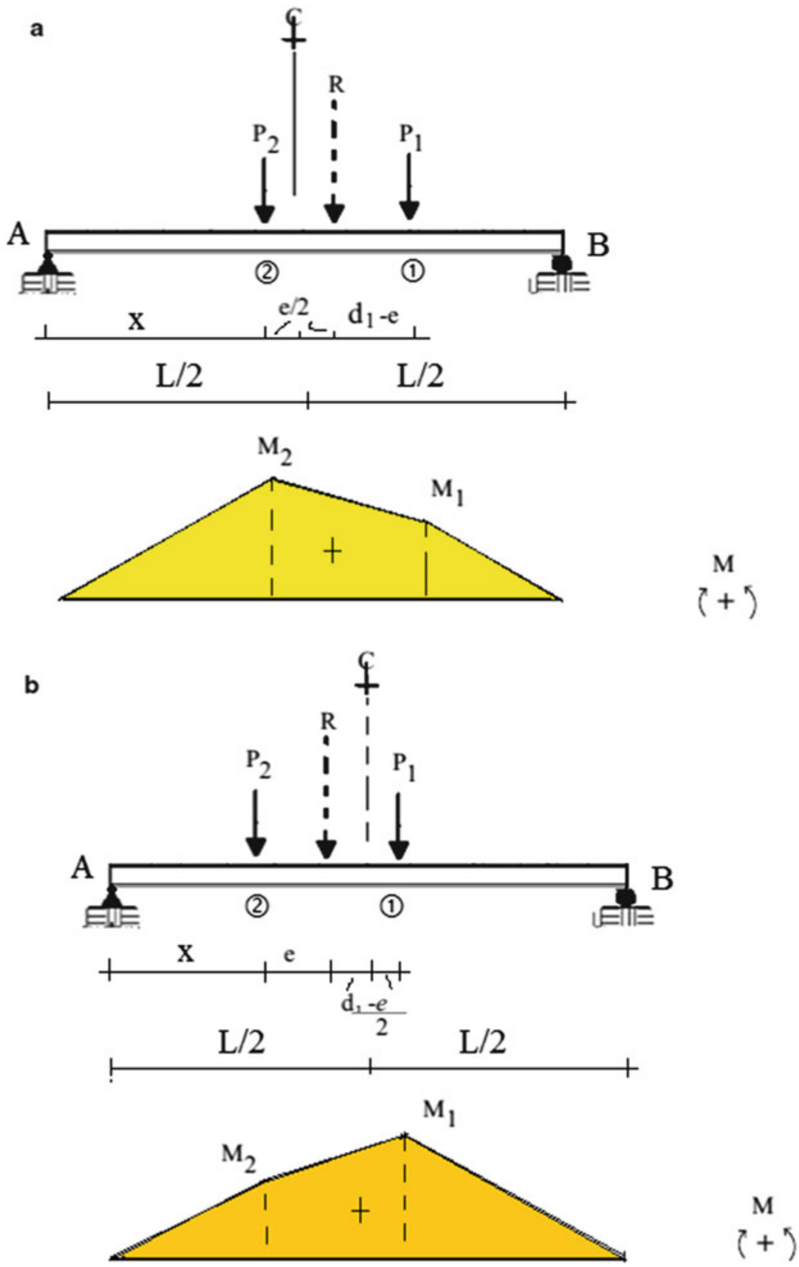
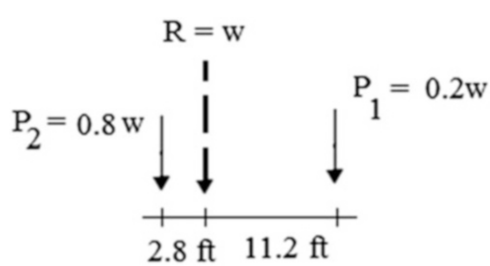


Fig. 3.53 Critical location of loading for maximum bending moments. (**a**) $x|_{M_{2\max}}$. (**b**) $x|_{M_{1\max}}$

Solution: The resultant is located $e = \frac{0.2W}{W}(14) = 2.8\,\text{ft}$ from P_2.



Using (3.60)

$$x|_{M_{2\max}} = \frac{L}{2} - \frac{e}{2} = 20 - 1.4 = 18.6\,\text{ft}$$

Using (3.59) and the above value for x, the reactions and bending moments are

$$R_A = 0.465W$$
$$R_B = 0.535W$$
$$M_1 = 3.96W$$
$$M_2 = 8.69W$$

The critical loading position for M_2 is shown in Fig. E3.29b.

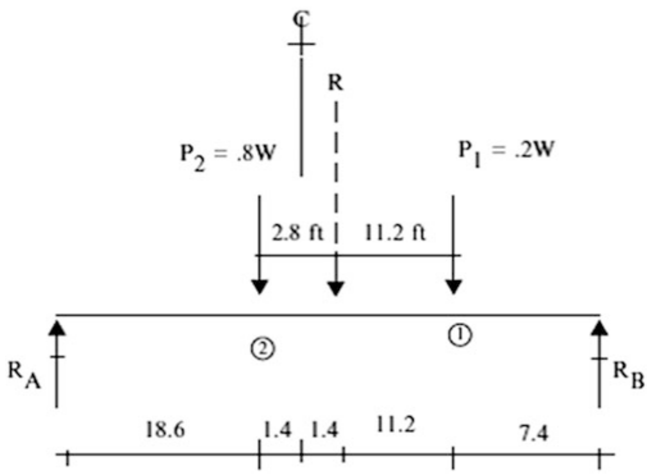


Fig. E3.29b

We compute M_1 in a similar way. The critical location is found using (3.61).

$$x|_{M_{1\max}} = \frac{L}{2} - \frac{d_1}{2} - \frac{e}{2} = 20 - 7 - 1.4 = 11.6\,\text{ft}$$

Next, we apply (3.59).

$$R_A = 0.64W$$
$$R_B = 0.36W$$
$$M_1 = 5.184W$$
$$M_2 = 7.424W$$

The critical loading position for M_1 is shown in Fig. E3.29c.

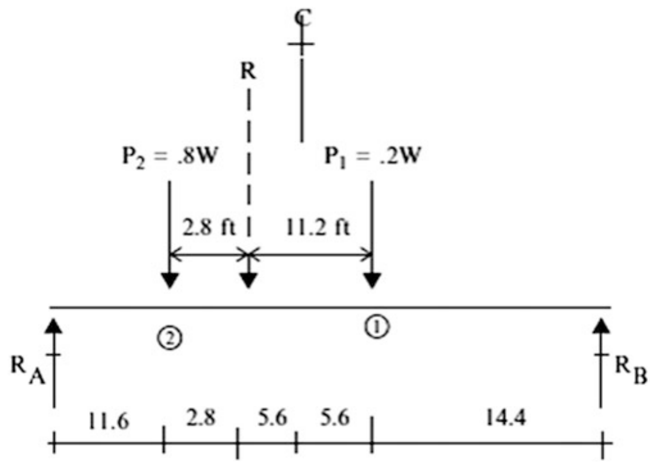


Fig. E3.29c

It follows that the absolute maximum live load moment occurs when P_2 is positioned 18.6 ft from the left support. This point is close to mid-span (Fig. E3.29d).

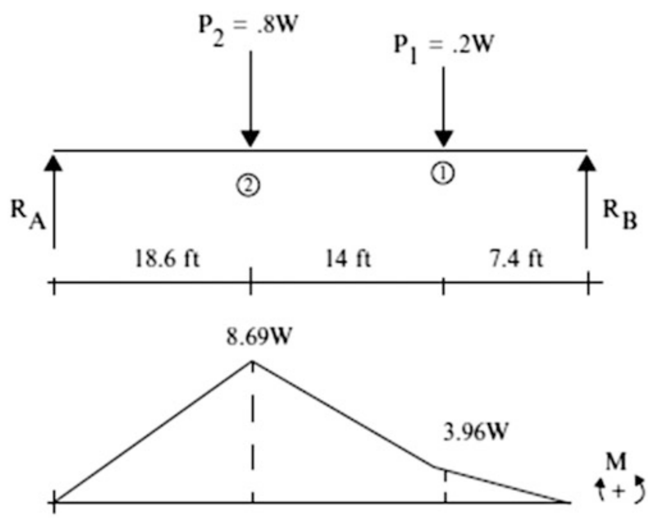


Fig. E3.29d

The analysis for the case of three concentrated loads proceeds in a similar way. Figure 3.54 shows the notation used to define the loading and the location of the resultant force. The moment diagram is piecewise linear with peaks at the point of application of the concentrated loads.

