10 The Displacement Method

Abstract

The previous chapter dealt with the force method, one of two procedures for analyzing statically indeterminate structures. In this chapter, we describe the second procedure, referred to as the displacement method. This method works with equilibrium equations expressed in terms of variables that correspond to displacement measures that define the position of a structure, such as translations and rotations of certain points on the structure. We start by briefly introducing the method specialized for frame-type structures and then apply it to truss, beam, and frame structures. Our focus in this chapter is on deriving analytical solutions and using these solutions to explain structural behavior trends. We also include a discussion of the effect of geometrically nonlinear behavior on the stiffness. Later in Chap. 12, we describe how the method can be transformed to a computer-based analysis procedure.

10.1 Introduction

The displacement method works with equilibrium equations expressed in terms of displacement measures. For truss and frame-type structures, which are composed of members connected at node points, the translations and rotations of the nodes are taken as the displacement measures.

Plane truss structures have two displacement measures per node. For example, the plane truss shown in Fig. 10.1a has two unknown displacements (u_2, v_2). The available equilibrium equations are the two force equilibrium equations for node 2.

Planar beam-type structures have two displacement measures per node, the transverse displacement and the cross-section rotation. The corresponding equations are the shear, and moment equilibrium equations for each node. For example, the planar beam shown in Fig. 10.1b has five unknown displacements (θ_1, θ_2, θ_3, θ_4, v_4).

Plane frame-type structures have three displacement measures per node: two translations and one rotation. One works with the force and moment equilibrium equations for each unrestrained node. In general, the number of node equilibrium equations will always be equal to the number of displacements. For example, the plane frame shown in Fig. 10.1c has six unknown displacements (u_2, v_2, θ_2, u_3, v_3, θ_3).

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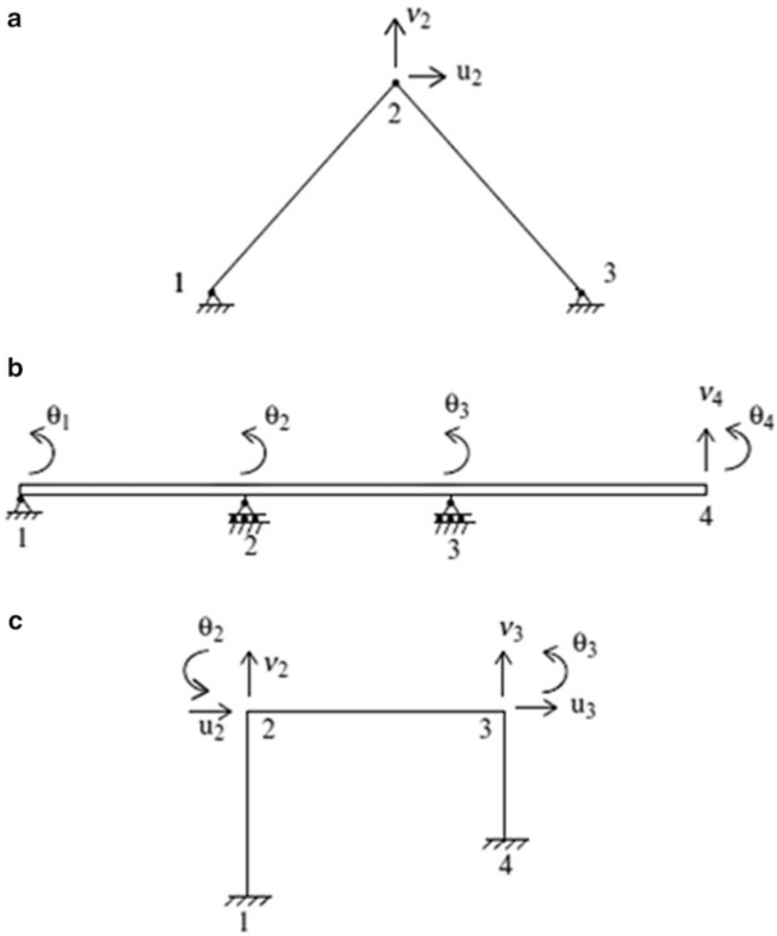


Fig. 10.1 (**a**) Plane truss. (**b**) Planar beam. (**c**) Plane frame

The approach followed to generate equations involves the following steps:

1. Firstly, we decompose the structure into nodes and members. Note that the forces applied by a member to the node at its end are equal in magnitude but oppose in sense to the forces acting on the end of the member. The latter are called end actions.
2. Secondly, we relate the end actions for a member to the displacement measures for the nodes at the ends of the member. We carry out this procedure for each member.
3. Thirdly, we establish the force equilibrium equations for each node. This step involves summing the applied external loads and the end actions for those members which are incident on the node.
4. Fourthly, we substitute for the member end actions expressed in terms of the nodal displacements. This leads to a set of equilibrium equations relating the applied external loads and the nodal displacements.
5. Lastly, we introduce the prescribed values of nodal displacements corresponding to the supports in the equilibrium equations. The total number of unknowns is now reduced by the number of prescribed displacements. We solve this reduced set of equations for the nodal displacements and then use these values to determine the member end actions.

The solution procedure is systematic and is applicable for both statically determinate and statically indeterminate structures. Applications of the method to various types of structure are described in the following sections.

10.2 Displacement Method Applied to a Plane Truss

Consider the truss shown in Fig. 10.2. We suppose nodes 2, 3, and 4 are unyielding. We analyzed this structure with the force method in Sect. 9.6. We include it here to provide a comparison between the two approaches. There are two displacement measures, the horizontal and vertical translations for node 1. The structure is statically indeterminate to the first degree, so it is a trade-off whether one uses the force method or the displacement method.

The first step is to develop the equations relating the member forces and the nodal displacements. We start by expressing the change in length, e, of each member in terms of the displacements for node 1. This analysis is purely geometrical and involves projecting the nodal displacements on the initial direction of the member. We define an extension as positive when the length is increased. Noting Fig. 10.3, the extensions of members (1), (2), and (3) due to nodal displacements are given by:

$$\begin{aligned} e_{(1)} &= u_1 \cos\theta + v_1 \sin\theta \\ e_{(2)} &= v_1 \\ e_{(3)} &= -u_1 \cos\theta + v_1 \sin\theta \end{aligned} \tag{10.1}$$

Next, we express the member force in terms of the corresponding extension using the stress–strain relation for the material. Noting Fig. 10.3b, the generic equations are:

Fig. 10.2 Truss geometry and loading

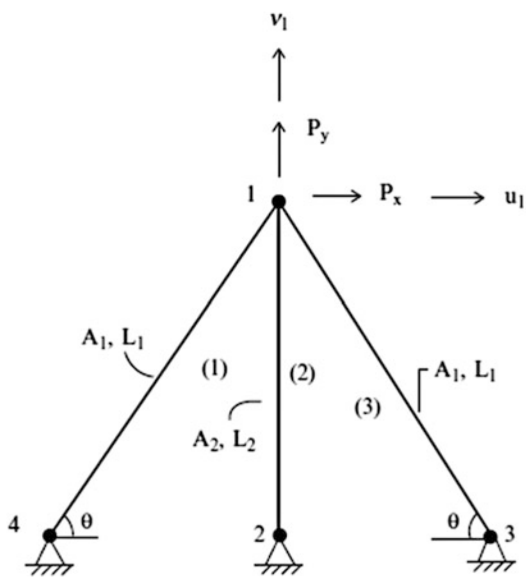
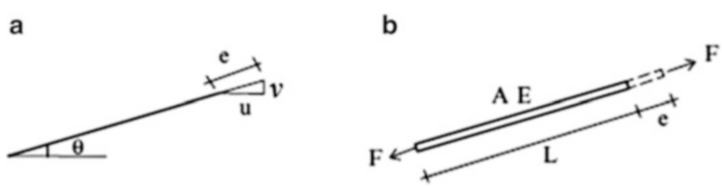


Fig. 10.3 Extension and force quantities—axial loaded member



$$\varepsilon_{\text{total}} = \varepsilon_0 + \frac{1}{E}\sigma = \frac{e}{L}$$
$$\sigma = \frac{F}{A}$$

where ε_0 is the initial strain due to temperature change and fabrication error. Then,

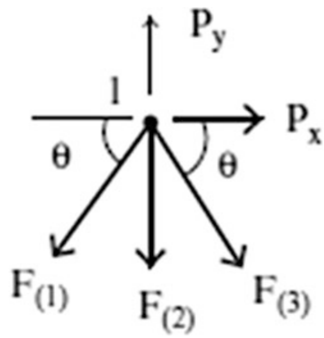
$$\begin{aligned} F &= \frac{AE}{L}e - AE\varepsilon_0 \\ &= \frac{AE}{L}e + F^{\mathrm{F}} \end{aligned} \tag{10.2}$$

where F^{F} is the magnitude of the member force due to initial strain.

Substituting for the extensions leads to the desired expressions relating the member forces and the corresponding nodal displacements.

$$\begin{aligned} F_{(1)} &= \frac{A_1 E}{L_1}\cos\theta u_1 + \frac{A_1 E}{L_1}\sin\theta v_1 + F_{(1)}^{\mathrm{F}} \\ F_{(2)} &= \frac{A_2 E}{L_2} v_1 = \frac{A_2 E}{L_1 \sin\theta} v_1 + F_{(2)}^{\mathrm{F}} \\ F_{(3)} &= -\frac{A_1 E}{L_1}\cos\theta u_1 + \frac{A_1 E}{L_1}\sin\theta v_1 + F_{(3)}^{\mathrm{F}} \end{aligned} \tag{10.3}$$

We generate the force equilibrium equations for node 1 using the free body diagram shown below.



$$\begin{aligned} \sum F_x = 0 \rightarrow \quad & P_x = \cos\theta\left(F_{(1)} - F_{(3)}\right) \\ \sum F_y = 0 \uparrow \quad & P_y = \sin\theta\left(F_{(1)} + F_{(3)}\right) + F_{(2)} \end{aligned} \tag{10.4}$$

Substituting for the member forces, one obtains a set of uncoupled equations for u_1 and v_1.

$$\begin{aligned} P_x &= \left\{\frac{2A_1 E}{L_1}\cos^2\theta\right\} u_1 + \cos\theta\left(F_{(1)}^{\mathrm{F}} - F_{(3)}^{\mathrm{F}}\right) \\ P_y &= \left\{\frac{A_2 E}{L_1 \sin\theta} + \frac{2A_1 E}{L_1}\sin^2\theta\right\} v_1 + \sin\theta\left(F_{(1)}^{\mathrm{F}} + F_{(3)}^{\mathrm{F}}\right) + F_{(2)}^{\mathrm{F}} \end{aligned} \tag{10.5}$$

One solves these equations for u_1 and v_1 and then determines the member forces using (10.3). The resulting expressions are:

$$F_{(1)} = \frac{P_x^*}{2\cos\theta} + P_y\left\{\frac{A_1 \sin\theta}{A_2/\sin\theta + 2A_1 \sin^2\theta}\right\} + F_{(1)}^{\mathrm{F}}$$
$$F_{(2)} = P_y^*\left\{\frac{A_2/\sin\theta}{A_2/\sin\theta + 2A_1 \sin^2\theta}\right\} + F_{(2)}^{\mathrm{F}} \qquad (10.6)$$
$$F_{(3)} = -\frac{P_x^*}{2\cos\theta} + P_y^*\left\{\frac{A_1 \sin\theta}{A_2/\sin\theta + 2A_1 \sin^2\theta}\right\} + F_{(3)}^{\mathrm{F}}$$

where

$$P_x^* = P_x - \cos\theta\left(F_{(1)}^{\mathrm{F}} - F_{(3)}^{\mathrm{F}}\right)$$
$$P_y^* = P_y - \sin\theta\left(F_{(1)}^{\mathrm{F}} + F_{(3)}^{\mathrm{F}}\right) + F_{(2)}^{\mathrm{F}}$$

For this example, it may seem like more effort is required to apply the displacement method vs. the force method (Sect. 9.6). However, the displacement method generates the complete solution, i.e., both the member forces and the nodal displacements. A separate computation is required to compute the displacements when using the force method.

10.3 Member Equations for Frame-Type Structures

The members in frame-type structures are subjected to both bending and axial actions. The key equations for bending behavior of a member are the equations which relate the shear forces and moments acting on the ends of a member to the deflection and rotation of each end. These equations play a very important role in the analysis of statically indeterminate beams and frames and also provide the basis for the matrix formulation of the displacement method for structural frames. In what follows, we develop these equations using the force method.

We consider the structure shown in Fig. 10.4a. We focus specifically on member AB. Both of its ends are rigidly attached to nodes. When the structure is loaded, the nodes displace and the member bends as illustrated in Fig. 10.4b. This motion produces a shear force and moment at each end. The positive sense of these quantities is defined in Figs. 10.4b, c.

We refer to the shear and moment acting at the ends as *end actions*. Our objective here is to relate the end actions (V_{B}, M_{B}, V_{A}, M_{A}) and the end displacements (v_{B}, θ_{B}, v_{A}, θ_{A}). Our approach is based on treating the external loading and end actions as separate loadings and superimposing their responses. We proceed as follows:

Step 1. Firstly, we assume the nodes at A and B are fixed and apply the external loading to member AB. This leads to a set of end actions that we call *fixed end actions*. This step is illustrated in Fig. 10.5.

Step 2. Next, we allow the nodes to displace. This causes additional bending of the member AB resulting in additional end actions (ΔV_{B}, ΔM_{B}, ΔV_{A}, ΔM_{A}). Figure 10.6 illustrates this notation.

Step 3. Superimposing the results obtained in these two steps leads to the final state shown in Fig. 10.7.

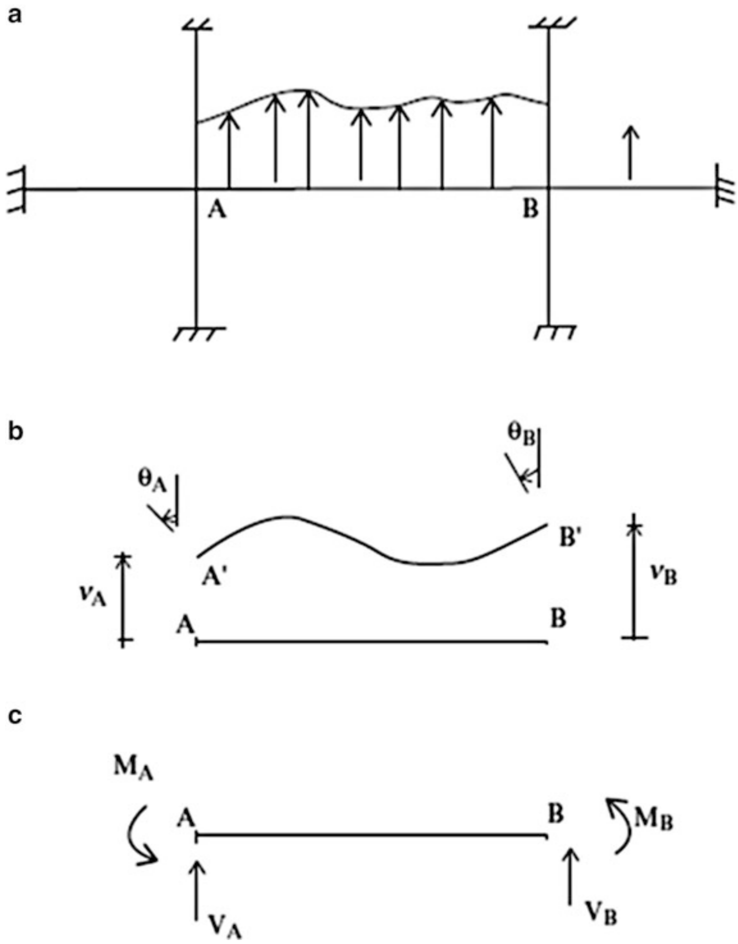


Fig. 10.4 Member deformation and end actions. (**a**) Initial geometry. (**b**) Deformed configuration for member AB. (**c**) Notation for end shear and moment

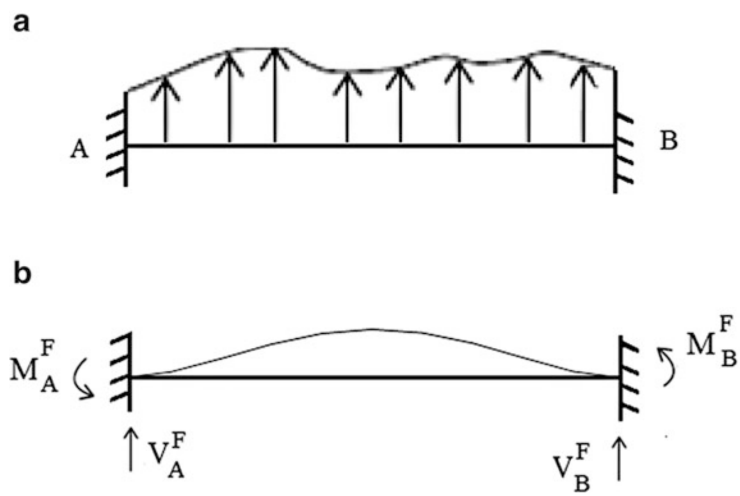


Fig. 10.5 Fixed end Actions. (**a**) Initial. (**b**) Deformed

Fig. 10.6 Response to nodal displacements

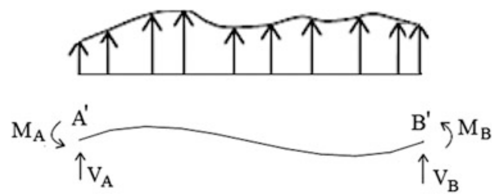


Fig. 10.7 Final state

$$
\begin{aligned}
M_{\mathrm{B}} &= M_{\mathrm{B}}^{\mathrm{F}} + \Delta M_{\mathrm{B}} \\
M_{\mathrm{A}} &= M_{\mathrm{A}}^{\mathrm{F}} + \Delta M_{\mathrm{A}} \\
V_{\mathrm{B}} &= V_{\mathrm{B}}^{\mathrm{F}} + \Delta V_{\mathrm{B}} \\
V_{\mathrm{A}} &= V_{\mathrm{A}}^{\mathrm{F}} + \Delta V_{\mathrm{A}}
\end{aligned}
$$

We determine the fixed end actions corresponding to the first step using the force method. Details are described in Chap. 9. Fixed end actions for various loading cases are listed in Table 9.1.

For the second step, we visualize the process as consisting of two substeps. First, we displace node B holding A fixed. Then, we displace node A, holding B fixed. Combining these cases result in the response shown in Fig. 10.8c. Superposition is valid since the behavior is linear.

These two substeps are similar and can be analyzed using the same procedure. We consider first case (a) shown in Fig. 10.8a. We analyze this case by considering AB to be a cantilever beam fixed at A and subjected to unknown forces, $\Delta V_{\mathrm{B}}{}^{(1)}$ and $\Delta M_{\mathrm{B}}{}^{(1)}$ at B (see Fig. 10.9a).

The displacements at B are (see Table 3.1):

$$
\begin{aligned}
v_{\mathrm{B}} &= \frac{\Delta V_{\mathrm{B}}^{(1)} L^3}{3EI} + \frac{\Delta M_{\mathrm{B}}^{(1)} L^2}{2EI} \\
\theta_{\mathrm{B}} &= \frac{\Delta V_{\mathrm{B}}^{(1)} L^2}{2EI} + \frac{\Delta M_{\mathrm{B}}^{(1)} L}{EI}
\end{aligned}
\tag{10.7}
$$

We determine $\Delta V_{\mathrm{B}}{}^{(1)}$ and $\Delta M_{\mathrm{B}}{}^{1}$ by requiring these displacements to be equal to the actual nodal displacements v_{B} and θ_{B}. Solving for $\Delta V_{\mathrm{B}}{}^{(1)}$ and $\Delta M_{\mathrm{B}}{}^{(1)}$ leads to

$$
\begin{aligned}
\Delta V_{\mathrm{B}}^{(1)} &= \frac{12EI}{L^3} v_{\mathrm{B}} - \frac{6EI}{L^2} \theta_{\mathrm{B}} \\
\Delta M_{\mathrm{B}}^{(1)} &= \frac{4EI}{L} \theta_{\mathrm{B}} - \frac{6EI}{L} v_{\mathrm{B}}
\end{aligned}
\tag{10.8}
$$

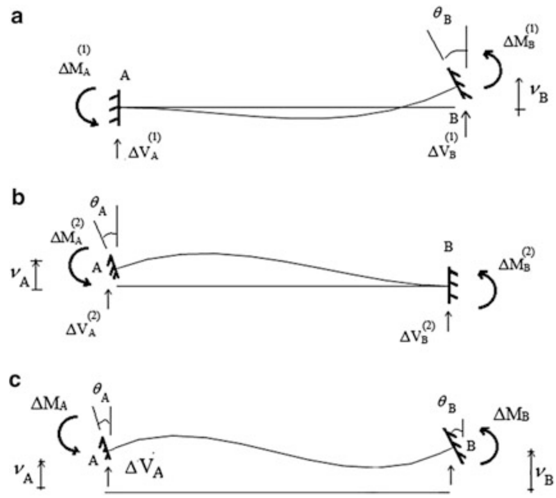


Fig. 10.8 Superposition of nodal motions. (**a**) Support A fixed. (**b**) Support B fixed. (**c**) Superimposed motions

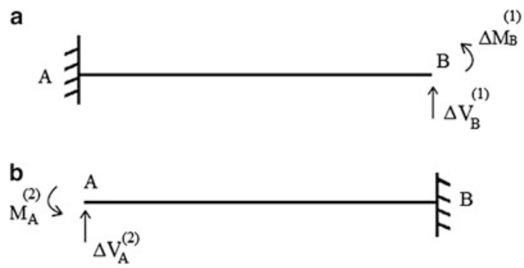


Fig. 10.9 (**a**) Support A fixed. (**b**) Support B fixed

The corresponding end actions at A are determined using the equilibrium conditions for the member.

$$\sum F_y = 0 \Rightarrow \Delta V_B^{(1)} + \Delta V_A^{(1)} = 0$$

$$\sum_{\text{at A}} M = 0 \Rightarrow \Delta M_B^{(1)} + \Delta M_A^{(1)} + L\Delta V_B^{(1)} = 0$$

Then

$$\Delta V_A^{(1)} = -\frac{12EI}{L^3} v_B + \frac{6EI}{L^2}\theta_B$$
$$\Delta M_A^{(1)} = -\frac{6EI}{L^2} v_B + \frac{2EI}{L}\theta_B \tag{10.9}$$

Equations (10.8) and (10.9) define the end actions due to the displacement of node B *with* A *fixed.*

Case (b) of Fig. 10.8 is treated in a similar way (see Fig. 10.9b). One works with a cantilever fixed at B and solves for $\Delta V_A{}^{(2)}$ and $\Delta M_A{}^{(2)}$. The result is

$$\begin{aligned}\Delta V_{\mathrm{A}}^{(2)} &= \frac{12EI}{L^3}v_{\mathrm{A}} + \frac{6EI}{L^2}\theta_{\mathrm{A}}\\ \Delta M_{\mathrm{A}}^{(2)} &= \frac{6EI}{L^2}v_{\mathrm{A}} + \frac{4EI}{L}\theta_{\mathrm{A}}\end{aligned} \tag{10.10}$$

The end actions at B follow from the equilibrium conditions for the member.

$$\begin{aligned}\Delta V_{\mathrm{B}}^{(2)} &= -\frac{12EI}{L^3}v_{\mathrm{A}} - \frac{6EI}{L^2}\theta_{\mathrm{A}}\\ \Delta M_{\mathrm{B}}^{(2)} &= \frac{6EI}{L^2}v_{\mathrm{A}} + \frac{4EI}{L}\theta_{\mathrm{A}}\end{aligned} \tag{10.11}$$

Equations (10.10) and (10.11) define the end actions due to the displacement of node A *with* B *fixed.*

The complete solution is generated by superimposing the results for these two loading conditions and the fixed end actions.

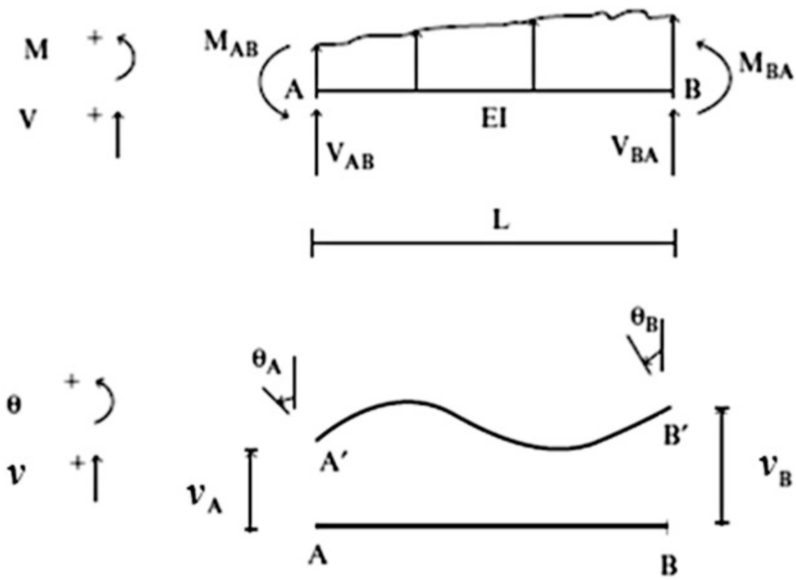
$$\begin{aligned}V_{\mathrm{B}} &= \Delta V_{\mathrm{B}}^{(1)} + \Delta V_{\mathrm{B}}^{(2)} + V_{\mathrm{B}}^{\mathrm{F}} = -\frac{6EI}{L^2}(\theta_{\mathrm{B}} + \theta_{\mathrm{A}}) + \frac{12EI}{L^3}(v_{\mathrm{B}} - v_{\mathrm{A}}) + V_{\mathrm{B}}^{\mathrm{F}}\\ M_{\mathrm{B}} &= \Delta M_{\mathrm{B}}^{(1)} + \Delta M_{\mathrm{B}}^{(2)} + M_{\mathrm{B}}^{\mathrm{F}} = +\frac{2EI}{L}(2\theta_{\mathrm{B}} + \theta_{\mathrm{A}}) - \frac{6EI}{L^2}(v_{\mathrm{B}} - v_{\mathrm{A}}) + M_{\mathrm{B}}^{\mathrm{F}}\\ V_{\mathrm{A}} &= \Delta V_{\mathrm{A}}^{(1)} + \Delta V_{\mathrm{A}}^{(2)} + V_{\mathrm{A}}^{\mathrm{F}} = +\frac{6EI}{L^2}(\theta_{\mathrm{B}} + \theta_{\mathrm{A}}) - \frac{12EI}{L^3}(v_{\mathrm{B}} - v_{\mathrm{A}}) + V_{\mathrm{A}}^{\mathrm{F}}\\ M_{\mathrm{A}} &= \Delta M_{\mathrm{A}}^{(1)} + \Delta M_{\mathrm{A}}^{(2)} + M_{\mathrm{A}}^{\mathrm{F}} = +\frac{2EI}{L}(\theta_{\mathrm{B}} + 2\theta_{\mathrm{A}}) - \frac{6EI}{L^2}(v_{\mathrm{B}} - v_{\mathrm{A}}) + M_{\mathrm{A}}^{\mathrm{F}}\end{aligned}$$

We rearrange these equations according to moment and shear quantities. The final form is written as

$$\begin{aligned}M_{\mathrm{AB}} &= \frac{2EI}{L}\left\{2\theta_{\mathrm{A}} + \theta_{\mathrm{B}} - 3\left(\frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{L}\right)\right\} + M_{\mathrm{AB}}^{\mathrm{F}}\\ M_{\mathrm{BA}} &= \frac{2EI}{L}\left\{\theta_{\mathrm{A}} + 2\theta_{\mathrm{B}} - 3\left(\frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{\mathrm{L}}\right)\right\} + M_{\mathrm{BA}}^{\mathrm{F}}\end{aligned} \tag{10.12a}$$

and

$$V_{AB} = +\frac{6EI}{L^2}\left\{\theta_A + \theta_B - 2\left(\frac{v_B - v_A}{L}\right)\right\} + V^F_{AB}$$
$$V_{BA} = -\frac{6EI}{L^2}\left\{\theta_A + \theta_B - 2\left(\frac{v_B - v_A}{L}\right)\right\} + V^F_{BA} \tag{10.12b}$$



Equations (10.12a, 10.12b) are referred to as the *slope-deflection equations*. They are based on the sign conventions and notation defined above.

10.4 The Displacement Method Applied to Beam Structures

In what follows, we first describe how the slope-deflection equations are employed to analyze horizontal beam structures, starting with two-span beams and then moving on to multi-span beams and frames. The displacement measures for beams are taken as the nodal rotations; the transverse displacements are assumed to be specified.

10.4.1 Two-Span Beams

We consider the two-span beam shown in Fig. 10.10a. One starts by subdividing the beam into two beam segments and three nodes, as indicated in Figs. 10.10b, c. There are only two rotations unknowns: the rotations at nodes A and B; the rotation at node C is considered to be zero.

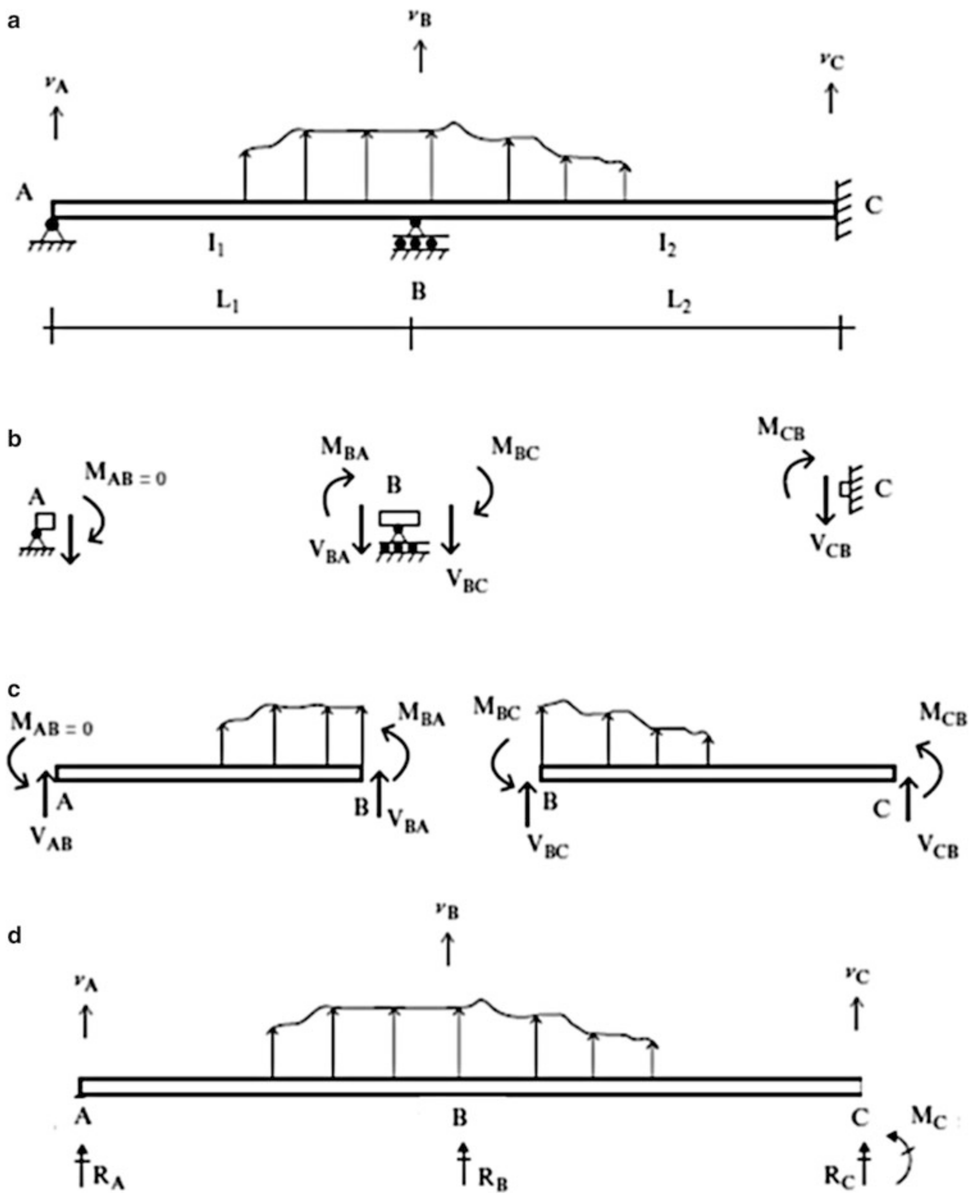


Fig. 10.10 Decomposition of two-span beam into beam segments and nodes. (**a**) Beam geometry and loading. (**b**) Segments and nodes. (**c**) Segments. (**d**) Reactions

Next we apply the slope-deflection equations (10.12a) to members AB and BC.

$$\begin{aligned} M_{AB} &= \frac{2EI_1}{L}\left\{2\theta_A + \theta_B - 3\left(\frac{v_B - v_A}{L_1}\right)\right\} + M^F_{AB} \\ M_{BA} &= \frac{2EI_1}{L_1}\left\{2\theta_B + \theta_A - 3\left(\frac{v_B - v_A}{L_1}\right)\right\} + M^F_{BA} \\ M_{BC} &= \frac{2EI_2}{L_2}\left\{2\theta_B - 3\left(\frac{v_C - v_B}{L_2}\right)\right\} + M^F_{BC} \\ M_{CB} &= \frac{2EI_2}{L_2}\left\{\theta_B - 3\left(\frac{v_C - v_B}{L_2}\right)\right\} + M^F_{CB} \end{aligned} \tag{10.13}$$

Then, we enforce moment equilibrium at the nodes. The corresponding equations are:

$$\begin{aligned} M_{AB} &= 0 \\ M_{BA} + M_{BC} &= 0 \end{aligned} \tag{10.14}$$

Substituting for the end moments in the nodal moment equilibrium equations yields

$$\begin{aligned} &\frac{4EI_1}{L_1}\theta_A + \frac{2EI_1}{L_1}\theta_B = \frac{6EI_1}{L_1}\left(\frac{v_B - v_A}{L_1}\right) - M^F_{AB} \\ &\frac{2EI_1}{L_1}\theta_A + \left(\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}\right)\theta_B = \frac{6EI_1}{L_1}\left(\frac{v_B - v_A}{L_1}\right) + \frac{6EI_2}{L_2}\left(\frac{v_C - v_B}{L_2}\right) - \left(M^F_{BA} + M^F_{BC}\right) \end{aligned} \tag{10.15}$$

Once the loading, support motion, and member properties are specified, one can solve for θ_B and θ_A. Substituting for the θs in (10.13) leads to the end moments. Lastly, we calculate the end shears. Since the end moments are known, we can determine the end shear forces using either the static equilibrium equations for the members AB and BC or by using (10.12b).

$$\begin{aligned} V_{AB} &= \frac{6EI_1}{L_1^2}(\theta_A + \theta_B) - \frac{12EI_1}{L_1^2}\left(\frac{v_B - v_A}{L_1}\right) + V^F_{AB} \\ V_{BA} &= -\frac{6EI_1}{L_1^2}(\theta_B + \theta_A) + \frac{12EI_1}{L_1^2}\left(\frac{v_B - v_A}{L_1}\right) + V^F_{BA} \\ V_{BC} &= \frac{6EI_2}{L_2^2}(\theta_B) - \frac{12EI_2}{L_2^2}\left(\frac{v_C - v_B}{L_2}\right) + V^F_{BC} \\ V_{BC} &= -\frac{6EI_2}{L_2^2}(\theta_B) + \frac{12EI_2}{L_2^2}\left(\frac{v_C - v_B}{L_2}\right) + V^F_{BC} \end{aligned} \tag{10.16}$$

The reactions are related to the end actions by (see Fig. 10.10d)

$$\begin{aligned} R_A &= V_{AB} \\ M_A &= M_{AB} = 0 \\ R_B &= V_{BA} + V_{BC} \\ R_C &= V_{CB} \\ M_C &= M_{CB} \end{aligned}$$

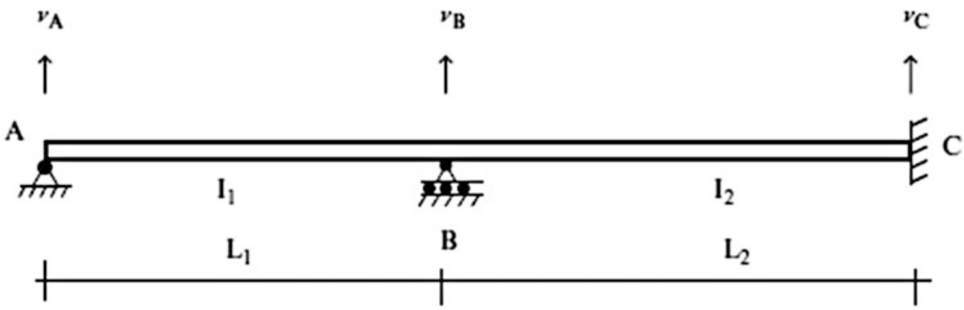


Fig. 10.11 Beam geometry and support settlements

Suppose the only external action on the above two-span beam is prescribed support settlements v_A, v_B, and v_C as shown in Fig. 10.11. We compute the corresponding chord rotation terms and include these terms in the slope-deflection equations. The chord rotations are

$$\rho_{AB} = \frac{v_B - v_A}{L_1}$$
$$\rho_{BC} = \frac{v_C - v_B}{L_2} \tag{10.17}$$

Noting (10.13), the chord rotation terms introduce additional end moments for each member connected to the support which experiences the settlement. The corresponding expressions for the end moments due to this support settlement are

$$M_{AB} = \frac{2EI_1}{L_1}\{2\theta_A + \theta_B - 3\rho_{AB}\}$$
$$M_{BA} = \frac{2EI_1}{L_1}\{2\theta_B + \theta_A - 3\rho_{AB}\}$$
$$M_{BC} = \frac{2EI_2}{L_2}\{2\theta_B - 3\rho_{BC}\}$$
$$M_{CB} = \frac{2EI_2}{L_2}\{\theta_B - 3\rho_{BC}\} \tag{10.18}$$

Substituting for the support movements, the nodal moment equilibrium equations reduce to

$$2\theta_A + \theta_B = 3\rho_{AB}$$
$$\frac{2EI_1}{L_1}\{2\theta_B + \theta_A\} + \frac{2EI_2}{L_2}\{2\theta_B\} = \frac{6EI_1}{L_1}\rho_{AB} + \frac{6EI_2}{L_2}\rho_{BC} \tag{10.19}$$

Note that the solution depends on the ratio of EI to L for each span. One specifies ρ for each member, solves (10.19) for the θs, and then evaluates the end actions.

Example 10.1

Given: The two-span beam defined in Fig. E10.1a. Assume the supports are unyielding. Take $E = 29{,}000$ ksi, $I = 428$ in.^{4}, and $L = 20$ ft.

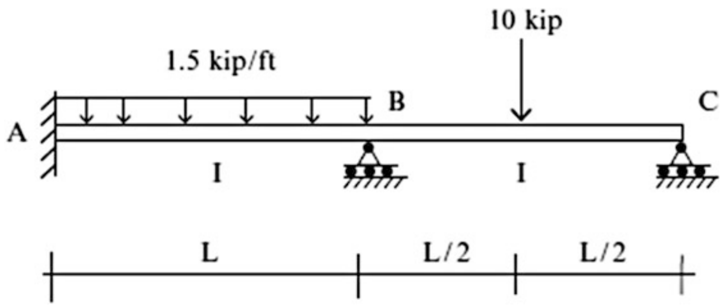
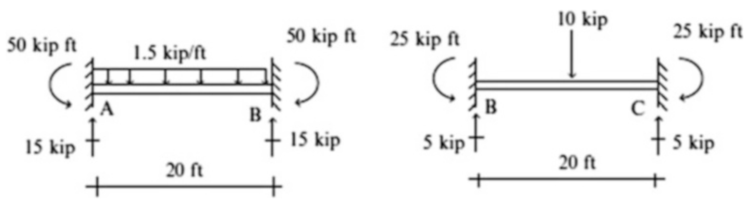


Fig. E10.1a

Determine: The end actions and the shear and moment diagrams due to the applied loading.

Solution: First, we compute the fixed end actions by using Table 9.1.



$$M_{AB}^{F} = \frac{1.5(20)^2}{12} = 50\,\text{kip ft} \qquad V_{AB}^{F} = \frac{1.5(20)}{2} = 15\,\text{kip}$$

$$M_{BA}^{F} = -50\,\text{kip ft} \qquad V_{AB}^{F} = \frac{1.5(20)}{2} = 15\,\text{kip}$$

$$M_{BC}^{F} = \frac{10(20)}{8} = 25\,\text{kip ft} \qquad V_{BC}^{F} = \frac{10}{2} = 5\,\text{kip}$$

$$M_{CB}^{F} = -25\,\text{kip ft} \qquad V_{CB}^{F} = \frac{10}{2} = 5\,\text{kip}$$

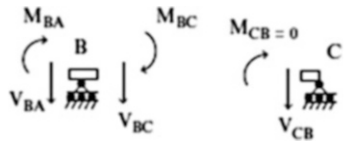
We define the relative member stiffness for each member as

$$k_{\text{members AB}} = k_{\text{members BC}} = \frac{EI}{L} = k_1 = \frac{29,000(428)}{20}\frac{1}{(12)^2} = 4310\,\text{kip ft}$$

Next, we generate the expressions for the end moments using the slope-deflection equation (10.12a) and noting that $\theta_A = 0$ and the supports are unyielding ($v_A = v_B = v_C = 0$).

$$M_{AB} = 2k_1(\theta_B) + 50$$
$$M_{BA} = 2k_1(2\theta_B) - 50$$
$$M_{BC} = 2k_1(2\theta_B + \theta_C) + 25$$
$$M_{CB} = 2k_1(\theta_B + 2\theta_C) - 25$$

Enforcing moment equilibrium at nodes B and C



$$M_{BA} + M_{BC} = 0$$
$$M_{CB} = 0$$

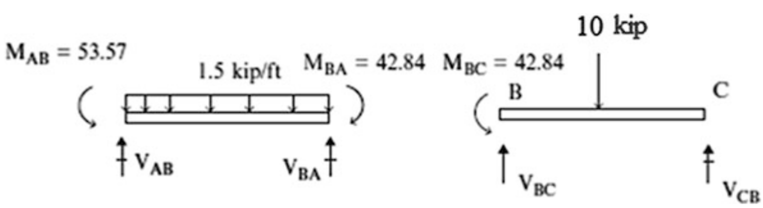
leads to

$$2k_1\theta_B + 4k_1\theta_C = 25$$
$$8k_1\theta_B + 2k_1\theta_C = 25$$
$$\Downarrow$$
$$k_1\theta_B = 1.786$$
$$k_1\theta_C = 5.357$$
$$\Downarrow$$
$$\theta_B = 0.0004\,\text{rad} \quad \text{counter clockwise}$$
$$\theta_C = 0.0012\,\text{rad} \quad \text{counter clockwise}$$

These rotations produce the following end moments

$$M_{AB} = 53.57\,\text{kip ft}$$
$$M_{BA} = -42.84\,\text{kip ft}$$
$$M_{BC} = +42.84\,\text{kip ft}$$
$$M_{CB} = 0$$

Since the end moments are known, we can determine the end shear forces either by using the static equilibrium equations for the members or by using (10.12b).



Noting (10.12b), we find

$$V_{AB} = \frac{6}{L}(k_1\theta_B) + V_{AB}^F = \frac{6}{20}(1.786) + 15 = 15.53\,\text{kip}$$

$$V_{AB} = -\frac{6}{L}(k_1\theta_B) + V_{AB}^F = -\frac{6}{20}(1.786) + 15 = 14.47\,\text{kip}$$

$$V_{BC} = \frac{6}{L}(k_1\theta_B + k_1\theta_C) + V_{BC}^F = \frac{6}{20}(1.786 + 5.357) + 5 = 7.14\,\text{kip}$$

$$V_{AB} = -\frac{6}{L}(k_1\theta_B + k_1\theta_C) + V_{CB}^F = -\frac{6}{20}(1.786 + 5.357) + 5 = 2.86\,\text{kip}$$

The reactions are:

$$R_A = V_{AB} = 15.53\,\text{kip} \uparrow$$

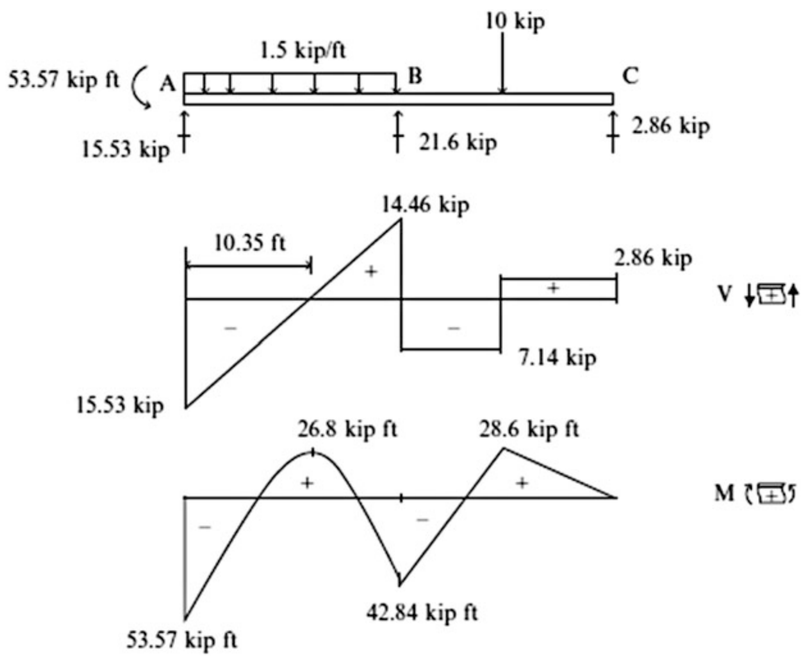
$$M_A = M_{AB} = 53.57\,\text{kip ft}$$

$$R_B = V_{BA} + V_{BC} = 21.6\,\text{kip} \uparrow$$

$$R_C = V_{CB} = 2.86\,\text{kip} \uparrow$$

$$M_C = M_{CB} = 0$$

Lastly, the shear and moment diagrams are plotted below.



Example 10.2: Two-Span Symmetrical Beam—Settlement of the Supports

Given: The symmetrical beam shown in Fig. E10.2a. Assume EI is constant. Take $L = 6$ m, $I = 180(10)^6$ mm^4, and $E = 200$ kN/mm^2.

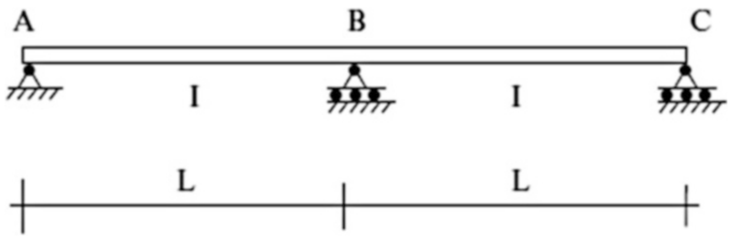


Fig. E10.2a

Case (i), the middle support settles an amount $v_B = 40$ mm.
Case (ii), the left support settles an amount $v_A = 40$ mm.

Determine: The end actions, the shear and bending moment diagrams.

Solution:

Case (i): Support settlement at B (Fig. E10.2b)

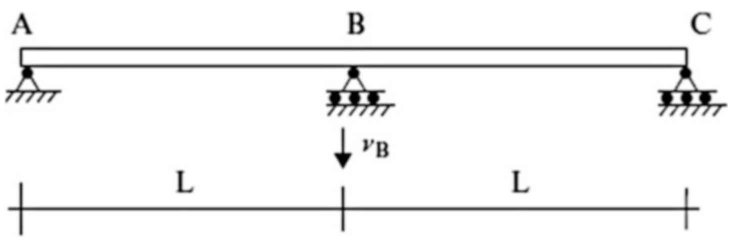


Fig. E10.2b Settlement at B

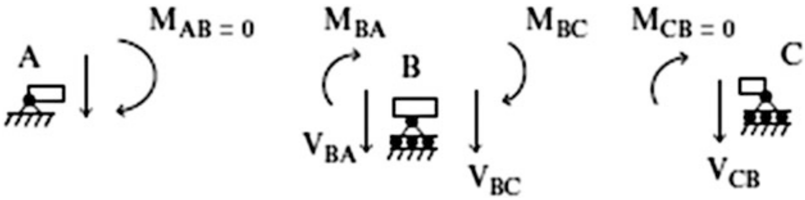
Noting (10.17), the chord rotations due to settlement at B are:

$$\rho_{AB} = \frac{v_B - v_A}{L} = -\frac{v_B}{L}$$
$$\rho_{BC} = \frac{v_C - v_B}{L} = +\frac{v_B}{L}$$

Substituting for ρ_{AB} and ρ_{BC}, the corresponding slope-deflection equation (10.12a) take the form

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) - \frac{6EI}{L}\rho_{AB}$$
$$M_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A) - \frac{6EI}{L}\rho_{AB}$$
$$M_{BC} = \frac{2EI}{L}(2\theta_B + \theta_C) - \frac{6EI}{L}\rho_{BC}$$
$$M_{CB} = \frac{2EI}{L}(2\theta_C + \theta_B) - \frac{6EI}{L}\rho_{BC}$$

We enforce moment equilibrium at nodes A, B, and C.



The corresponding equations are:

$$M_{\mathrm{AB}} = 0 \Rightarrow 2\theta_{\mathrm{A}} + \theta_{\mathrm{B}} = -3\frac{v_{\mathrm{B}}}{L}$$
$$M_{\mathrm{BA}} + M_{\mathrm{BC}} = 0 \Rightarrow \theta_{\mathrm{A}} + 4\theta_{\mathrm{B}} + \theta_{\mathrm{C}} = 0$$
$$M_{\mathrm{CB}} = 0 \Rightarrow 2\theta_{\mathrm{C}} + \theta_{\mathrm{B}} = 3\frac{v_{\mathrm{B}}}{L}$$

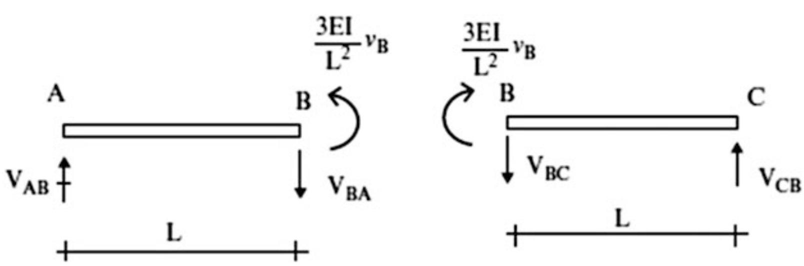
Solving for the θs leads to

$$\theta_{\mathrm{B}} = 0$$
$$\theta_{\mathrm{A}} = -\frac{3v_{\mathrm{B}}}{2L}$$
$$\theta_{\mathrm{C}} = +\frac{3v_{\mathrm{B}}}{2L}$$

The corresponding end moments are:

$$M_{\mathrm{BA}} = \frac{2EI}{L}\left(-\frac{3v_{\mathrm{B}}}{2\mathrm{L}}\right) - \frac{6EI}{L}\left(-\frac{v_{\mathrm{B}}}{L}\right) = +\frac{3EI}{L^2}v_{\mathrm{B}} = \frac{3(200)(180)10^6}{(6000)^2}(40)$$
$$= 120{,}000\,\mathrm{kN\,mm} = 120\mathrm{kN\,m}$$
$$M_{\mathrm{BC}} = \frac{2EI}{L}\left(\frac{3v_{\mathrm{B}}}{2L}\right) - \frac{6EI}{L}\left(\frac{v_{\mathrm{B}}}{L}\right) = -\frac{3EI}{L^2}v_{\mathrm{B}} = -120\mathrm{kN\,m}$$

Next, we determine the end shear forces using the static equilibrium equations for the members.



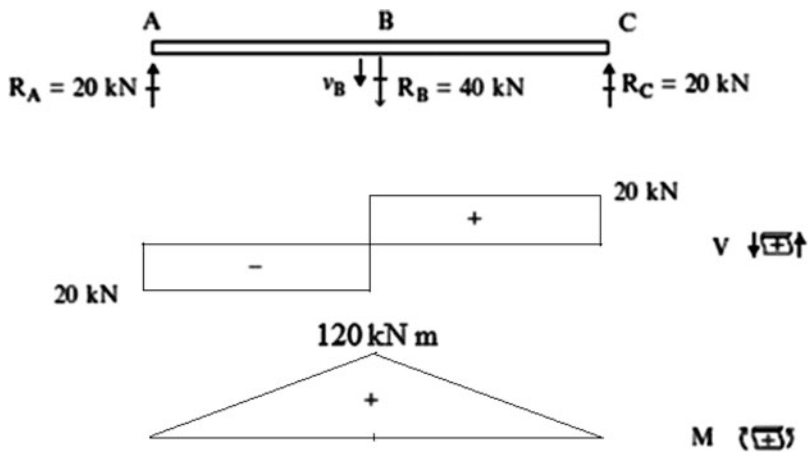
$$V_{\mathrm{AB}} = V_{\mathrm{CB}} = +\frac{3EI}{L^3}v_{\mathrm{B}} = \frac{3(200)(180)10^6}{(6000)^3}(40) = 20\,\mathrm{kN}\uparrow$$

$$V_{\mathrm{BA}} = V_{\mathrm{BC}} = -\frac{3EI}{L^3}v_{\mathrm{B}} = 20\,\mathrm{kN}\downarrow$$

The corresponding reactions are:

$$R_A = V_{AB} = 20\,\text{kN} \uparrow$$
$$R_B = V_{BA} + V_{BC} = 40\,\text{kN} \downarrow$$
$$R_C = V_{CB} = 20\,\text{kN} \uparrow$$

One should expect that $\theta_B = 0$ because of symmetry. The shear and moment diagrams are plotted below.



Case (ii): Support settlement at A (Fig. E10.2c)

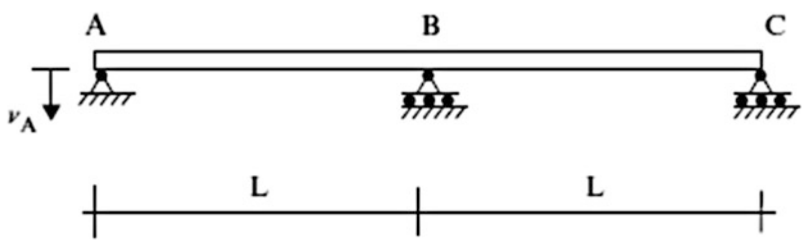


Fig. E10.2c Settlement at A

Settlement at A produces chord rotation in member AB only. The chord rotation for member AB due to settlement of node A is $\rho_{AB} = v_A/L$. Substituting for ρ_{AB}, the corresponding slope-deflection equation (10.12a) take the form

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) - \frac{6EI}{L}\rho_{AB}$$
$$M_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A) - \frac{6EI}{L}\rho_{AB}$$
$$M_{BC} = \frac{2EI}{L}(2\theta_B + \theta_C)$$
$$M_{CB} = \frac{2EI}{L}(2\theta_C + \theta_B)$$

Setting $M_{AB} = M_{CB} = 0$ and $M_{BA} + M_{BC} = 0$ leads to

$$2\theta_A + \theta_B = 3\rho_{AB}$$
$$2\theta_C + \theta_B = 0$$
$$4\theta_B + \theta_A + \theta_C = 3\rho_{AB}$$

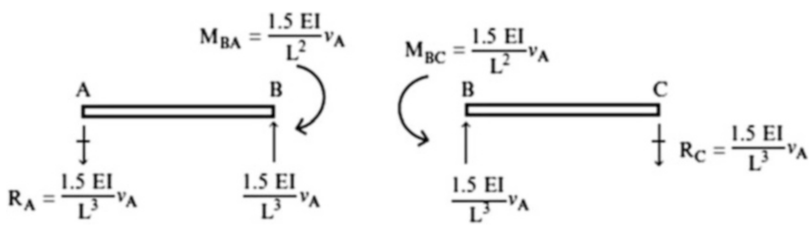
Solving for the θs leads to

$$\theta_A = \frac{5}{4}\rho_{AB}$$
$$\theta_B = \frac{1}{2}\rho_{AB}$$
$$\theta_C = -\frac{1}{4}\rho_{AB}$$

Finally, the bending moment at B due to support settlement at A is:

$$M_{BA} = \frac{2EI}{L}\left(\frac{v_A}{L} + \frac{5v_A}{4L}\right) - \frac{6EIv_A}{L^2} = -\frac{1.5EI}{L^2}v_A = \frac{1.5(200)(180)10^6}{(6000)^2}(40)$$
$$= -60,000\,\text{kN mm} = -60\,\text{kN m}$$
$$M_{BC} = -M_{BA} = 60\,\text{kN m} \quad \text{counterclockwise}$$

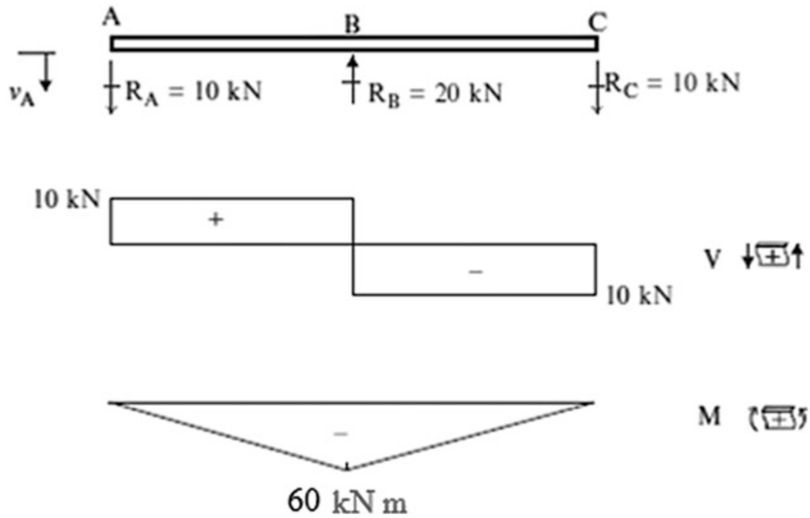
Next, we determine the end shear forces using the static equilibrium equations for the members.



Then,

$$R_A = R_C = -\frac{1.5EI}{L^3}v_A = -\frac{1.5(200)(180)10^6}{(6000)^3}(40) = -10\,\text{kN}$$
$$R_B = \frac{1.5EI}{L^3}v_A + \frac{1.5EI}{L^3}v_A = +20\,\text{N}$$

The shear and moment diagrams are plotted below.



Example 10.3: Two-Span Beam with Overhang

Given: The beam shown in Fig. E10.3a. Assume EI is constant.

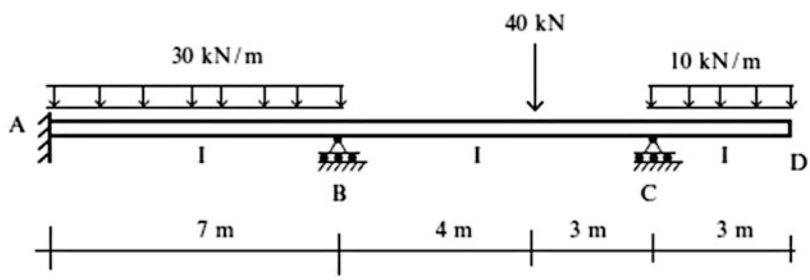
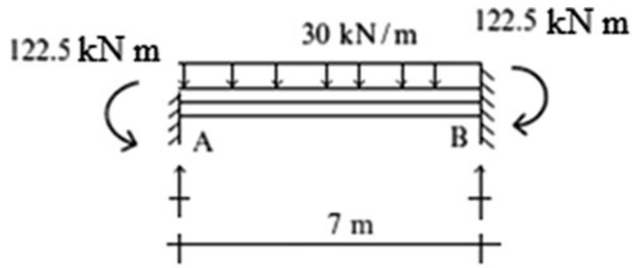
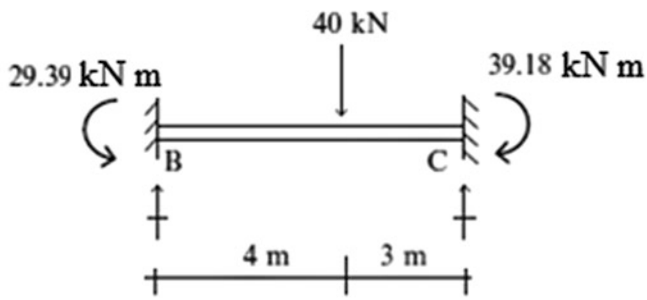


Fig. E10.3a

Determine: The end actions and the shear and moment diagrams.

Solution: First, we compute the fixed end moments by using Table 9.1.





$$M_{AB}^{F} = +\frac{30(7)^2}{12} = +122.5\,\mathrm{kN\,m}$$

$$M_{BA}^{F} = -122.5\,\mathrm{kN\,m}$$

$$M_{BC}^{F} = \frac{40(4)(3)^2}{(7)^2} = +29.39\,\mathrm{kN\,m}$$

$$M_{CB}^{F} = -\frac{40(4)^2(3)}{(7)^2} = -39.18\,\mathrm{kN\,m}$$

We define the relative member stiffness for each member as

$$k_{\text{member AB}} = k_{\text{member BC}} = \frac{EI}{L} = k_1$$

Noting that $\theta_A = 0$ and the supports are unyielding ($v_A = v_B = v_C = 0$), the corresponding slope-deflection equation (10.12a) take the form

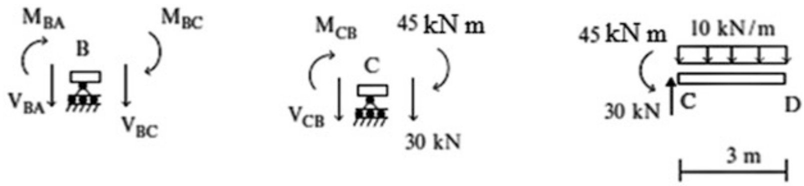
$$M_{AB} = 2k_1(\theta_B) + 122.5$$

$$M_{BA} = 2k_1(2\theta_B) - 122.5$$

$$M_{BC} = 2k_1(2\theta_B + \theta_C) + 29.39$$

$$M_{CB} = 2k_1(2\theta_C + \theta_B) - 39.18$$

We enforce moment equilibrium at the nodes B and C.



The corresponding equations are:

$$M_{BA} + M_{BC} = 0 \Rightarrow 2k_1\theta_C + 6k_1\theta_B = 93.11$$
$$M_{CB} + 45 = 0 \quad \Rightarrow 4k_1\theta_C + 2k_1\theta_B = -5.82$$

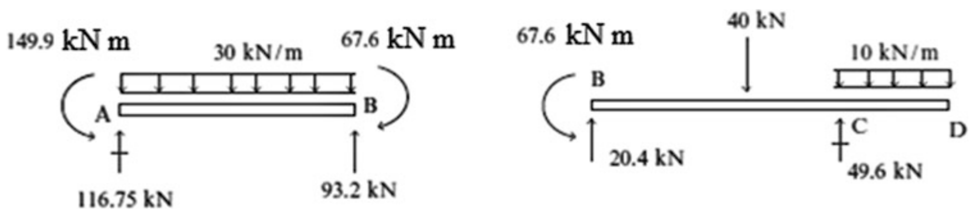
Solving these equations leads to

$$k_1\theta_B = 13.71$$
$$k_1\theta_C = -8.31$$

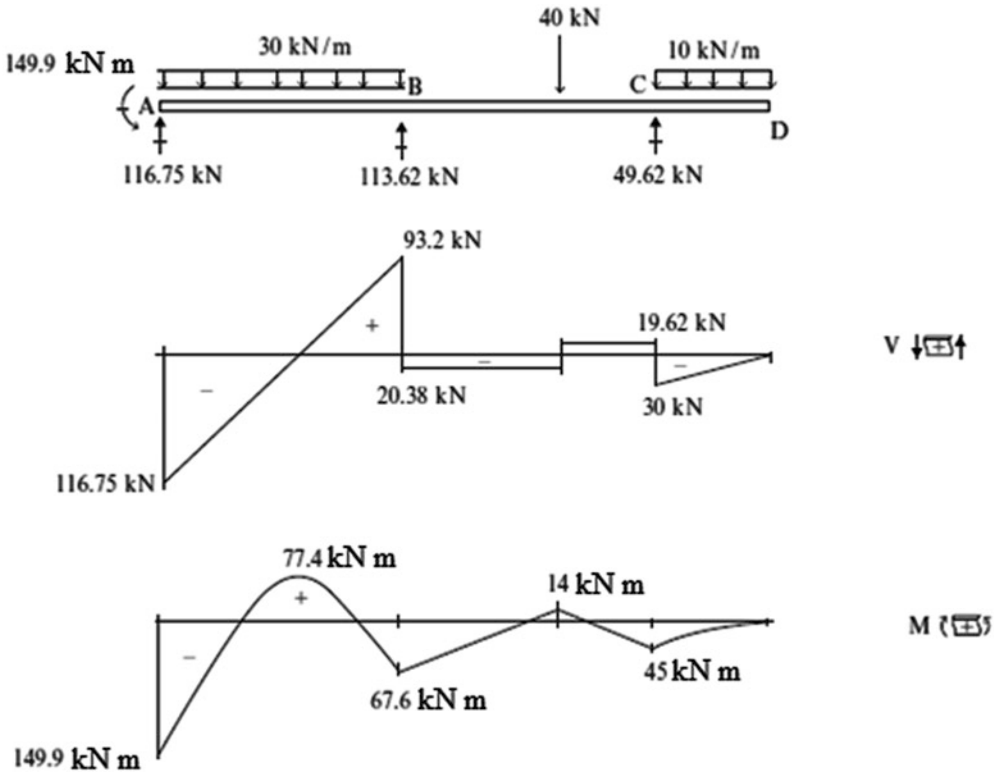
The corresponding end moments are:

$$M_{AB} = +149.9\,\text{kN m}$$
$$M_{BA} = -67.6\,\text{kN m}$$
$$M_{BC} = +67.6\,\text{kN m}$$
$$M_{CB} = -45\,\text{kN m}$$

Next, we determine the end shear forces using the static equilibrium equations for the members.



The shear and moment diagrams are plotted below.



10.4.2 Multi-Span Beams

In what follows, we modify the slope-deflection equations for the end members of a multi-span continuous beam when they have either a pin or roller support. Consider the three-span beam shown in Fig. 10.12a. There are three beam segments and four nodes. Since the end nodes have zero moment, we can simplify the slope-deflection equations for the end segments by eliminating the end rotations. We did this in the previous examples, as part of the solution process. Now, we formalize the process and modify the slope-deflection equations before setting up the nodal moment equilibrium equations for the interior nodes.

Consider member AB. The end moment of A is zero, and we use this fact to express θ_A in terms of θ_B. Starting with the expression for M_{AB},

$$M_{AB} = \frac{2EI_1}{L_1}\left(2\theta_A + \theta_B - 3\left(\frac{v_B - v_A}{L_1}\right)\right) + M_{AB}^F = 0$$

and solving for θ_A leads to

$$\theta_A = -\frac{1}{2}\theta_B + \frac{3}{2}\left(\frac{v_B - v_A}{L_1}\right) - \frac{L_1}{4EI_1}M_{AB}^F$$

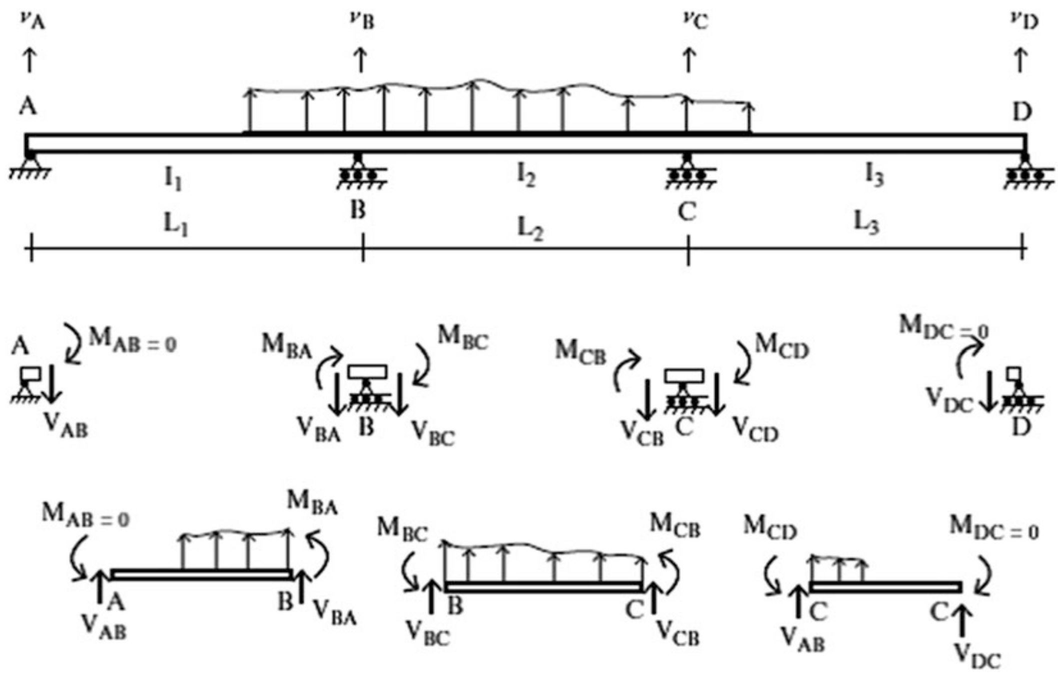


Fig. 10.12 Three-span beam

Then, we substitute for θ_A in the expression M_{BA},

$$M_{BA} = \frac{2EI_1}{L_1}\left(2\theta_B + \theta_A - 3\frac{v_B - v_A}{L_1}\right) + M^F_{BA}$$

and obtain the following form,

$$M_{BA_{\text{modified}}} = \frac{3EI_1}{L_1}\left(\theta_B - \left(\frac{v_B - v_A}{L}\right)\right) + \left(M^F_{BA} - \frac{1}{2}M^F_{AB}\right) \tag{10.20}$$

Note that the presence of a pin or roller at A *reduces* the rotational stiffness at B from $4EI/L$ to $3EI/L$. Substituting for θ_A in the expression V_{AB} and V_{BA} leads to the following expressions,

$$V_{AB_{\text{modified}}} = +\frac{6EI}{L^2}\left\{\frac{1}{2}\theta_B - \frac{1}{2}\left(\frac{v_B - v_A}{L}\right)\right\} + V^F_{AB} - \frac{3}{2}\frac{M^F_{AB}}{L}$$

$$V_{BA_{\text{modified}}} = -\frac{6EI}{L^2}\left\{\frac{1}{2}\theta_B - \frac{1}{2}\left(\frac{v_B - v_A}{L}\right)\right\} + V^F_{BA} + \frac{3}{2}\frac{M^F_{AB}}{L}$$

For member BC, we use the general unchanged form

$$M_{BC} = \frac{2EI_2}{L_2}\left(2\theta_B + \theta_C - 3\frac{v_C - v_B}{L_2}\right) + M^F_{BC}$$

$$M_{CB} = \frac{2EI_2}{L_2}\left(2\theta_C + \theta_B - 3\frac{v_C - v_B}{L_2}\right) + M^F_{CB}$$

The modified form for member CD is

$$M_{\mathrm{DC}} = 0$$
$$M_{\mathrm{CD}} = \frac{3EI_3}{L_3}\left(\theta_{\mathrm{C}} - \frac{v_{\mathrm{D}} - v_{\mathrm{C}}}{L_3}\right) + \left(M^{\mathrm{F}}_{\mathrm{CD}} - \frac{1}{2}M^{\mathrm{F}}_{\mathrm{DC}}\right)$$

Nodal moment equilibrium equations

Now, we return back to Fig. 10.12. If we use the modified form of the moment expressions for members AB and CD, we *do not* have to enforce moment equilibrium at nodes A and D since we have already employed this condition to modify the equations. Therefore, we need only to consider nodes B and C. Summing moments at these nodes,

$$M_{\mathrm{BA}} + M_{\mathrm{BC}} = 0$$
$$M_{\mathrm{CB}} + M_{\mathrm{CD}} = 0$$

and substituting for the end moments expressed in terms of θ_{B} and θ_{C} leads to

$$\begin{aligned}
&\theta_{\mathrm{B}}\left\{\frac{3EI_1}{L_1} + \frac{4EI_2}{L_2}\right\} + \theta_{\mathrm{C}}\left\{\frac{2EI_2}{L_2}\right\} - \left\{\frac{6EI_2}{L_2}\left(\frac{v_{\mathrm{C}} - v_{\mathrm{B}}}{L_2}\right) + \frac{3EI_1}{L_1}\left(\frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{L_1}\right)\right\}\\
&+\left\{M^{\mathrm{F}}_{\mathrm{BC}} + \left(M^{\mathrm{F}}_{\mathrm{BA}} - \frac{1}{2}M^{\mathrm{F}}_{\mathrm{AB}}\right)\right\} = 0\\
&\theta_{\mathrm{B}}\left\{\frac{2EI_2}{L_2}\right\} + \theta_{\mathrm{C}}\left\{\frac{4EI_2}{L_2} + \frac{3EI_3}{L_3}\right\} - \left\{\frac{6EI_2}{L_2}\left(\frac{v_{\mathrm{C}} - v_{\mathrm{B}}}{L_2}\right) + \frac{3EI_3}{L_3}\left(\frac{v_{\mathrm{D}} - v_{\mathrm{C}}}{L_3}\right)\right\}\\
&+\left\{\left(M^{\mathrm{F}}_{\mathrm{CD}} - \frac{1}{2}M^{\mathrm{F}}_{\mathrm{DC}}\right) + M^{\mathrm{F}}_{\mathrm{CB}}\right\} = 0
\end{aligned} \tag{10.21}$$

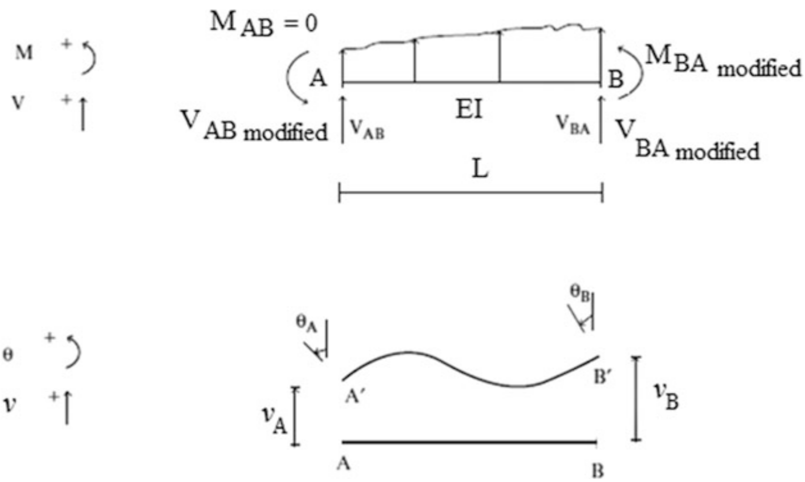
Given the nodal fixed end moments due to the loading and the chord rotations due to support settlement, one can solve the above simultaneous equations for θ_{B} and θ_{C} and determine the end moments by back substitution. Note that the solution depends on the relative magnitudes of the ratio, I/L, for each member.

In what follows, we list the modified slope-deflection equations for an end member with a pin or roller support.

End member AB (exterior pin or roller at A end):

$$\begin{aligned}
&M_{\mathrm{AB}} = 0\\
&M_{\mathrm{BA_{modified}}} = \frac{3EI}{L}\left\{\theta_{\mathrm{B}} - \left(\frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{L}\right)\right\} + \left(M^{\mathrm{F}}_{\mathrm{BA}} - \frac{1}{2}M^{\mathrm{F}}_{\mathrm{AB}}\right)
\end{aligned} \tag{10.22a}$$

$$\begin{aligned}
&V_{\mathrm{AB_{modified}}} = \frac{3EI}{L^2}\left\{\theta_{\mathrm{B}} - \left(\frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{L}\right)\right\} + V^{\mathrm{F}}_{\mathrm{AB}} - \frac{3}{2}\frac{M^{\mathrm{F}}_{\mathrm{AB}}}{L}\\
&V_{\mathrm{BA_{modified}}} = -\frac{3EI}{L^2}\left\{\theta_{\mathrm{B}} - \left(\frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{L}\right)\right\} + V^{\mathrm{F}}_{\mathrm{BA}} + \frac{3}{2}\frac{M^{\mathrm{F}}_{\mathrm{AB}}}{L}
\end{aligned} \tag{10.22b}$$



Equations ([10.22a](#), [10.22b](#)) are referred to as the *modified slope-deflection equations*.

Example 10.4: Two-Span Beam with Moment Releases at Both Ends

Given: The two-span beam shown in Fig. E10.4a. Assume *EI* is constant.

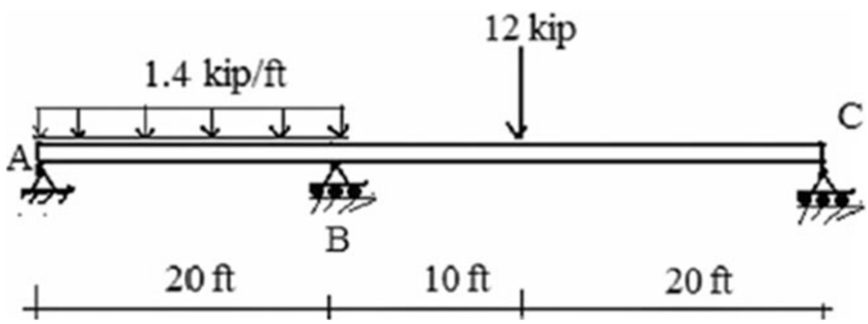


Fig. E10.4a

Determine: The end actions and the shear and moment diagrams.

Solution: The fixed end moments are (see Table 9.1):

$$M_{\mathrm{AB}}^{\mathrm{F}} = \frac{1.4(20)^2}{12} = 46.67\,\mathrm{kip\,ft}$$

$$M_{\mathrm{BA}}^{\mathrm{F}} = -46.67\,\mathrm{kip\,ft}$$

$$M_{\mathrm{BC}}^{\mathrm{F}} = \frac{12(10)(20)^2}{(30)} = 53.33\,\mathrm{kip\,ft}$$

$$M_{\mathrm{CB}}^{\mathrm{F}} = -\frac{12(20)(10)^2}{(30)^2} = -26.67\,\mathrm{kip\,ft}$$

We define the relative member stiffness for each member as

$$k_{\text{member BC}} = \frac{EI}{L_{BC}} = k_1$$

$$k_{\text{member AB}} = \frac{EI}{L_{AB}} = 1.5k_1$$

Next, we generate the expressions for the end moments using the modified slope-deflection equation (10.22a).

$$M_{AB} = 0$$

$$M_{BA} = M_{BA_{\text{modified}}} = 3(1.5k_1)(\theta_B) + \left(M^F_{BA} - \frac{1}{2}M^F_{AB}\right) = 3(1.5k_1)(\theta_B)$$

$$+ \left\{-46.67 - \frac{1}{2}(46.67)\right\} = 4.5k_1\theta_B - 70$$

$$M_{BC} = M_{BC_{\text{mdified}}} = 3(k_1)(\theta_B) + \left(M^F_{BC} - \frac{1}{2}M^F_{CB}\right) = 3(k_1)(\theta_B)$$

$$+ \left\{+53.33 - \frac{1}{2}(-26.67)\right\} = 3k_1\theta_B + 66.66$$

$$M_{CB} = 0$$

The moment equilibrium equation for node B expands to

$$M_{BA} + M_{BC} = 0$$

$$\Downarrow$$

$$7.5k_1\theta_B - 3.34 = 0$$

$$\Downarrow$$

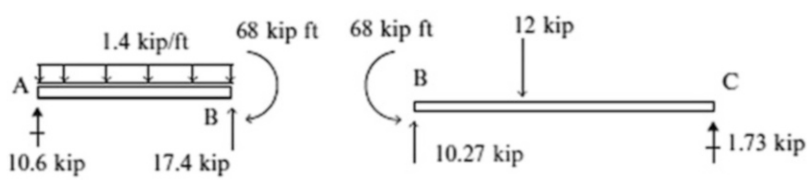
$$k_1\theta_B = 0.4453$$

Finally, the bending moment at B is

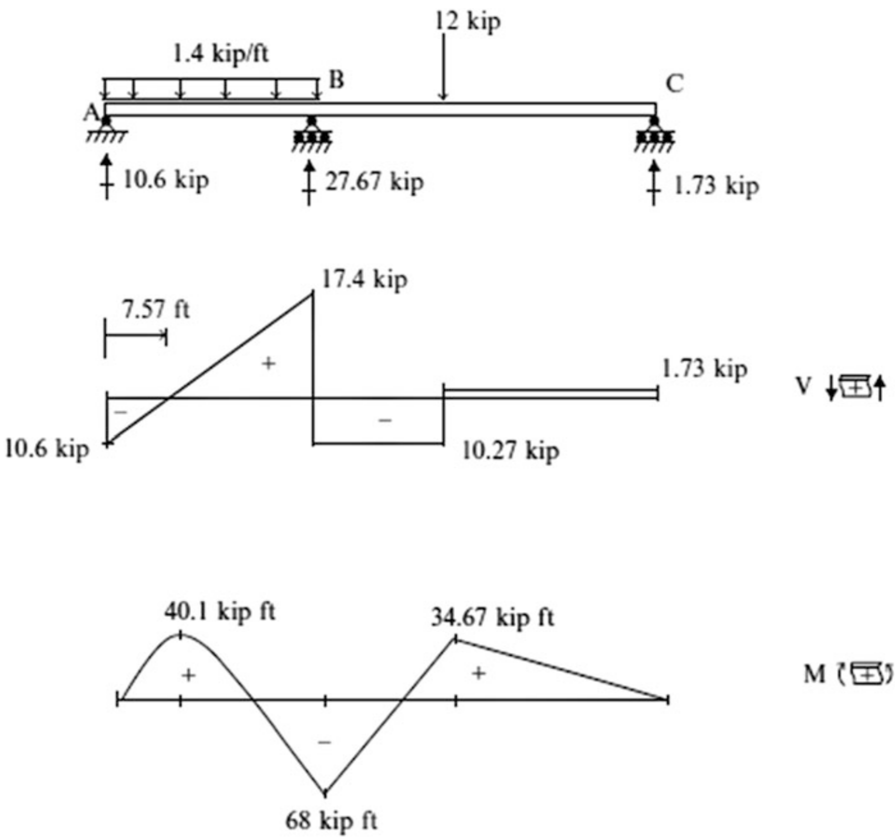
$$M_{BA} = -68 \text{ kip ft}$$

$$M_{BC} = -M_{BA} = 68 \text{ kip ft}$$

Noting the free body diagrams shown below, we find the remaining end actions.