We generate expressions for the bending moments at points ①, ①, and ① for an arbitrary position of the loading defined by x and then determine the locations of maximum moment by differentiating these expressions. First, we locate the resultant force

$$\begin{aligned} R &= P_1 + P_2 + P_3 \\ e &= \frac{d_2 P_2 + (d_1 + d_2) P_1}{R} \end{aligned} \tag{3.62}$$

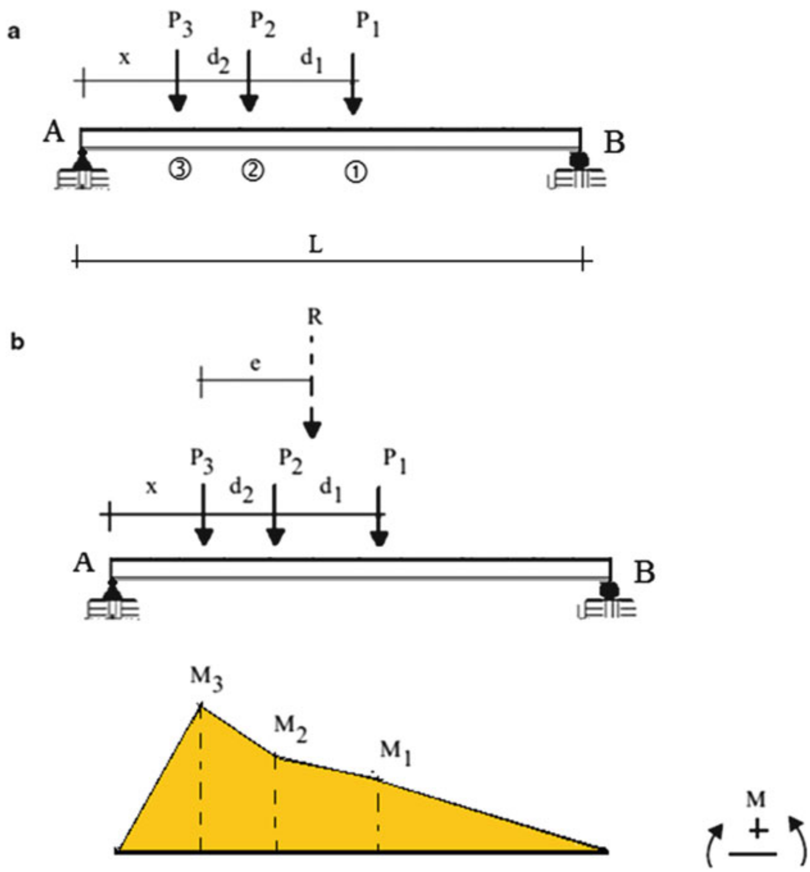


Fig. 3.54 Notation and moment diagram—three concentrated loads

The moments at locations 1, 2, and 3 are functions of x.

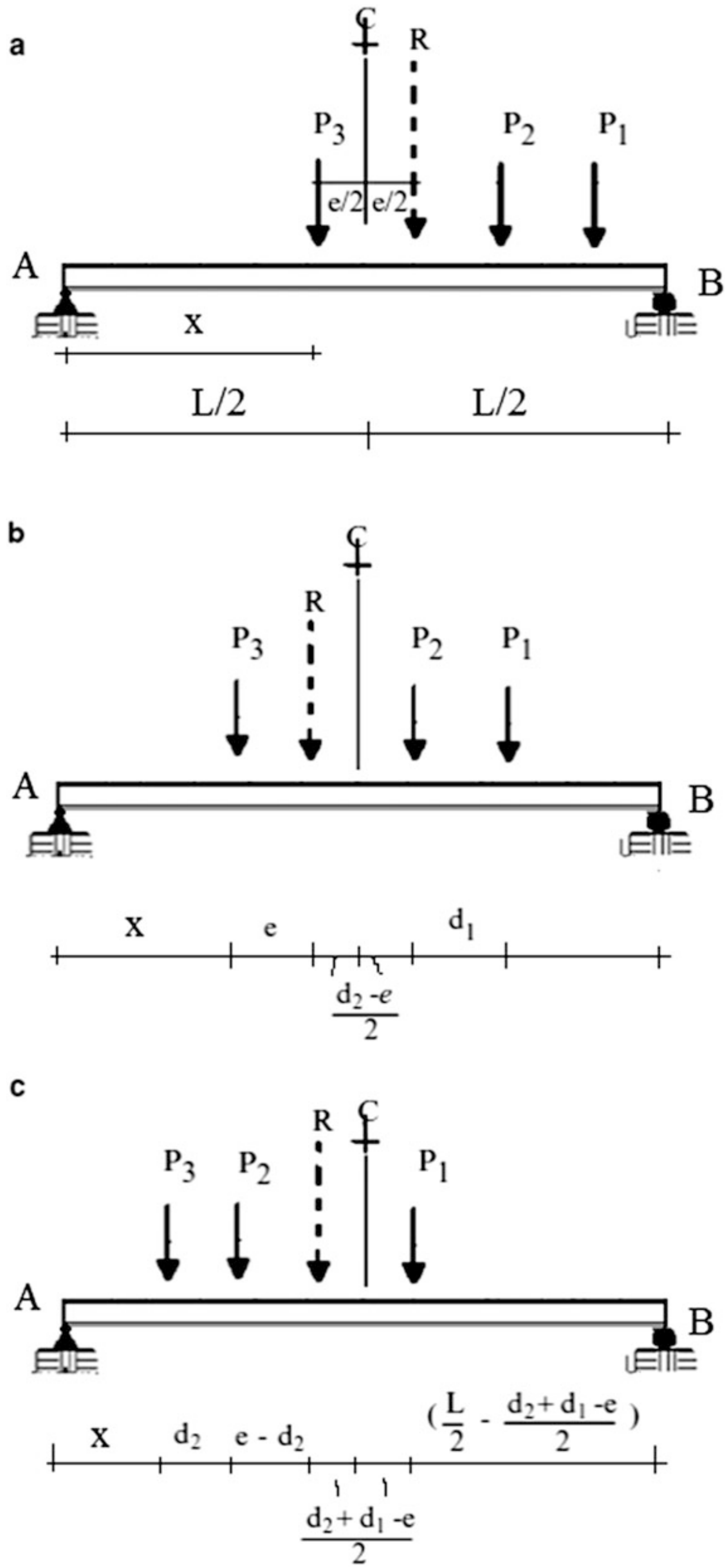
$$
\begin{aligned}
M_3 &= \frac{R}{L}(L - x - e)x \\
M_2 &= \frac{R}{L}(L - x - e)(x + d_2) - P_3 d_2 \\
M_1 &= \frac{R}{L}(L - x - d_2 - d_1)(x + e)
\end{aligned}
\tag{3.63}
$$

Differentiating each expression with respect to x and equating the result to zero leads to the equations for the critical values of x that correspond to relative maximum values of the moments.

$$
\begin{aligned}
&\text{For} \quad M_3|_{\max} \quad x = \frac{1}{2}(L - e) \\
&\text{For} \quad M_2|_{\max} \quad x = \frac{1}{2}(L - e - d_2) \\
&\text{For} \quad M_1|_{\max} \quad x = \frac{1}{2}(L - d_2 - d_1 - e)
\end{aligned}
\tag{3.64}
$$

The positions of the loading corresponding to these three values of x are plotted in Fig. 3.55. Note that the results are similar to the two concentrated load case. We need to evaluate (3.63) for each value of x in order to establish the absolute maximum value of the bending moment.

Fig. 3.55 Possible locations of loading for maximum moment



Example 3.30

Given: The beam shown in Fig. E3.30a.

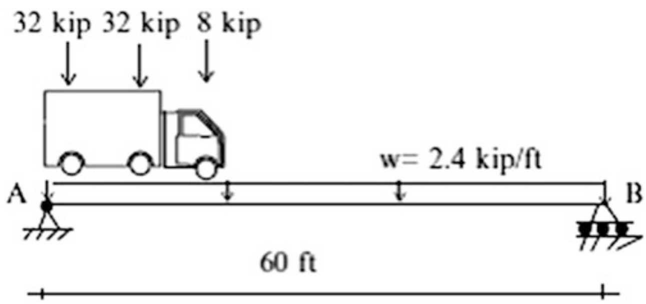


Fig. E3.30a

Determine: The maximum possible moment in the beam caused by

1. A truck moving across the span (Fig. E3.30b).

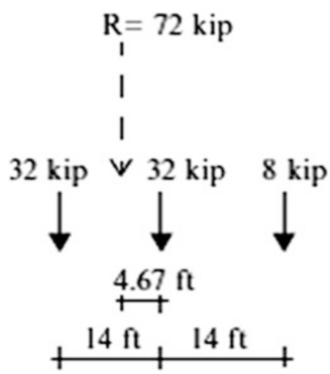


Fig. E3.30b

2. A uniformly distributed dead load of $w = 2.4$ kip/ft in addition to the truck loading.

Solution:

Part (1): The critical truck loading position is defined by Fig. E3.30c. The corresponding bending moment diagram is plotted below; the maximum moment occurs 2.3 ft from the center of the span. $M_{\max} = 806.7$ kip ft.