The shear and moment diagrams are plotted below.



Example 10.5: Three-Span Beam

Given: The three-span beam shown in Figs. E10.5a, E10.5b, E10.5c.

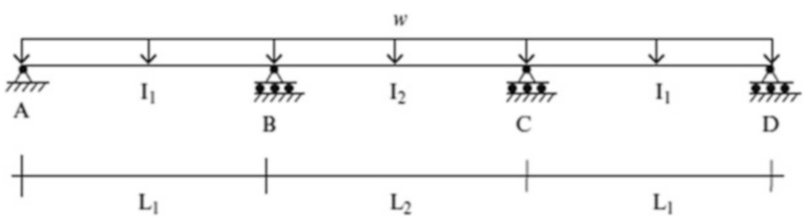


Fig. E10.5a Uniform load

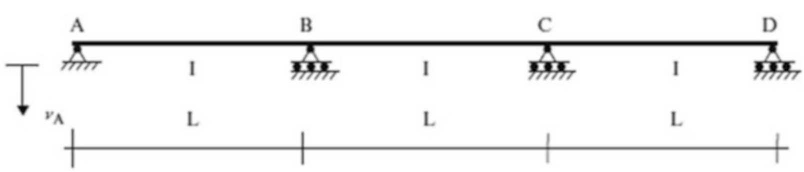


Fig. E10.5b Settlement at A

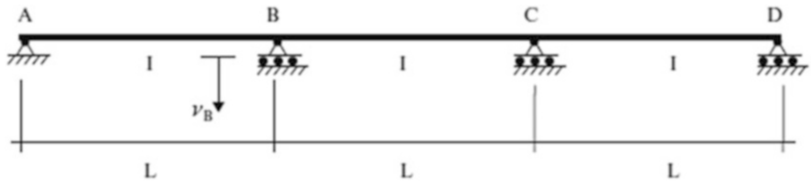


Fig. E10.5c Settlement at B

Determine: The end moments and draw the moment diagram for

Case (i): uniform load w. No support settlement.
Case (ii): No loading. Support settlement at A. Consider I and L are constants.
Case (iii): No loading. Support settlement at B. Consider I and L are constants.

Solution:

Case (i): Uniform loading
The supports are unyielding. Therefore $v_A = v_B = v_C = 0$. The fixed end moments due to the uniform loading are (see Table 9.1)

$$M_{AB}^F = +\frac{wL_1^2}{12} \quad M_{BA}^F = -\frac{wL_1^2}{12}$$
$$M_{BC}^F = +\frac{wL_2^2}{12} \quad M_{CB}^F = -\frac{wL_2^2}{12}$$
$$M_{CD}^F = +\frac{wL_1^2}{12} \quad M_{DC}^F = -\frac{wL_1^2}{12}$$

We use (10.22a) for members AB and CD and (10.12a) for member BC.

$$M_{AB} = 0$$
$$M_{BA} = M_{BA_{\text{modified}}} = \frac{3EI_1}{L_1}\theta_B + \left(M_{BA}^F - \frac{1}{2}M_{AB}^F\right) = \frac{3EI_1}{L_1}\theta_B - \frac{wL_1^2}{8}$$
$$M_{BC} = \frac{2EI_2}{L_2}\{2\theta_B + \theta_C\} + M_{BC}^F = \frac{2EI_2}{L_2}\{2\theta_B + \theta_C\} + \frac{wL_2^2}{12}$$
$$M_{CB} = \frac{2EI_2}{L_2}\{\theta_B + 2\theta_C\} + M_{CB}^F = \frac{2EI_2}{L_2}\{\theta_B + 2\theta_C\} - \frac{wL_2^2}{12}$$
$$M_{CD} = M_{CD_{\text{modified}}} = \frac{3EI_1}{L_1}\theta_C + \left(M_{CD}^F - \frac{1}{2}M_{DC}^F\right) = \frac{3EI_1}{L_1}\theta_C + \frac{wL_1^2}{8}$$
$$M_{DC} = 0$$

The nodal moment equilibrium equations are

$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CD} = 0$$

Substituting for the end moments, the above equilibrium equations expand to

$$\theta_B\left\{\frac{3EI_1}{L_1}+\frac{4EI_2}{L_2}\right\}+\theta_C\left\{\frac{2EI_2}{L_2}\right\}=-\frac{wL_2^2}{12}+\frac{wL_1^2}{8}$$

$$\theta_B\left\{\frac{2EI_2}{L_2}\right\}+\theta_C\left\{\frac{4EI_2}{L_2}+\frac{3EI_1}{L_1}\right\}=-\frac{wL_1^2}{8}+\frac{wL_2^2}{12}$$

$$\Downarrow$$

$$\frac{EI_2}{L_2}\theta_B=\frac{wL_2^2}{12}\left\{\frac{-1+\frac{3}{2}(L_1/L_2)^2}{2+3(I_1/I_2)(L_2/L_1)}\right\}$$

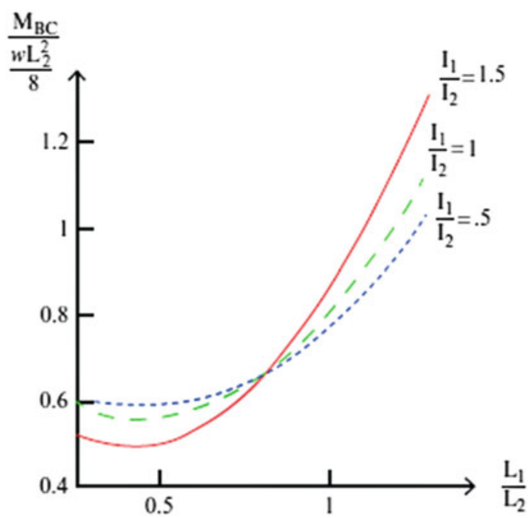
$$\theta_C=-\theta_B$$

The corresponding moments are

$$M_{BC}=\frac{wL_2^2}{8}\left\{\frac{(L_1/L_2)^2+(I_1/I_2)(L_2/L_1)}{1+3/2(I_1/I_2)(L_2/L_1)}\right\}$$

$$M_{CB}=-M_{BC}$$

We note that the moments are a function of (I_1/I_2) and (L_1/L_2). The sensitivity of M_{BC} to the ratio (L_1/L_2) is plotted below for various values of (I_1/I_2).



When I and L are constants for all the spans, the solution is

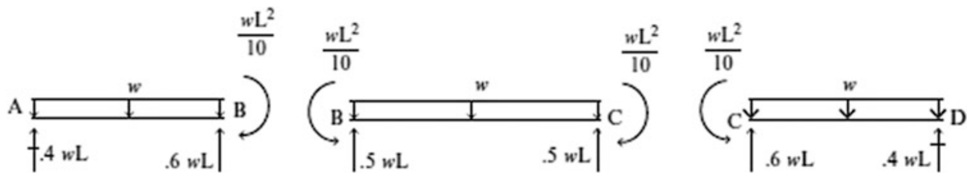
$$\theta_B=\frac{wL^3}{120EI}$$

$$\theta_C=-\frac{wL^3}{120EI}$$

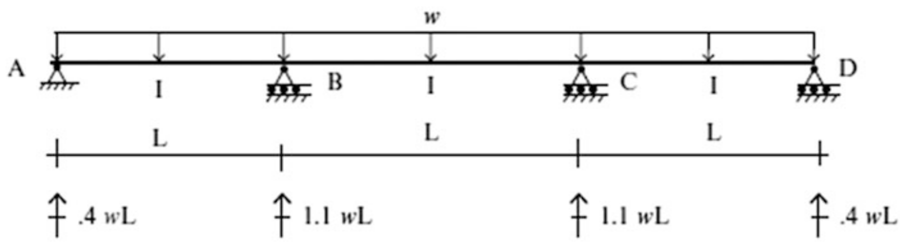
and

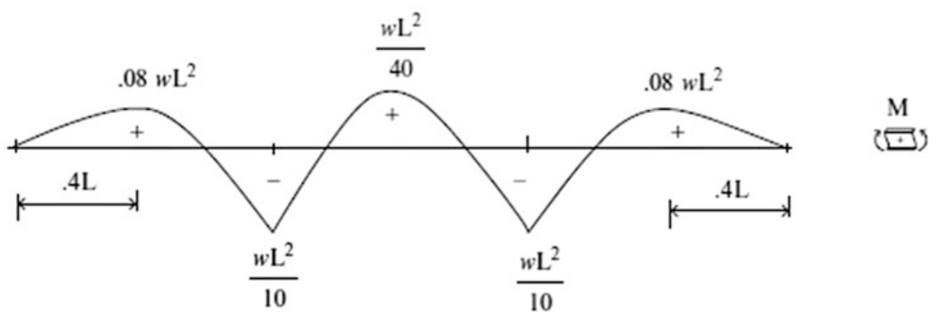
$$M_{BA} = M_{CB} = -\frac{wL^2}{10}$$
$$M_{BC} = M_{CD} = \frac{wL^2}{10}$$

We determine the end shear forces using the static equilibrium equations for the members.



The moment diagram is plotted below.





Case (ii): Support settlement at A, no loading, I and L are constants

The chord rotations are

$$\rho_{AB} = +\frac{v_A}{L}$$
$$\rho_{AB} = \rho_{CD} = 0$$

Specializing (10.22a) for members AB and CD and (10.12a) for member BC for I and L constant, and the above notation results in

$$M_{BA_{\text{modified}}} = \frac{3EI}{L}\left\{\theta_B - \frac{v_A}{L}\right\}$$
$$M_{BC} = \frac{2EI}{L}\{2\theta_B + \theta_C\}$$
$$M_{CB} = \frac{2EI}{L}\{\theta_B + 2\theta_C\}$$
$$M_{CD_{\text{modified}}} = \frac{3EI}{L}\{\theta_C\}$$

The nodal moment equilibrium equations are

$$\begin{aligned} M_{BA} + M_{BC} &= 0 \\ M_{CB} + M_{CD} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 7\theta_B + 2\theta_C &= \frac{3v_A}{L} \\ 2\theta_B + 7\theta_C &= 0 \end{aligned}$$

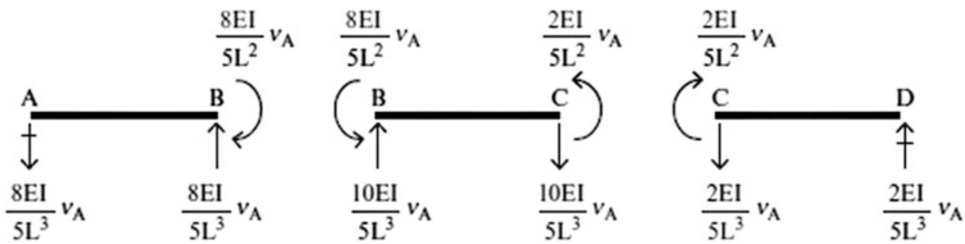
The solution is

$$\theta_B = \frac{7v_A}{15L}$$
$$\theta_C = -\frac{2v_A}{15L}$$

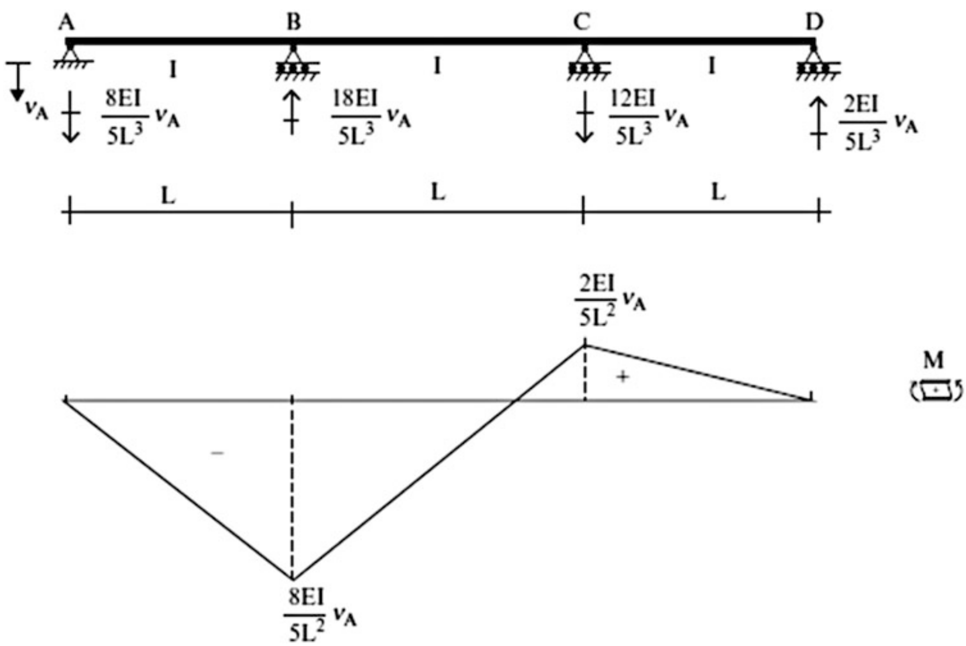
and the corresponding moments are

$$M_{BA} = -\frac{8EI}{5L^2}v_A$$
$$M_{CD} = -\frac{2EI}{5L^2}v_A$$

We determine the end shear forces using the static equilibrium equations for the members.



The moment diagram is plotted below.



Case (iii): Support settlement at B, no loading, I and L are constants

The chord rotations are

$$\rho_{AB} = -\frac{v_B}{L}$$
$$\rho_{BC} = +\frac{v_B}{L}$$
$$\rho_{CD} = 0$$

Specializing (10.22a) for members AB and CD and (10.12a) for member BC for I and L constant, and the above notation results in

$$M_{BA_{modified}} = \frac{3EI}{L}\left\{\theta_B + \left(\frac{v_B}{L}\right)\right\}$$
$$M_{BC} = \frac{2EI}{L}\left\{2\theta_B + \theta_C - 3\frac{v_B}{L}\right\}$$
$$M_{CB} = \frac{2EI}{L}\left\{\theta_B + 2\theta_C - 3\frac{v_B}{L}\right\}$$
$$M_{CD_{modified}} = \frac{3EI}{L}\{\theta_C\}$$

The nodal moment equilibrium equations are

$$\begin{matrix} M_{BA} + M_{BC} = 0 \\ M_{CB} + M_{CD} = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 7\theta_B + 2\theta_C = \dfrac{3v_B}{L} \\ 2\theta_B + 7\theta_C = \dfrac{6v_B}{L} \end{matrix}$$

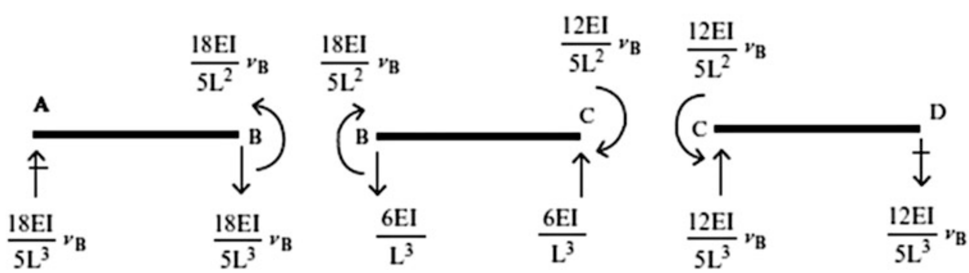
The solution is

$$\theta_B = \frac{v_B}{5L}$$
$$\theta_C = \frac{4v_B}{5L}$$

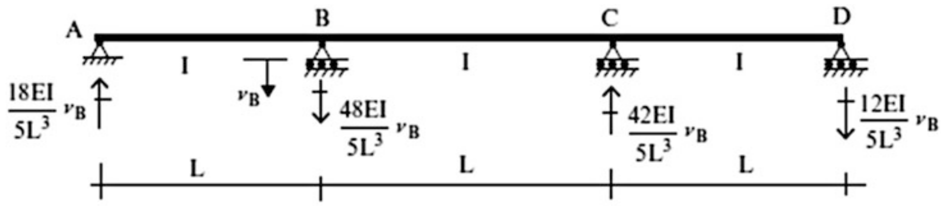
and the corresponding moments are

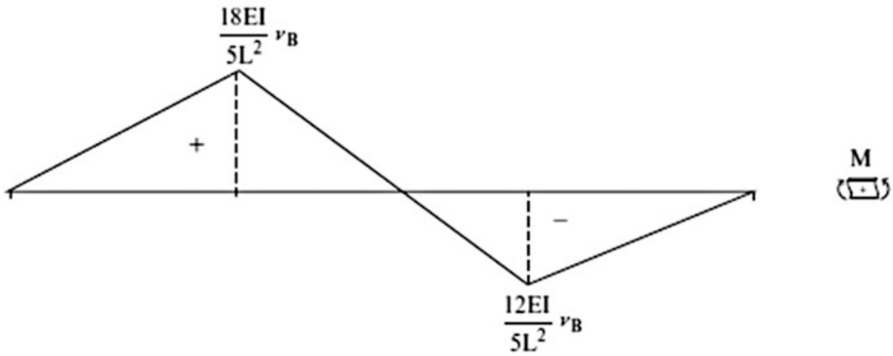
$$M_{BA} = \frac{18EI}{5L^2}v_B$$
$$M_{CD} = \frac{12EI}{5L^2}v_B$$

We determine the end shear forces using the static equilibrium equations for the members.



Noting the free body diagrams, we find the reactions. The moment diagram is plotted below.





Example 10.6: Uniformly Loaded Three-Span Symmetrical Beam—Fixed Ends

Given: The three-span symmetrical fixed end beam defined in Fig. E10.6a. This model is representative of an integral bridge with very stiff abutments at the ends of the beam.

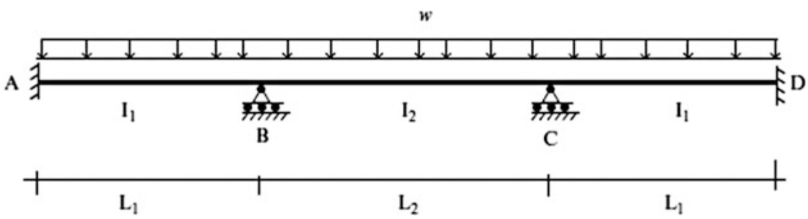


Fig. E10.6a

Determine: The end moments.

Solution: The slope-deflection equations for unyielding supports, $\theta_A = \theta_D = 0$ and symmetry $\theta_B = -\theta_C$ are

$$M_{AB} = -M_{DC} = \frac{2EI_1}{L_1}(\theta_B) + M_{AB}^F$$

$$M_{BA} = \frac{2EI_1}{L_1}(2\theta_B) + M_{BA}^F$$

$$M_{BC} = -M_{CB} = \frac{2EI_2}{L_2}(\theta_B) + M_{BC}^F$$

where

$$M_{AB}^{F} = M_{CD}^{F} = +\frac{wL_1^2}{12}$$

$$M_{BA}^{F} = M_{DC}^{F} - \frac{wL_1^2}{12}$$

$$M_{BC}^{F} = +\frac{wL_2^2}{12}$$

$$M_{CB}^{F} = -\frac{wL_2^2}{12}$$

Summing end moments at node B

$$M_{BA} + M_{BC} = 0$$

$$\theta_B\left\{\frac{4EI_1}{L_1} + \frac{2EI_2}{L_2}\right\} = -\left(M_{BA}^{F} + M_{BC}^{F}\right)$$

and solving for θ_B leads to

$$\theta_B = -\theta_C = \frac{\left(\left(wL_1^2/12\right) - \left(wL_2^2/12\right)\right)}{\left((4EI_1/L_1) + (2EI_2/L2)\right)}$$

Suppose I and L are constants. The end rotations corresponding to this case are

$$\theta_B = \theta_C = 0$$

It follows that the end moments are equal to the fixed end moments.

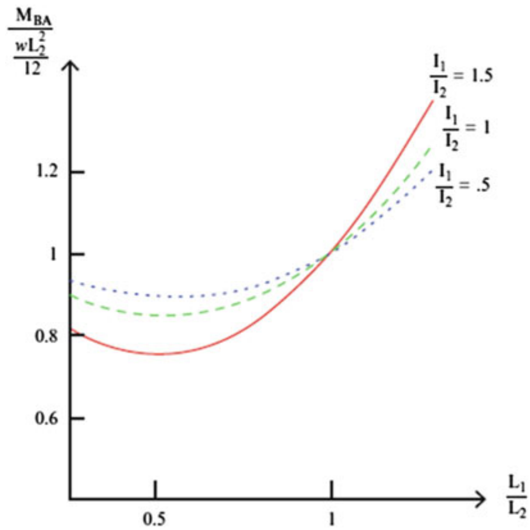
$$M_{AB} = \frac{wL^2}{12}$$

$$M_{BA} = -M_{BC} = -\frac{wL^2}{12}$$

The general solution for the moment at B follows by substituting for θ_B in either the expression for M_{BA} or M_{BC}. After some algebraic manipulation, the expression for M_{BA} reduces to

$$M_{BA} = \frac{wL_2^2}{12}\frac{\left\{(L_1/L_2)^2 + 2(I_1/I_2)(L_2/L_1)\right\}}{(1 + 2(I_1/I_2)(L_2/L_1))}$$

We note that the moments are a function of (I_1/I_2) and (L_1/L_2). The sensitivity of M_{BA} to the ratio (L_1/L_2) is plotted below for various values of (I_1/I_2).



10.5 The Displacement Method Applied to Rigid Frames

The essential difference between the analysis of beams and frames is the choice of the nodal displacements. The nodal variables for a beam are taken as the rotations. When there is support movement, we prescribe the nodal translation and compute the corresponding fixed end moments. In this way, the equilibrium equations always involve only rotation variables. Rigid frames are considered to be an assemblage of members rigidly connected at nodes. Since frame structures are formed by joining members at an arbitrary angle, the members rotate as well as bend. When this occurs, we need to include the chord rotation terms in the slope-deflection equations, and work with both translation and rotation variables. Using these relations, we generate a set of equations relating the nodal translations and rotations by enforcing equilibrium for the nodes. The approach is relatively straightforward when there are not many displacement variables. However, for complex structures involving many displacement unknowns, one would usually employ a computer program which automates the generation and solution of the equilibrium equations.

The term "sideway" is used to denote the case where some of the members in a structure experience chord rotation resulting in "sway" of the structure. Whether sideway occurs depends on how the members are arranged and also depends on the loading applied. For example, consider the frame shown in Fig. 10.13a. Sideway is not possible because of the horizontal restraint. The frame shown in Fig. 10.13b is symmetrical and also loaded symmetrically. Because of symmetry, there will be no sway. The frame shown in Fig. 10.13c will experience sideway. The symmetrical frame shown in Fig. 10.13d will experience sideway because of the unsymmetrical loading. All three members will experience chord rotation for the frame shown in Fig. 10.13e.

When starting an analysis, one first determines whether sideway will occur in order to identify the nature of the displacement variables. The remaining steps are relatively straightforward. One establishes the free body diagram for each node and enforces the equilibrium equations. The essential difference is that now one needs to consider force equilibrium as well as moment equilibrium. We illustrate the analysis process with the following example.

Consider the frame shown in Fig. 10.14. Under the action of the applied loading, nodes B and C will displace horizontally an amount Δ. Both members AB and CD will have chord rotation. There are

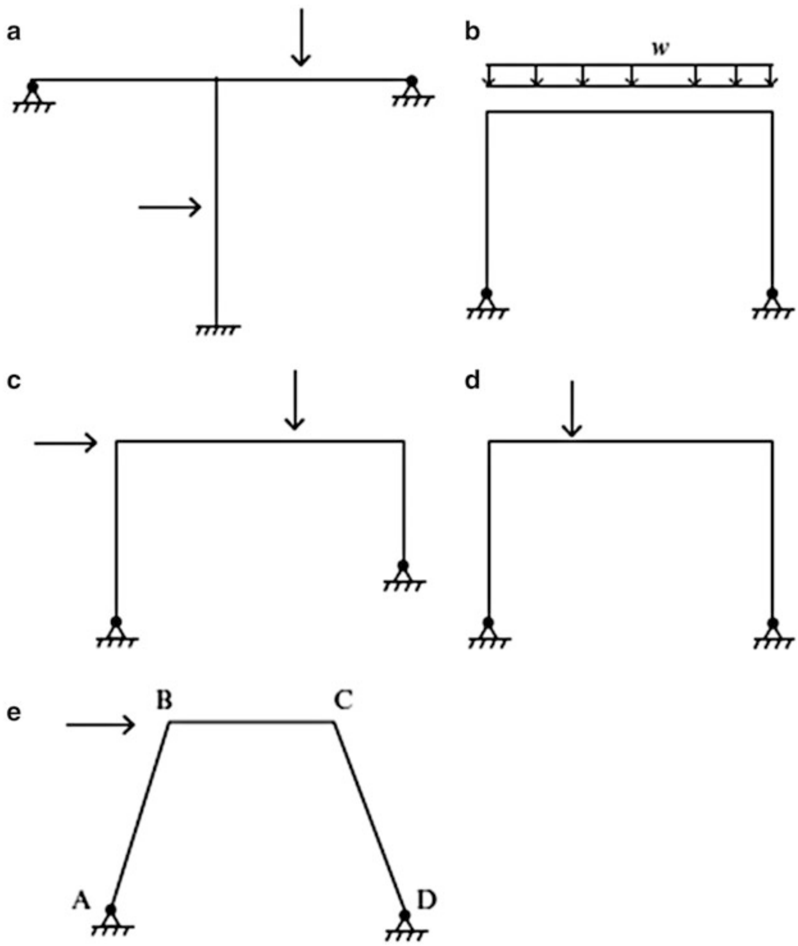


Fig. 10.13 Examples of sideway

three displacement unknowns θ_{B}, θ_{C}, and Δ. In general, we neglect the axial deformation. The free body diagrams for the members and nodes are shown in Fig. 10.15. We take the positive sense of the members to be from A → B, B → C, and C → D. Note that this fixes the sense of the shear forces. The end moments are always positive when counterclockwise.

Moment equilibrium for nodes B and C requires

$$\begin{aligned} \sum M_{\mathrm{B}} &= 0 \Rightarrow M_{\mathrm{BA}} + M_{\mathrm{BC}} = 0 \\ \sum M_{\mathrm{C}} &= 0 \Rightarrow M_{\mathrm{CB}} + M_{\mathrm{CD}} = 0 \end{aligned} \tag{10.23a}$$

We also need to satisfy horizontal force equilibrium for the entire frame.

$$\sum F_x = 0 \rightarrow + \Rightarrow -V_{\mathrm{AB}} + V_{\mathrm{DC}} + \sum F_x = 0 \tag{10.23b}$$

where $\sum F_x = P_1 + P_2 + \frac{1}{2} w_1 h_1$.

The latter equation is associated with sideway.

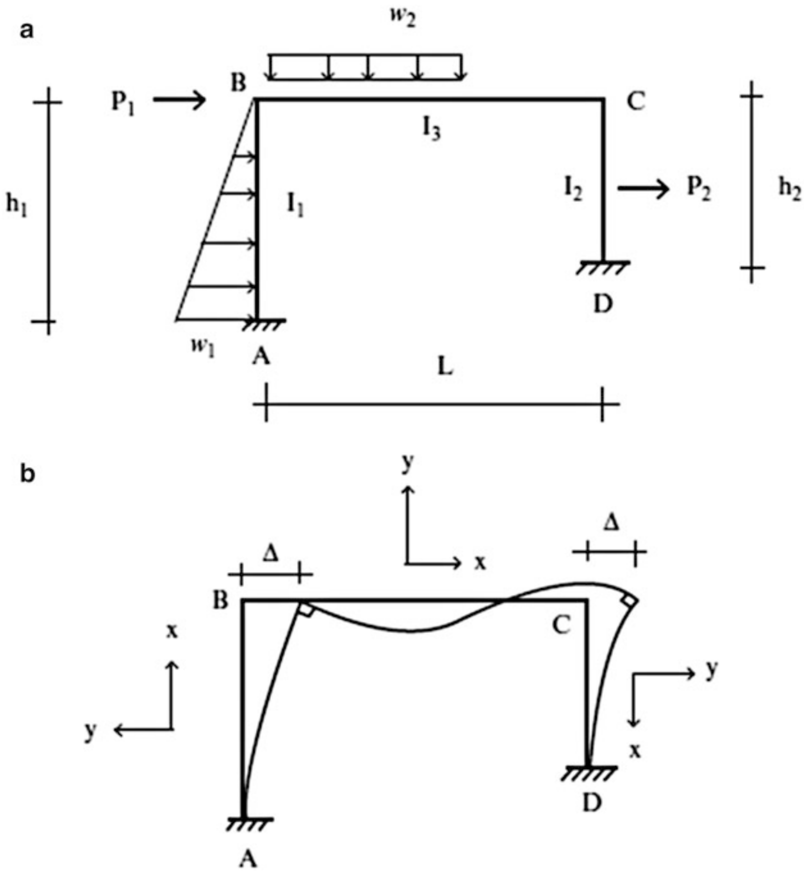
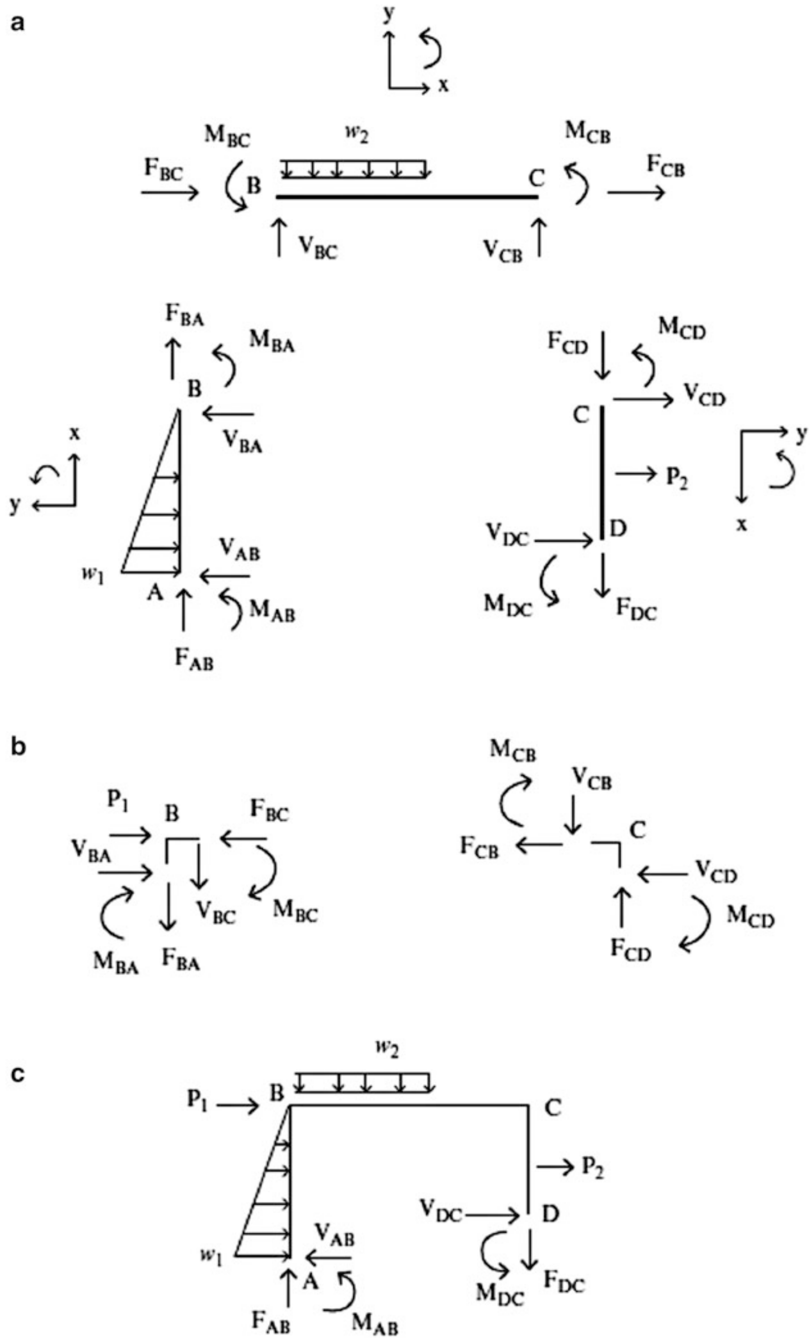


Fig. 10.14 (**a**) Loading. (**b**) Deflected shape

Noting that $\theta_A = \theta_D = 0$, $v_A = v_D = 0$, $v_B = -\Delta$, and $v_C = +\Delta$, the slope-deflection equations (10.12a, 10.12b) simplify to

$$\begin{aligned}
M_{AB} &= \frac{2EI_1}{h_1}\left\{\theta_B - 3\left(\frac{-\Delta}{h_1}\right)\right\} + M_{AB}^F \\
M_{BA} &= \frac{2EI_1}{h_1}\left\{2\theta_B - 3\left(\frac{-\Delta}{h_1}\right)\right\} + M_{BA}^F \\
M_{BC} &= \frac{2EI_3}{L}\{2\theta_B + \theta_C\} + M_{BC}^F \\
M_{CB} &= \frac{2EI_3}{L}\{2\theta_C + \theta_B\} + M_{CB}^F \\
M_{CD} &= \frac{2EI_2}{h_2}\left\{2\theta_C - 3\left(\frac{-\Delta}{h_2}\right)\right\} + M_{CD}^F \\
M_{DC} &= \frac{2EI_2}{h_2}\left\{\theta_C - 3\left(\frac{-\Delta}{h_2}\right)\right\} + M_{DC}^F \\
V_{AB} &= \frac{6EI_1}{h_1^2}\left\{\theta_B + 2\frac{\Delta}{h_1}\right\} + V_{AB}^F \\
V_{DC} &= -\frac{6EI_2}{h_2^2}\left\{\theta_C + 2\frac{\Delta}{h_2}\right\} + V_{DC}^F
\end{aligned} \tag{10.24}$$

Fig. 10.15 Free body diagrams for members and nodes of the frame. (**a**) Members. (**b**) Nodes. (**c**) Reactions



Substituting for the end moments and shear forces in (10.23a, 10.23b) leads to

$$\begin{aligned}
&\left(\frac{4EI_1}{h_1}+\frac{4EI_3}{L}\right)\theta_B+\frac{2EI_3}{L}\theta_C+\frac{6EI_1}{h_1}\left(\frac{\Delta}{h_1}\right)+\left(M^F_{BC}+M^F_{BA}\right)=0\\
&\frac{2EI_3}{L}\theta_B+\left(\frac{4EI_3}{L}+\frac{4EI_2}{h_2}\right)\theta_C+\frac{6EI_2}{h_2}\left(\frac{\Delta}{h_2}\right)+\left(M^F_{CD}+M^F_{CB}\right)=0\\
&-\frac{6EI_1}{h_1^2}\theta_B-\frac{6EI_2}{h_2^3}\theta_C+\left(-\frac{12EI_1}{h_1^3}-\frac{12EI_2}{h_2^3}\right)\Delta-V^F_{AB}+V^F_{DC}+\sum F_x=0
\end{aligned} \tag{10.25}$$

Once the loading and properties are specified, one can solve (10.25) for θ_B, θ_C, and Δ. The end actions are then evaluated with (10.24).

10.5.1 Portal Frames: Symmetrical Loading

Consider the symmetrical frame defined in Fig. 10.16. When the loading is also symmetrical, nodes B and C do not displace laterally, and therefore there is no chord rotation for members AB and CD. Also, the rotations at B and C are equal in magnitude but opposite in sense ($\theta_B = -\theta_C$). With these simplifications, the expressions for the end moments reduce to

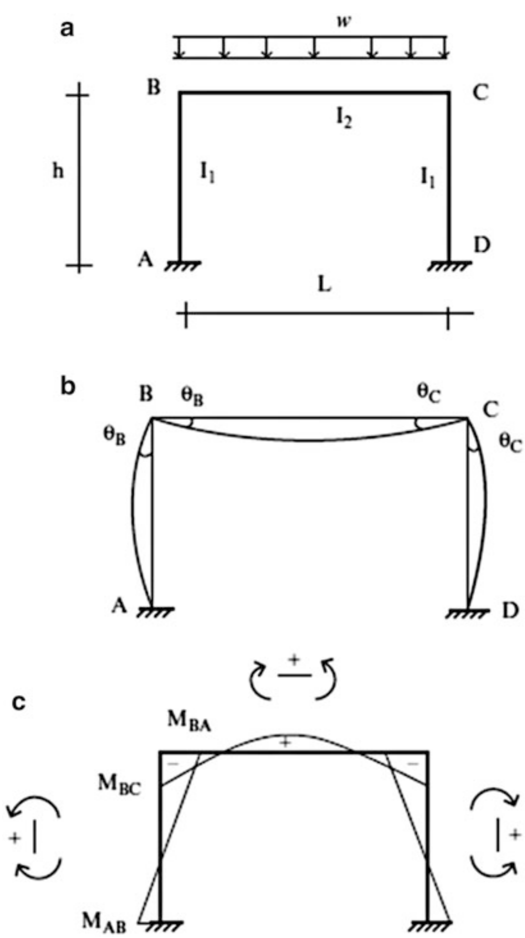


Fig. 10.16 Portal frame—symmetrical loading. (**a**) Loading. (**b**) Deflected shape. (**c**) Moment diagram

$$M_{BC} = \frac{2EI_2}{L}(\theta_B) + M_{BC}^F$$
$$M_{BA} = \frac{2EI_1}{h}(2\theta_B) \tag{10.26}$$
$$M_{AB} = \frac{1}{2}M_{BA}$$

Moment equilibrium for node B requires

$$M_{BC} + M_{BA} = 0 \tag{10.27}$$

Substituting for the moments, the equilibrium equation expands to

$$2E\theta_B\left\{2\frac{I_1}{h} + \frac{I_2}{L}\right\} = -M_{BC}^F$$

We solve for θ_B and then evaluate the end moments.

$$M_{BA} = 2M_{AB} = -M_{BC} = \frac{-1}{1 + (I_2/L)/2(I_1/h)}M_{BC}^F \tag{10.28}$$

The bending moment diagram is plotted in Fig. 10.16c.