Part (2): The bending moment diagram for uniform loading is parabolic, with a maximum value at mid-span.

$$M_{\text{dead}}(x) = 72x - 1.2x^2 \quad 0 \le x \le 60$$

We estimate the peak moment due to the combined loading by adding corresponding moment values from Figs. E3.30c and E3.30d.

$$\left.\begin{aligned} M_{\text{combined}} &= (M_{\text{dead}} + M_{\text{truck}})_{\text{at}\,x=32.33\,\text{ft}} = 1073.5 + 806.7 \approx 1880\,\text{kip ft} \\ M_{\text{combined}} &= (M_{\text{dead}} + M_{\text{truck}})_{\text{at}\,x=30\,\text{ft}} = 1080 + 791 \approx 1871\,\text{kip ft} \end{aligned}\right\} M_{\max} = 1880\,\text{kip ft}$$

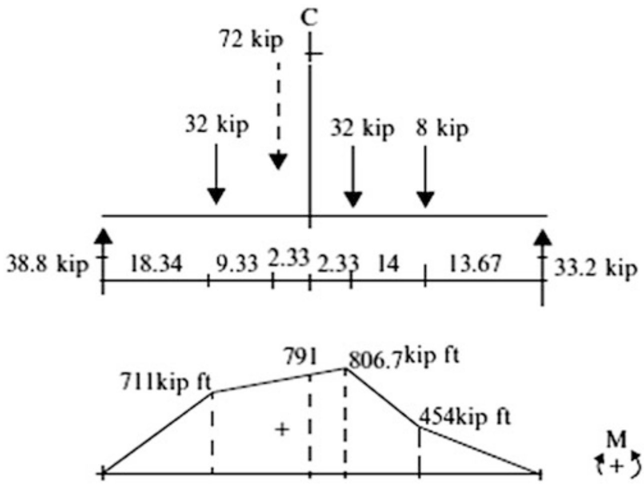


Fig. E3.30c Moment distribution for moving truck load

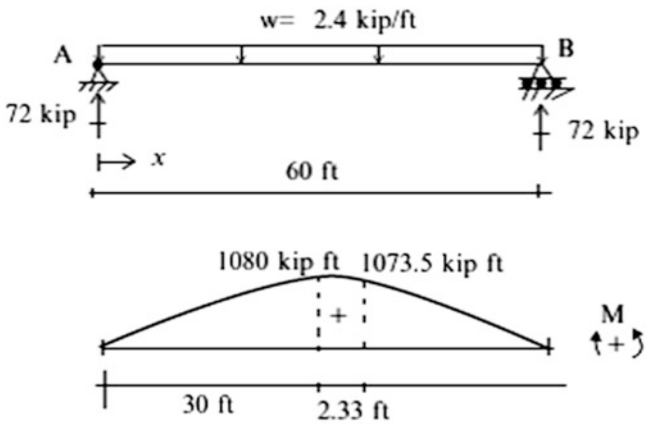


Fig. E3.30d Moment distribution for dead load

When there are multiple loadings, it is more convenient to generate discrete moment envelope using a computer-based analysis system. The discrete moment envelope for the combined (dead + truck) loading is plotted below (Fig. E3.30e). Scanning the envelope shows that the maximum moment occurs at $x = 30.9$ ft and $M_{\max} = 1882.6$ kip ft. This result shows that it was reasonable to superimpose the moment diagrams in this example.

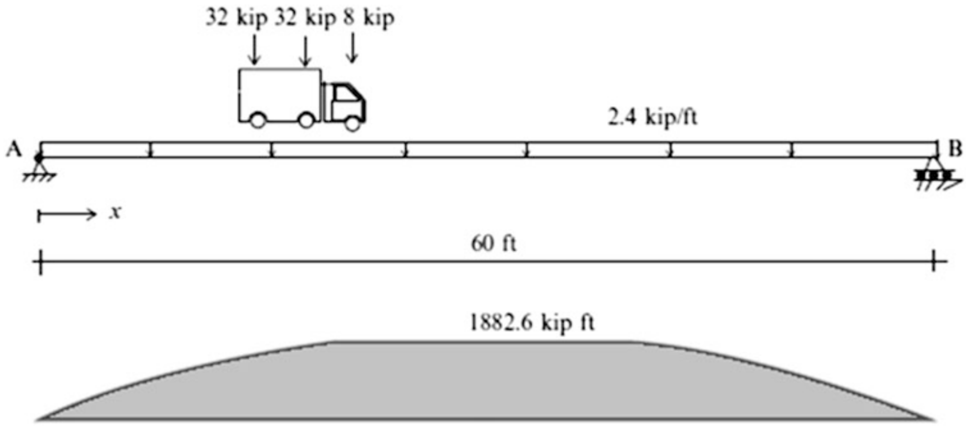


Fig. E3.30e Discrete moment envelope for the combined (dead + truck) loading

3.11 Summary

3.11.1 Objectives of the Chapter

- To develop analytical and computational methods for quantifying the behavior of beams subjected to transverse loading. Of particular interest are the reactions, the internal forces (shear, bending, and twisting moments), and the displacements.

- To introduce the concepts of influence lines and force envelopes which are needed to establish design values for beam cross-sections.

3.11.2 Key Facts and Concepts

- A stable statically determinate beam requires three nonconcurrent displacement restraints. There are three reaction forces which are determined using the static equilibrium equations.

- External loads are resisted by internal forces acting on a cross-section. For planar loading, these quantities consist of an axial force, F, a transverse shear force, V, and a bending moment, M. One can establish the magnitude of these variables using the static equilibrium equations. Alternatively, one can start with the following differential equilibrium equations,

$$\frac{\mathrm{d}V}{\mathrm{d}x} = w$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -V$$

Integrating between points 1 and 2 leads to

$$V_2 - V_1 = \int_{x_1}^{x_2} w\,\mathrm{d}x$$
$$M_2 - M_1 = -\int_{x_1}^{x_2} V\,\mathrm{d}x$$

The first equation states that the difference in shear is equal to the area under the loading diagram. The second equation states that the change in moment is equal to minus the area under the shear diagram.

- Planar bending results in a transverse displacement, $v(x)$. When the beam is slender, these variables are related by

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{M}{EI}$$

where I is the second moment of area for the section. Given $M(x)$, one determines $v(x)$ by integrating this expression and noting the two boundary conditions on v.

- The transverse displacement at a particular point can also be determined using the Principle of Virtual Forces specialized for planar bending of slender beams.

$$d\,\delta P = \int_L \frac{M}{EI}\delta M\,\mathrm{d}x$$

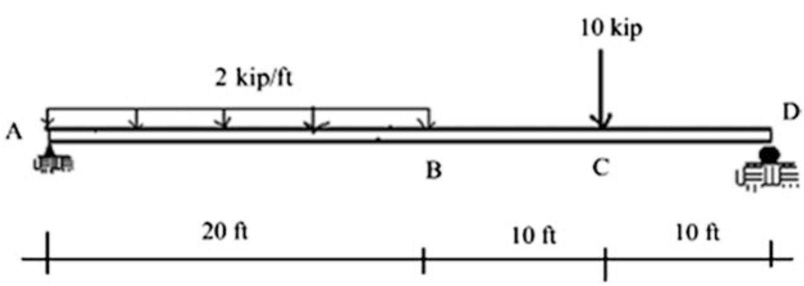
Here, d is the desired displacement, δP is a virtual force in the direction of d, and δM is the virtual moment corresponding to δP. One usually employs numerical integration when the integral is complex.

- An influence line is a plot of the magnitude of a particular internal force quantity, say the bending moment at a specific location, vs. the position of a unit concentrated load as it moves across the span. It is useful for establishing the peak magnitude of the force quantity at that location.
- A force envelope is a plot of the maximum positive and negative values of a force quantity, say the bending moment, at different sections along the beam. This data is used to determine cross-sectional properties.

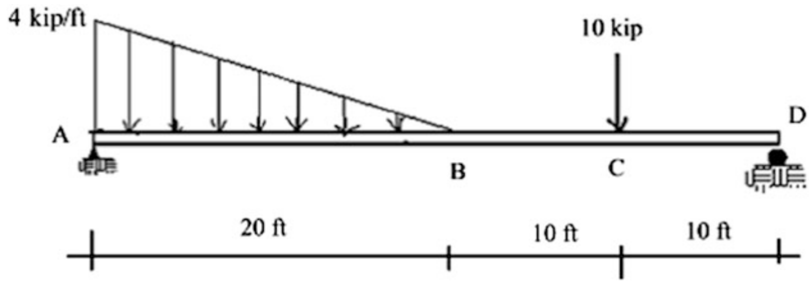
3.12 Problems

For the beams defined in Problems 3.1–3.20, compute the reactions and draw the shear and moment diagrams.