10.5.2 Portal Frames: Anti-symmetrical Loading

Lateral loading produces anti-symmetrical behavior, as indicated in Fig. 10.17, and chord rotation for members AB and CD. In this case, the nodal rotations at B and C are equal in both magnitude and sense ($\theta_B = \theta_C$). The chord rotation is related to the lateral displacement of B by

$$\rho_{AB} = -\frac{v_B}{h} \tag{10.29}$$

Note that the chord rotation sign convention for the slope-deflection equations (10.12a, 10.12b) is positive when counterclockwise. Therefore for this choice of the sense of v_B, the chord rotation for AB is negative. The corresponding expressions for the end moments are

$$M_{BC} = 2E\frac{I_2}{L}(3\theta_B)$$
$$M_{BA} = 2E\frac{I_1}{h}\left(2\theta_B + \frac{3v_B}{h}\right) \tag{10.30}$$
$$M_{AB} = 2E\frac{I_1}{h}\left(\theta_B + \frac{3v_B}{h}\right)$$

Equilibrium requires

$$M_{BC} + M_{BA} = 0$$
$$-V_{AB} + \frac{P}{2} = 0 \tag{10.31}$$

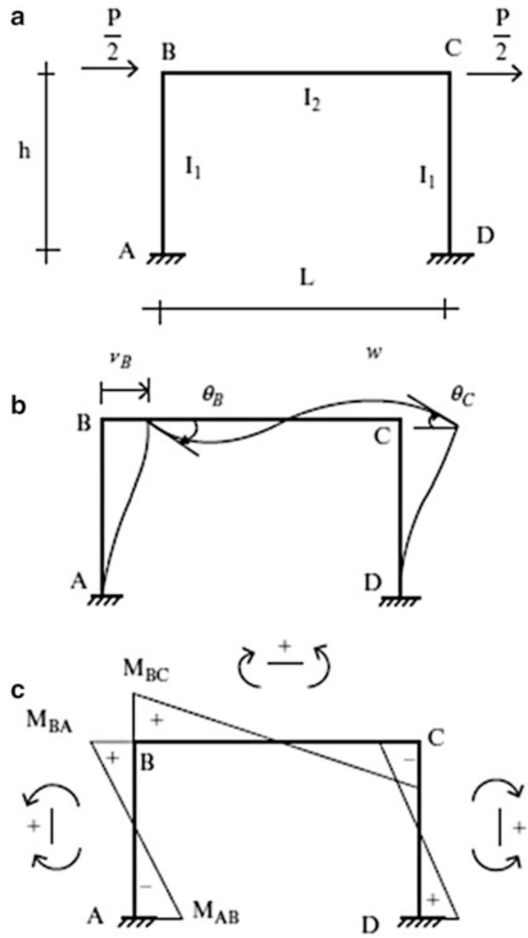
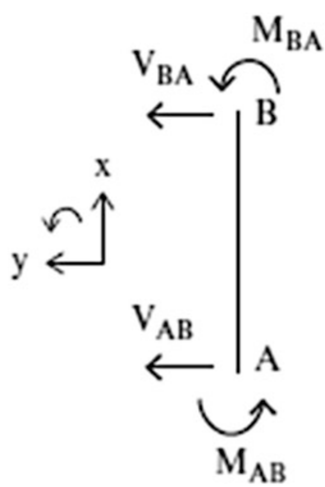


Fig. 10.17 Portal frame—anti-symmetric loading. (**a**) Loading. (**b**) Deflected shape. (**c**) Moment diagram

We determine the end shear with the moment equilibrium equation for member AB.

$$V_{\mathrm{AB}} = \frac{M_{\mathrm{BA}} + M_{\mathrm{AB}}}{h} = \frac{2EI_1}{h^2}\left(3\theta_{\mathrm{B}} + 6\,\frac{v_{\mathrm{B}}}{h}\right) \tag{10.32}$$



Substituting for the moments and shear in (10.31) leads to two equations with two unknowns, θ_B and v_B. The solution has the following form

$$\theta_B = \frac{-h^2 P}{4EI_1}\frac{1}{1+6(I_2/L)/(I_1/h)}$$
$$v_B = \frac{h^3 P}{24EI_1}\left(\frac{1+\frac{2}{3}((I_1/h)/(I_2/L))}{1+(I_1/h)/6(I_2/L)}\right) \tag{10.33}$$

We evaluate M_{BA} and M_{AB} using (10.30).

$$M_{BA} = \frac{Ph}{4}\frac{1}{1+\frac{1}{6}(I_1/h)/(I_2/L)}$$
$$M_{AB} = \frac{Ph}{4}\frac{1+1/3(I_1/h)/(I_2/L)}{1+1/6(I_1/h)/(I_2/L)} \tag{10.34}$$

A typical moment diagram is shown in Fig. 10.17c. Note the sign convention for bending moment. When the girder is very stiff with respect to the column, $I_2/L \gg I_1/h$ the solution approaches

$$\theta_B \rightarrow 0$$
$$v_B \rightarrow \frac{h^3 P}{24EI_1}$$
$$M_{BA} \rightarrow \frac{Ph}{4}$$
$$M_{AB} \rightarrow \frac{Ph}{4} \tag{10.35}$$

Example 10.7: Frame with No Sideway

Given: The frame defined in Fig. E10.7a.

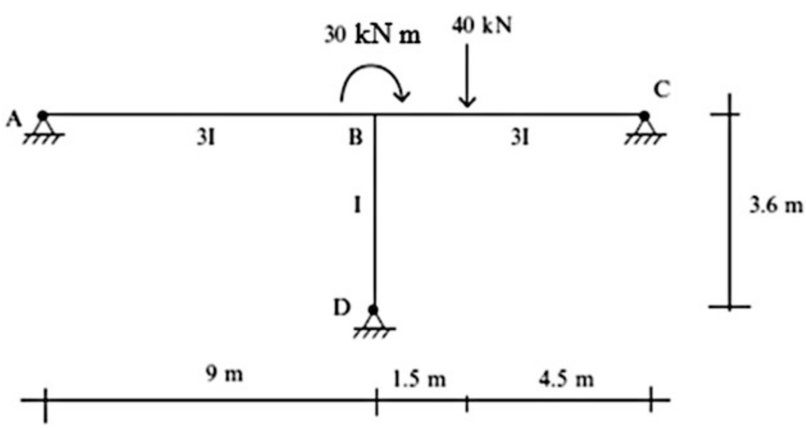


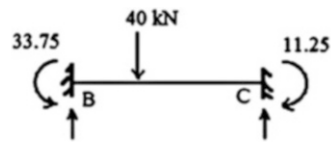
Fig. E10.7a

Determine: The end actions.

Solution: The fixed end moments are (see Table 9.1)

$$M_{\mathrm{BC}}^{\mathrm{F}} = \frac{40(1.5)(4.5)^2}{(6)^2} = 33.75\,\mathrm{kN\,m}$$

$$M_{\mathrm{CB}}^{\mathrm{F}} = \frac{40(4.5)(1.5)^2}{(6)^2} = -11.25\,\mathrm{kN\,m}$$



The modified slope-deflection equations (10.22a) which account for moment releases at A, C, and D are

$$M_{\mathrm{AB}} = M_{\mathrm{DB}} = M_{\mathrm{CB}} = 0$$

$$M_{\mathrm{BA}} = M_{\mathrm{BA_{modified}}} = 3\frac{E(3I)}{9}(\theta_{\mathrm{B}}) = EI\theta_{\mathrm{B}}$$

$$M_{\mathrm{BD}} = M_{\mathrm{BD_{modified}}} = 3\frac{E(I)}{3.6}(\theta_{\mathrm{B}}) = (0.83)EI\theta_{\mathrm{B}}$$

$$M_{\mathrm{BC}} = M_{\mathrm{BC_{modified}}} = 3\frac{E(3I)}{6}(\theta_{\mathrm{B}}) + \left\{M_{\mathrm{BC}}^{\mathrm{F}} - \frac{1}{2}M_{\mathrm{CB}}^{\mathrm{F}}\right\} = (1.5)EI\theta_{\mathrm{B}} + 39.375$$

Moment equilibrium for node B requires (Fig. E10.7b)

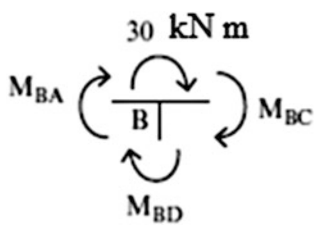


Fig. E10.7b

$$M_{\mathrm{BA}} + M_{\mathrm{BC}} + M_{\mathrm{BD}} + 30 = 0$$

$$\Downarrow$$

$$EI\theta_{\mathrm{B}} + (0.83)EI\theta_{\mathrm{B}} + (1.5)EI\theta_{\mathrm{B}} + 39.375 + 30 = 0$$

$$\Downarrow$$

$$EI\theta_{\mathrm{B}} = -20.83$$

The final bending moments at B are

$$M_{BA} = -20.8 \Rightarrow M_{BA} = 20.8\,\text{kN m clockwise}$$
$$M_{BD} = -17.3 \Rightarrow M_{BD} = 17.3\,\text{kN m clockwise}$$
$$M_{BC} = 8.1 \quad \Rightarrow M_{BC} = 8.1\,\text{kN m counterclockwise}$$

Noting the free body diagrams below, we find the remaining end actions (Figs. E10.7c and E10.7d).

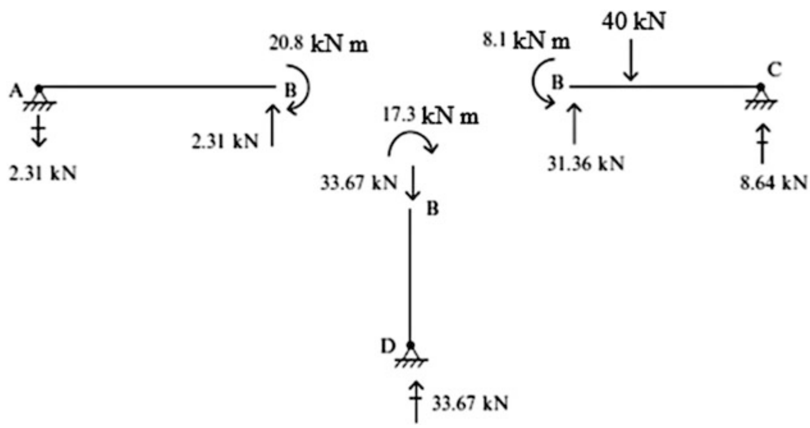


Fig. E10.7c Free body diagrams

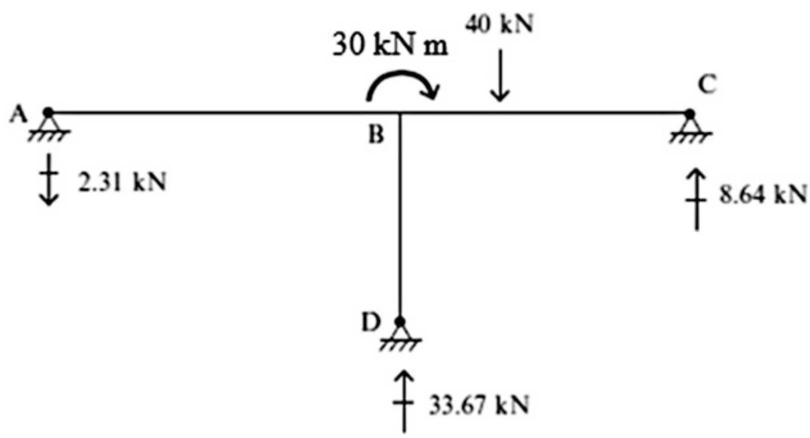


Fig. E10.7d Reactions

Example 10.8: Frame with Sideway

Given: The frame defined in Fig. E10.8a.

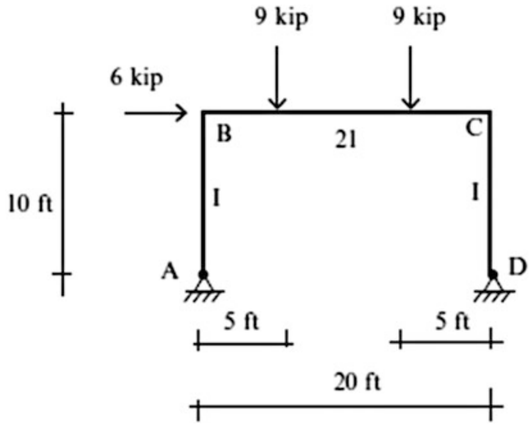


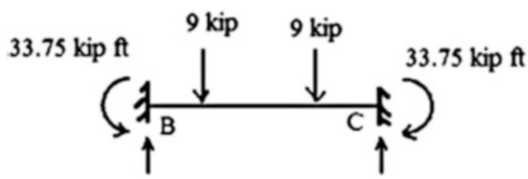
Fig. E10.8a

Determine: The end actions.

Solution: The fixed end moments are (see Table 9.1)

$$M_{BC}^{F} = \frac{9(5)(15)^2}{20^2} + \frac{9(15)(5)^2}{20^2} = 33.75\,\text{kip ft}$$

$$M_{CB}^{F} = -M_{BC}^{F} = -33.75\,\text{kip ft}$$



The chord rotations follow from the sketch below (Fig. E10.8b):

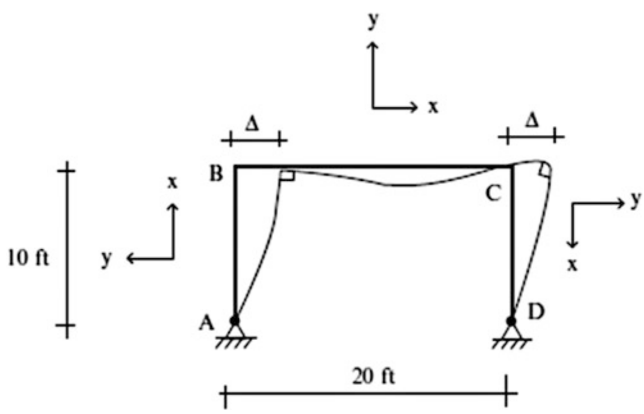


Fig. E10.8b

$$\rho = \rho_{AB} = \rho_{CD} = -\frac{\Delta}{10}$$

$$\rho_{BC} = 0$$

Substituting for the chord rotations in the slope-deflection equations [(10.12a, 10.12b) and (10.22a, 10.22b)] results in (Figs. E10.8c and E10.8d)

$$M_{AB} = M_{DC} = 0$$

$$M_{BA} = M_{BA_{modified}} = \frac{3E(I)}{10}\{\theta_B + \rho\} = 0.3EI\theta_B + 0.3EI\,\rho$$

$$M_{BC} = \frac{2E(2I)}{20}\{2\theta_B + \theta_C\} + 33.75 = 0.4EI\theta_B + 0.2EI\theta_C + 33.75$$

$$M_{CB} = \frac{2E(2I)}{20}\{\theta_B + 2\theta_C\} - 33.75 = 0.2EI\theta_B + 0.4EI\theta_C - 33.75$$

$$M_{CD} = M_{CD_{modified}} = \frac{3E(I)}{10}\{\theta_C + \rho\} = 0.3EI\theta_C + 0.3EI\,\rho$$

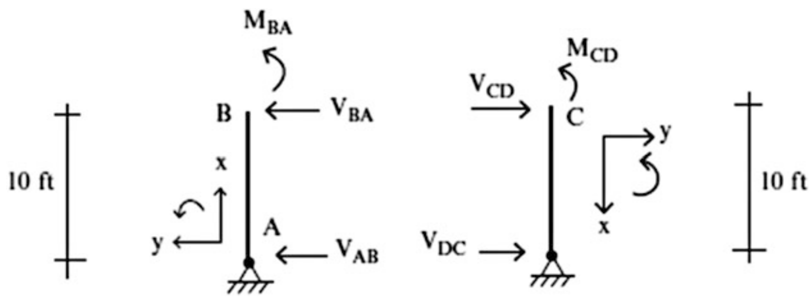


Fig. E10.8c

Also

$$V_{AB} = \frac{M_{BA}}{10}$$

$$V_{DC} = -\frac{M_{CD}}{10}$$

The end actions are listed in Fig. E10.8c.
Enforcing equilibrium at nodes B and C yields two equations,

$$M_{BC} + M_{BA} = 0 \rightarrow 0.7EI\theta_B + 0.2EI\theta_C + 0.3EI\,\rho + 33.75 = 0$$

$$M_{CB} + M_{CD} = 0 \rightarrow 0.2EI\theta_B + 0.7EI\theta_C + 0.3EI\,\rho - 33.75 = 0$$

Summing horizontal forces for the entire frame leads to an additional equation,

$$\sum F_x = 0 \quad \rightarrow + \qquad -V_{AB} + V_{DC} + 6 = 0$$

$$\Downarrow$$

$$-0.3EI\theta_B - 0.3EI\theta_C - 0.6EI\,\rho + 60 = 0$$

Solving these three equations, one obtains

$$\begin{cases} EI\theta_B = -117.5\,\text{kip ft}^2 \\ EI\theta_C = 17.5\,\text{kip ft}^2 \\ EI\,\rho = 150\,\text{kip ft}^2 \end{cases}$$

and then

$$\begin{cases} M_{BA} = +9.75 \\ M_{BC} = -9.75 \\ M_{CB} = -50.25 \\ M_{CD} = +50.25 \\ V_{AB} = .975 \\ V_{DC} = -5.25 \end{cases} \Rightarrow \begin{cases} M_{BA} = 9.75\,\text{kip ft} & \text{counterclockwise} \\ M_{BC} = 9.75\,\text{kip ft} & \text{clockwise} \\ M_{CB} = 50.25\,\text{kip ft} & \text{clockwise} \\ M_{CD} = 50.25\,\text{kip ft} & \text{counterclockwise} \\ V_{AB} = .975\,\text{kip} & \leftarrow \\ V_{DC} = 5.25\,\text{kip} & \leftarrow \end{cases}$$

Noting the free body diagrams below, we find the remaining end actions (Figs. E10.8d and E10.8e).

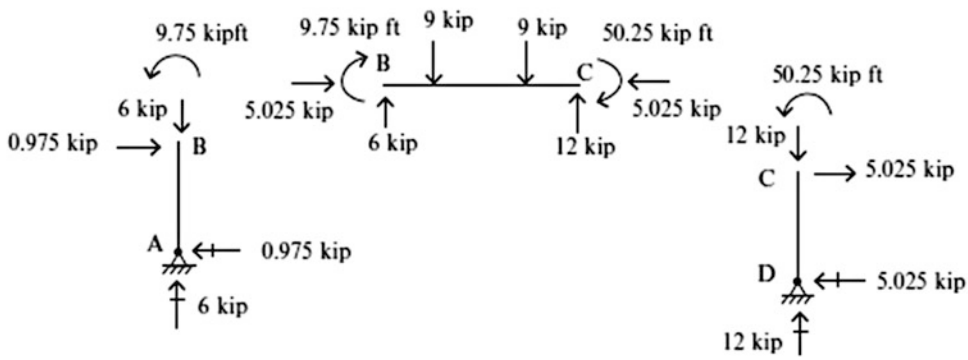


Fig. E10.8d Free body diagrams

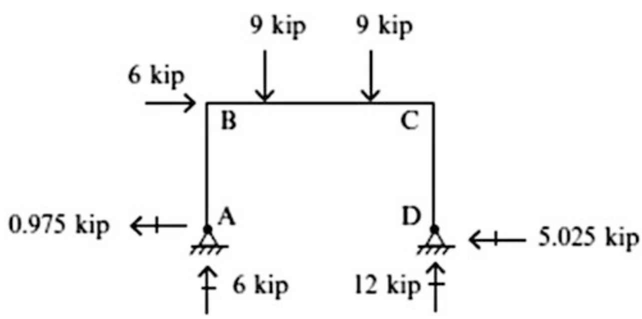


Fig. E10.8e Reactions

10.6 The Moment Distribution Solution Procedure for Multi-span Beams

10.6.1 Introduction

In the previous sections, we developed an analysis procedure for multi-span beams that is based on using the slope-deflection equations to establish a set of simultaneous equations relating the nodal rotations. These equations are equivalent to the nodal moment equilibrium equations. We generated the solution by solving these equations for the rotations and then, using these values, we determined the end moments and end shears. The solution procedure is relatively straightforward from a

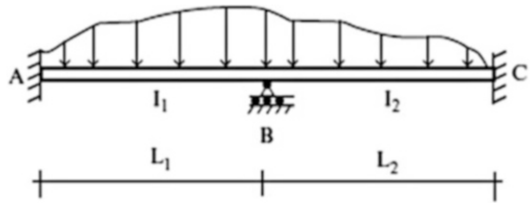


Fig. 10.18 Two-span beam with fixed ends

mathematical perspective, but it is difficult to gain some physical insight as to how the structure is responding during the solution process. This is typical of mathematical procedures which involve mainly number crunching and are ideally suited for computer-based solution schemes.

The Moment Distribution Method is a solution procedure developed by Structural Engineers to solve the nodal moment equilibrium equations. The method was originally introduced by Cross [1] and has proven to be an efficient hand-based computational scheme for beam- and frame-type structures. Its primary appeal is its computational simplicity.

The solution is generated in an iterative manner. Each iteration cycle involves only two simple computations. Another attractive feature is the fact that one does not have to formulate the nodal equilibrium equations expressed in terms of the nodal displacements. The method works directly with the end moments. This feature allows one to assess convergence by comparing successive values of the moments as the iteration progresses. In what follows, we illustrate the method with a series of beam-type examples. Later, we extend the method to frame-type structures.

Consider the two-span beam shown in Fig. 10.18. Supports A and C are fixed, and we assume that there is no settlement at B.

We assume initially that there is no rotation at B. Noting Fig. 10.19, the net unbalanced clockwise nodal moment at B is equal to the sum of the fixed end moments for the members incident on node B.

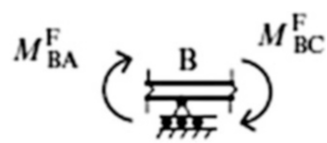


Fig. 10.19 Nodal moments

$$+\circlearrowright \; M_{\mathrm{B,net}} = +\sum M^{\mathrm{F}}_{\mathrm{B}} = +M^{\mathrm{F}}_{\mathrm{BA}} + M^{\mathrm{F}}_{\mathrm{BC}} \tag{10.36}$$

This unbalanced moment will cause node B to rotate until equilibrium is restored. Using the slope-deflection equations, we note that the increment in the end moment for a member which is incident on B due to a counterclockwise rotation at B is proportional to the relative stiffness I/L for the member.

The moments acting on the node are of opposite sense, i.e., clockwise, from Newton's law. The equilibrium state for the node is shown in Fig. 10.20.

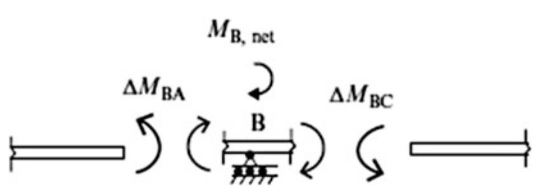


Fig. 10.20 Moment equilibrium for node B

Equilibrium requires the moment sum to vanish.

$$\Delta M_{\mathrm{BA}} + \Delta M_{\mathrm{BC}} + M_{\mathrm{B,net}} = 0$$

Substituting for the moment increments yields an equation for θ_{B}

$$\left(\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}\right)\theta_{\mathrm{B}} = -M_{\mathrm{B,net}}$$
$$\Downarrow \tag{10.37}$$
$$\theta_{\mathrm{B}} = \frac{1}{((4EI_1/L_1) + (4EI_2/L_2))}(-M_{\mathrm{B,net}})$$

Lastly, we use this value of θ_{B} to evaluate the incremental end moments.

$$\begin{aligned}\Delta M_{\mathrm{BA}} &= \frac{(4EI_1/L_1)}{((4EI_1/L_1) + (4EI_2/L_2))}(-M_{\mathrm{B,net}})\\ \Delta M_{\mathrm{BC}} &= \frac{(4EI_2/L_2)}{((4EI_1/L_1) + (4EI_2/L_2))}(-M_{\mathrm{B,net}})\end{aligned} \tag{10.38}$$

The form of the solution suggests that we define a dimensionless factor, DF, for each member as follows:

$$\begin{aligned}\mathrm{DF}_{\mathrm{BA}} &= \frac{I_1/L_1}{(I_1/L_1) + (I_2/L_2)}\\ \mathrm{DF}_{\mathrm{BC}} &= \frac{I_2/L_2}{(I_1/L_1) + (I_2/L_2)}\end{aligned} \tag{10.39}$$

Note that $\mathrm{DF}_{\mathrm{BA}} + \mathrm{DF}_{\mathrm{BC}} = 1.0$. With this notation, the expressions for the incremental end moments reduce to

$$\begin{aligned}\Delta M_{\mathrm{BA}} &= -\mathrm{DF}_{\mathrm{BA}}\left(M_{\mathrm{B,net}}\right)\\ \Delta M_{\mathrm{BC}} &= -\mathrm{DF}_{\mathrm{BC}}\left(M_{\mathrm{B,net}}\right)\end{aligned} \tag{10.40}$$

One distributes the unbalanced fixed end moment to the members incident on the node according to their distribution factors which depend on their relative stiffness.

The nodal rotation at B produces end moments at A and C. Again, noting the slope-deflection equations, these incremental moments are related to θ_{B} by

$$\begin{aligned}\Delta M_{\mathrm{AB}} &= \frac{2EI_1}{L_1}\theta_{\mathrm{B}} = \frac{2EI_1/L_1}{\{(4EI_1/L_1) + (4EI_2/L_2)\}}(-M_{\mathrm{B,net}}) = -\frac{1}{2}\mathrm{DF}_{\mathrm{BA}}(M_{\mathrm{B,net}})\\ \Delta M_{\mathrm{CB}} &= \frac{2EI_2}{L_2}\theta_{\mathrm{B}} = \frac{2EI_2/L_2}{\{(4EI_1/L_1) + (4EI_2/L_2)\}}(-M_{\mathrm{B,net}}) = -\frac{1}{2}\mathrm{DF}_{\mathrm{BC}}(M_{\mathrm{B,net}})\end{aligned} \tag{10.41}$$

Comparing (10.41) with (10.40), we observe that the incremental moments at the far end are ½ the magnitude at the distributed moments at the near end.

$$\Delta M_{AB} = \frac{1}{2}\Delta M_{BA}$$
$$\Delta M_{CB} = \frac{1}{2}\Delta M_{BC} \qquad (10.42)$$

We summarize the moment distribution procedure for this example. The steps are:

1. Determine the distribution factors at each free node (only node B in this case)
2. Determine the fixed end moments due to the applied loading and chord rotation for the beam segments.
3. Sum the fixed end moments at node B. This sum is equal to the unbalanced moment at node B.
4. Distribute the unbalanced nodal moment to the members incident on node B.
5. Distribute one half of the incremental end moment to the other end of each member incident on node B.

Executing these steps is equivalent to formulating and solving the nodal moment equilibrium equations at node B. Moment distribution avoids the operation of setting up and solving the equations. It reduces the effort to a series of simple computations.

Example 10.9: Moment Distribution Method Applied to a Two-Span Beam

Given: The two-span beams shown in Fig. E10.9a.

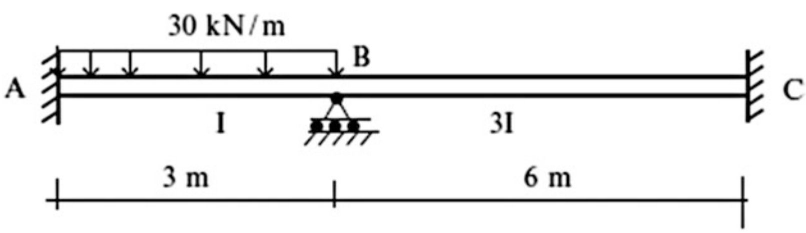


Fig. E10.9a

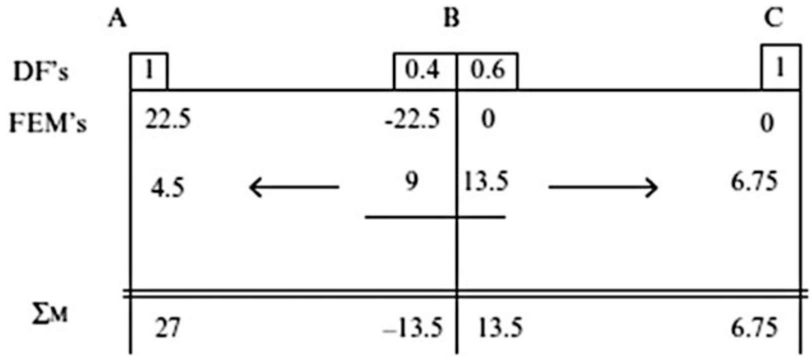
Determine: The end moments using moment distribution.

Solution: The fixed end moments and the distribution factors for node B are listed below.

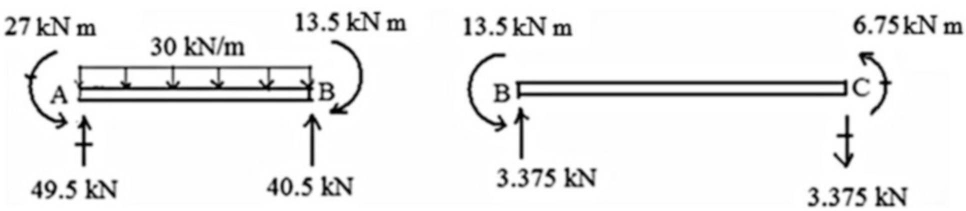
$$M^F_{AB} = 30\frac{(3)^2}{12} = 22.5\,\text{kN m} \qquad M^F_{BC} = 0$$
$$M^F_{BA} = -22.5\,\text{kN m} \qquad M^F_{CB} = 0$$

$$\text{At joint B}\begin{cases}\sum\frac{I}{L} = \frac{1}{3} + \frac{3I}{6} = \frac{5I}{6}\\ \text{DF}_{BA} = \frac{I/3}{5I/6} = 0.4 \quad \text{DF}_{BC} = \frac{3I/6}{5I/6} = 0.6\end{cases}$$

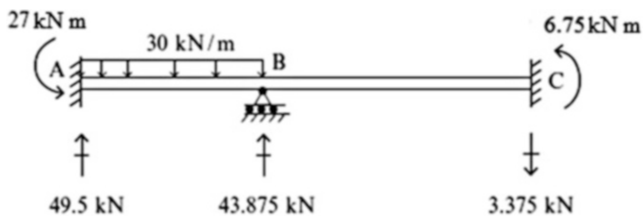
It is convenient to list the end moments and distribution factors on a sketch superimposed on the multi-span beam. A typical sketch is shown below. We distribute the 22.5 kN m unbalanced moment at B and carry over the moments to A and C. After one distribution, moment equilibrium at B is restored.



Since the end moments are known, one can determine the end shear forces using the static equilibrium equations for the member.



Lastly, the reactions are listed below.



10.6.2 Incorporation of Moment Releases at Supports

We consider next the case where an end member has a moment release, as shown in Fig. 10.21. We work with the modified slope-deflection equation for member AB developed in Sect. 10.3. The end moments are given by (10.22a, 10.22b) which is listed below for convenience.

$$M_{\mathrm{BA}_{\mathrm{modified}}} = \frac{3EI_1}{L_1}\theta_{\mathrm{B}} + \left\{ M^{\mathrm{F}}_{\mathrm{BA}} - \frac{1}{2} M^{\mathrm{F}}_{\mathrm{AB}} \right\} = \frac{3EI_1}{L_1}\theta_{\mathrm{B}} + M^{\mathrm{F}}_{\mathrm{BA}_{\mathrm{modified}}}$$

$$M_{\mathrm{AB}} = 0$$

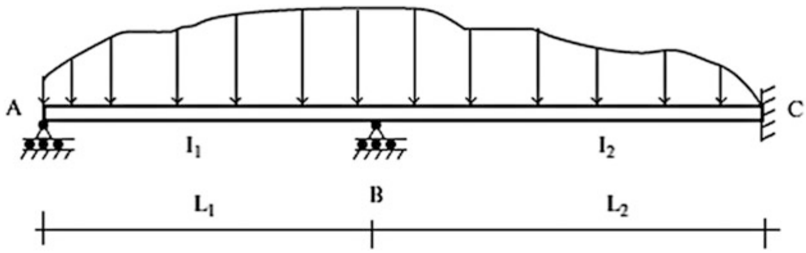


Fig. 10.21 Two-span beam with a moment release at a support

Then, the increment in moment for member BA due to a rotation at B is

$$\Delta M_{\mathrm{BA}} = 4E\left(\frac{3I_1}{4L_1}\right)\theta_{\mathrm{B}} \qquad \Delta M_{\mathrm{AB}} = 0 \tag{10.43}$$

We use a *reduced relative rigidity* factor $(3/4)I_1/L_1$ when computing the distribution factor for node B. Also, we use a *modified* fixed end moment (see Table 9.2). There is *no* carry-over moment to A.

Example 10.10: Two-Span Beam with a Moment Release at One End

Given: The beam shown in Fig. E10.10a.

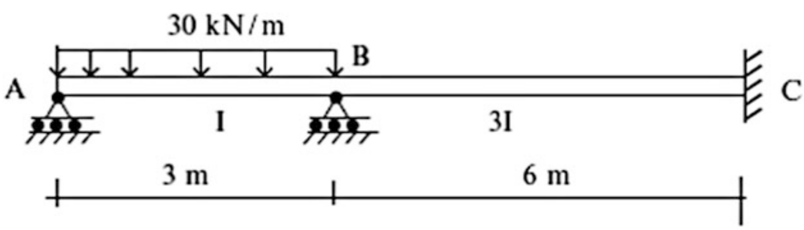
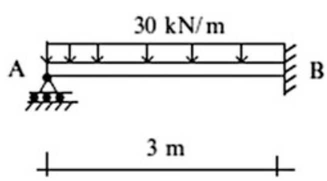


Fig. E10.10a

Determine: The end actions.

Solution: Since member AB has a moment release, we work with the modified slope-deflection equation for member AB. The computational details are listed below.

The modified fixed end moments (see Table 9.2):



$$M^{\mathrm{F}}_{\mathrm{BA_{modified}}} = M^{\mathrm{F}}_{\mathrm{BA}} - \frac{1}{2}M^{\mathrm{F}}_{\mathrm{AB}} = -\frac{30(3)^2}{8} = -33.75\,\mathrm{kN\,m}$$

$$M^{\mathrm{F}}_{\mathrm{AB}} = 0$$

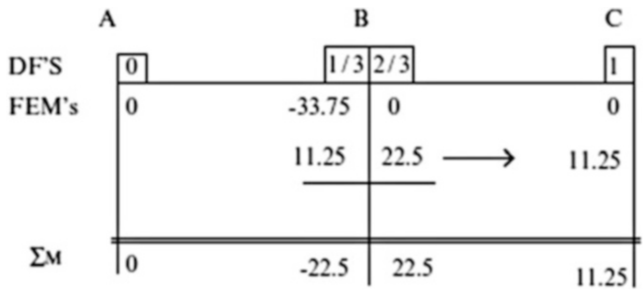
The modified distribution factors for node B:

$$\sum_{\text{jointB}} \frac{1}{L} = \frac{3}{4}\left(\frac{I}{3}\right) + \left(\frac{3I}{6}\right) = \frac{3I}{4}$$

$$\mathrm{DF}_{\mathrm{BA}} = \frac{(3/4)(I/3)}{(3I/4)} = \frac{1}{3}$$

$$\mathrm{DF}_{\mathrm{BC}} = 1 - \mathrm{DF}_{\mathrm{BA}} = \frac{2}{3}$$

The distribution details are listed below.



Noting the free body diagrams below, we find the remaining end actions (Figs. E10.10b and E10.10c).

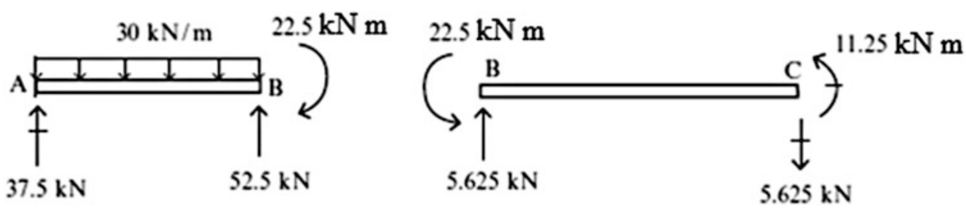


Fig. E10.10b Free body diagrams

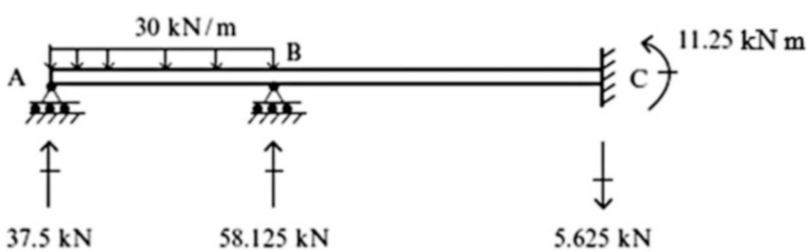


Fig. E10.10c Reactions

10.6.3 Moment Distribution for Multiple Free Nodes

The previous examples have involved only a single free node. We now extend the method for multiple free nodes. The overall approach is the same. We just have to incorporate an *iterative* procedure for successively balancing the nodal moments.

Consider the three-span beam shown in Fig. 10.22. We assume nodes B and C are fixed, determine the fixed end moments for the members, and compute the unbalanced nodal moments at nodes B and C. If these moments are not equal to zero, the nodes will rotate until equilibrium is restored. Allowing a node, such as B, to rotate produces incremental end moments in members AB and BC equal to

$$\begin{aligned} \Delta M_{BA} &= \frac{4EI_1}{L_1}\theta_B \quad \Delta M_{AB} = \frac{1}{2}\Delta M_{BA} \\ \Delta M_{BC} &= \frac{4EI_2}{L_2}\theta_B \quad \Delta M_{CB} = \frac{1}{2}\Delta M_{BC} \end{aligned} \tag{10.44}$$

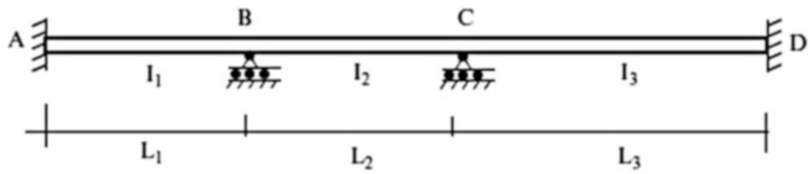


Fig. 10.22 Three span beam

Similarly, a rotation at node C produces incremental end moments in segment BC and CD.

$$\begin{aligned} \Delta M_{CB} &= \frac{4EI_2}{L_2}\theta_C \quad \Delta M_{BC} = \frac{1}{2}\Delta M_{CB} \\ \Delta M_{CD} &= \frac{4EI_3}{L_3}\theta_C \quad \Delta M_{DC} = \frac{1}{2}\Delta M_{CD} \end{aligned} \tag{10.45}$$

The distribution and carry-over process is the same as described previously. One evaluates the distribution factors using (10.39) and takes the carry-over factor as ½. Since there is more than one node, we start with the node having the *largest unbalanced moment*, distribute this moment, and carry over the distributed moment to the adjacent nodes. This operation changes the magnitudes of the remaining unbalanced moments. We then select the node with the "largest" new unbalanced moment and execute a moment distribution and carry-over at this node. The solution process proceeds by successively eliminating residual nodal moments at various nodes throughout the structure. At any step, we can assess the convergence of the iteration by examining the nodal moment residuals. Usually, only a few cycles of distribution and carry-over are sufficient to obtain reasonably accurate results.

Example 10.11: Moment Distribution Method Applied to a Three-Span Beam

Given: The three-span beam defined in Fig. E10.11a.

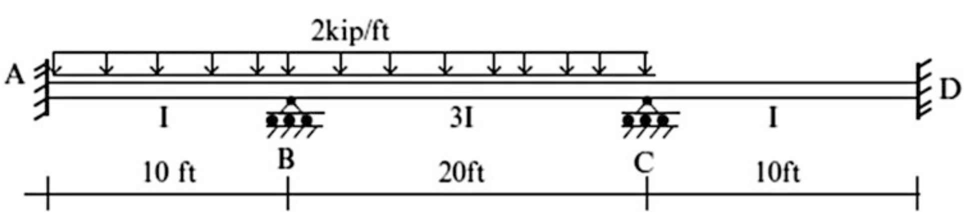


Fig. E10.11a

Determine: The end actions.

Solution: The sequence of nodal moment balancing is at the following nodes: C, B, C, B, C. We stop when the unbalanced nodal moment is approximately less than 0.5 kip ft.

The computations and distribution details are listed below.

$$M^{\mathrm{F}}_{\mathrm{AB}} = 2\frac{(10)^2}{12} = 16.67\,\text{kip ft} \quad M^{\mathrm{F}}_{\mathrm{BA}} = -16.67\,\text{kip ft}$$

$$M^{\mathrm{F}}_{\mathrm{BC}} = 2\frac{(20)^2}{12} = 66.67\,\text{kip ft} \quad M^{\mathrm{F}}_{\mathrm{CB}} = -66.67\,\text{kip ft}$$

$$M^{\mathrm{F}}_{\mathrm{CD}} = M^{\mathrm{F}}_{\mathrm{DC}} = 0$$

$$\text{At joint B or C} \begin{cases} \sum_{\text{modified}} \dfrac{I}{L} = \dfrac{I}{10} + \dfrac{3I}{20} = \dfrac{5I}{20} \\ \mathrm{DF_{BA}} = \mathrm{DF_{CD}} = \dfrac{I/10}{5I/20} = 0.4 \\ \mathrm{DF_{CB}} = \mathrm{DF_{BC}} = 1 - 0.4 = 0.6 \end{cases}$$

$$\mathrm{DF_{DC}} = \mathrm{DF_{AB}} = 1$$

	A	B	B	C	C	D
DF's	1	.4	.6	.4	.6	1
FEM's	**16.67**	**-16.67**	**66.67**	**-66.67**	**0**	**0**
			20 ←	40	26.67 →	13.33
	-14 ←	-28	-42 →	-21		
			6.3 ←	12.6	8.4 →	4.2
	-1.26 ←	-2.52	-3.78 →	-1.89		
			.56 ←	1.13	.75 →	.37
		-.22	-.34			
ΣM	1.4	-47.4	+47.4	-35.8	35.8	17.9

Noting the free body diagrams below, we find the remaining end actions (Figs. E10.11b and E10.11c).