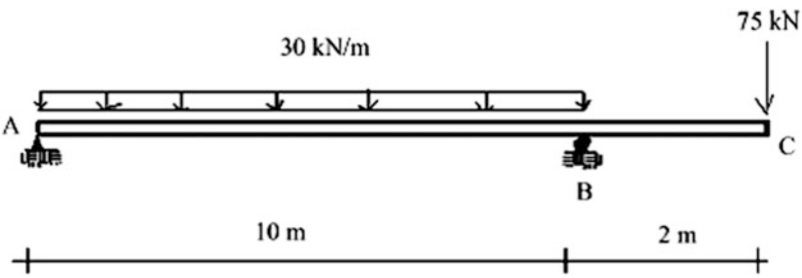
Problem 3.1



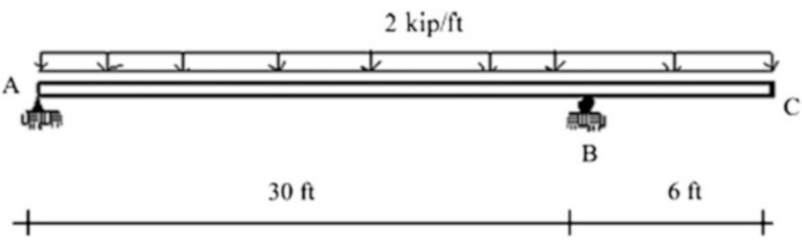
Problem 3.2



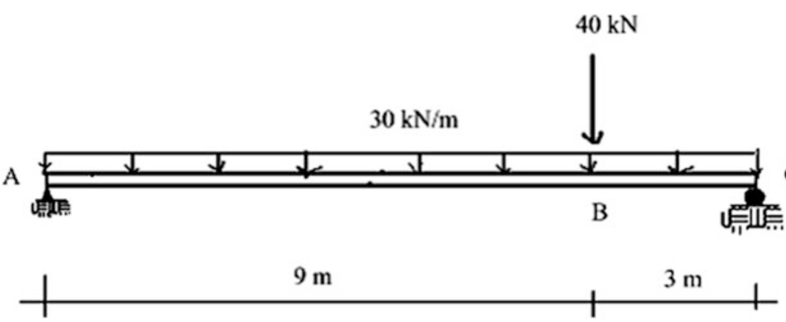
Problem 3.3



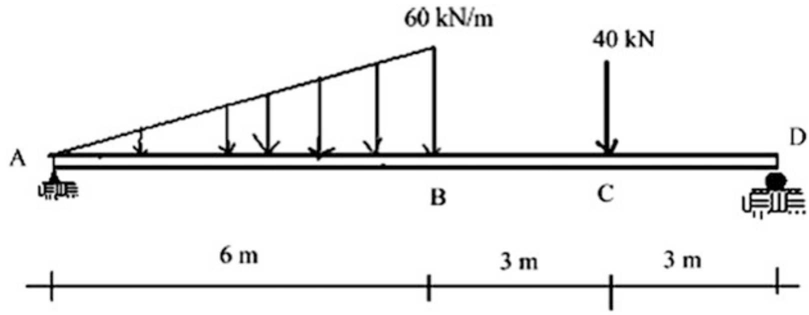
Problem 3.4



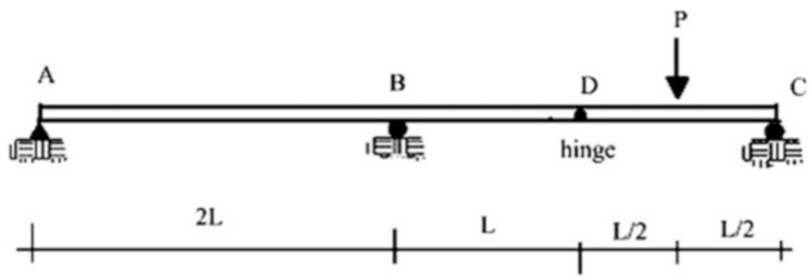
Problem 3.5



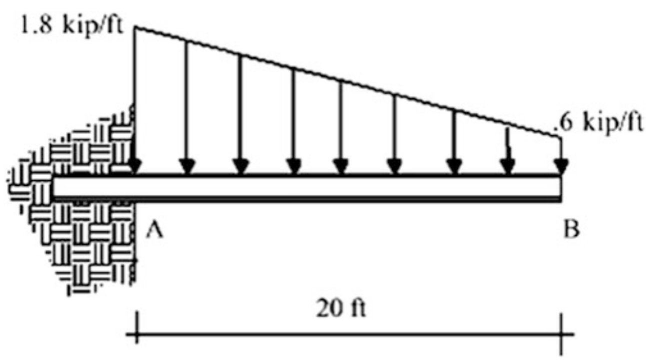
Problem 3.6



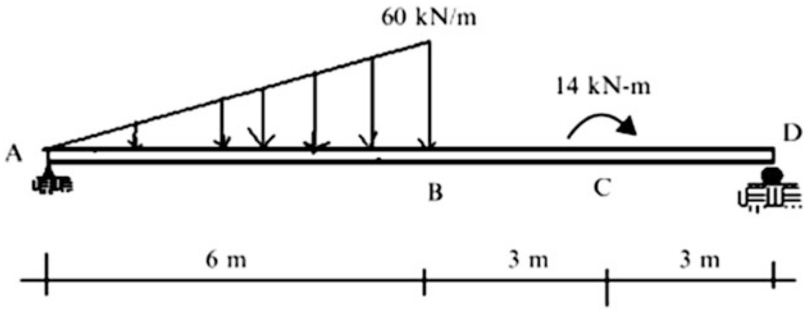
Problem 3.7



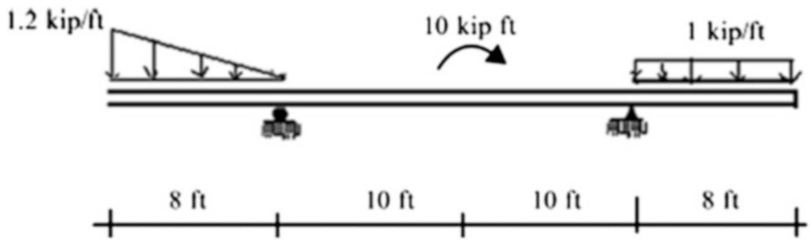
Problem 3.8



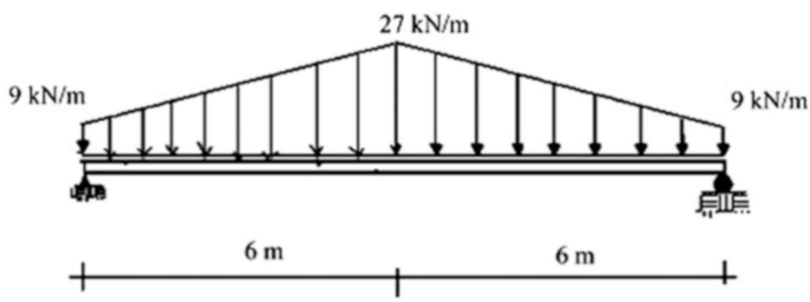
Problem 3.9



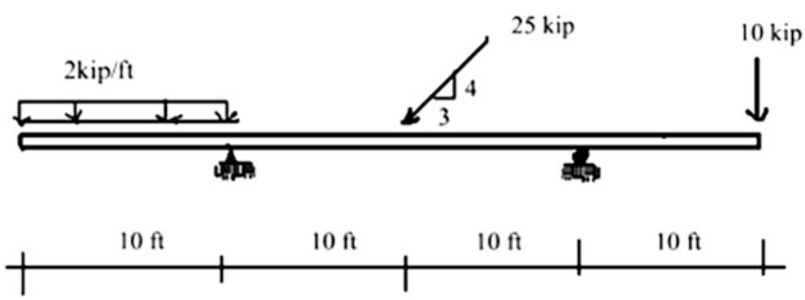
Problem 3.10



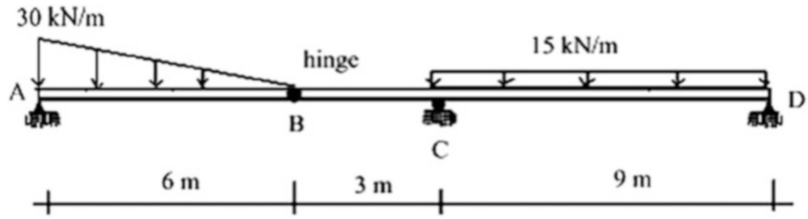
Problem 3.11



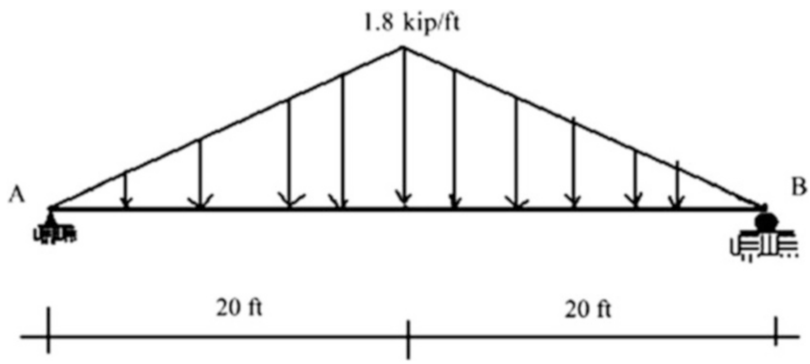
Problem 3.12



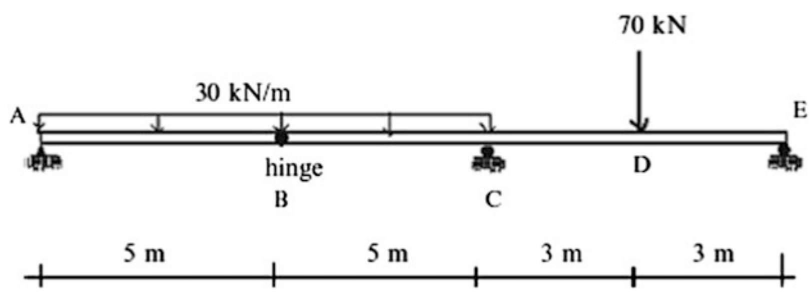
Problem 3.13



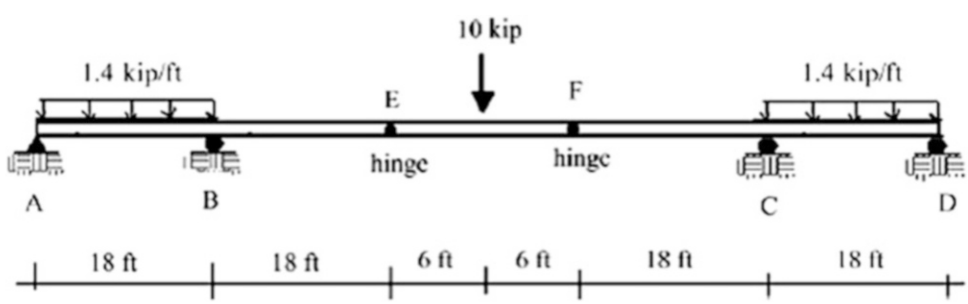
Problem 3.14



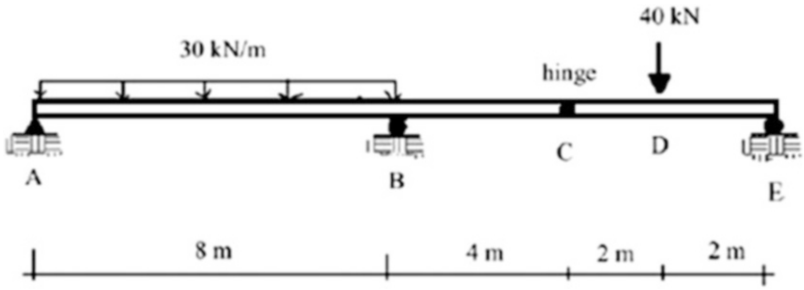
Problem 3.15



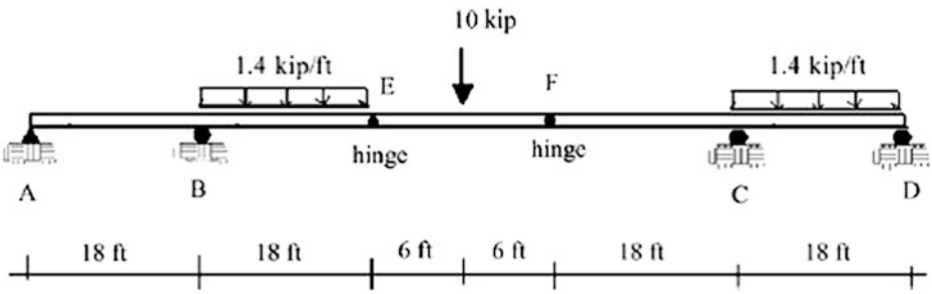
Problem 3.16



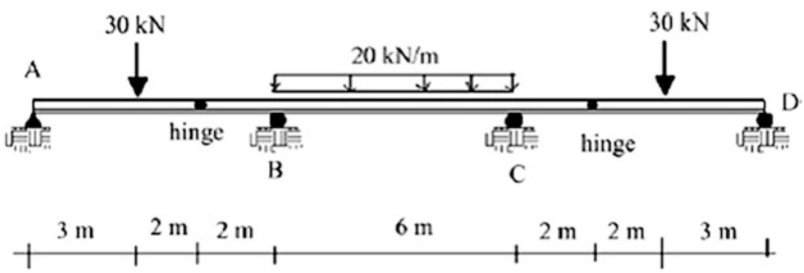
Problem 3.17



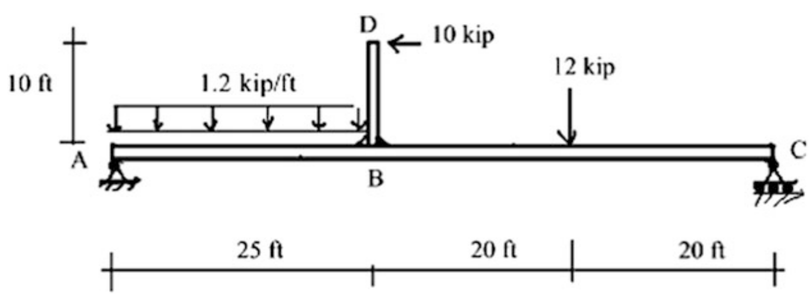
Problem 3.18



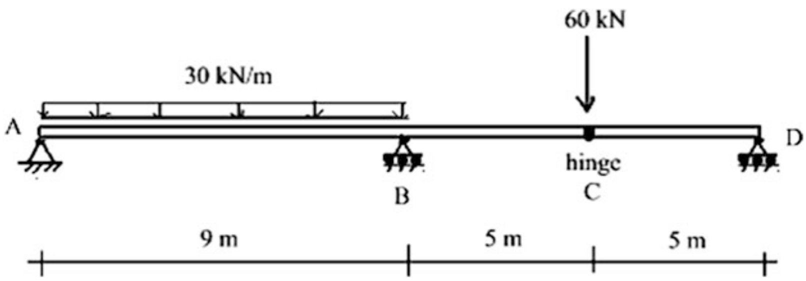
Problem 3.19



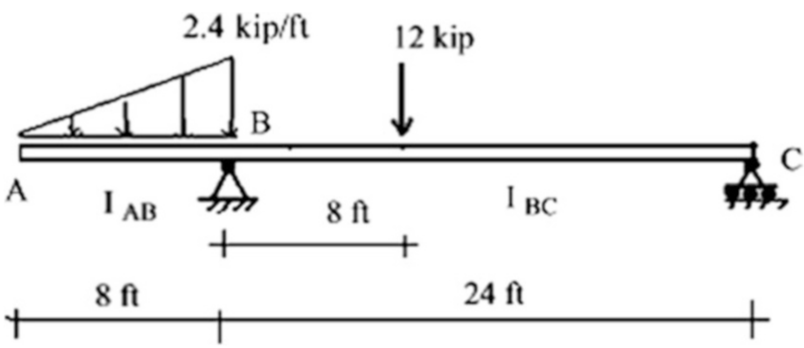
Problem 3.20 Member BD is rigidly attached to the beam at B.



Problem 3.21 Determine the maximum bending moment. Assume EI is constant.



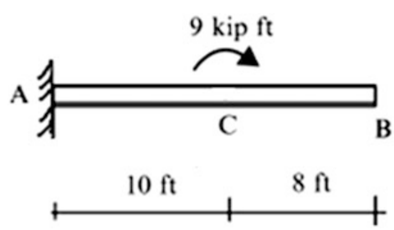
Problem 3.22 Determine the maximum bending moment. Does the bending moment distribution depend on either E or I? Justify your response.



For the beams defined in Problems 3.23–3.26, use the Table 3.1 to determine the vertical deflection and rotation measures indicated. Assume EI is constant.