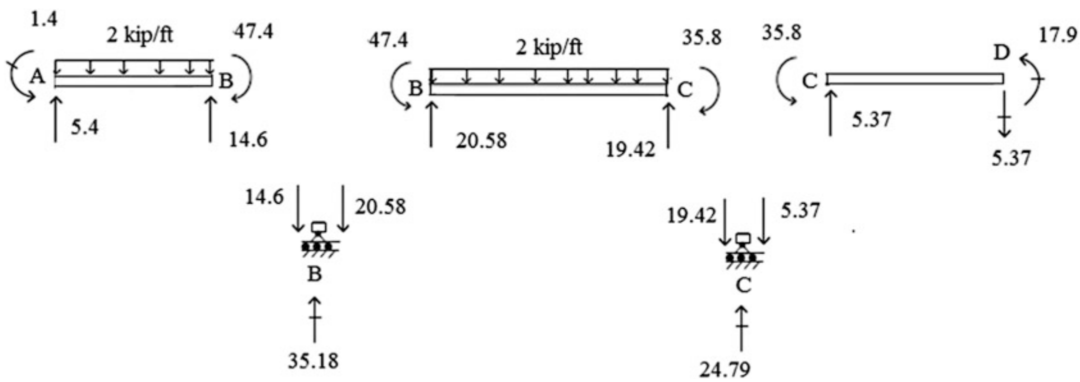


Fig. E10.11b Free body diagrams

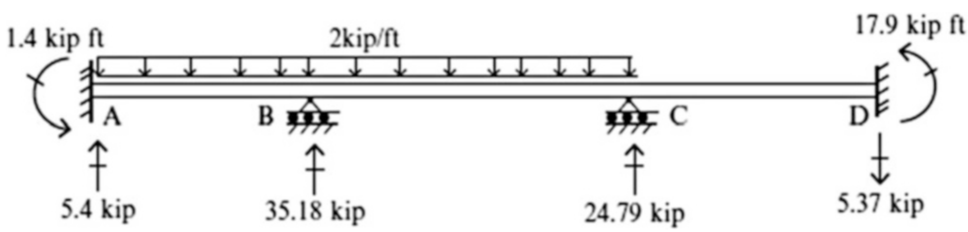


Fig. E10.11c Reactions

Example 10.12: Example 10.11 with Moment Releases at the End Supports

Given: A three-span beam with moment releases at its end supports (Figs. E10.12a, E10.12b, E10.12c).

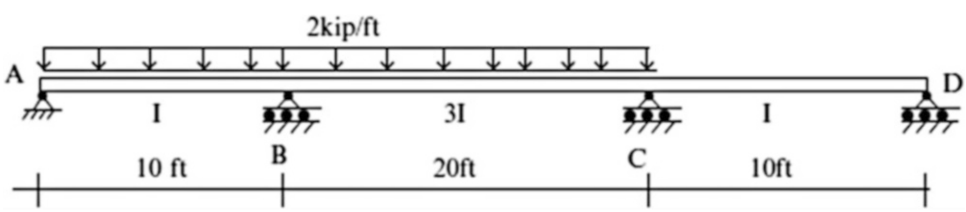
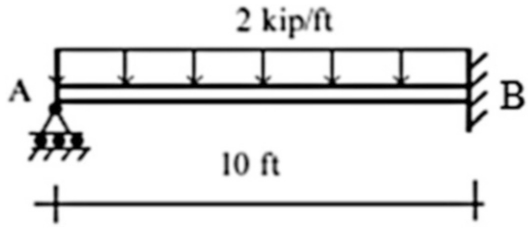


Fig. E10.12a

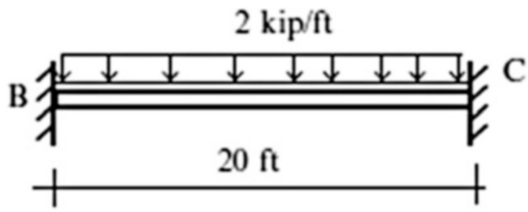
Determine: The end actions.

Solution: We rework Example 10.11 with moment releases at A and D. We use reduced relative rigidities for members AB and CD, and a modified fixed end moment for AB. There is no carry-over from B to A or from C to D. Details are listed below.



$$M^{F}_{BA_{modified}} = M^{F}_{BA} - \frac{1}{2} M^{F}_{AB} = -\frac{2(10)^2}{8} = -25\,\text{kip ft}$$

$$M^{F}_{AB_{modified}} = 0$$



$$M^{F}_{BC} = \frac{2(20)^2}{12} = 66.67\,\text{kip ft}$$

$$M^{F}_{CB} = -66.67\,\text{kip ft}$$

$$M^{F}_{CD} = M^{F}_{DC} = 0$$

$$\text{At joint B or C} \begin{cases} \sum_{\text{modified}} \frac{I}{L} = \frac{3}{4}\left(\frac{I}{10}\right) + \left(\frac{3I}{20}\right) = \frac{9I}{40} \\ \text{DF}_{BA_{modified}} = \text{DF}_{CD_{modified}} = \frac{3I/40}{9I/40} = \frac{1}{3} \\ \text{DF}_{CB} = \text{DF}_{BC} = 1 - \frac{1}{3} = \frac{2}{3} \end{cases}$$

$$\text{DF}_{DC} = \text{DF}_{AB} = 0$$

The distribution details and end actions are listed below.

	A	B	B		C	C	D
DF's	1	1/3	2/3		2/3	1/3	0
FEM's	0	-25	66.67		-66.67	0	0
			22.22	←	44.44	22.22	
		- 21.29	-42.59	→	- 21.29		
			7.1	←	14.2	7.1	
		-2.36	-4.74	→	-2.37		
			.79	←	1.56	.79	
		-.26	-.52	→	-.26		
					.17	.09	
ΣM	0	-48.9	48.9		-30.2	+30.2	0

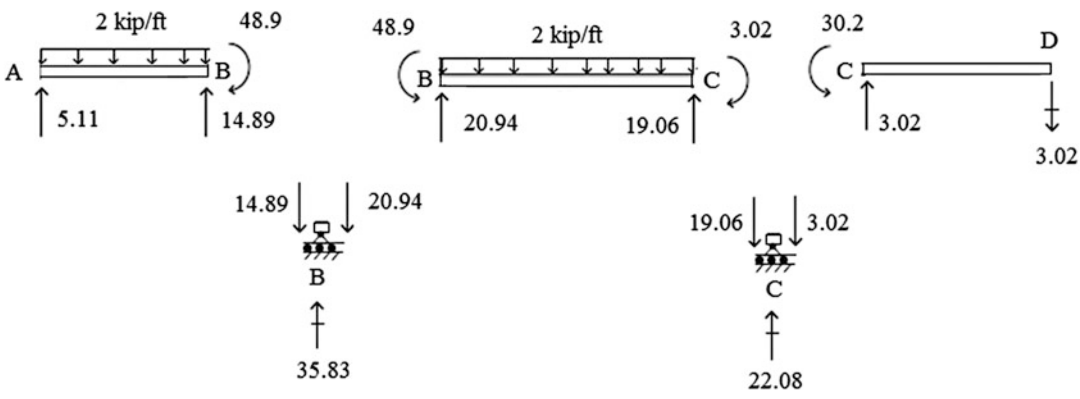


Fig. E10.12b Free body diagrams

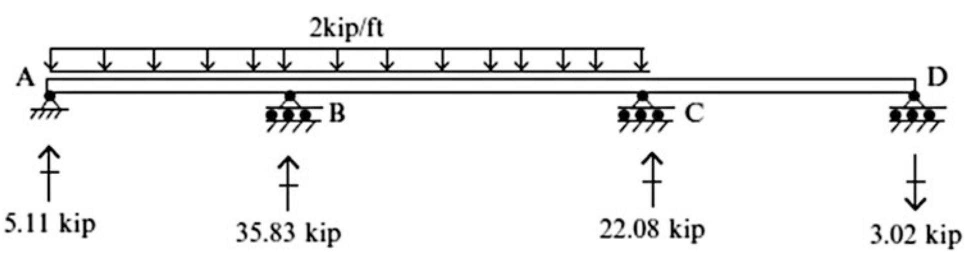


Fig. E10.12c Reactions

10.7 Moment Distribution: Frame Structures

10.7.1 Frames: No Sideway

Sideway does not occur if there is a lateral restraint. Frames with no sideway are treated in a similar way as beams. The following examples illustrate the process.

Example 10.13: Moment Distribution Method for a Frame with No Sideway

Given: The frame shown in Fig. E10.13a.

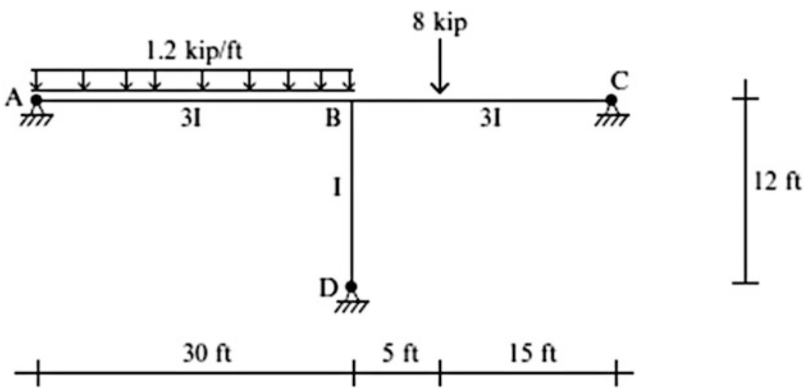


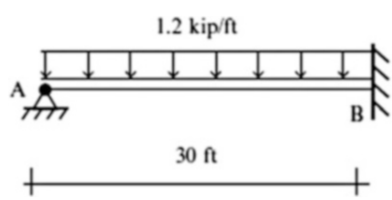
Fig. E10.13a

Determine: The end actions.

Solution: The distribution details and the fixed end moments and end actions are listed below.

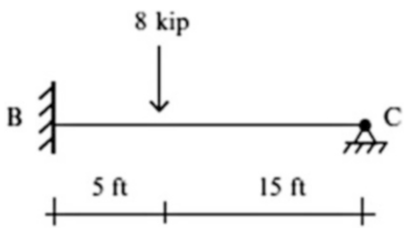
$$M^{\mathrm{F}}_{\mathrm{BA}_{\text{modified}}} = M^{\mathrm{F}}_{\mathrm{BA}} - \frac{1}{2}M^{\mathrm{F}}_{\mathrm{AB}} = -\frac{1.2(30)^2}{8} = -135\,\text{kip ft}$$

$$M^{\mathrm{F}}_{\mathrm{AB}_{\text{modified}}} = 0$$



$$M^{\mathrm{F}}_{\mathrm{BC}_{\text{modified}}} = M^{\mathrm{F}}_{\mathrm{BC}} - \frac{1}{2}M^{\mathrm{F}}_{\mathrm{CB}} = +\frac{21PL}{128} = 26.25\,\text{kip ft}$$

$$M^{\mathrm{F}}_{\mathrm{CB}_{\text{modified}}} = 0$$



$$\text{At joint B}\begin{cases} \sum_{\text{modified}} \frac{I}{L} = \frac{3}{4}\left(\frac{3I}{30} + \frac{3I}{20} + \frac{I}{12}\right) = \frac{I}{4} \\ DF_{BA_{\text{modified}}} = \frac{3/4(3I/30)}{I/4} = 0.3 \\ DF_{BC_{\text{modified}}} = \frac{3/4(3I/20)}{I/4} = 0.45 \\ DF_{BC_{\text{modified}}} = \frac{3/4(I/12)}{I/4} = 0.25 \end{cases}$$

The distribution details are listed in Fig. E10.13b.

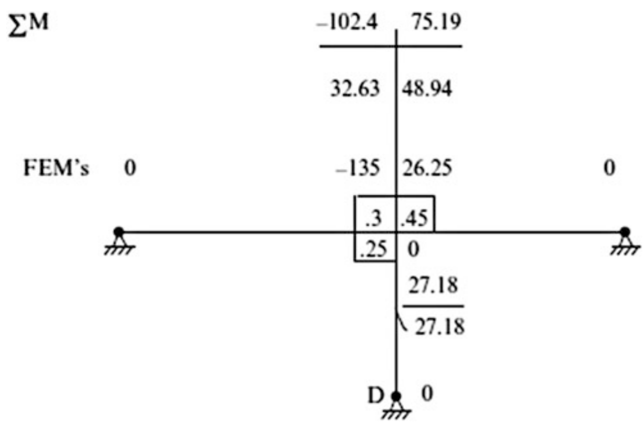


Fig. E10.13b

Noting the free body diagrams below, we find the remaining end actions (Fig. E10.13c).

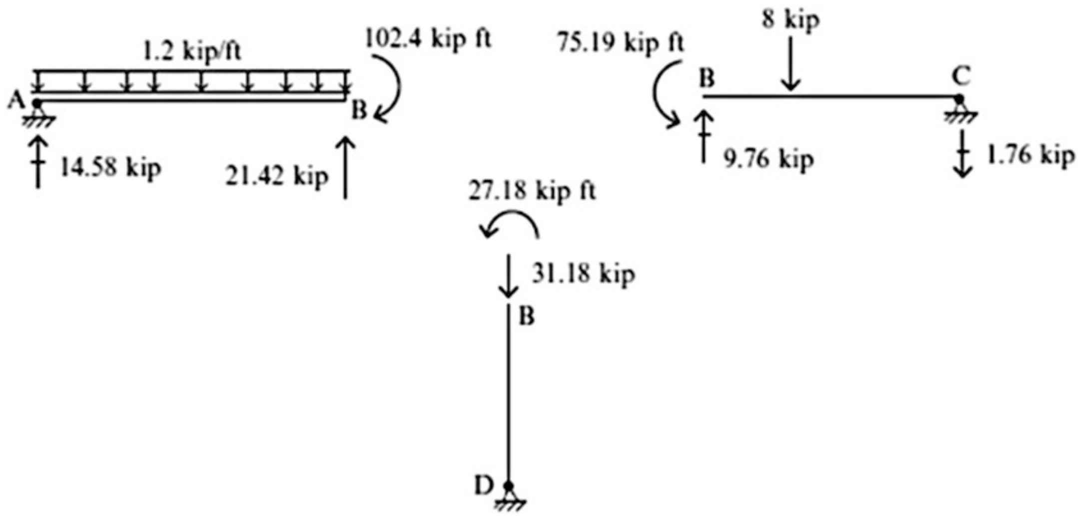


Fig. E10.13c End actions

Example 10.14: Symmetrical Two-Bay Portal Frame—Symmetrical Loading

Given: The two-bay frame defined in Fig. E10.14a.

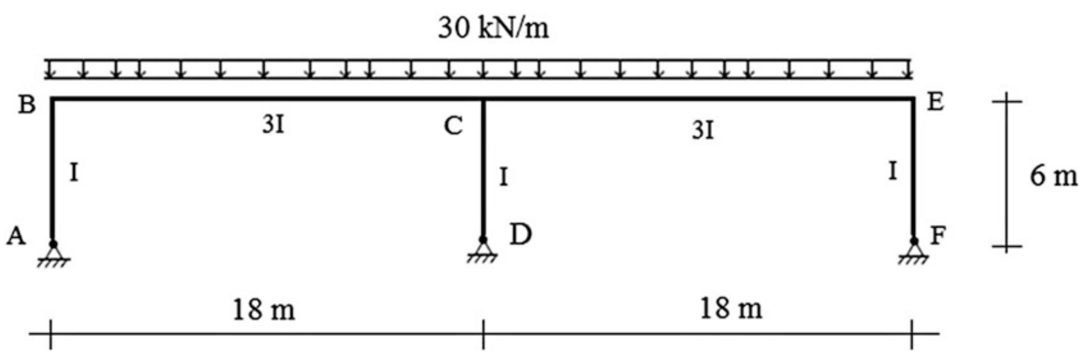


Fig. E10.14a

Determine: The bending moment distribution and end actions using moment distribution.

Solution: We use reduced rigidity factors for the column members and *no* carry-over to the hinged ends at nodes A, D, and F (Fig. E10.14b).

$$\text{At node B or E}\begin{cases}\sum \dfrac{I}{L} = \left(\dfrac{I}{6}\right)\dfrac{3}{4} + \left(\dfrac{3I}{18}\right) \\ \mathrm{DF}_{\mathrm{BA}_{\mathrm{modified}}} = \mathrm{DF}_{\mathrm{EF}_{\mathrm{modified}}} = \dfrac{(I/6)3/4}{(I/6)3/4 + (3I/18)} = \dfrac{3}{7} \\ \mathrm{DF}_{\mathrm{BC}} = \mathrm{DF}_{\mathrm{EC}} = 1 - \dfrac{3}{7} = \dfrac{4}{7}\end{cases}$$

$$\text{At node C}\begin{cases}\sum \dfrac{I}{L} = \left(\dfrac{I}{6}\right)\dfrac{3}{4} + \dfrac{3I}{18} + \dfrac{3I}{18} = \dfrac{11}{24}I \\ \mathrm{DF}_{\mathrm{CD}_{\mathrm{modified}}} = \dfrac{(I/6)3/4}{(11/24)I} = \dfrac{3}{11} \\ \mathrm{DF}_{\mathrm{CB}} = \mathrm{DF}_{\mathrm{CE}} = \dfrac{1}{2}\left(1 - \dfrac{3}{11}\right) = \dfrac{4}{11}\end{cases}$$

The fixed end moments are

$$M^{\mathrm{F}}_{\mathrm{BC}} = -M^{\mathrm{F}}_{\mathrm{CB}} = M^{\mathrm{F}}_{\mathrm{CE}} = -M^{\mathrm{F}}_{\mathrm{EC}} = +\frac{30(18)^2}{12} = +810\,\mathrm{kN\,m}$$

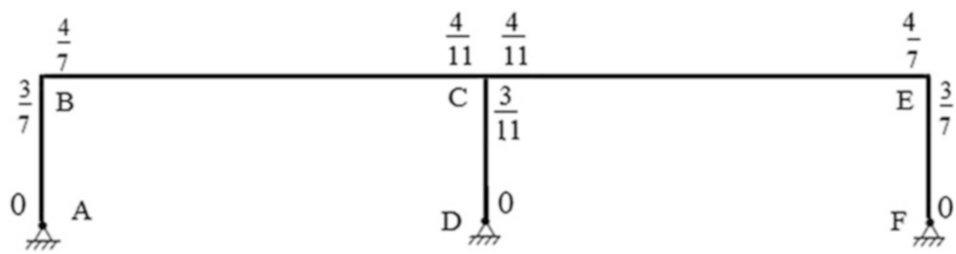


Fig. E10.14b Distribution factors

The moment distribution sequence is listed in Fig. E10.14c. Note that there is never any redistribution at node C because of symmetry (Fig. E10.14d).

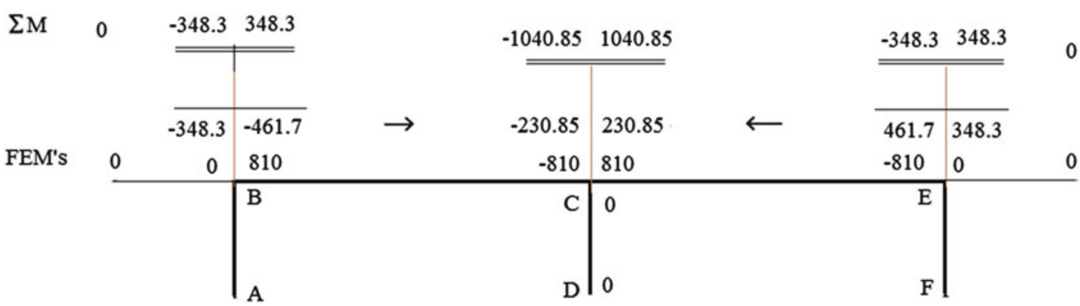


Fig. E10.14c

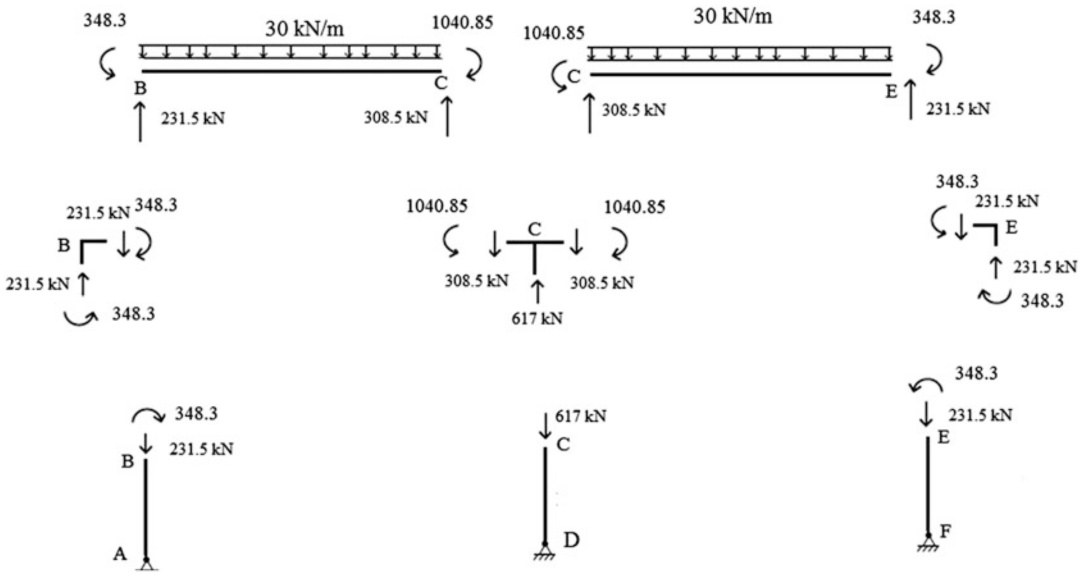


Fig. E10.14d Free body diagram

The final bending moment distributions are plotted in Fig. E10.14e.

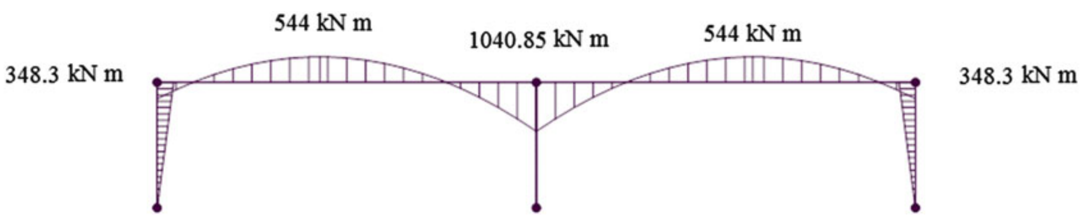


Fig. E10.14e

Example 10.15: Two-Bay Portal Frame—Support Settlement

Given: The frame shown in Fig. E10.15a. Consider Support D to experience a downward settlement of $\delta = 1$ in.

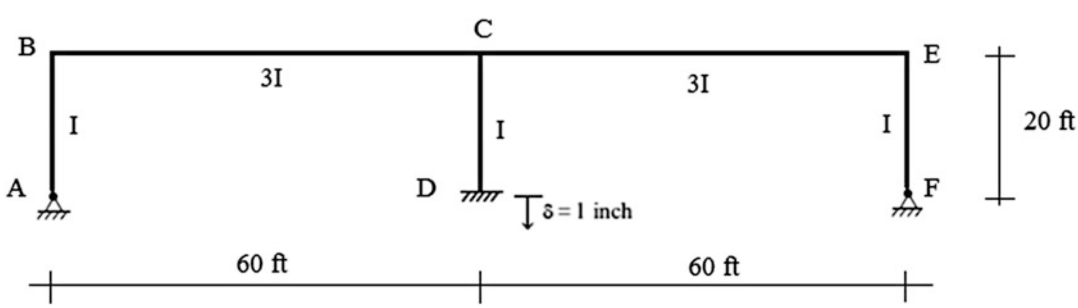


Fig. E10.15a

Determine: The end moments. Take $E = 29{,}000$ ksi and $I = 2000\ \text{in.}^4$

Solution: We use reduced factors for the column members AB and EF and no carry-over to the hinged ends. The distribution factors are listed on the following sketch (Fig. E10.15b).

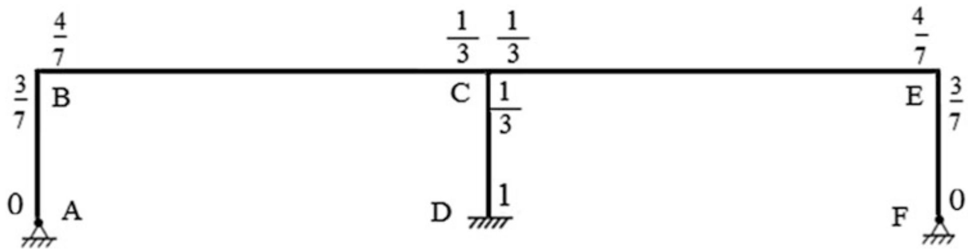


Fig. E10.15b Distribution factors

$$\text{At node B or E}\begin{cases}\sum\frac{I}{L}=\left(\frac{I}{20}\right)\frac{3}{4}+\left(\frac{3I}{60}\right)=\frac{7I}{80}\\ \text{DF}_{\text{BA}_{\text{modified}}}=\text{DF}_{\text{EF}_{\text{modified}}}=\frac{(I/20)3/4}{7I/80}=\frac{3}{7}\\ \text{DF}_{\text{BC}}=\text{DF}_{\text{EC}}=1-\frac{3}{7}=\frac{4}{7}\end{cases}$$

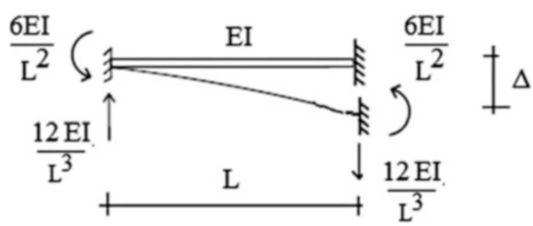
$$\text{At node C}\begin{cases}\sum\frac{I}{L}=\frac{I}{20}+\frac{3I}{60}+\frac{3I}{60}=\frac{3I}{20}\\ \text{DF}_{\text{CD}}=\text{DF}_{\text{CB}}=\text{DF}_{\text{CE}}=\frac{I/20}{3I/20}=\frac{1}{3}\end{cases}$$

Settlement at D produces chord rotation in members BC and CE. The corresponding rotations for a 1 in. settlement are

$$\rho_{\text{BC}}=-\frac{\delta}{L}$$

$$\rho_{\text{CE}}=+\frac{\delta}{L}$$

These rotations produce the following fixed end moments (see Table 9.1),



$$M^{\text{F}}_{\text{BC}}=M^{\text{F}}_{\text{CB}}=\frac{6E(3I)\delta}{L^2}=+\frac{18EI\delta}{L^2}=\frac{18(29,000)(2000)(1)}{(60)^2}\frac{1}{(12)^3}=167.8\,\text{kip ft}$$

$$M^{\text{F}}_{\text{CE}}=M^{\text{F}}_{\text{EC}}=-\frac{6E(3I)\delta}{L^2}=-\frac{18EI\delta}{L^2}=-167.8\,\text{kip ft}$$

These moments are distributed at nodes B and E. Note that no unbalanced moment occurs at node C (Fig. E10.15c).

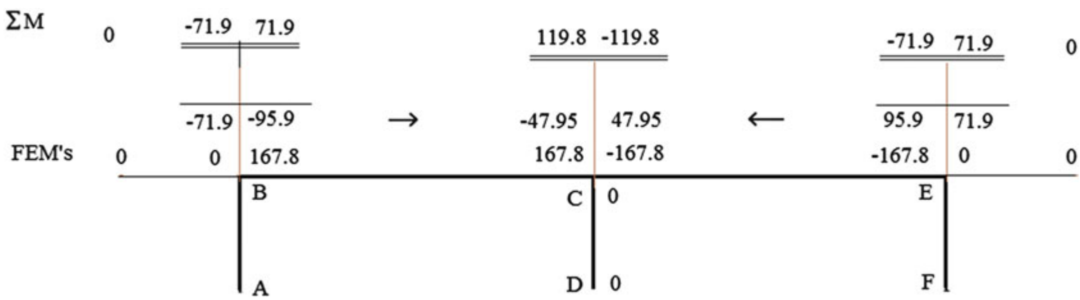


Fig. E10.15c

The final bending moment distributions are plotted in Fig. E10.15d.

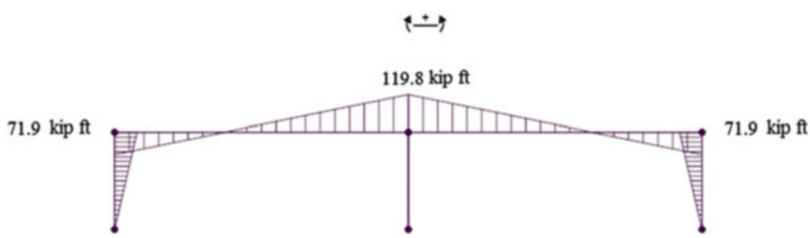


Fig. E10.15d

Example 10.16: Two-Bay Portal Frame—Temperature Increase

Given: The frame shown in Fig. E10.16a. Consider members BC and CE to experience a temperature increase of ΔT.

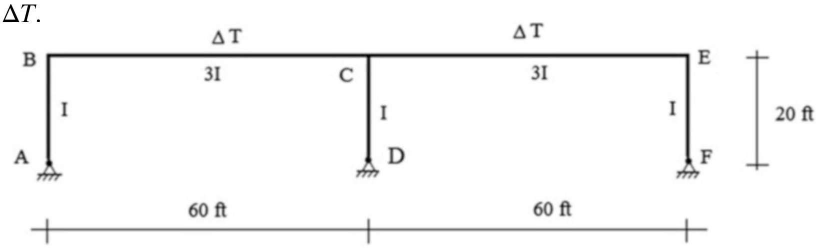


Fig. E10.16a

Determine: The end moments.

Solution: The top members will expand, causing members AB and EF to rotate. Member CD will not rotate because of symmetry. Noting Fig. E10.16b, the rotations are

$$\rho_{AB} = \frac{u/2}{L_{AB}}$$

$$\rho_{EF} = -\frac{u/2}{L_{EF}}$$

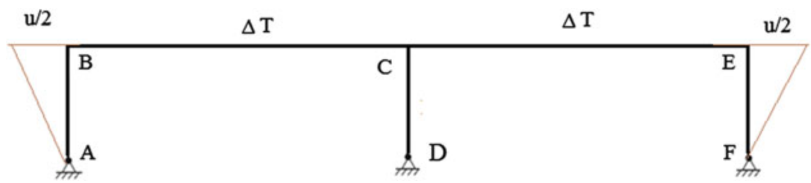


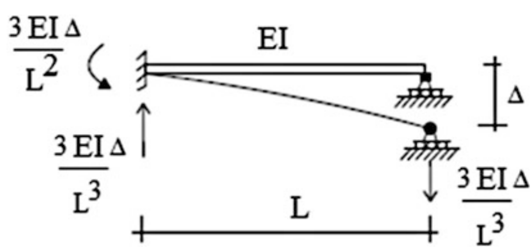
Fig. E10.16b

Assuming a uniform temperature increase over the total span, u is equal to

$$u = (\alpha \Delta T)\sum L = 120\alpha\Delta T$$

This motion is symmetrical and known. Therefore, there will be no additional displacement (therefore no additional sideway).

Noting Table 9.2, the fixed end actions corresponding to the case where there is a hinge at one end are



$$M^{F}_{BAmodified} = -\frac{3EI}{L_{AB}}\rho_{AB} = -\frac{3EI}{20(12)}(3\alpha\Delta T) = -\frac{3}{80}EI\alpha\Delta T$$

$$M^{F}_{EFmodified} = +\frac{3}{80}EI\alpha\Delta T$$

We assume the material is steel ($E = 3 \times 10^4$ ksi, $\alpha = 6.6 \times 10^{-6}/°F$), $\Delta T = 120$ °F, and $I = 2000$ in.^{4}.

The corresponding fixed end moments are

$$M^{F}_{BAmodified} = -1782\,\text{kip in.} = -148.5\,\text{kip ft}$$

$$M^{F}_{EFmodified} = +1782\,\text{kip in.} = +148.5\,\text{kip ft}$$

The distribution factors are

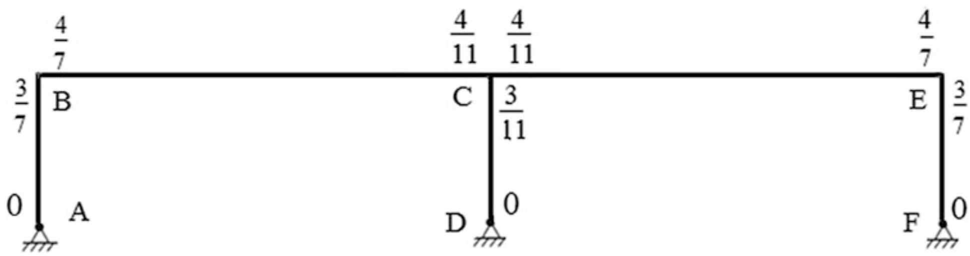
At node B or E

$$\begin{cases} \sum \dfrac{I}{L} = \left(\dfrac{I}{20}\right)\dfrac{3}{4} + \left(\dfrac{3I}{60}\right) = \dfrac{7I}{80} \\ DF_{BA_{modified}} = DF_{EF_{modified}} = \dfrac{(I/20)\dfrac{3}{4}}{7I/80} = 0.43 \\ DF_{BC} = DF_{EC} = 1 - 0.43 = 0.57 \end{cases}$$

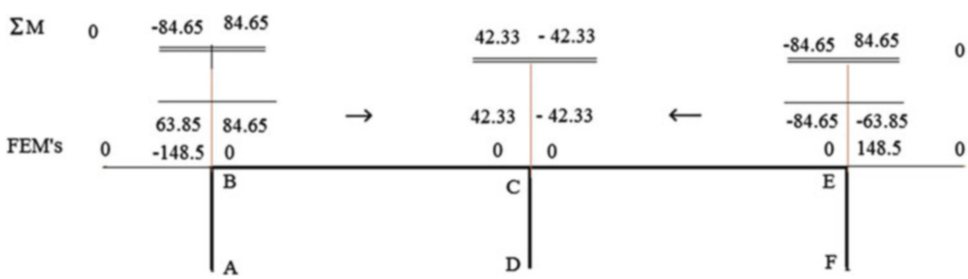
At node C

$$\begin{cases} \sum \dfrac{I}{L} = \left(\dfrac{I}{20}\right)\dfrac{3}{4} + \dfrac{3I}{60} + \dfrac{3I}{60} = \dfrac{11I}{80} \\ DF_{CD_{modified}} = \dfrac{(I/20)\dfrac{3}{4}}{11I/80} = 0.28 \\ DF_{CB} = DF_{CE} = \dfrac{1 - 0.28}{2} = 0.36 \end{cases}$$

The distribution factors are listed on the following sketch.



The distribution details are listed below.



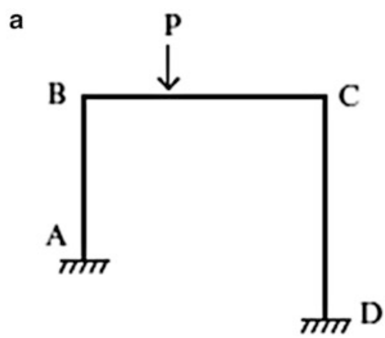
The final bending moment distributions are plotted in the following figure.

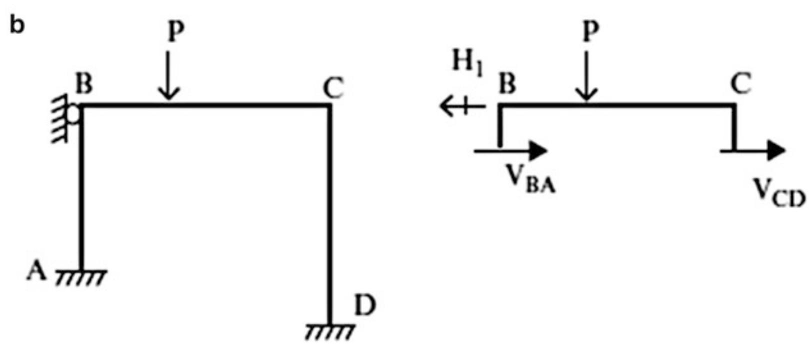
10.7.2 Frames with Sideway

Given a frame structure, one needs to identify whether there will be chord rotation due to lateral displacement. If sideway is possible, we introduce "holding" forces applied at certain nodes to prevent this motion and carry out a conventional moment distribution based on distribution and carry-over factors. Once the fixed end moments are distributed, we can determine the member shear forces, and using these values, establish the magnitude of the holding forces. This computation is illustrated in Fig. 10.23. There is one degree of sideway, and we restrain node B. The corresponding lateral force is H_1. Note that we generally neglect axial deformation for framed structures so fixing B also fixes C.

The next step involves introducing an arbitrary amount of the lateral displacement that we had restrained in Step 1, computing the chord rotations and corresponding fixed end moments, applying

Fig. 10.23 (**a**) Frame with sideway. (**b**) Sideway restraining force—case I





the holding force again, and then distributing the fixed end moments using the conventional distribution procedure. The holding force produced by this operation is illustrated in Fig. 10.24. *One combines the two solutions such that the resulting sideway force is zero*.

$$\text{Final solution} = \text{case I} + \left(\frac{H_1}{H_2}\right)\text{case II} \tag{10.46}$$

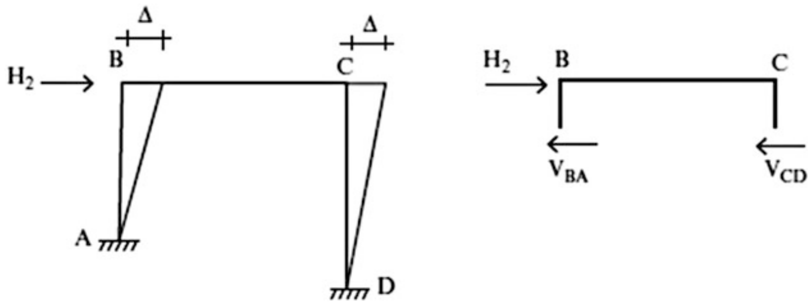


Fig. 10.24 Sideway introduced—case II

The fixed end moments due to the chord rotation produced by the horizontal displacement, Δ, are (see Table 9.1)

$$M_{BA}^{F} = M_{AB}^{F} = \pm\frac{6EI_{AB}}{L_{AB}^{2}}\Delta = \pm\frac{6EI_{AB}}{L_{AB}}\rho_{AB}$$

$$M_{DC}^{F} = M_{CD}^{F} = \pm\frac{6EI_{CD}}{L_{CD}^{2}}\Delta = \pm\frac{6EI_{CD}}{L_{CD}}\rho_{CD}$$

where moment quantities are counterclockwise when positive. Following this approach, one works only with chord rotation quantities and converts these measures into equivalent fixed end moments. The standard definition equations for the distribution and carry-over factors are employed to distribute the moments.

Example 10.17: Portal Bent—Sideway Analysis

Given: The portal frame defined in Fig. E10.17a.

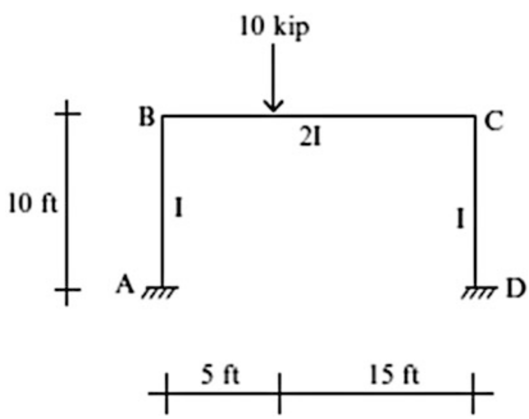


Fig. E10.17a

Determine: The end actions.

Solution: Since the loading is not symmetrical, there will be lateral motion (sideway). We restrain node B as indicated in Fig. E10.17b. The distribution factors are also indicated in the sketch.

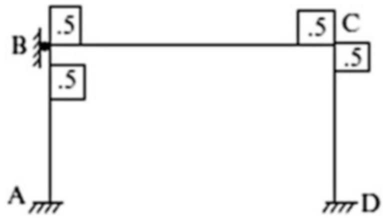


Fig. E10.17b

We compute the fixed end moments due to the 10 kip load.

$$M_{BC}^{F} = \frac{10(5)(15)^2}{20^2} = +28.13\,\text{kip ft}$$

$$M_{CB}^{F} = -\frac{10(15)(5)^2}{20^2} = -9.38\,\text{kip ft}$$

Details of the moment distribution and the end moments for case I are listed below (Fig. E10.17c). The holding force is determined by summing the shear forces in the columns and is equal to 1.1 kip.

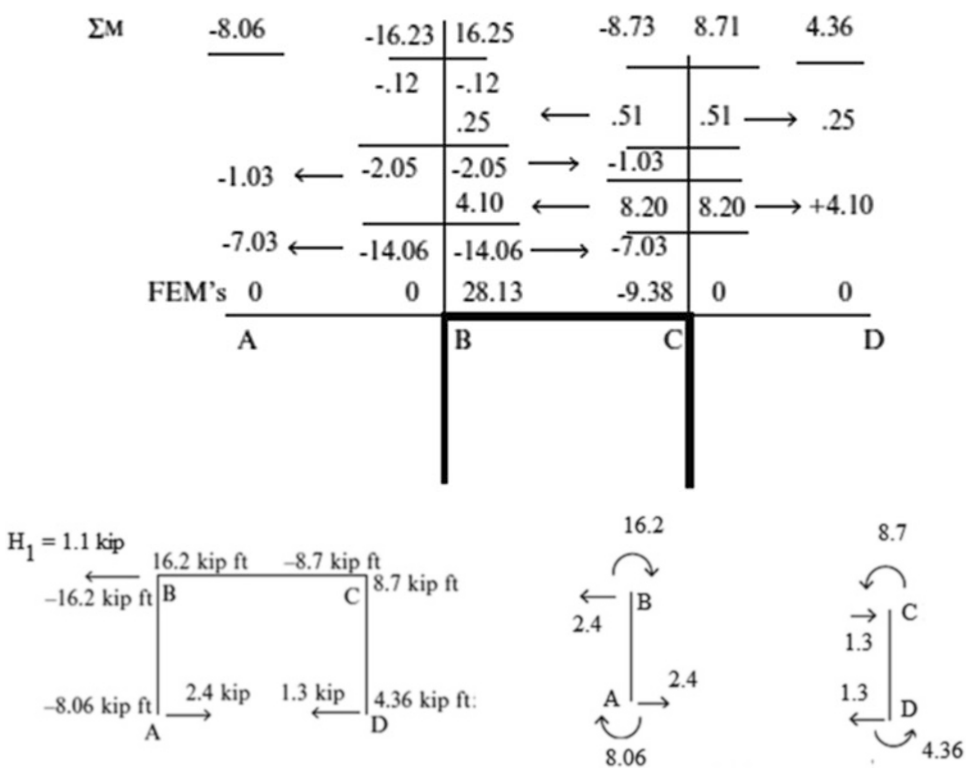


Fig. E10.17c Case I—end moments and column shear

Next, we introduce a lateral displacement to the left equal to Δ. Figure E10.17d shows this operation.

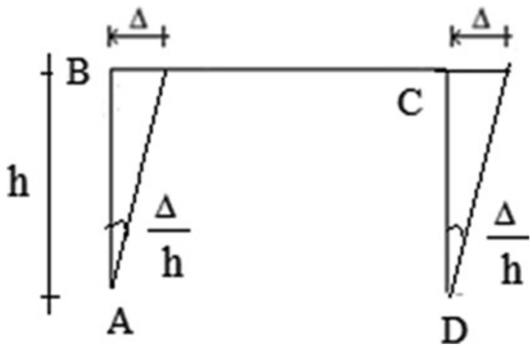


Fig. E10.17d Case II—sideway introduced

The chord rotations and corresponding fixed end moments are

$$M_{\mathrm{BA}}^{\mathrm{F}} = M_{\mathrm{AB}}^{\mathrm{F}} = M_{\mathrm{DC}}^{\mathrm{F}} = M_{\mathrm{CD}}^{\mathrm{F}} = -\frac{6EI\Delta}{h^2}$$

Since we are interested only in relative moments, we take $EI\Delta/h^2 = 1$. Details of the moment distribution and the end moments for case II are listed below (Fig. E10.17e).

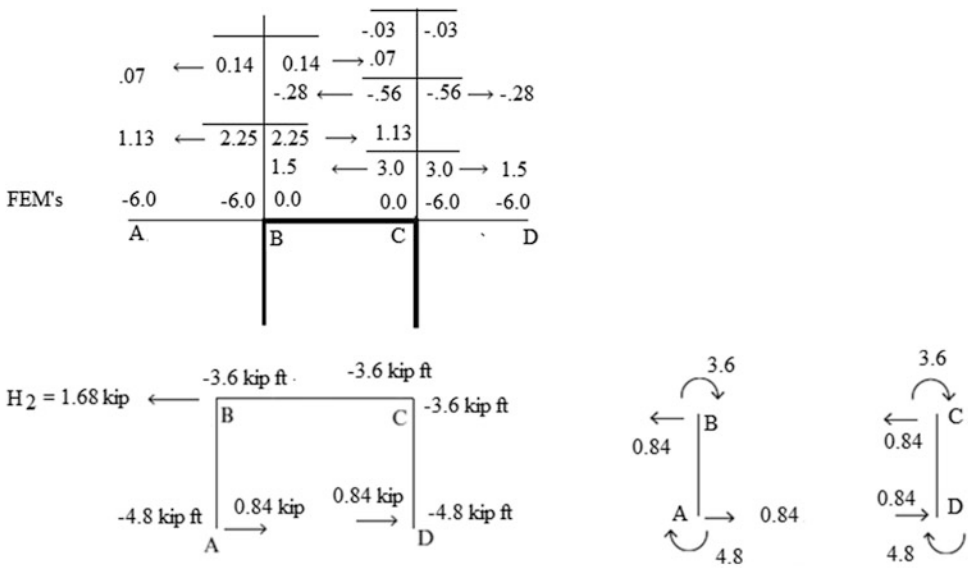


Fig. E10.17e Case II—end moments and column shear

We scale this solution by $H_1/H_2 = -1.1/1.68$ and then combine these scaled results with the results for case I.

Final end moments = end moments case I + end moments case II (H_1/H_2).
The final moments are summarized in Fig. E10.17f followed by free body diagrams.

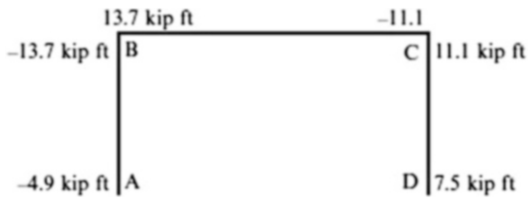
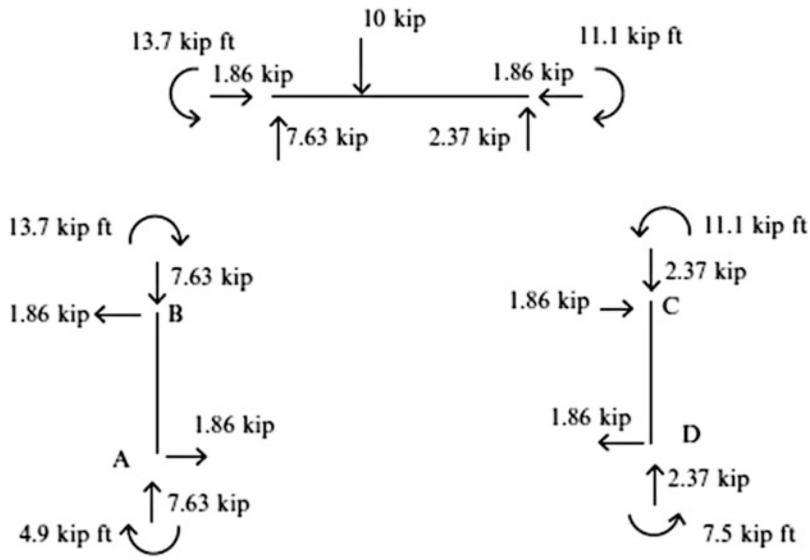


Fig. E10.17f Final end moments

Using these moments, we find the axial and shear forces.



Example 10.18: Frame with Inclined Legs

Given: The frame shown in Fig. E10.18a.

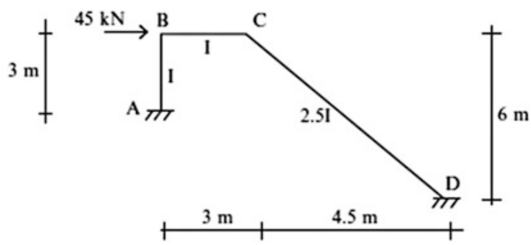


Fig. E10.18a

Determine: The end actions.

Solution: The distribution factors are listed in the sketch below (Fig. E10.18b).

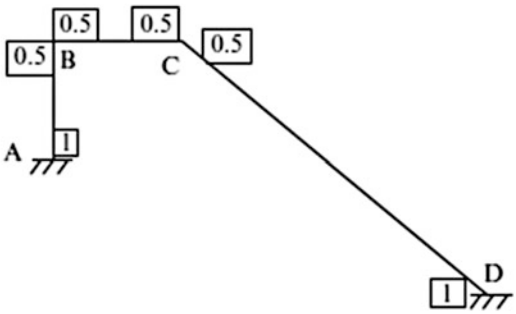
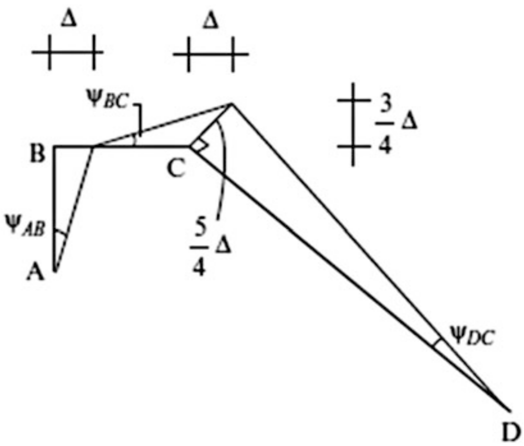


Fig. E10.18b

There are no fixed end moments due to member loads. However, we need to carry out a sideway analysis (case II). We introduce a horizontal displacement at B and compute the corresponding rotation angles.



The rotation of members BC and CD is determined by requiring the horizontal displacement of node C to be equal to Δ. The angles follow from the above sketch

$$\psi_{AB} = \frac{\Delta}{3}$$

$$\psi_{BC} = \frac{3/4\Delta}{3} = \frac{\Delta}{4}$$

$$\psi_{DC} = \frac{5/4\Delta}{7.5} = \frac{\Delta}{6}$$

Finally, the chord rotations are (note: positive sense is counterclockwise)

$$\rho_{AB} = -\frac{\Delta}{3}$$

$$\rho_{BC} = +\frac{\Delta}{4}$$

$$\rho_{DC} = -\frac{\Delta}{6}$$

Using these values, we compute the fixed end moments due to chord rotation.

$$M_{AB}^{F} = M_{BA}^{F} = \frac{6(EI)}{(3)}\left(\frac{\Delta}{3}\right) = +\frac{2}{3}(EI\Delta)$$

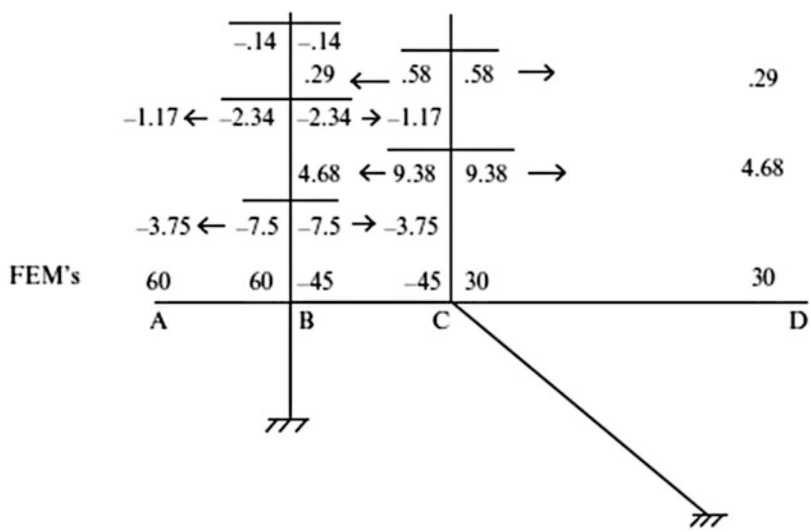
$$M_{BC}^{F} = M_{CB}^{F} = -\frac{6(EI)}{(3)}\left(\frac{\Delta}{4}\right) = -\frac{1}{2}(EI\Delta)$$

$$M_{CD}^{F} = M_{DC}^{F} = \frac{6(2.5EI)}{(7.5)}\left(\frac{\Delta}{6}\right) = +\frac{1}{3}(EI\Delta)$$

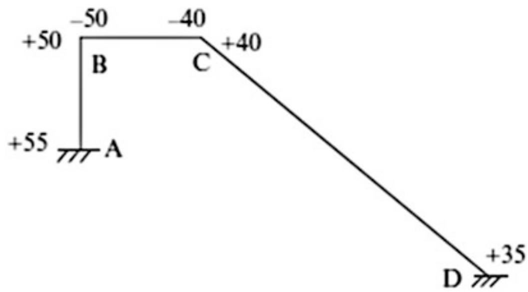
Since we need only relative moments, we take $EI\Delta = 90$
Then

$$M_{AB}^{F} = M_{BA}^{F} = +60\,\text{kN m}$$
$$M_{CB}^{F} = M_{BC}^{F} = -45\,\text{kN m}$$
$$M_{DC}^{F} = M_{CD}^{F} = +30\,\text{kN m}$$

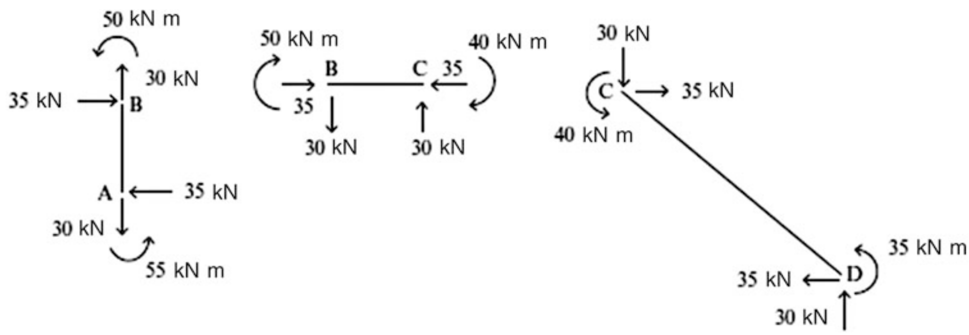
Next, we distribute the moments as shown below



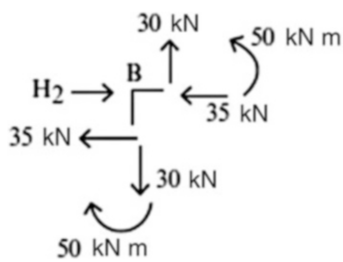
The end moments values are



Using these moments, we find the axial and shear forces.



Note that one needs the axial force in member BC in order to determine H_2.
Summing horizontal force components at B leads to H_2.



Therefore

$$H_2 = 35 + 35 = 70\,\text{kN}$$

Given that the actual horizontal force is 45 kN, we scale the sideway moments by $H_1/H_2 = 45/70 = 9/14$.

$$\text{Final end moments} = \text{end moments case II}\left(\frac{H_1}{H_2}\right)$$

The final end moments (kN m) are listed below (Fig. E10.18c).

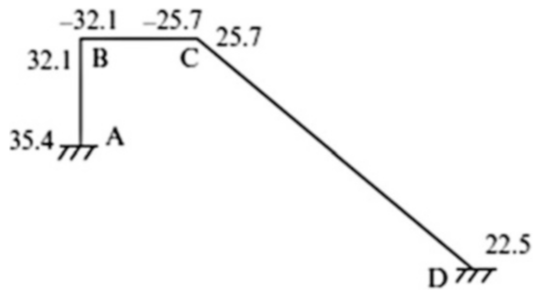
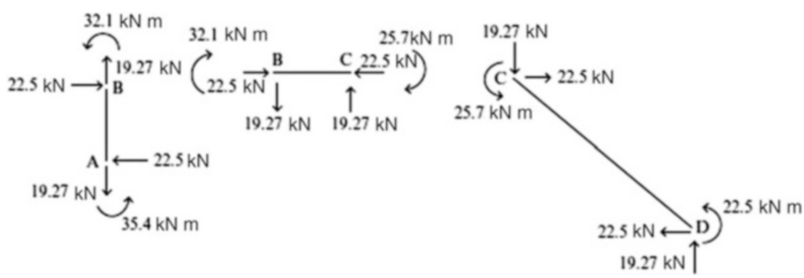


Fig. E10.18c Final end moments

Using these moments, we find the axial and shear forces.



Example 10.19: Computer-Based Analysis—Frame with Inclined Legs

Given: The frame shown in Fig. E10.19a.

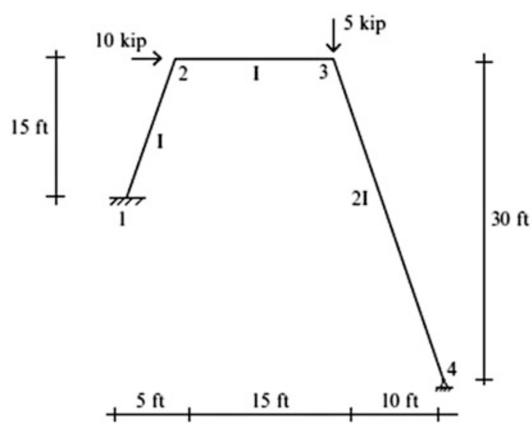


Fig. E10.19a

Determine: The displacement components at nodes 2 and 3, the bending moment distribution, and the reactions. Consider a range of values for I (I = 100, 200, and 400 in.^{4}). Take $A = 20$ in.^{2}. Use computer software.

Solution: The computer generated deflection profiles and the reactions and moment diagram are listed below (Figs. E10.19b, E10.19c, E10.19d, E10.19e, E10.19f). Hand computation is not feasible for this task.

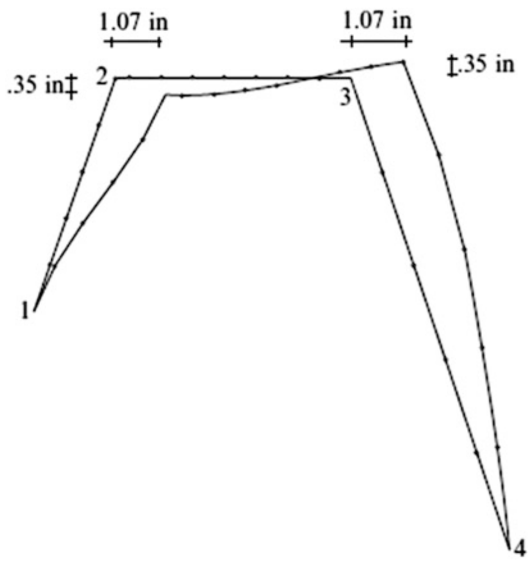


Fig. E10.19b Deflection profile—$I = 100$ in.^{4}

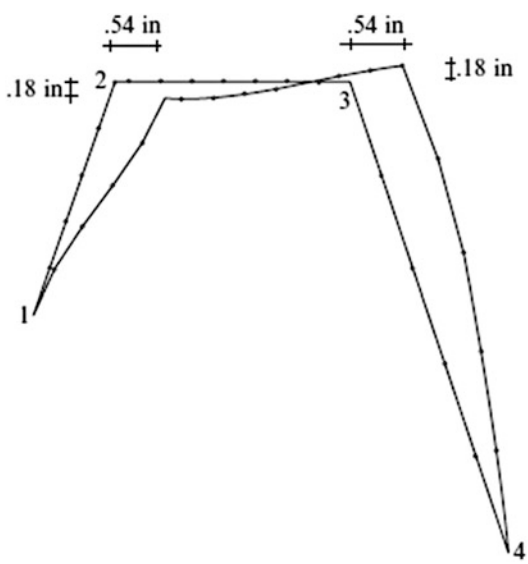


Fig. E10.19c Deflection profile—$I = 200$ in.^{4}

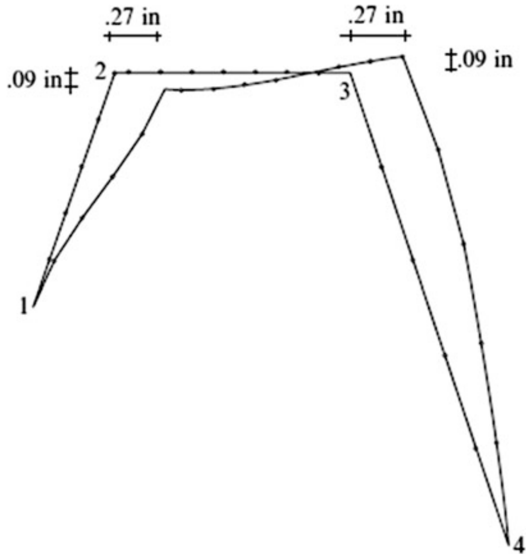


Fig. E10.19d Deflection Profile—$I = 400\ \text{in.}^4$

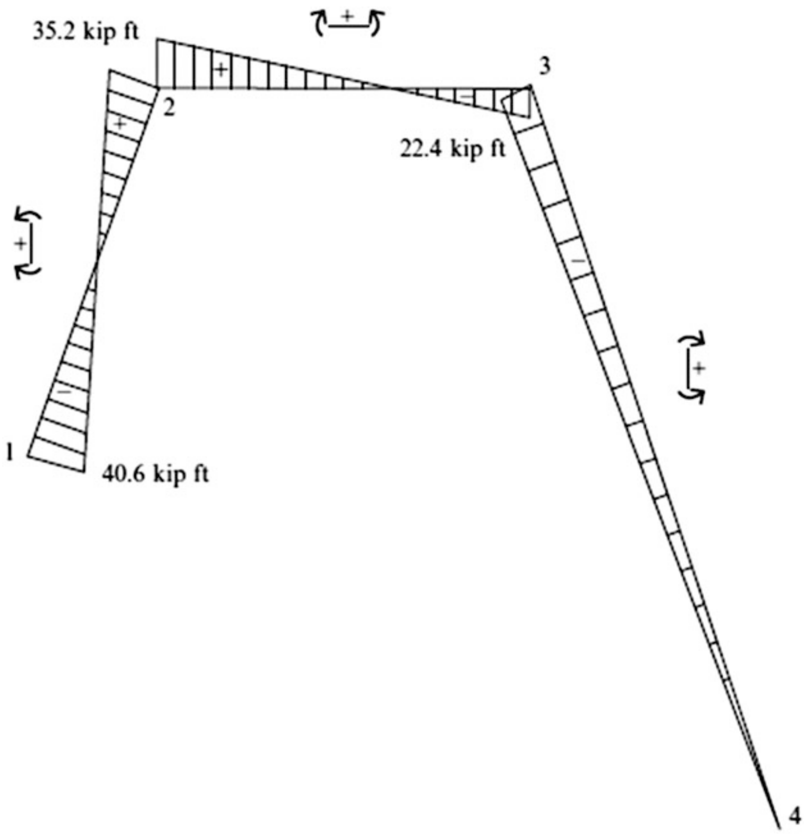


Fig. E10.19e Moment diagram

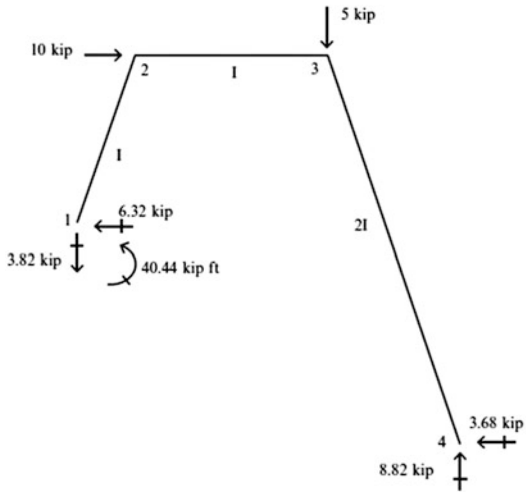


Fig. E10.19f Reactions

Note that the member forces are *invariant* since the relative stiffness of the members is the same. Also, the displacement varies linearly with *I*.

10.8 Plane Frames: Out of Plane Loading

We discussed this case briefly in Chap. 4 when we dealt with statically determinate plane frame structures loaded normal to the plane such as highway signs. We extend the analysis methodology here to deal with statically indeterminate cases. Our strategy is based on the displacement method, i.e., we use generalized slope-deflection equations for the members and enforce equilibrium at the nodes. This approach is more convenient than the force method and has the additional advantage that it can be readily adopted for digital computation.

10.8.1 Slope-Deflection Equations: Out of Plane Loading

Consider the prismatic member shown in Fig. 10.25a. We assume that the member is loaded in the X–Z plane (note that all the previous discussions have assumed the loading is in the X–Y plane). The relevant displacement measures for this loading are the rotation θ_x, the rotation θ_y, and the transverse displacement v_z. Figure 10.25b defines the positive sense for these quantities and the corresponding end actions at B.

Following the procedure described in Sect. 10.3, one can establish the equations relating the end actions at A and B to the end displacements at A and B. Their form is

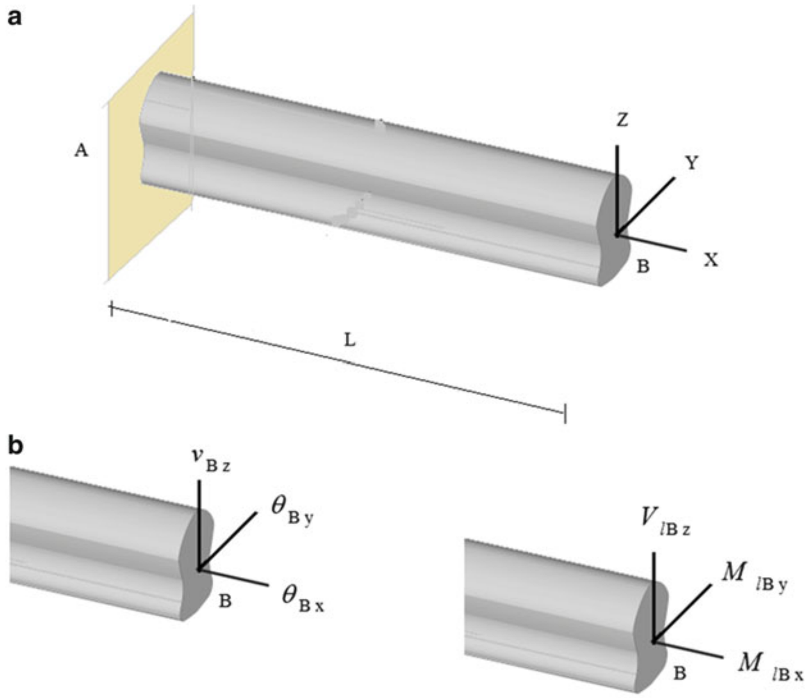


Fig. 10.25 (**a**) Prismatic member (**b**) Positive sense

$$
\begin{aligned}
V_{lAz} &= \frac{6EI_y}{L^2}\left(\theta_{By}+\theta_{Ay}\right)-\frac{12EI_y}{L^3}\left(v_{Bz}-v_{Az}\right)+V_{lAz}^{F}\\
M_{lAx} &= -\frac{GJ}{L}\left(\theta_{Bx}-\theta_{Ax}\right)+M_{lAx}^{F}\\
M_{lAy} &= \frac{2EI_y}{L}\left(\theta_{By}+2\theta_{Ay}\right)-\frac{6EI_y}{L^2}\left(v_{Bz}-v_{Az}\right)+M_{lAy}^{F}\\
V_{lBz} &= -\frac{6EI_y}{L^2}\left(\theta_{By}+\theta_{Ay}\right)+\frac{12EI_y}{L^3}\left(v_{Bz}-v_{Az}\right)+V_{lBz}^{F}\\
M_{lBx} &= \frac{GJ}{L}\left(\theta_{Bx}-\theta_{Ax}\right)+M_{lBx}^{F}\\
M_{lBy} &= \frac{2EI_y}{L}\left(2\theta_{By}+\theta_{Ay}\right)-\frac{6EI_y}{L^2}\left(v_{Bz}-v_{Az}\right)+M_{lBy}^{F}
\end{aligned}
\tag{10.47}
$$

where GJ is the torsional rigidity for the cross section, and I_y is the second moment of area with respect to y-axis.

$$I_y = \int_A z^2 \mathrm{d}A \tag{10.48}$$

The remaining steps are essentially the same as for the planar case. One isolates the members and nodes and enforces equilibrium at the nodes. In what follows, we illustrate the steps involved.

Consider the structure shown in Fig. 10.26. We suppose the supports are rigid. There are three unknown nodal displacement measures, θ_x, θ_y, and v_z at node 1.

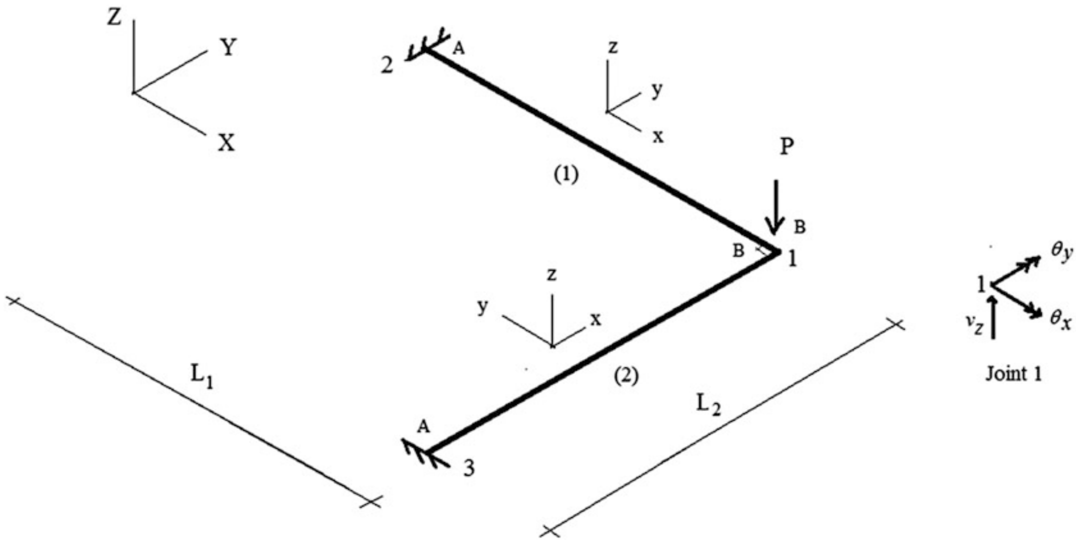


Fig. 10.26 Plane grid

Free body diagrams for the members incident on node 1 are shown below in Fig. 10.27. Requiring equilibrium at node 1 leads to the following equations:

$$\begin{aligned} V_{\mathrm{B}z}^{(1)} + V_{\mathrm{B}z}^{(2)} + P &= 0 \\ M_{\mathrm{B}x}^{(1)} - M_{\mathrm{B}y}^{(2)} &= 0 \\ M_{\mathrm{B}y}^{(1)} + M_{\mathrm{B}x}^{(2)} &= 0 \end{aligned} \tag{10.49}$$

Noting the relationship between the variables,

$$\begin{aligned} \theta_{\mathrm{B}x}^{(1)} &= \theta_{1x} \\ \theta_{\mathrm{B}x}^{(2)} &= \theta_{1y} \\ \theta_{\mathrm{B}y}^{(1)} &= \theta_{1y} \\ \theta_{\mathrm{B}y}^{(2)} &= -\theta_{1x} \\ v_{\mathrm{B}z}^{(1)} &= v_{\mathrm{B}z}^{(2)} = v_{1z} \end{aligned} \tag{10.50}$$

the member equations take the following form,

$$\begin{aligned} M_{\mathrm{B}x}^{(1)} &= \frac{GJ_1}{L_1}\theta_{1x} \\ M_{\mathrm{B}y}^{(1)} &= \frac{4EI_1}{L_1}\theta_{1y} + \frac{6EI_1}{L_1^2}v_{1z} \\ V_{\mathrm{B}z}^{(1)} &= \frac{12EI_1}{L_1^3}v_{1z} - \frac{6EI_1}{L_1^2}\theta_{1y} \\ &\text{and} \\ M_{\mathrm{B}x}^{(2)} &= \frac{GJ_2}{L_2}\theta_{1y} \\ M_{\mathrm{B}y}^{(2)} &= \frac{4EI_2}{L_2}(-\theta_{1x}) + \frac{6EI_2}{L_2^2}v_{1z} \\ V_{\mathrm{B}z}^{(2)} &= \frac{12EI_2}{L_2^3}v_{1z} - \frac{6EI_2}{L_2^2}(-\theta_{1x}) \end{aligned} \tag{10.51}$$

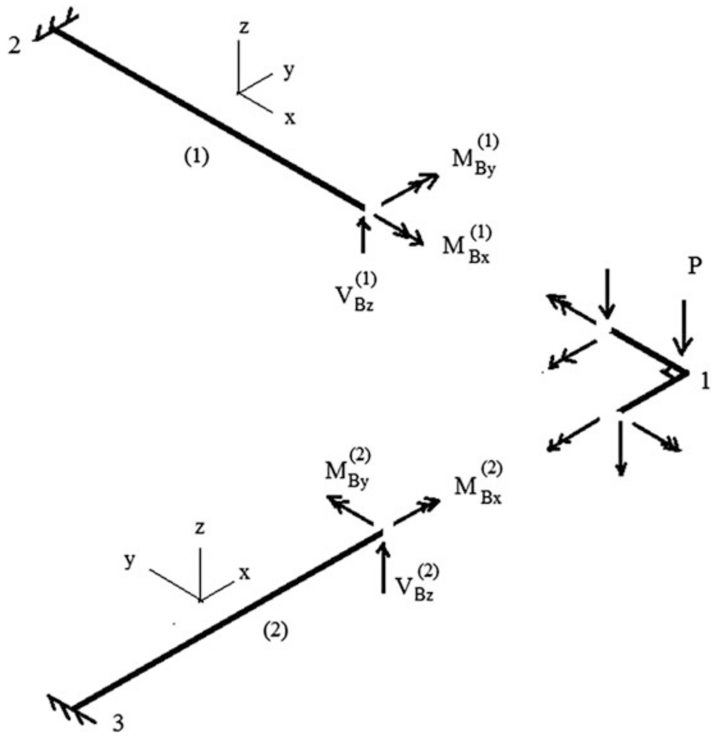


Fig. 10.27 Free body diagrams

Lastly, we substitute for the end actions in the equilibrium equations (10.49) leading to

$$
\begin{aligned}
&12E\left(\frac{I_1}{L_1^3}+\frac{I_2}{L_2^3}\right)v_{1z}+\frac{6EI_2}{L_2^2}\theta_{1x}-\frac{6EI_1}{L_1^2}\theta_{1y}+P=0\\
&-\frac{6EI_2}{L_2^2}v_{1z}+\left(\frac{GJ_1}{L_1}+\frac{4EI_2}{L_2}\right)\theta_{1x}=0\\
&\frac{6EI_1}{L_1^2}v_{1z}+\left(\frac{GJ_2}{L_2}+\frac{4EI_1}{L_1}\right)\theta_{1y}=0
\end{aligned}
\tag{10.52}
$$

The solution is

$$
\begin{aligned}
&\theta_{1x}=\frac{6EI_2/L_2^2}{(GJ_1/L_1+4EI_2/L_2)}v_{1z}\\
&\theta_{1y}=\frac{-6EI_1/L_1^2}{(GJ_2/L_2+4EI_1/L_1)}v_{1z}\\
&\left\{12E\left(\frac{I_1}{L_1^3}+\frac{I_2}{L_2^3}\right)+\frac{\left(6EI_1/L_1^2\right)^2}{(GJ_2/L_2+4EI_1/L_1)}+\frac{\left(6EI_2/L_2^2\right)^2}{(GJ_1/L_1+4EI_2/L_2)}\right\}v_{1z}=-P
\end{aligned}
\tag{10.53}
$$

When the member properties are equal,

$$I_1 = I_2$$
$$L_1 = L_2$$
$$J_1 = J_2$$

the solution reduces to

$$v_{1z} = \frac{-P}{\left(\frac{6EI}{L^3}\right)\left(1 + \frac{\frac{12EI}{L^2}}{(GJ+4EI)}\right)}$$

$$\theta_{1y} = -\theta_{1x} = \frac{6EI/L^3}{(GJ+4EI)} v_{1z} = \frac{-P}{(GJ+4EI) + \left(\frac{12EI}{L^2}\right)} \tag{10.54}$$

$$\text{end shear forces} \quad V_{Bz}^{(1)} = V_{Bz}^{(2)} = \frac{P}{2}$$

Note that even for this case, the vertical displacement depends on both I and J. In practice, we usually use a computer-based scheme to analyze grid-type structures.

Example 10.20: Grid Structure

Given: The grid structure defined in Fig. E10.20a. The members are rigidly connected at all the nodes. Assume the members are steel and the cross-sectional properties are constant. $I = 100$ in.^{4}, $J = 160$ in.^{4}.

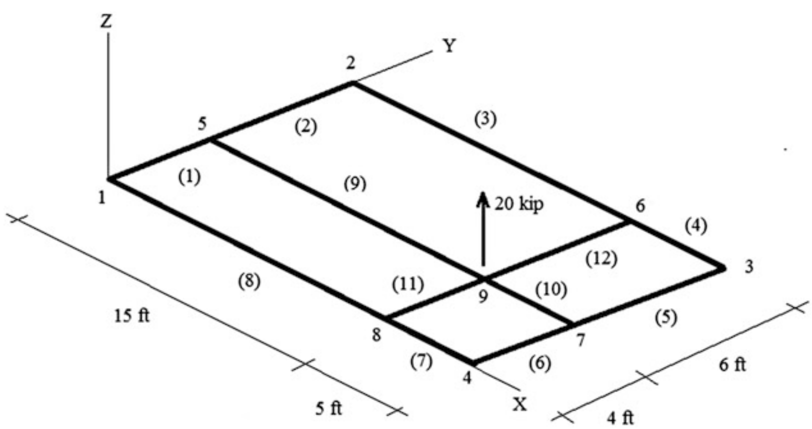


Fig. E10.20a

The nodal displacement restraints are as follows:

Node 1: x, y, z translation
Node 2: z translation
Node 3: z translation
Node 4: y, z translation
Determine: The displacement measures and member end forces at node 9. Use computer software.

Solution: The computer output data for this structure is
Displacement measures at node 9:

$$\text{Nade } 9\begin{cases} w = 0.189\,\text{in.} \\ \theta_x = 0.00051\,\text{rad} \\ \theta_y = 0.00192\,\text{rad} \end{cases}$$

Member end forces at node 9:

$$\text{Member }(9)\quad \begin{cases} V_Z = 1.31\,\text{kip} \\ M_x = 0.25\,\text{kip ft} \\ M_y = 14.5\,\text{kip ft} \end{cases} \qquad \text{Member }(10)\quad \begin{cases} V_Z = 5.2\,\text{kip} \\ M_x = 0.55\,\text{kip ft} \\ M_y = -16.6\,\text{kip ft} \end{cases}$$

$$\text{Member }(11)\quad \begin{cases} V_Z = 8.2\,\text{kip} \\ M_x = 0.86\,\text{kip ft} \\ M_y = 27\text{kip ft} \end{cases} \qquad \text{Member }(12)\quad \begin{cases} V_Z = 5.3\,\text{kip} \\ M_x = 1.2\,\text{kip ft} \\ M_y = -26.2\,\text{kip ft} \end{cases}$$

One checks the results by noting that the sum of the end shears at node 9 must equal the applied load of 20 kip.

10.9 Nonlinear Member Equations for Frame-Type Structures

10.9.1 Geometric Nonlinearity

Although we did not mention it explicitly, when dealing with equilibrium equations, we always showed the forces acting on the initial geometry position of the structure. However, the geometry changes due to deformation under the action of the loading, and this assumption is justified only when the change in geometry (deformation) is negligible. This is true in most cases. However, there are exceptions, and it is of interest to explore the consequence of accounting for geometric change when establishing the equilibrium equations. This approach is referred to as geometric nonlinear analysis since the additional geometric terms result in nonlinear equations. In what follows, we illustrate this effect for different types of structures.

Consider the two-member truss shown in Fig. 10.28a. We suppose the angle θ is small, say about 15°. When a vertical force is applied, the structure deforms as shown in Fig. 10.28b. The load is resisted by the member forces generated by the deformation resulting from the displacement, v.