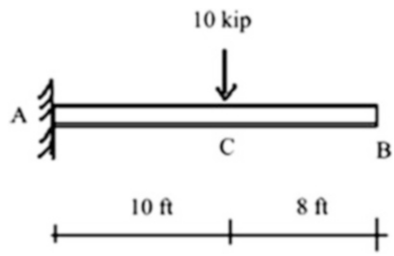
Problem 3.23

$$\theta_B, v_B I = 200\,\text{in.}^4, \quad E = 29,000\,\text{kip/in}^2$$



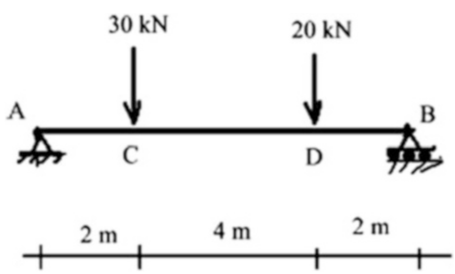
Problem 3.24

$$\theta_B, v_B \quad I = 200\,\text{in.}^4, \quad E = 29,000\,\text{ksi}$$



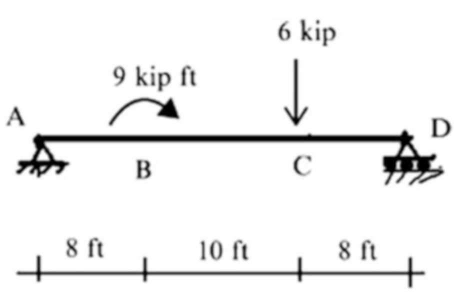
Problem 3.25

$$\theta_A, v_C \quad I = 80\left(10^6\right)\text{mm}^4, \quad E = 200\,\text{GPa}$$



Problem 3.26

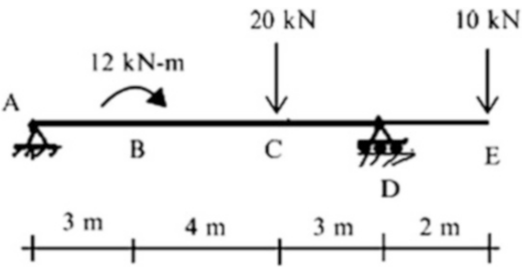
$$\theta_D, v_C \quad I = 200\,\text{in.}^4, \quad E = 29,000\,\text{ksi}$$



For the beams defined in Problems 3.27–3.35, use the virtual force method to determine the vertical deflection and rotation measures indicated.

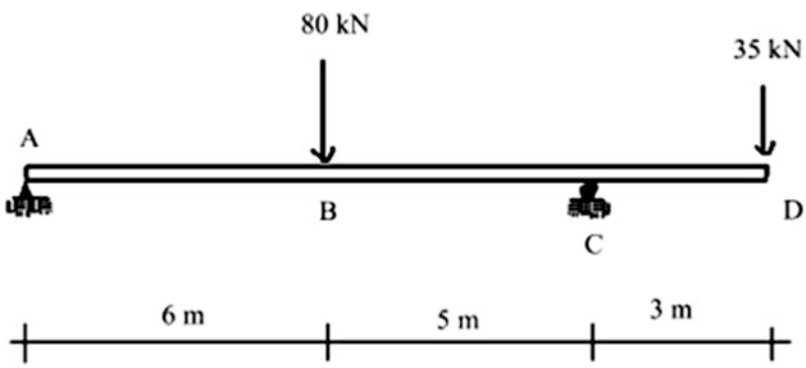
Problem 3.27

$$\theta_{\mathrm{D}}, v_{\mathrm{C}} I = 80\left(10^{6}\right)\mathrm{mm}^{4}, \quad E = 200\,\mathrm{GPa}$$



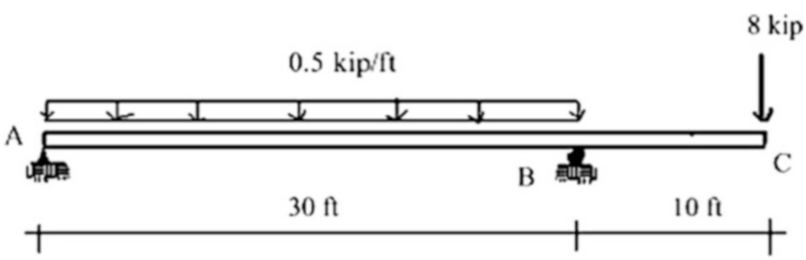
Problem 3.28

$$\theta_{\mathrm{B}}, v_{\mathrm{D}} I = 120\left(10^{6}\right)\mathrm{mm}^{4}, \quad E = 200\,\mathrm{GPa}$$



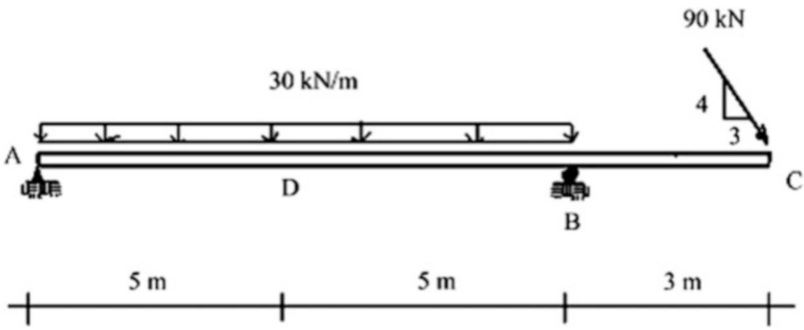
Problem 3.29

$$\theta_{\mathrm{A}}, v_{\mathrm{C}} I = 300\,\mathrm{in.}^{4}, \quad E = 29,000\,\mathrm{ksi}$$



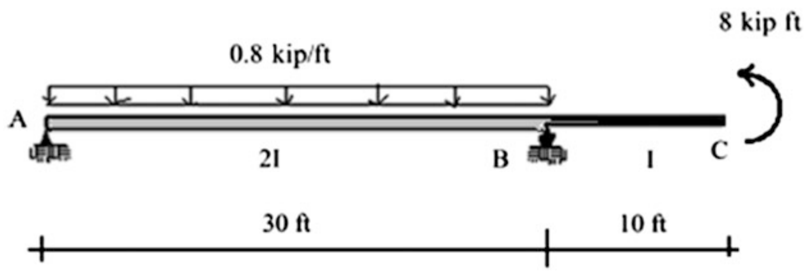
Problem 3.30

$$\theta_C, v_D I = 120\left(10^6\right)\text{mm}^4, \quad E = 200\,\text{GPa}$$



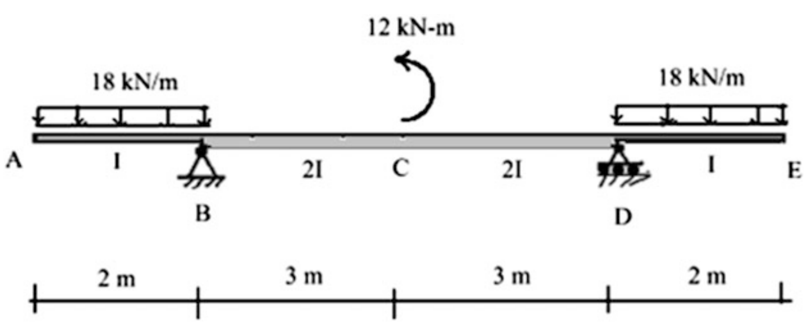
Problem 3.31

$$\theta_C, v_C I = 200\,\text{in.}^4, \quad E = 29,000\,\text{ksi}$$



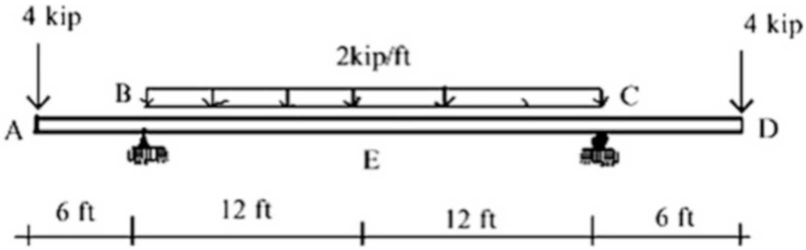
Problem 3.32

$$\theta_C, v_C I = 100\left(10^6\right)\text{mm}^4, \quad E = 200\,\text{GPa}$$



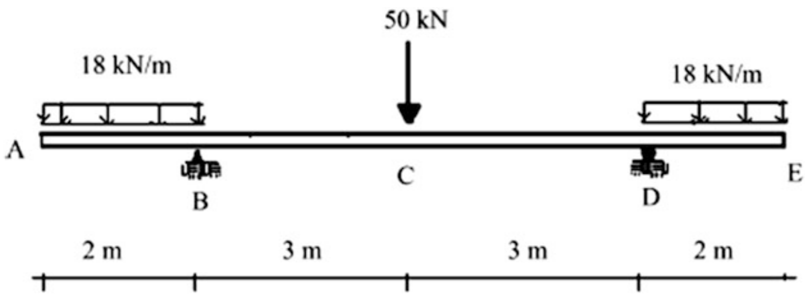
Problem 3.33

$$\theta_B, v_E I = 300\,\text{in.}^4, \quad E = 29,000\,\text{ksi}$$



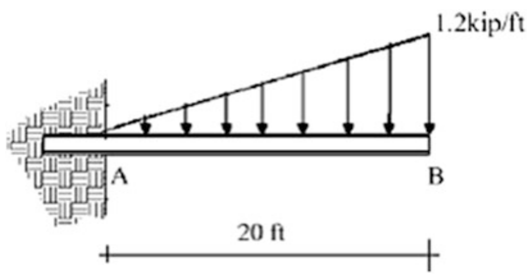
Problem 3.34

$$\theta_C, \theta_E, \text{ and } v_E I = 160\left(10^6\right)\text{mm}^4, \quad E = 200\,\text{GPa}$$



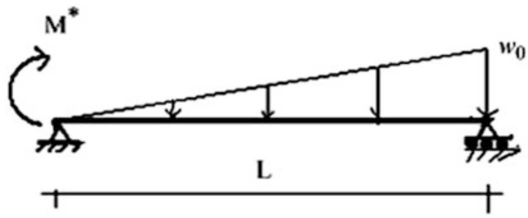
Problem 3.35

$$\theta_B, v_B I = 120\,\text{in.}^4, \quad E = 29,000\,\text{ksi}$$

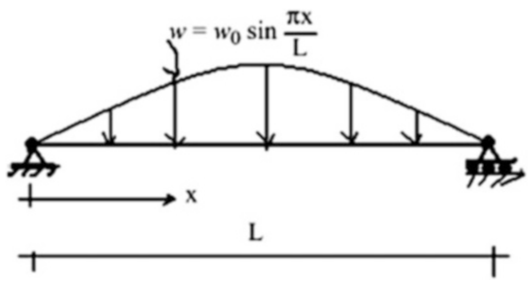


Determine the analytical solutions for the deflected shape for the beams defined in Problems 3.36–3.39. Assume *EI* is constant.