Due to the displacement, the angle changes from θ to β, where β is a function of v.

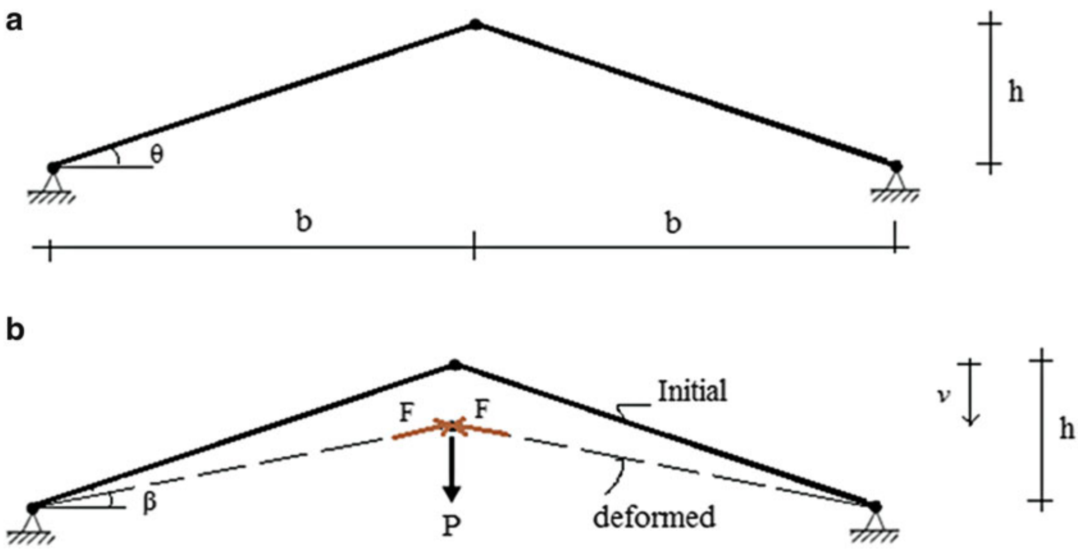


Fig. 10.28 Nonlinear truss example

$$\tan\beta = \frac{h - v}{b} \tag{10.55}$$

The equilibrium equation also depends on v.

$$P = 2F\sin\beta \tag{10.56}$$

When v is small with respect to h, β can be approximated as

$$\tan\beta \approx \frac{h}{b} \equiv \tan\theta \tag{10.57}$$

and it follows that

$$P \approx 2F\sin\theta \tag{10.58}$$

which is the "linearized" form of the equilibrium equation. One cannot neglect the change in angle when h is also small which is the case when θ is on the order of 15°.

Another example of geometric nonlinearity is a beam subjected to axial compression and transverse loading. Fig. 10.29 shows the loading condition and deformed geometry.

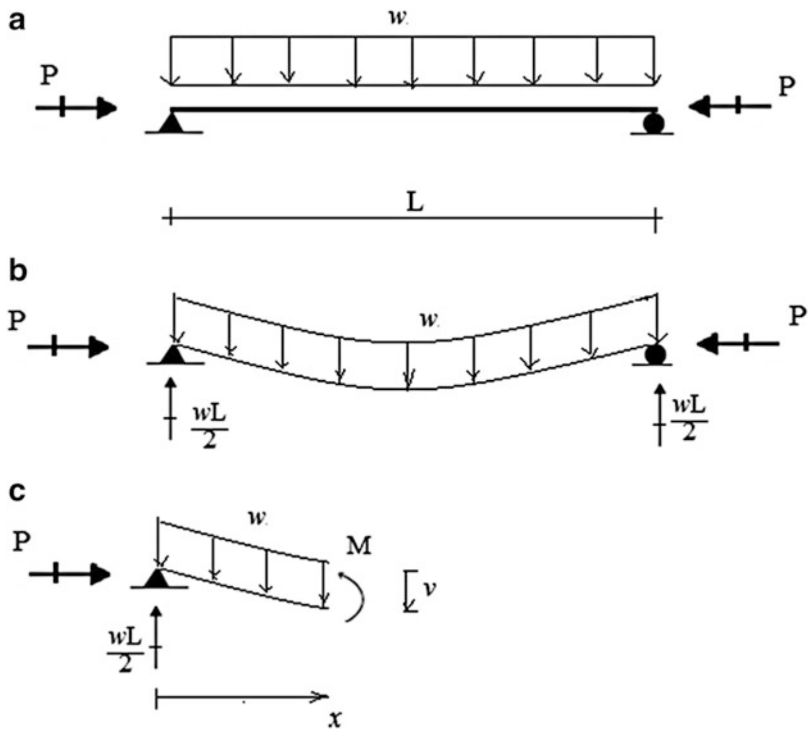


Fig. 10.29 Nonlinear beam example

Noting Fig. 10.29c, the bending moment at location x is

$$M = \left(\frac{wL}{2}\right)x - \frac{wx^2}{2} + Pv \tag{10.59}$$

The last term is due to accounting for the displacement from the initial position of the beam. We will show later that this term has a destabilizing effect on the response, i.e., it magnifies the response.

Up to this point in the text, we have neglected the geometric term and always worked with the initial undeformed geometry. For example, we have been taking M as

$$M \approx \left(\frac{wL}{2}\right)x - \frac{wx^2}{2} \tag{10.60}$$

When P is compressive, and the beam is flexible, this linearized expression is not valid and one needs to use the nonlinear form, (10.59). If P is a tensile force, the free body diagram shown in Fig. 10.30 now applies and the appropriate expression for M is

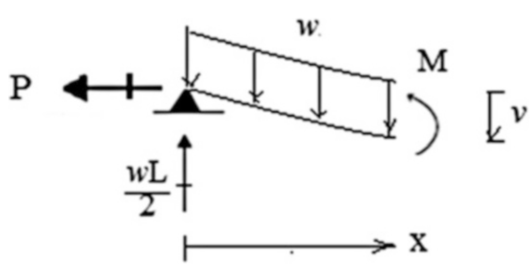


Fig. 10.30 Nonlinear beam – axial tension

$$M = \left(\frac{wL}{2}\right)x - \frac{wx^2}{2} - Pv \tag{10.61}$$

In this case, the nonlinear contribution has a *stabilizing* effect.

Generalizing these observations, whenever a member is subjected to a *compressive* axial load, one needs to consider the potential destabilizing effect of geometric nonlinearity on the axial stiffness of the member. This type of behavior is usually referred to as "buckling." From a design perspective, buckling must be avoided. This is achieved by appropriately dimensioning the cross section and providing bracing to limit transverse displacement.

In what follows, we extend the planar beam bending formulation presented in Sects. 3.5 and 3.6 to account for geometric nonlinearities. This revised formulation is applied to establish the nonlinear form of the member equations described in Sect. 10.3. Lastly, these equations are used to determine the nonlinear behavior of some simple frame structures.

10.9.2 Geometric Equations Accounting for Geometric Nonlinearity

Figure 10.31a shows the initial and deformed position of a differential element experiencing planar bending in the x–y plane. The geometric variables are the axial displacement, u; the transverse displacement, v; and the rotation of the cross section, β.

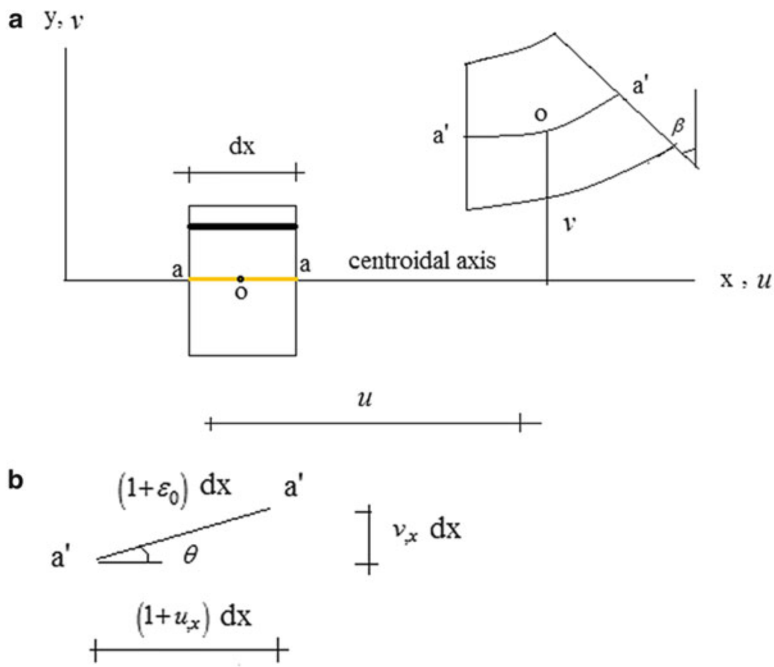


Fig. 10.31 (**a**) Initial and deformed positions (**b**) Position of tangent

Figure 10.31b defines the deformed position of line a-a; θ denotes the rotation of the centroidal axis. Assuming $\theta = 0$ leads to the linearized expression for the strain, ε_0.

$$\theta \approx 0 \Rightarrow \varepsilon_0 = u_{,x} \tag{10.62}$$

The next level of approximation is based on assuming θ is sufficiently small such that $\theta^2 \ll 1$. This leads to

$$\begin{aligned} &\tan\theta \approx \sin\theta \approx \theta \\ &\cos\theta \approx 1 \end{aligned} \tag{10.63}$$

Then, the exact expressions for ε_0 and tan θ

$$\begin{aligned} (1+\varepsilon_0)^2 &= (1+u_{,x})^2 + (v_{,x})^2 \\ \tan\theta &= \frac{v_{,x}}{1+u_{,x}} \end{aligned} \tag{10.64}$$

reduce to

$$\begin{aligned} \varepsilon_0 &= u_{,x} + \frac{1}{2}(v_{,x})^2 \\ \theta &\approx v_{,x} \end{aligned} \tag{10.65}$$

Equation (10.65) applies for small strain, i.e., $\varepsilon \ll 1$. Note that the nonlinearity involves the rotation of the centroidal axis.

The remaining steps are similar to those followed for the linear case. We assume the cross section remains a plane and neglect the transverse shear deformation. These assumptions lead to (see Fig. 10.32):

$$\begin{aligned} &\gamma = 0 \quad \Rightarrow \quad \beta \approx \theta = \frac{\mathrm{d}v}{\mathrm{d}x} \\ &\text{and} \\ &\varepsilon = \varepsilon_0 - y\chi \\ &\chi = \frac{\mathrm{d}\beta}{\mathrm{d}x} = \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \end{aligned} \tag{10.66}$$

Given the strains, one can determine the internal axial force and moment using the linear elastic

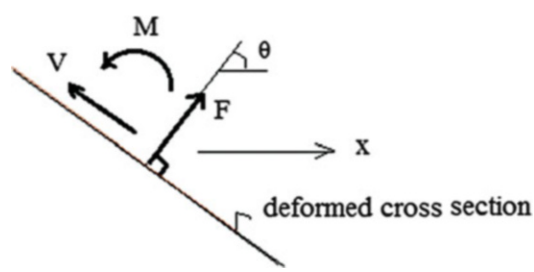


Fig. 10.32 Orientation of deformed cross section

stress–strain relations.

$$\begin{aligned} F &= \int \sigma \mathrm{d}A = AE\varepsilon_0 \\ M &= \int -y\sigma \mathrm{d}A = EI\chi \end{aligned} \tag{10.67}$$

Note that F acts at an angle θ with respect to the x-axis. Since we have neglected the transverse shear deformation, V has to be determined using an equilibrium requirement.

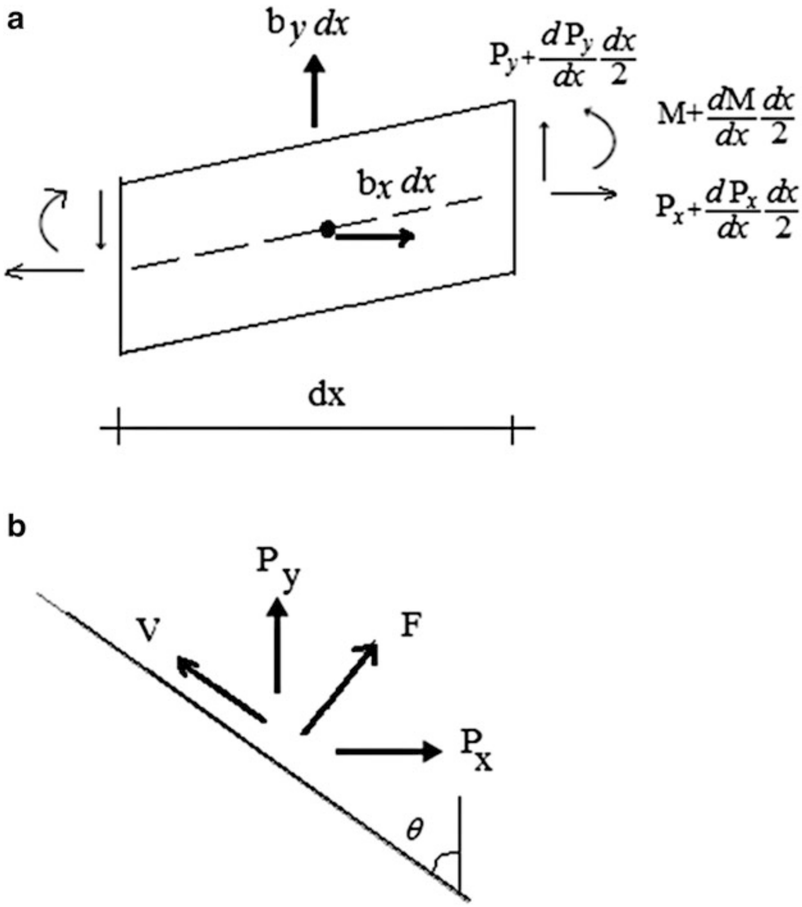


Fig. 10.33 (**a**) Differential element (**b**) Cartesian components

The last step involves enforcing the equilibrium condition. We work with the differential element shown in Fig. 10.33a; b_x and b_y are the loads per unit length. The Cartesian components are related to the internal forces in terms of the rotation angle, θ.

$$
\begin{aligned}
P_x &= F\cos\theta - V\sin\theta \\
P_y &= F\sin\theta + V\cos\theta
\end{aligned}
\tag{10.68}
$$

Noting Fig. 10.33b, it follows that

$$
\begin{aligned}
F &= P_x\cos\theta + P_y\sin\theta \\
V &= -P_x\sin\theta + P_y\cos\theta
\end{aligned}
$$

Assuming θ is small, we simplify Eq. (10.68) to

$$
\begin{aligned}
P_x &\approx F \\
P_y &\approx F\cdot\theta + V = Fv_{,x} + V
\end{aligned}
\tag{10.69}
$$

The equilibrium equations for the element are

$$\begin{aligned}&\frac{d}{dx}P_x + b_x = 0\\&\frac{d}{dx}P_y + b_y = 0\\&\frac{dM}{dx} + P_y - v_{,x}P_x = 0\end{aligned} \tag{10.70}$$

Substituting for P_x and P_y, Equation (10.70) reduce to

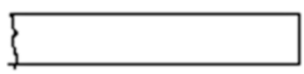
$$\begin{aligned}&\frac{\mathrm{d}F}{\mathrm{d}x} + b_x = 0\\&\frac{\mathrm{d}}{\mathrm{d}x}\left(F\frac{\mathrm{d}v}{\mathrm{d}x}\right) + \frac{\mathrm{d}V}{\mathrm{d}x} + b_y = 0\\&\frac{\mathrm{d}M}{\mathrm{d}x} + V = 0\end{aligned} \tag{10.71}$$

The nonlinearity is present in the transverse equilibrium equation in the form of a coupling between axial and bending actions. Lastly, the boundary conditions are:

$$\left.\begin{array}{lll} u \text{ or } P_x = F & \text{prescribed} \\ v \text{ or } P_x = Fv_{,x} + V & \text{prescribed} \\ \theta \text{ or } M & \text{prescribed}\end{array}\right\}\text{at each end} \tag{10.72}$$

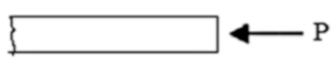
We illustrate the boundary condition for various types of supports.

Case 1: Free end



$$P_x = P_y = M = 0 \Rightarrow F = V = M = 0$$

Case 2: Axial load

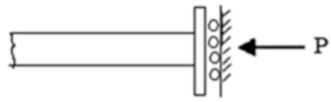


$$\left\{\begin{array}{l} P_x = -P \\ P_y = 0 \\ M = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} F = -P \\ V = v_{,x}P \\ M = 0 \end{array}\right.$$

Case 3: Roller support

$$\left\{\begin{array}{l} P_x = -P \\ v = 0 \\ M = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} F = -P \\ v = M = 0 \end{array}\right.$$

Case 4: Rotation restraint



$$\begin{cases} P_x = -P \\ P_y = 0 \\ \theta = 0 \end{cases} \Rightarrow \begin{cases} F = -P \\ V = 0 \\ \theta = 0 \end{cases}$$

10.9.3 Solution for Compressive Axial Load

We consider here the case where the axial load is compressive (e.g., see Fig. 10.29). Taking

$$F = -P = \text{constant} \tag{10.73}$$

the remaining equations in (10.71) reduce to

$$\begin{aligned} V &= -\frac{dM}{dx} \\ \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + P\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} &= b_y \end{aligned} \tag{10.74}$$

The corresponding boundary conditions are:

$$\left.\begin{matrix} v \,\text{or}\, V - P\dfrac{\mathrm{d}v}{\mathrm{d}x} & \text{prescribed} \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} \,\text{or}\, M & \text{prescribed} \end{matrix}\right\} \text{at each end} \tag{10.75}$$

Noting (10.67), the expression for M expands to

$$M = EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \tag{10.76}$$

Integrating the second equation in (10.74) leads to

$$M + Pv = \int_x \int_x b_y \,\mathrm{d}x + c_1 x + c_2$$

where c_1 and c_2 are integration constants. Then, substituting for M, we obtain the governing equation for v.

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + Pv = \int_x \int_x b_y \mathrm{d}x + c_1 x + c_2 \tag{10.77}$$

We suppose b_y and EI are constant. For this case,

$$\int_x \int_x b_y \,\mathrm{d}x = \frac{1}{2} b_y x^2 \equiv \frac{1}{2} b x^2 \tag{10.78}$$

and the corresponding solution of (10.77) is:

$$v = \frac{1}{\mu^2}\left\{\frac{b}{2EI}\left(x^2 - \frac{2}{\mu^2}\right) + \frac{1}{EI}(c_1 x + c_2)\right\} + c_3 \cos \mu x + c_4 \sin \mu x \quad (10.79)$$

where

$$\mu^2 = \frac{P}{EI}$$

The integration constants are determined using the boundary conditions, (10.75).

Example 10.21

Given: The axially loaded member shown in Fig. E10.21a.

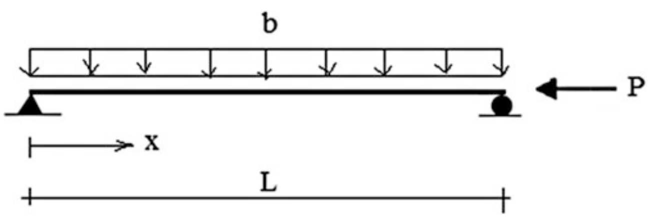


Fig. E10.21a

Determine: The transverse displacement as a function of the axial load, P.

Solution: The boundary conditions for the simply supported axially loaded member shown in Fig. E10.21a are

$$v(0) = v(L) = 0$$
$$M(0) = M(L) = 0$$

Substituting for v in (10.76) leads to

$$M = -\mu^2\{c_3 \cos \mu x + c_4 \sin \mu x\} + \frac{b}{\mu^2 EI}$$

Enforcing the boundary conditions, the corresponding integration constants are:

$$c_1 = -\frac{bL}{2}$$
$$c_2 = 0$$
$$c_3 = \frac{b}{\mu^4 EI}$$
$$c_4 = \frac{b}{\mu^4 EI}\left(\frac{1 - \cos \mu L}{\sin \mu L}\right)$$

Using these values, the solution for v expands to

$$v = \frac{b}{\mu^4 EI}\left\{\cos\mu x + \sin\mu x\left(\frac{1-\cos\mu L}{\sin\mu L}\right)\right\} + \frac{b}{\mu^2 EI}\left\{\frac{1}{2}\left(x^2 - \frac{2}{\mu^2}\right) - \frac{xL}{2}\right\}$$

The nonlinear behavior is generated by the axial force parameter, μ.To illustrate this effect, we evaluate v at $x = L/2$.

$$v\left(\frac{L}{2}\right) = \frac{b}{\mu^4 EI}\left\{\cos\frac{\mu L}{2} + \sin\frac{\mu L}{2}\left(\frac{1-\cos\mu L}{\sin\mu L}\right)\right\} + \frac{b}{\mu^2 EI}\left\{-\frac{1}{\mu^2} - \frac{L^2}{8}\right\}$$

The linear (i.e., $P = 0$) solution is

$$v\left(\frac{L}{2}\right) = \frac{5}{384}\frac{bL^4}{EI}$$

Figure E10.21b shows the variation of the ratio $\frac{v}{v_{\text{linear}}}$ vs. $\frac{P}{P_{\text{cr}}}$, where P_{cr} is the value of P for which $\sin\mu L = 0$.

$$\sin\mu L = 0 \Rightarrow \mu L = \pi$$

$$\mu L = \left(\frac{P}{EI}\right)^{\frac{1}{2}} L = \pi$$

$$P_{\text{cr}} = \frac{\pi^2 EI}{L^2}$$

The effect of axial load becomes pronounced when P approaches P_{cr}.

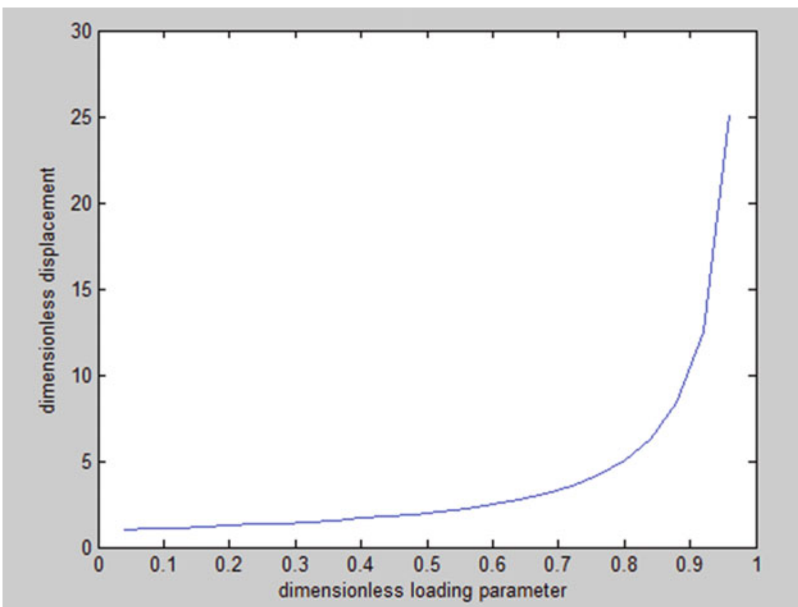


Fig. E10.21b

10.9.4 Nonlinear Member End Actions–End Displacement Equations

In order to deal with framed structures, one needs the set of equations relating the forces acting on the ends of a member and the displacement measures for the ends. The linear form of these equations is developed in Sect. 10.3. We derive the nonlinear form here using the general solution represented by (10.79).

Figures 10.34a, b define the notation for the displacement measures and the end actions. The only difference for the nonlinear case is the presence of the axial load, P. All quantities are referred to the local member frame. Note that the end actions act on the deformed configuration.

Equation (10.79) defines the solution for the case of a uniform load. To allow for an arbitrary load, we express the solution as

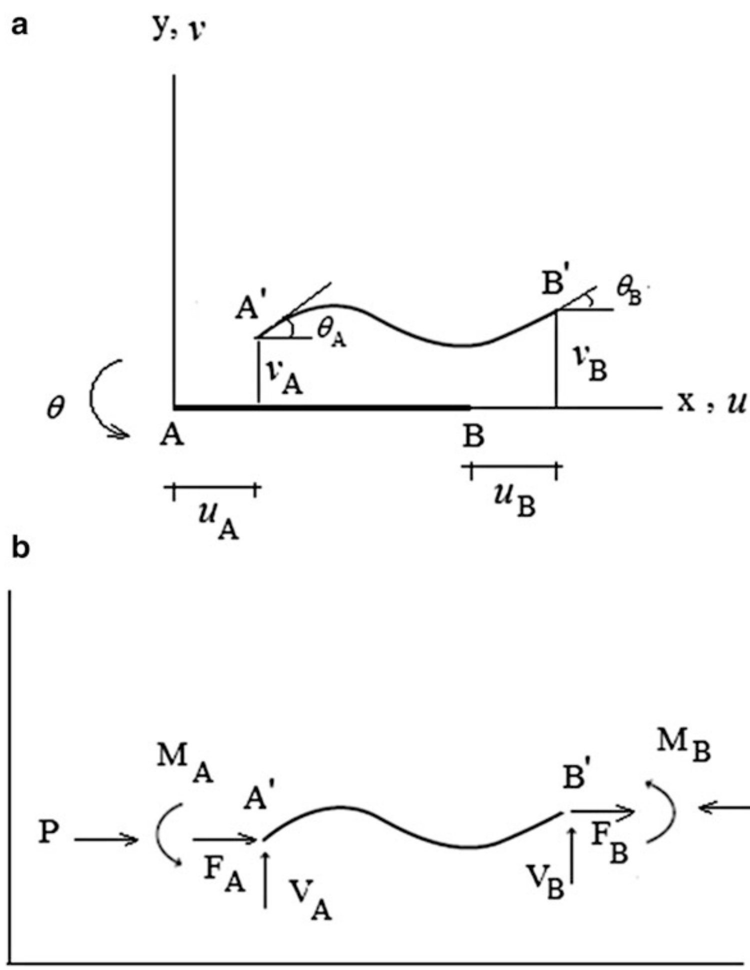


Fig. 10.34 Notation for nonlinear case. (**a**) Displacements. (**b**) End actions

$$v = v_p + \frac{1}{\mu^2 EI}(C_1 x + C_2) + C_3 \cos \mu x + C_4 \sin \mu x \tag{10.80}$$

The boundary conditions now involve the displacement measures at $x = 0$ and L.

$$v(0) = v_{\mathrm{A}} \quad \frac{dv}{dx}(0) = \omega_{\mathrm{A}}$$
$$v(L) = v_{\mathrm{B}} \quad \frac{dv}{dx}(L) = \omega_{\mathrm{B}} \tag{10.81}$$

Specializing (10.80) for these conditions leads to expressions for the integration constants.

$$\begin{aligned}
C_1 &= \omega_A - \omega_{A,p} - \mu C_4 \\
C_2 &= u_A - u_{A,p} - C_3 \\
C_3 &= -C_4 \frac{1 - \cos\mu L}{\sin\mu L} - \frac{\omega_B - \omega_A - \omega_{B,p} + \omega_{A,p}}{\mu \sin\mu L} \\
C_4 &= \frac{1}{D}\Big\{ \left[u_B - u_{B,p} - u_A + u_{A,p} - \left(\omega_A + \omega_{A,p}\right)L \right] \sin\mu L \\
&\qquad - \frac{1 - \cos\mu L}{\mu}\left[\omega_B - \omega_{B,p} - \omega_A + \omega_{A,p}\right] \Big\} \\
D &= 2(1 - \cos\mu L) - \mu L \sin\mu L
\end{aligned} \tag{10.82}$$

Note that $D \to 0$ as $\mu L \to 2\pi$. Then $C_4 \to \infty$ and it follows that $v \to \infty$ for any arbitrary loading. The limiting value of P is

$$P\big|_{\max} = P_{\mathrm{cr}} = \frac{4\pi^2 EI}{L^2} \tag{10.83}$$

Given v, one can evaluate the end actions. The bending moment is defined as

$$M = EI\frac{d^2v}{dx^2} \tag{10.84}$$

Noting Fig.10.34b, the end actions are related to the bending moment by

$$\begin{aligned}
V_{\mathrm{B}} &= \frac{-1}{L}(M_{\mathrm{A}} + M_{\mathrm{B}}) - P\frac{(v_{\mathrm{B}} - v_{\mathrm{A}})}{L} \\
V_{\mathrm{A}} &= -V_{\mathrm{B}}
\end{aligned} \tag{10.85}$$

The second term in the expression for V_{B} is due to the rotation of the chord connecting A and B. This term is neglected in the linear formulation. We will show later that it leads to a loss in lateral stiffness (commonly referred to as the P-delta effect).

Using the above equations, we express the final equations as:

$$\begin{aligned}
M_{\mathrm{A}} &= M_{\mathrm{A}}^{\mathrm{F}} + \frac{EI}{L}\left[\phi_1\omega_{\mathrm{A}} + \phi_2\omega_{\mathrm{B}} - \frac{\phi_3}{L}(v_{\mathrm{B}} - v_{\mathrm{A}})\right] \\
M_{\mathrm{B}} &= M_{\mathrm{B}}^{\mathrm{F}} + \frac{EI}{L}\left[\phi_1\omega_{\mathrm{B}} + \phi_2\omega_{\mathrm{A}} - \frac{\phi_3}{L}(v_{\mathrm{B}} - v_{\mathrm{A}})\right] \\
V_{\mathrm{A}} &= V_{\mathrm{A}}^{\mathrm{F}} + \frac{\phi_3 EI}{L^2}\left[\omega_{\mathrm{B}} + \omega_{\mathrm{A}} - \frac{2}{L}(v_{\mathrm{B}} - v_{\mathrm{A}})\right] + \frac{P}{L}(v_{\mathrm{B}} - v_{\mathrm{A}}) \\
V_{\mathrm{B}} &= V_{\mathrm{B}}^{\mathrm{F}} - \frac{\phi_3 EI}{L^2}\left[\omega_{\mathrm{B}} + \omega_{\mathrm{A}} - \frac{2}{L}(v_{\mathrm{B}} - v_{\mathrm{A}})\right] - \frac{P}{L}(v_{\mathrm{B}} - v_{\mathrm{A}})
\end{aligned} \tag{10.86}$$

where

$$D\phi_1 = \mu L(\sin \mu L - \mu L \cos \mu L)$$
$$D\phi_2 = \mu L(\mu L - \sin \mu L) \tag{10.87}$$
$$\phi_3 = \phi_1 + \phi_2$$

The ϕ functions were introduced by Livesley [2]. Figure 10.35 shows the variation with μL. For small μL, the coefficients reduce to the corresponding linear values

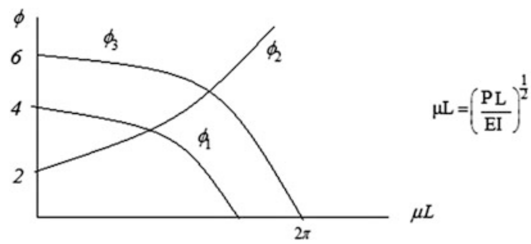


Fig. 10.35 ϕ functions

$$\mu L \to 0$$
$$\phi_1 \to 4$$
$$\phi_2 \to 2 \tag{10.88}$$
$$\phi_3 \to 6$$

For large μL, the functions behave in a nonlinear manner

$$\mu L \to 2\pi$$
$$\phi_1 \to -\infty$$
$$\phi_2 \to +\infty \tag{10.89}$$
$$\phi_3 \to 0$$

One can assume linear behavior and use these results to obtain an initial estimate for the axial load. Equations (10.86) are applicable for those members which have a compressive axial load. As the external loading is increased, the internal axial loads also increase, resulting in a reduction in stiffness and eventually to large displacements similar to the behavior shown in Fig. E10.21b. This trend is clearly evident in the expressions for V_A and V_B listed in (10.86). As P increases, ϕ_3 decreases, and the overall stiffness decreases. The following examples illustrate this effect.

Example 10.22

Given: The portal frame shown in Fig. E10.22a.

Determine: The effect of axial load on the lateral stiffness.

Solution: We assume $I_g \gg I_c$ so that member BC just translates under the action of the horizontal load. We also assume there is a gravity loading which creates compression in the columns. Of interest is the interaction between the gravity loading and the lateral loading. Due to the compressive nature of the gravity loading, we should expect a reduction in lateral stiffness, leading eventually to an unstable condition.

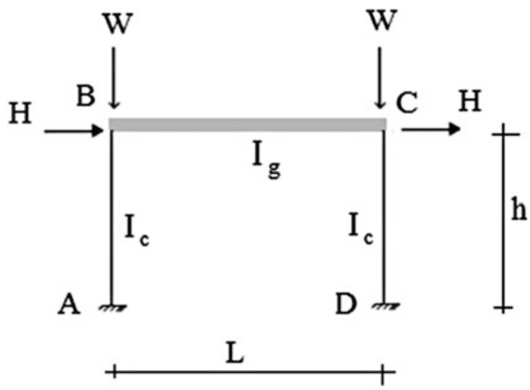


Fig. E10.22a

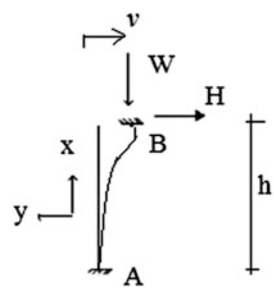


Fig. E10.22b

Noting the free body diagram shown in Fig. E10.22b, the end conditions for member AB are

$$\theta_{\mathrm{A}} = v_{\mathrm{A}} = 0 \quad \theta_{\mathrm{B}} = 0 \quad \overline{V}_{\mathrm{B}} = -H$$

v_{B} =-v $\quad P = W$

Using these values, the expression for $\overline{V}_{\mathrm{B}}$ follows from Equation (10.86).

$$V_{\mathrm{B}} = \frac{-2EI_c\phi_3}{h^3}v + \frac{P}{h}v$$

Then

$$H = \left\{\frac{2EI_c\phi_3}{h^3} - \frac{P}{h}\right\}v \equiv kv$$

Figure E10.22c shows how k degrades with increasing P. The P-delta term dominates this process; it leads to an 80 % reduction at the critical loading, $\alpha = 1$. As the load approaches this load level, the lateral stiffness approaches zero, resulting in large displacement, and eventual failure due to excessive inelastic deformation.

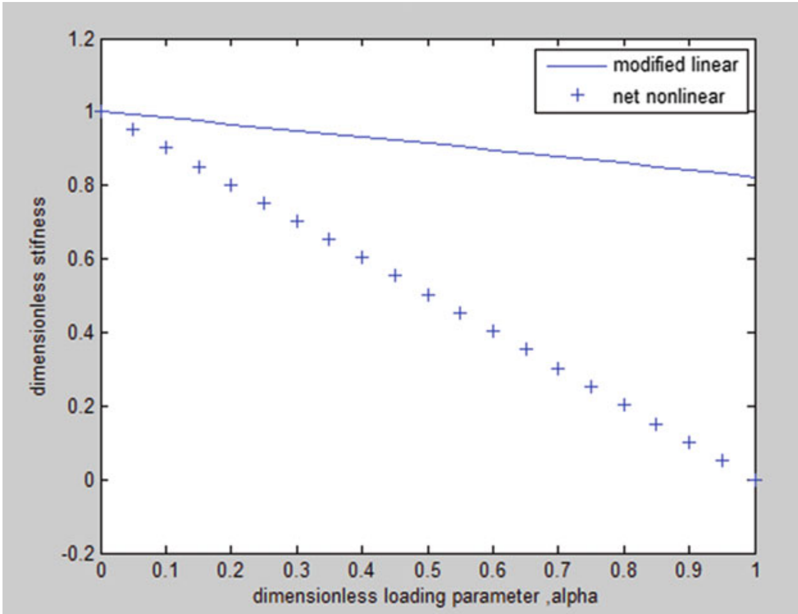


Fig. E10.22c

Example 10.23

Given: The pin-ended portal frame shown in Fig. E10.23a.

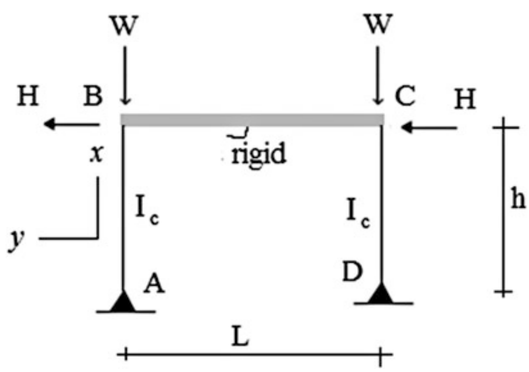


Fig. E10.23a

Determine: (a) The effect of axial load on the lateral stiffness. (b) The additional stiffness provided by diagonal bracing (Fig. E10.23c).

Solution:

Part(a)
The response is anti-symmetrical so one has only to analyze member AB. Noting Fig. E10.23a, the end conditions are

$$v_{\mathrm{A}} = M_{\mathrm{A}} = 0 \quad P = W$$
$$\theta_{\mathrm{B}} = 0 \quad V_{\mathrm{B}} = +H$$

Setting $M_{\mathrm{A}} = 0$ leads to

$$\frac{EI_{\mathrm{c}}}{h}\left[\phi_1\theta_{\mathrm{A}} + \phi_2\theta_{\mathrm{B}} - \phi_3\left(\frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{h}\right)\right] = 0$$

Then, noting the end conditions, and solving for θ_{A} yields

$$\theta_{\mathrm{A}} = \frac{\phi_3}{\phi_1}\frac{EI_{\mathrm{c}}}{h}\left(\frac{v_{\mathrm{B}}}{h}\right)$$

Substituting for θ_{A}, the expression for V_{B} becomes

$$V_{\mathrm{B}} = -\frac{EI_{\mathrm{c}}}{h}\phi_3\left[\left(\frac{\phi_3}{\phi_1} - 2\right)\left(\frac{v_{\mathrm{B}}}{h}\right)\right] - \frac{P}{h}v_{\mathrm{B}}$$

Lastly, we require $V_{\mathrm{B}} = +H$.

Then

$$H = \frac{EI_{\mathrm{c}}}{h^3}\left[\phi_3\left(2 - \frac{\phi_3}{\phi_1}\right)\right]v_{\mathrm{B}} - \frac{P}{h}v_{\mathrm{B}}$$

We express H as

$$H = kv_{\mathrm{B}}$$
$$k = \frac{3EI_{\mathrm{c}}}{h^3}\left[\frac{1}{3}\phi_3\left(2 - \frac{\phi_3}{\phi_1}\right) - \frac{\alpha\pi^2}{12}\right] = \frac{3EI_c}{h^3}(k_1 - k_2)$$

where

$$\alpha = \frac{P}{P_{\mathrm{cr}}} = \frac{P}{\frac{\pi^2 EI_{\mathrm{c}}}{4h^2}}$$

$$k_2 = \frac{Ph^2}{3EI_{\mathrm{c}}}$$

Note that k_2 represents the P-delta effect on stiffness. Figure E10.23b shows that k_2 dominates the stiffness reduction due to the axial compression in the columns.

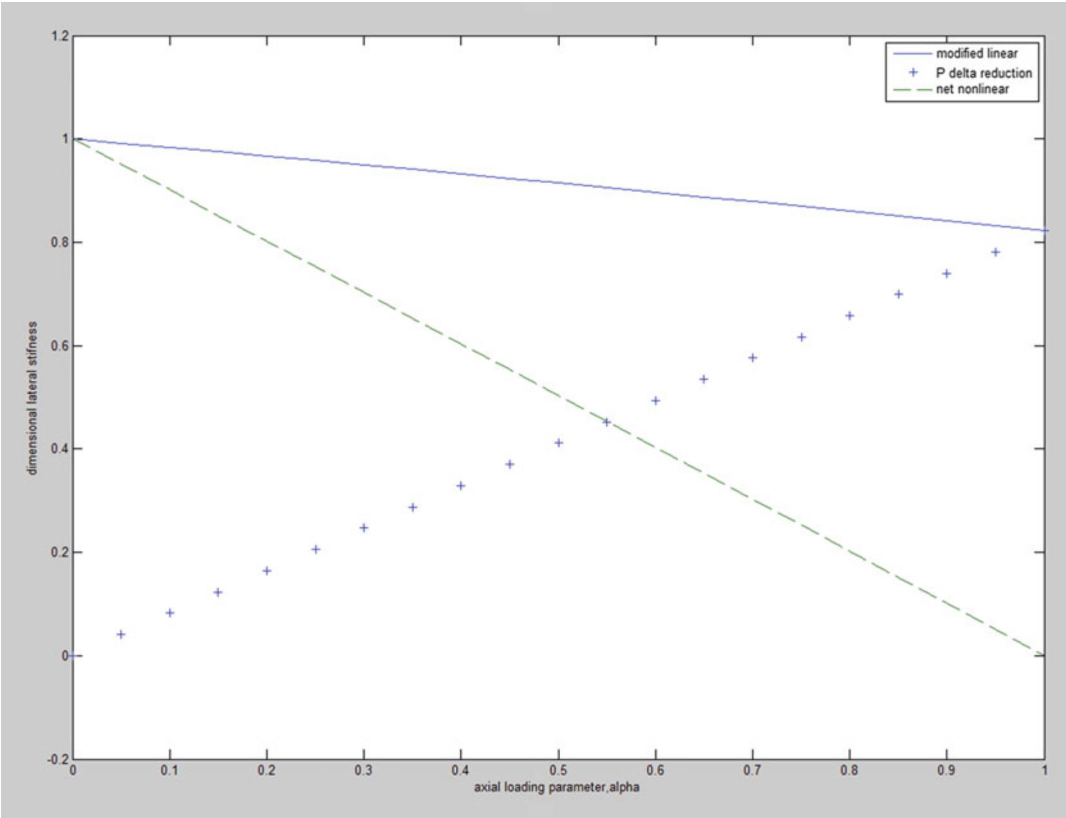


Fig. E10.23b

Part(b)

Suppose the diagonal braces shown in Fig. E10.23c are added. They provide additional stiffness which offsets the loss in stiffness due to P-delta effect. Noting the expression for V_{B} derived above, the force H is now equal to the sum of V_{B} and the horizontal component of the bracing force.

$$H = V_{\mathrm{B}} + F_{\mathrm{brace}} \cos\theta = (k + k_{\mathrm{brace}}) v_{\mathrm{B}}$$

where

$$k_{\mathrm{brace}} = \frac{A_{\mathrm{brace}} E}{L_{\mathrm{brace}}} (\cos\theta)^2 = \frac{A_{\mathrm{brace}} E}{h} (\cos\theta)^2 \sin\theta$$

One selects k_{brace} such that

$$k_{\mathrm{brace}} = \frac{A_{\mathrm{brace}} E}{h} (\cos\theta)^2 \sin\theta = \frac{3EI_{\mathrm{c}}}{h^3} = K\big|_{P=0}$$

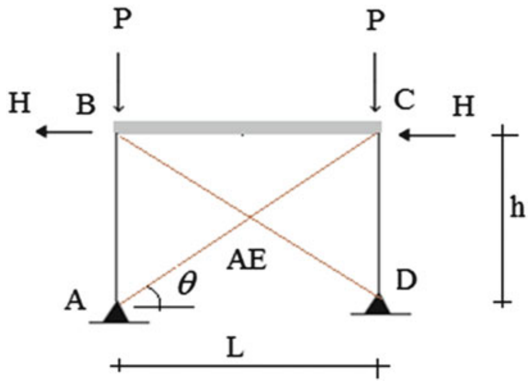


Fig. E10.23c

10.10 Summary

10.10.1 Objectives

- To describe the displacement method of analysis specialized for frame-type structures.
- To develop the slope-deflection equations for planar bending of beams
- To illustrate how to apply the displacement method for beams and rigid frame systems using the slope-deflection equations.
- To formulate the moment distribution procedure and demonstrate its application to indeterminate beams and rigid frames.
- To determine the effect of geometric nonlinearity and to formulate the geometric nonlinear form of the slope-deflection equations.

10.10.2 Key Factors and Concepts

- The displacement method works with nodal Force Equilibrium Equations expressed in terms of displacements
- The slope-deflection equations relate the end shears and moments to the end translations and rotations. Their general linear form for planar bending of a prismatic member AB is

$$M_{\mathrm{AB}} = \frac{2EI}{L}\{2\theta_{\mathrm{A}} + \theta_{\mathrm{B}}\} + \frac{6EI}{L}\left(\frac{v_{\mathrm{A}} - v_{\mathrm{B}}}{L}\right) + M^{\mathrm{F}}_{\mathrm{AB}}$$
$$M_{\mathrm{BA}} = \frac{2EI}{L}\{2\theta_{\mathrm{B}} + \theta_{\mathrm{A}}\} + \frac{6EI}{L}\left(\frac{v_{\mathrm{A}} - v_{\mathrm{B}}}{L}\right) + M^{\mathrm{F}}_{\mathrm{BA}}$$
$$V_{\mathrm{AB}} = \frac{6EI}{L^2}(\theta_{\mathrm{A}} + \theta_{\mathrm{B}}) + \frac{12EI}{L^3}(v_{\mathrm{A}} - v_{\mathrm{B}}) + V^{\mathrm{F}}_{\mathrm{AB}}$$
$$V_{\mathrm{BA}} = -\frac{6EI}{L^2}(\theta_{\mathrm{B}} + \theta_{\mathrm{A}}) - \frac{12EI}{L^3}(v_{\mathrm{A}} - v_{\mathrm{B}}) + V^{\mathrm{F}}_{\mathrm{BA}}$$

- Moment distribution is a numerical procedure for distributing the unbalanced nodal moments into the adjacent members based on relative stiffness. If one continued the process until the moment residuals are reduced to zero, one would obtain the exact solution. Normally, the process is terminated when the residuals are relatively small.

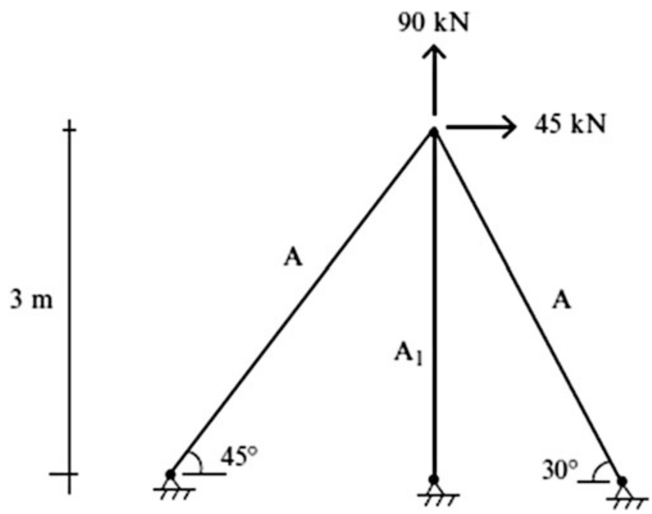
- The slope-deflection equations provide the basis for the computer-based analysis procedure described in Chap. 12.
- Geometric nonlinear behavior is due to the coupling between compressive axial load and transverse displacement. It results in a loss of stiffness and leads to unstable behavior.

10.11 Problems

Problem 10.1 Determine the displacements and member forces for the truss shown. Consider the following values for the areas:

(a) $A_1 = \frac{1}{2}A$
(b) $A_1 = 2A$
(c) Check your results with computer-based analysis.

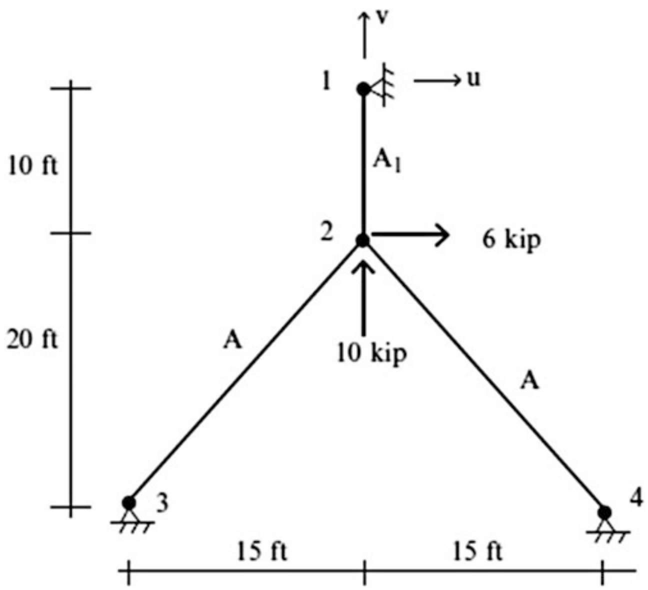
Take $E = 200$ GPa and $A = 2000$ mm^2



Problem 10.2 For the truss shown below, determine the member forces for:

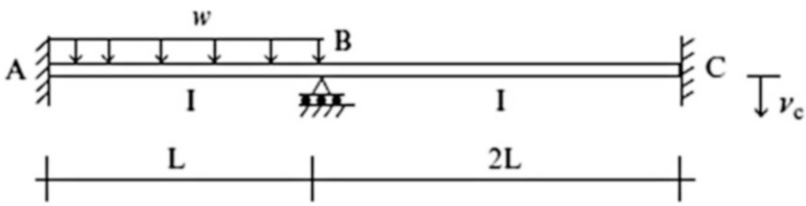
(a) The loading shown
(b) Support #1 moves as follows: $u = \frac{1}{8}$ in. → and $v = \frac{1}{2}$ in. ↑

Take $A = 0.1$ in.^{2}, $A_1 = 0.4$ in.^{2}, and $E = 29{,}000$ ksi.



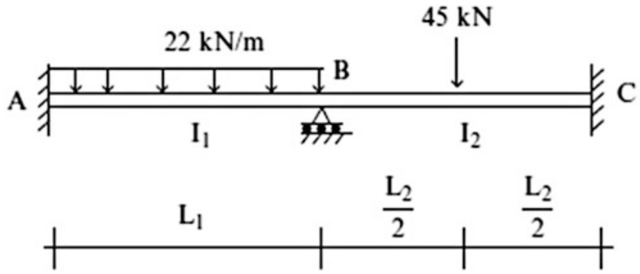
For the following beams and frames defined in Problems 10.3–10.18, determine the member end moments using the slope-deflection equations.

Problem 10.3 Assume $E = 29{,}000$ ksi, $I = 200$ in.^{4}, $L = 30$ ft, $v_C = 0.6$ in. ↓, and $w = 1.2$ kip/ft.



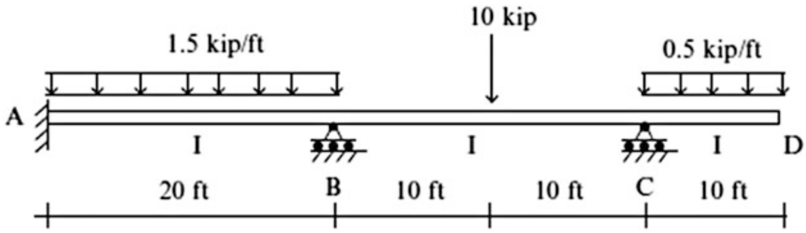
Problem 10.4

(a) $I_1 = I_2$ and $L_1 = L_2$
(b) $I_1 = 2I_2$ and $L_1 = L_2$
(c) $I_1 = 2I_2$ and $L_1 = 2\ L_2$



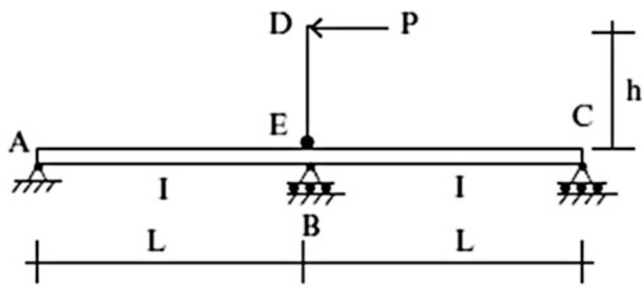
Assume $E = 200$ GPa, $I_2 = 80(10)^6$ mm^4, and $L_2 = 6$ m.

Problem 10.5

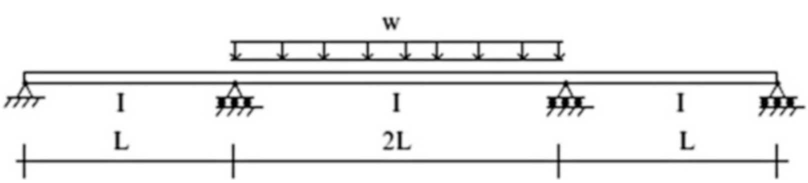


$E = 29{,}000$ ksi and $I = 300$ in.^{4}

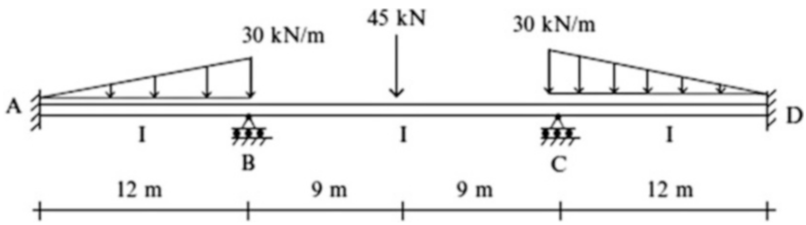
Problem 10.6 Assume $E = 200$ GPa, $I = 80(10)^6$ mm^4, $P = 45$ kN, $h = 3$ m, and $L = 9$ m.



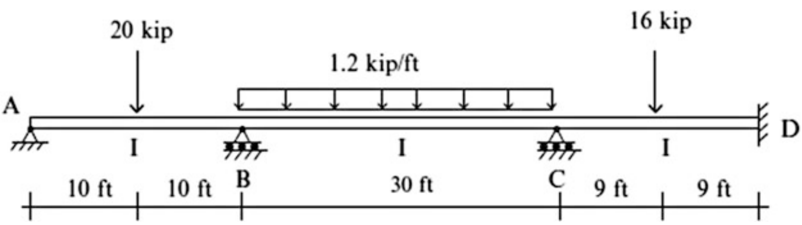
Problem 10.7 Assume $E = 29{,}000$ ksi, $I = 200$ in.^{4}, $L = 18$ ft, and $w = 1.2$ kip/ft.



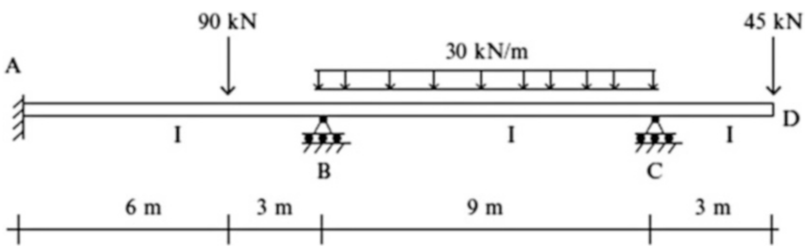
Problem 10.8 Assume $E = 200$ GPa and $I = 80(10)^6$ mm^4.



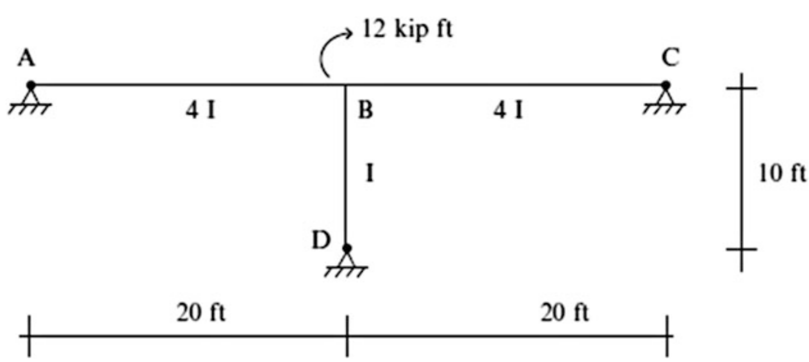
Problem 10.9 Assume $E = 29{,}000$ ksi and $I = 400$ in.^{4}



Problem 10.10 Assume $E = 200$ GPa and $I = 100(10)^6$ mm^4.

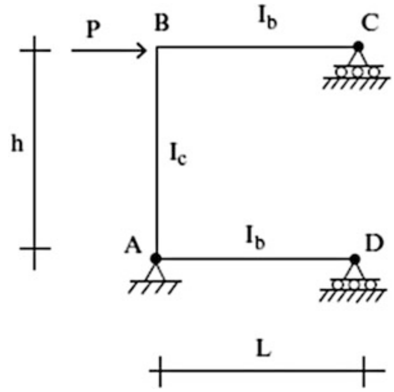


Problem 10.11 Assume $E = 29{,}000$ ksi and $I = 100$ in.^{4}



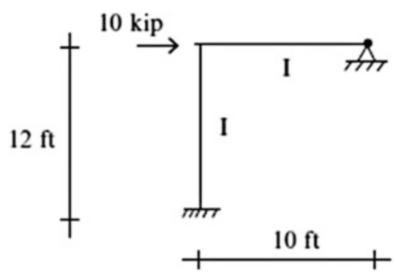
Problem 10.12

(a) $I_b = I_c$
(b) $I_b = 1.5I_c$

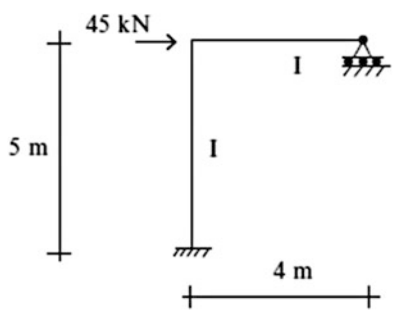


Assume $E = 200$ GPa, $I_c = 120(10)^6$ mm^4, $L = 8$ m, $h = 4$ m, and $P = 50$ kN.

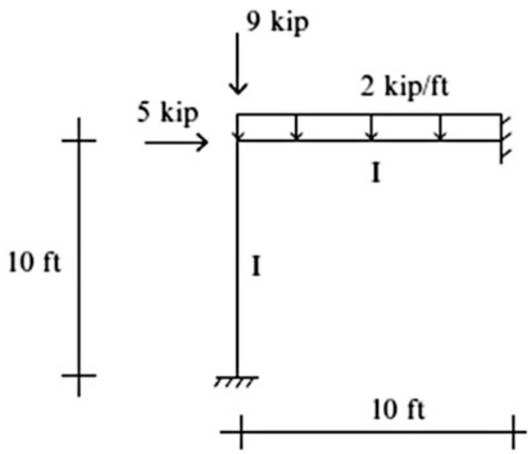
Problem 10.13 Assume $E = 29{,}000$ ksi and $I = 200$ in.^{4}



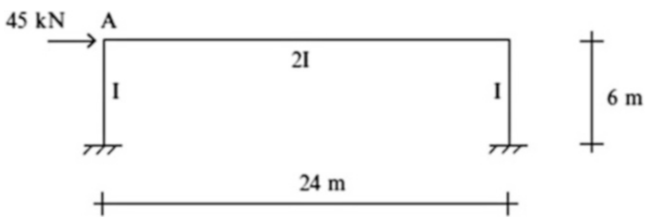
Problem 10.14 Assume $E = 200$ GPa and $I = 80(10)^6$ mm^4.



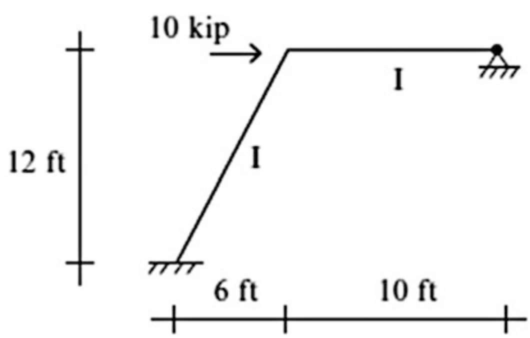
Problem 10.15 $I = 600$ in.^{4}
$E = 29{,}000$ kip/in.^{2}



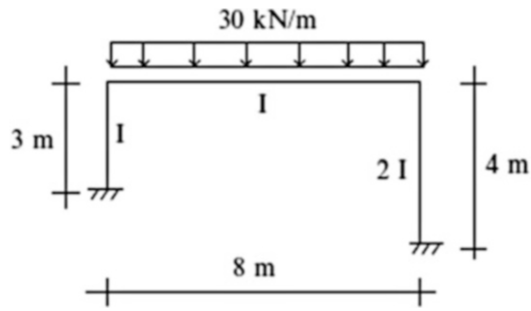
Problem 10.16 Assume $E = 200$ GPa and $I = 120(10)^6$ mm^4.



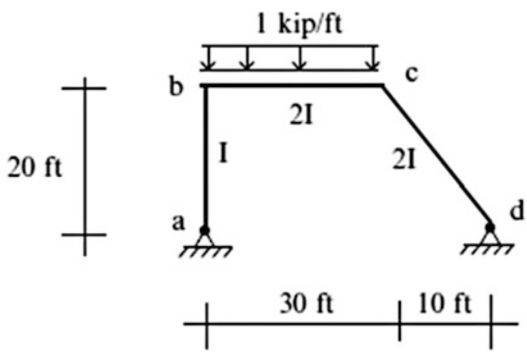
Problem 10.17 Assume $E = 29{,}000$ ksi and $I = 200$ in.^{4}



Problem 10.18 Assume $E = 200$ GPa and $I = 80(10)^6$ mm^4.



Problem 10.19 For the frame shown below, use computer software to determine the moment diagram and displacement profile. Assume $E = 29{,}000$ ksi and $I = 200$ in.^{4}

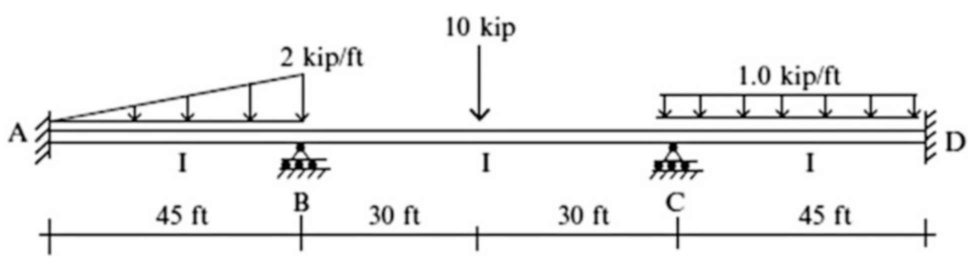


For the following beams and frames defined in Problems 10.20–10.34, determine the member end moments using moment distribution.

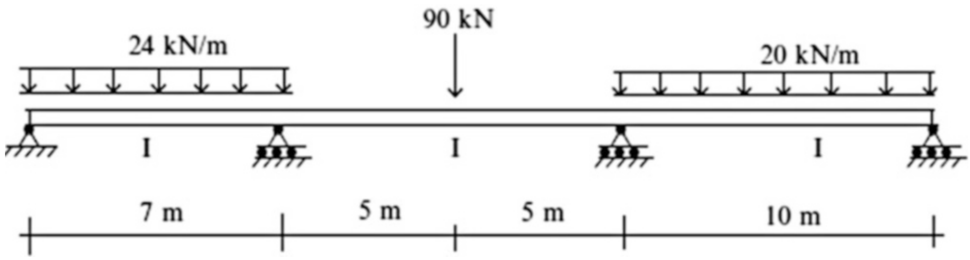
Problem 10.20 The loading shown

(a) A support settlement of .5 in. downward at joint B in addition to the loading
(b) Check your results with computer-based analysis.

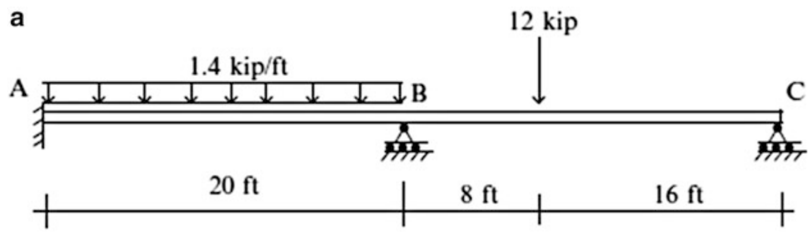
$$E = 29{,}000 \text{ ksi},$$
$$I = 300 \text{ in.}^4$$

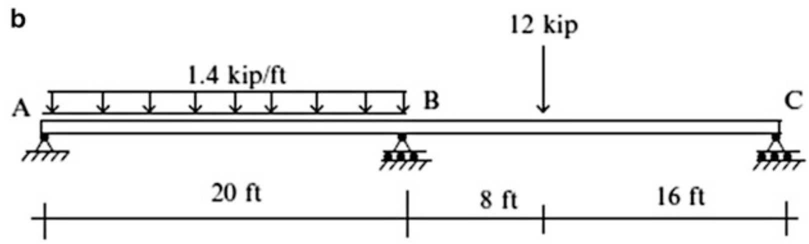


Problem 10.21 Compute the end moments and reactions. Draw the shear and moment diagrams. Check your results with computer analysis. Assume $E = 200$ GPa and $I = 75(10)^6$ mm^4.

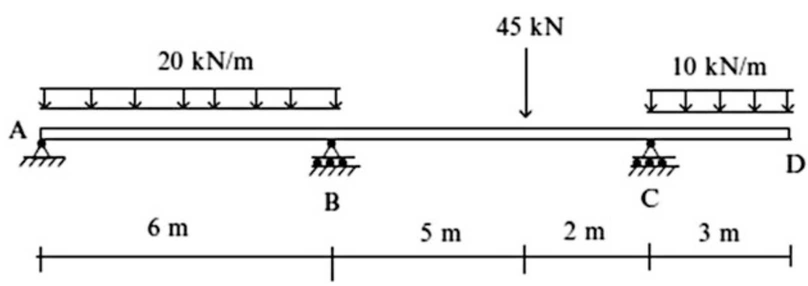


Problem 10.22 Determine the bending moments and the reactions for the following cases. Assume *EI* is constant

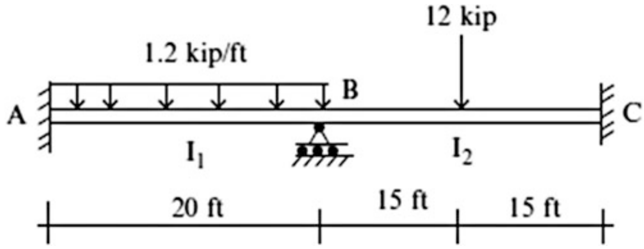




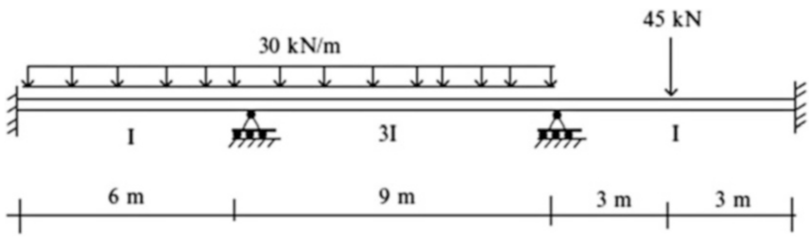
Problem 10.23 Determine the bending moment distribution for the beam shown below. Assume *EI* is constant.



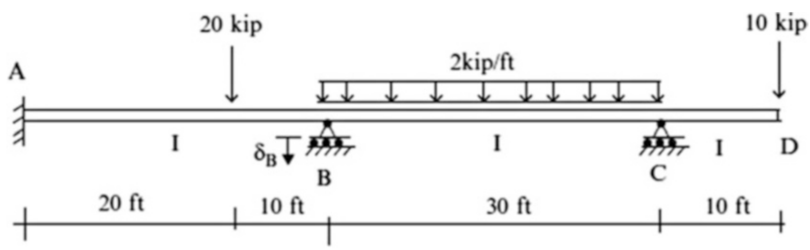
Problem 10.24 Determine the bending moment distribution. Assume $I_1 = 1.4I_2$.



Problem 10.25 Determine the bending moment distribution.



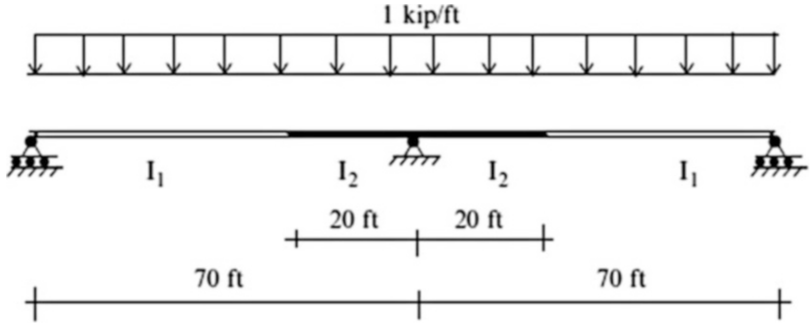
Problem 10.26 Solve for the bending moments. $\delta_B = 0.4$ in. ↓, $E = 29{,}000$ ksi, and $I = 240$ in.^{4}.



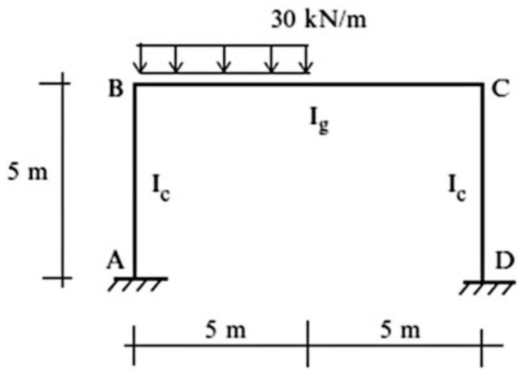
Problem 10.27 Determine the bending moment distribution and the deflected shape. $E = 29{,}000$ ksi

(a) Take $I_1 = I_2 = 1000$ in.^{4}
(b) Take $I_1 = 1.5I_1$. Use computer analysis.

Discuss the difference in behavior between case (a) and (b).



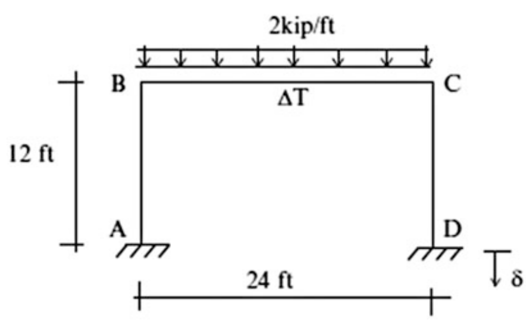
Problem 10.28 Determine the axial, shear, and bending moment distributions. Take $I_g = 2I_c$



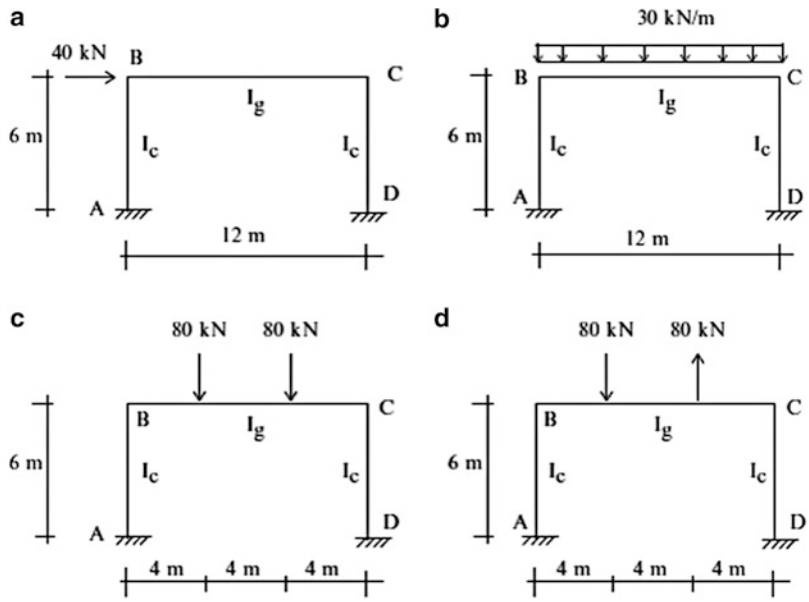
Problem 10.29 Determine the member forces and the reactions.

(a) Consider only the uniform load shown
(b) Consider only the support settlement of joint D ($\delta = 0.5$ in. ↓)
(c) Consider only the temperature increase of $\Delta T = 80$ °F for member BC.

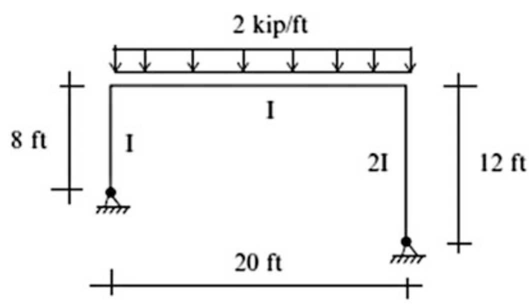
$$E = 29{,}000 \text{ ksi}$$
$$I_{AB} = I_{CD} = 100 \text{ in.}^4$$
$$I_{BC} = 400 \text{ in.}^4$$
$$\alpha = 6.5 \times 10^{-6}/^\circ\text{F}$$



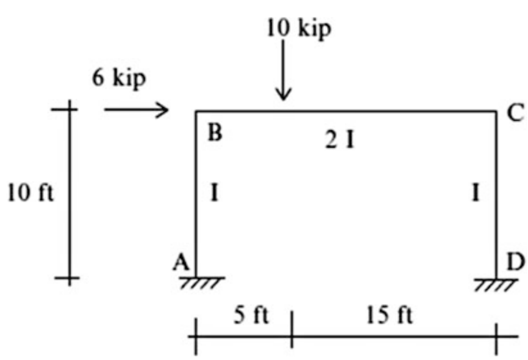
Problem 10.30 Determine the bending moment distribution for the following loadings. Take $I_g = 5I_c$.



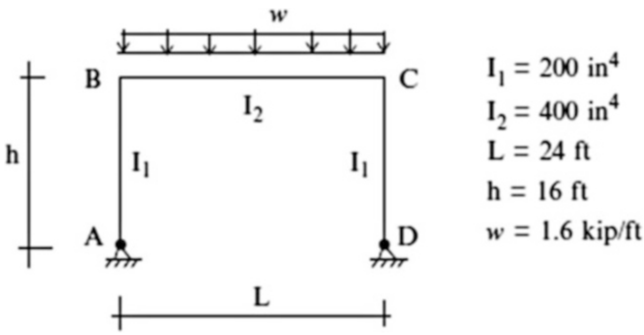
Problem 10.31 Solve for the bending moments.



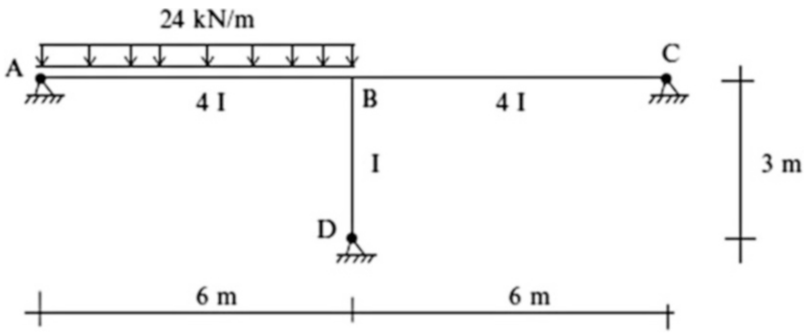
Problem 10.32 Determine the bending moment distribution.



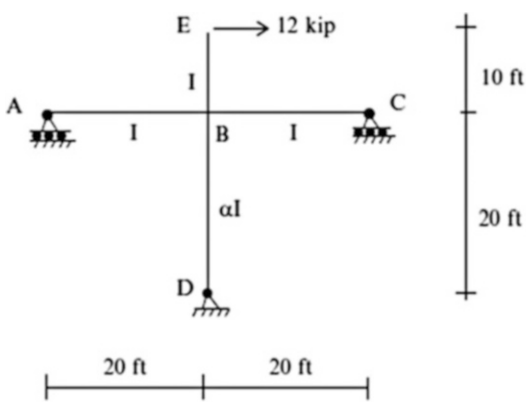
Problem 10.33 Solve for the bending moments.



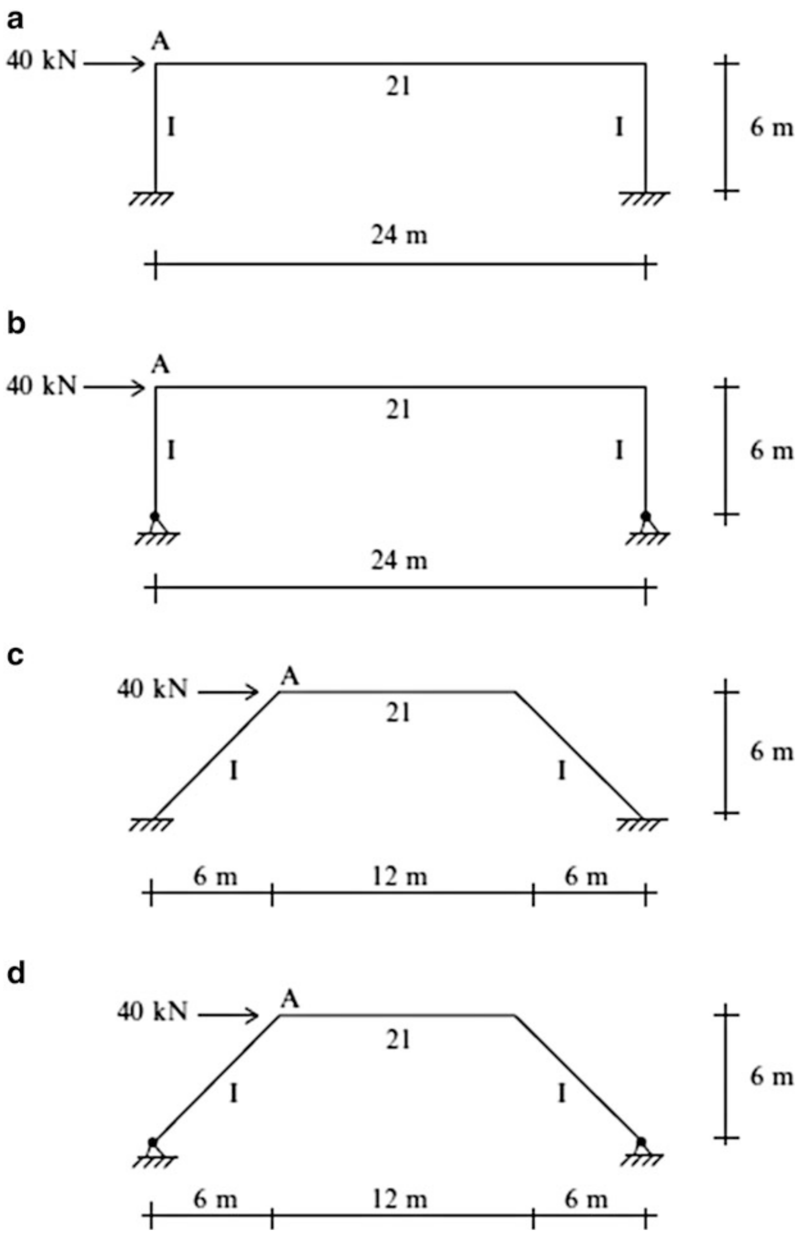
Problem 10.34 For the frame shown, determine the end moments and the reactions. Assume $E = 200$ GPa and $I = 40(10)^6$ mm^4.



Problem 10.35 Determine analytic expression for the rotation and end moments at B. Take $I = 1000$ in.^{4}, $A = 20$ in.^{2} for all members, and $\alpha = 1.0$, 2.0, 5.0. Is there an upper limit for the end moment, M_{BD}?



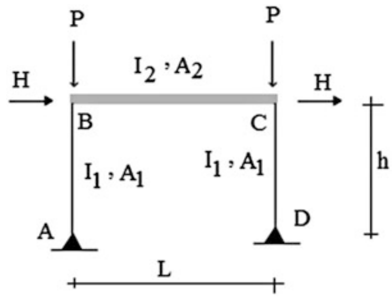
Problem 10.36 Compare the end moments and horizontal displacement at A for the rigid frames shown below. Check your results for parts (c) and (d) with a computer-based analysis. Take $E = 200$ GPa and $I = 120(10)^6$ mm^4. $A = 10{,}000$ mm^2 for all members.



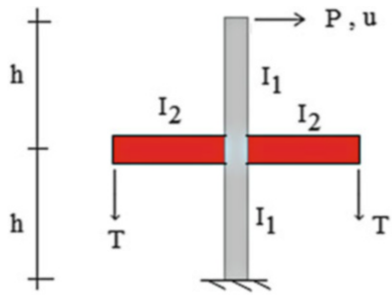
Problem 10.37 Compute displacement at node C for

(a) No P-delta effect
(b) With P-delta effect included

Take $I_1 = 881\,\text{in.}^4$, $A_1 = 24\,\text{in.}^2$, $I_2 = 2960\,\text{in.}^4$, $A_2 = 35.9\,\text{in.}^2$, $H = 100$ kip, $P = 200$ kip, $E = 29{,}000$ ksi, $L = 20\,\text{ft}$, and $h = 10\,\text{ft}$.



Problem 10.38 Generate the plots of P vs. u for various values of T. Starting at $T = 0$ and increasing to $T = \frac{\pi^2 EI_1}{4h^2}$. Take $I_1 = 400\text{in}^4$, $I_2 = \infty$, and $h = 10$ ft.



References

1. Cross H. Analysis of continuous frames by distributing fixed-end moments. Trans ASCE. 1932;96:1–10. Paper 1793.
2. Livesley RK. Matrix methods of structural analysis. London: Pergamon; 1964.