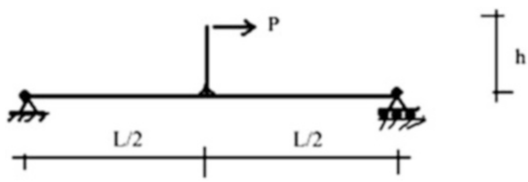
Problem 3.36



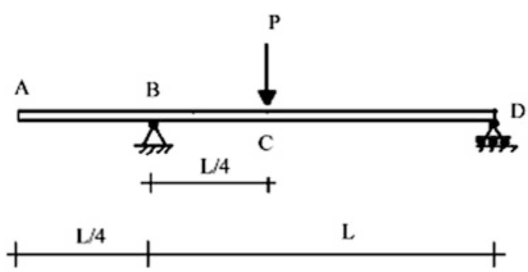
Problem 3.37



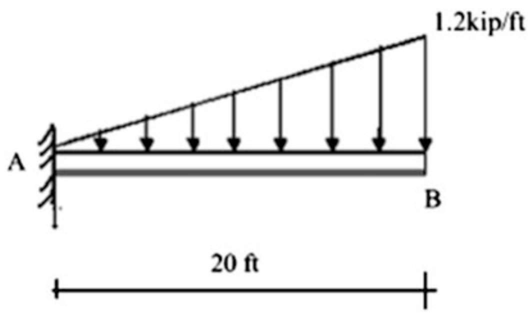
Problem 3.38



Problem 3.39



Problem 3.40 Determine the value of I require to limit the vertical deflection at B to 1/2 in. $E = 29{,}000$ ksi.

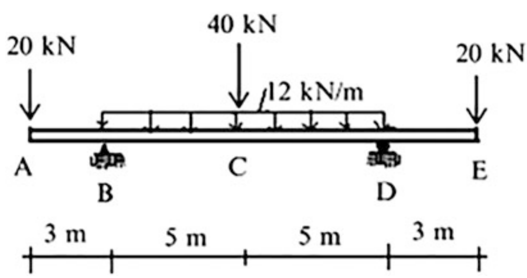


Problem 3.41

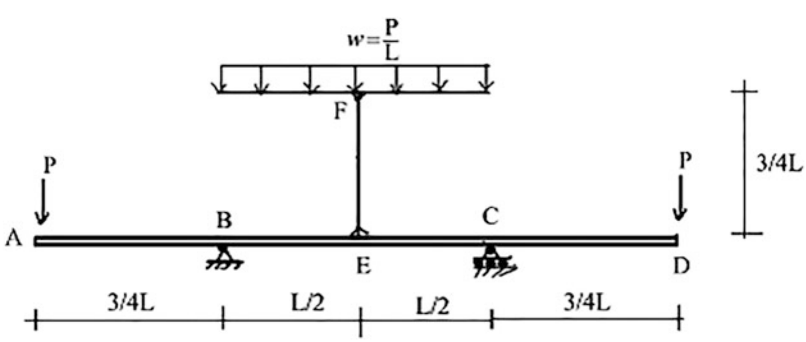
(a) Solve Problem 3.39 using computer software. Consider different sets of values for EI. Show that the magnitude of the deflection varies as $1/EI$. Assume $P = 100$ kN, and $L = 8$ m.

(b) Suppose the peak deflection is specified. How would you determine the appropriate value of I?

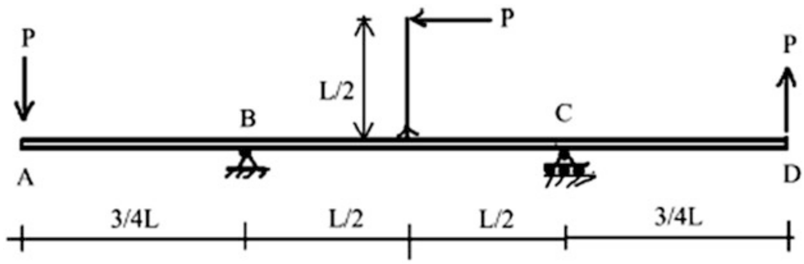
Problem 3.42 Utilize symmetry to sketch the deflected shape. EI is constant. Assume $E = 200$ GPa and $I = 160(10)^6$ mm^4.



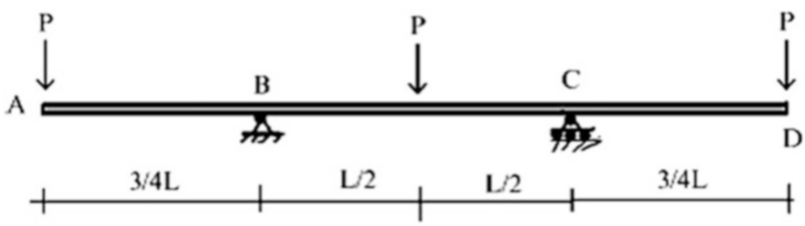
Problem 3.43 Determine the vertical deflection of point A. Sketch the deflected shape of the beam. EI is constant.



Problem 3.44 Determine the vertical deflection of point A. Sketch the deflected shape of the beam. *EI* is constant.

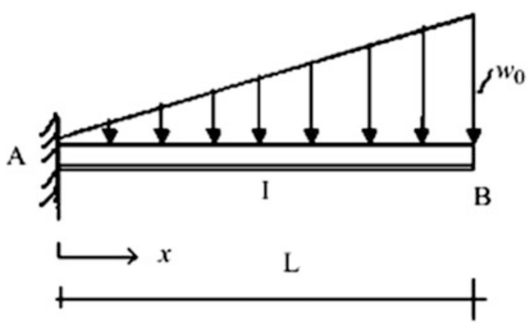


Problem 3.45 Determine the vertical deflection of point A. Sketch the deflected shape. *EI* is constant.

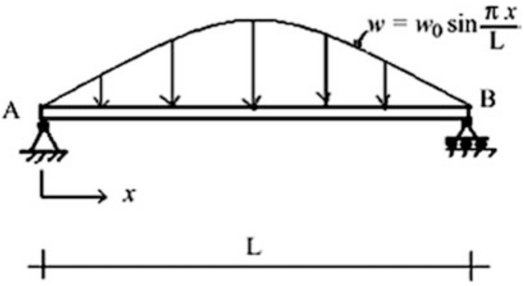


Problem 3.46 Consider the cantilever beam shown below. Determine the displacement at B due to the loading. Use the principle of Virtual Forces and evaluate the corresponding integral with the trapezoidal rule.

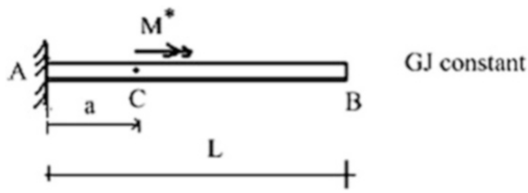
$$\text{Take} \quad w_0 = 10\,\text{kip/ft}, \quad L = 20\,\text{ft}, \quad I_0 = 1000\,\text{in.}^4, E = 29{,}000\,\text{ksi}, \quad I = I_0\left(1 + \cos\frac{\pi x}{2L}\right).$$



Problem 3.47 Assume AB is a "deep" beam. I and A are constant. Determine the analytical solution for β (the rotation of the cross-section) and v.



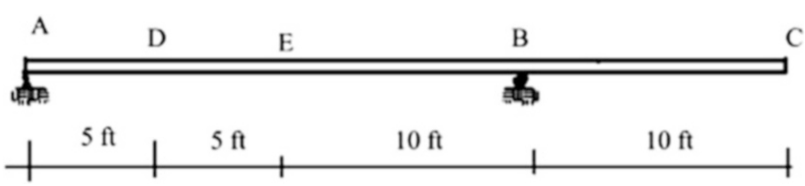
Problem 3.48



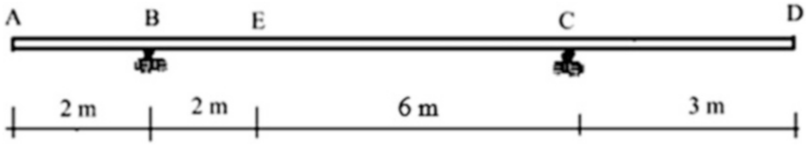
1. Determine β_t (the rotation of the cross-section about the longitudinal axis) at B due to the concentrated torque at C.
2. Suppose a distribution torque, m_t, is applied along A–B. Determine. $M_t(x)$. Take $m_t = \sin\frac{\pi x}{2L}$
3. Determine β_t at B due to the distributed torsional loading.

Problem 3.49 Draw the influence lines for:

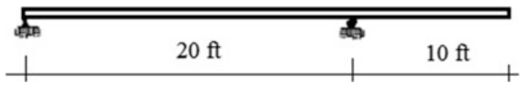
(a) Reaction at A
(b) Moment at E
(c) Shear at D



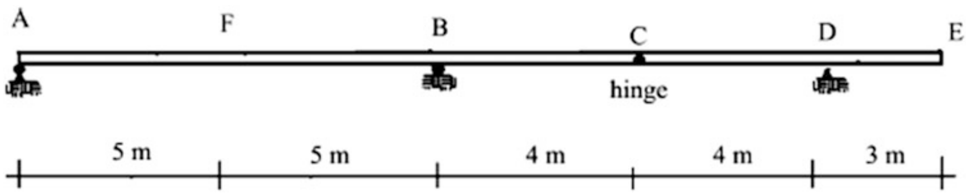
Problem 3.50
Draw the influence lines for the moment and shear at E.



Problem 3.51
For the beams shown, determine the moment envelope corresponding to a single concentrated load moving across the span.

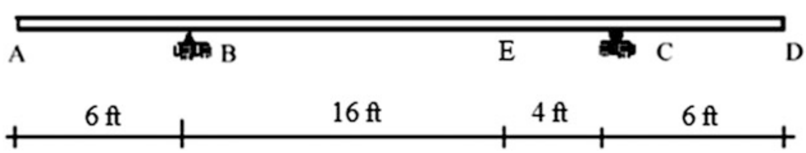


Problem 3.52
(a) Draw the influence lines for moment at F and moment at B.
(b) Draw the moment envelope.

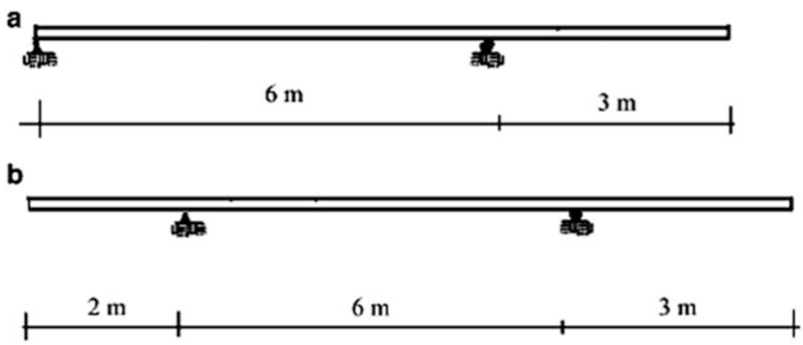


Suppose a uniformly distributed dead load of 18 kN/m and uniformly distributed live load of 30 kN/m are placed on the beam. Use the above results for influence lines to determine the maximum values for the moment at point F and point B. Also show the position of the live load on the beam for these limiting cases.

Problem 3.53 Suppose a uniformly distributed live load of 1.2 kip/ft and uniformly distributed dead load of 0.8 kip/ft are placed on the beam. Determine the critical loading pattern that results in the maximum and minimum values of moment at E.

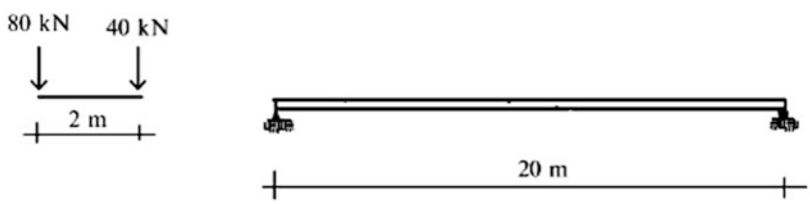


Problem 3.54 For the beams shown, determine the moment envelope corresponding to a single concentrated load moving across the span.

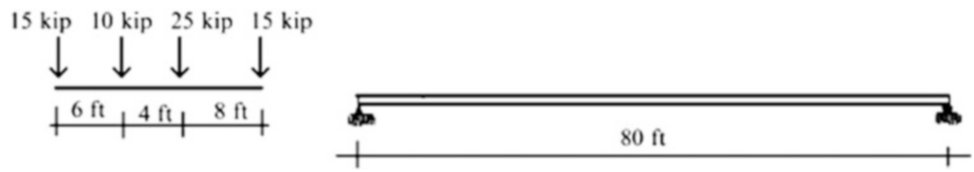


Problem 3.55 Determine the maximum possible moment in the 40 ft span beam as the loading system shown moves across the span. Use either the analytical approach or the moment envelope corresponding to the loading.

Problem 3.56 Determine the location of the maximum possible moment in the 20 m span beam as the loading system shown moves across the span.

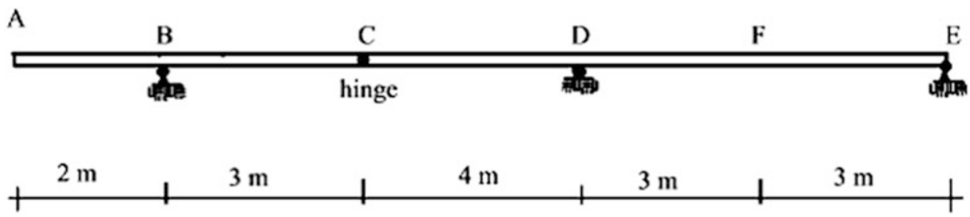


Problem 3.57 Determine the maximum possible moment in a 80 ft span beam as the loading system shown moves across the span. Assume a uniform load of 2 kip/ft also acts on the span. Use computer software.

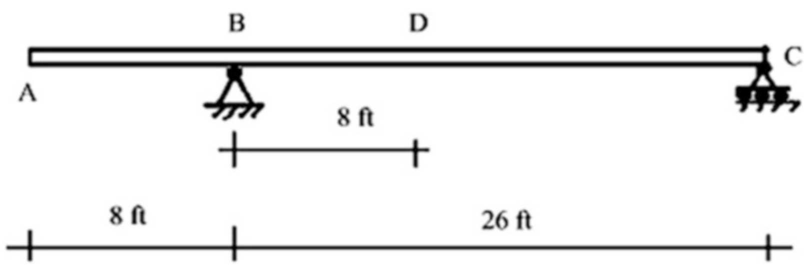


Problem 3.58 For the beam shown:

(a) Draw the influence lines for the vertical reaction at support D, and the moment at point F.
(b) For a uniformly distributed live load of 20 kN/m, use the above results to determine the maximum values of the reaction at D, and the moment at F. Also show the position of the live load on the beam.
(c) Establish the moment envelope corresponding to a single concentrated vertical load.



Problem 3.59 For the beam shown below



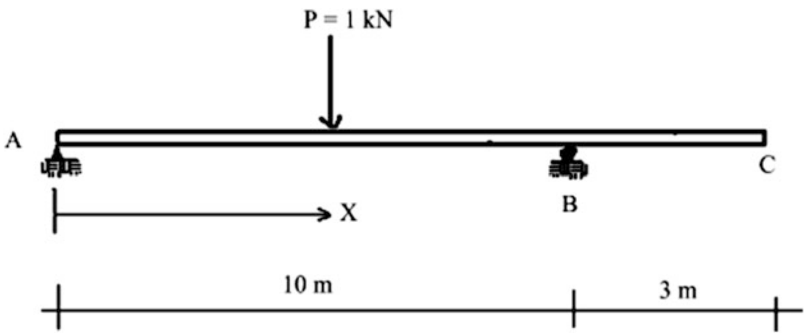
Determine the influence line for:

(a) The vertical reaction at C
(b) The moment at D

If a uniformly distributed live load of 1.8 kip/ft and uniformly distributed dead load of 1.4 kip/ft are placed on the beam, use the above results to determine the maximum and minimum values of

(a) The vertical reaction at C
(b) The moment at D

Problem 3.60 Using computer software, determine the influence line for the vertical displacement at $x = 5$ m. Assume EI is constant.



Hint: Apply a unit load at $x = 5$ m and determine the deflected shape. This is a scaled version of the influence line. Verify by moving the load and recomputing the displacement at $x = 5$ m.

References

1. Tauchert TR. Energy principles in structural mechanics. New York: McGraw-Hill; 1974.
2. Matlab 7.13, Engineering calculations software.