Debt Valuation

5

Debt is one of the possible sources of capital for the company. When the debt is issued on the market, it takes the form of bonds and complements the capital obtained as debt from banks.

Bond markets are very liquid and generally show high volumes of trading, and bonds of many types are traded every day, in order to satisfy the needs of capital of the corporations and the investment needs of investors.

The main issue when valuing the debt of a corporation is to calculate and interpret the actual value of the bonds issued by the firm and grasp the riskiness of the business and how the market perceives it.

Valuation of bonds goes through valuation of the cash flows generated and can be performed according to various approaches. All of them are meant to successfully compute the market value of debt.

The understanding of debt value for a firm is a primary task when looking at the overall average cost of capital and possibly to a valuation of the company. Mathematical models help with the task.

According to different forms assumed for the short rate at infinitesimal frequencies, the term structure used to discount bond cash flows assumes different shapes and provides several possible actual values of the cash flows involved.

After studying this chapter, you will be able to answer the following questions, among others:

1. What types of bonds are available in financial markets, and how do they differ from each other?
2. What is the relationship between the price and yield of a bond, and what does the yield curve represents?
3. How can duration and convexity measures be used to assess the riskiness of a bond?
4. What types of stochastic models for the short rate are available?

5. How can models for the short rate be used to build a term structure and price a bond?

The first section of the chapter is an introduction to the various types of bonds in the market. The second section deals with the relationship between the price and yield of a bond and presents the term structure and yield curve concepts. The third section gives a description of the stochastic models of the short rate and how they can be applied to the building of the term structure and bond pricing.

5.1   Bond Markets

Learning Outcomes

- Learn the foundations of bond theory.
- Explain the different types of bonds available on the market.
- Understand the role of risk in fixed income investment.

5.1.1   Bond Fundamentals

For a company, the cost of debt is represented by the interest rate it has to pay on new loans, like bank loans or bond issues. In relation to the bond issuance, the rate is clearly not the same as the coupon rate on the firm's existing debt, which reflects the interest rate the firm had to offer at the time the debt was issued.

When a company has already issued bonds, the existing debt trades on the market and the price reflect the normal fluctuations of the economic environment and modifications in the risk level.

A yield to maturity can be implied from bond prices, reflecting the actual valuation of the risk associated to the company, in terms of a yield that adds up a premium to the current risk-free level.

The yield to maturity indicates what debt holders would earn if they held the debt to maturity and received all of the payments as promised. Thus, the yield to maturity is the firm's current cost of debt.

Example 5.1   A company has bonds due in 36 months with a coupon rate of 5% and a market price of 95.72. The yield to maturity corresponding to these characteristics is 3.30%. In reality there is no need to compute the yield to maturity because prices and their implied yields to maturity are always quoted together in the bond market.

As opposed to equity capital, the cost of debt is such that the total expense is always lower than the actual return obtained by debt holders (Cochrane and Piazzesi 2005). This is possible because the interest paid on debt is a tax-deductible expense, whereas dividends paid to equity are not. In case of debt financing, the tax savings partially offset the interest payments.

The tax deductibility of interest lowers the effective cost of debt financing for the firm. More generally, with tax-deductible interest and denoting the corporate tax rate as TC, the effective after-tax borrowing rate is

$$r_D(1 - \tau_C)$$

This makes the effective cost of debt on a principal L equal to

$$r_D(1 - \tau_C)L$$

Example 5.2 A company borrows 10,000 € at a rate of 5%, and the corporate tax rate in the economy is 33%. Then its net cost at the end of the year is calculated as in the table.

Interest expense	$0.05 \times 10,000$	500
Tax savings	$(0.05 \times 10,000 \times 0.33)$	(165)
After-tax expense	$0.05 \times (1 - 0.33) \times 10,000$	335

The effective cost of the debt $\tilde{r}_D$ after taxes is

$$\tilde{r}_D = \frac{335}{10,000} = 3.35\%$$

5.1.2 Types of Bonds

The market of bonds is populated by several types of securities that can be differentiated based on the issuer, the structure, and the cash flows. The main distinction is between government bonds and corporate bonds.

Government bonds are issued by sovereign governments and can be zero-coupon bonds or coupon-paying bonds. As for most bonds, it promises to pay periodic interest payments and to repay the face value on the maturity date.

The currency of denomination of government bonds is generally the domestic currency of the country of issuance. In case the bonds are issued in a foreign currency, they are named sovereign bonds, but the term is often used as a synonymous of government bonds, regardless of other factors.

In terms of the terminology used by the media, it is common when a government or sovereign is close to default, to refer to such a situation as a sovereign debt crisis, again showing no relationship with the currency of issuance of the bonds.

The terms on which a government can sell bonds depend on how creditworthy the market considers it to be. International credit rating agencies will provide ratings for the bonds, but market participants will make up their own minds about this.

Bond markets are very old, with the first known issuance of government securities dating back to 1517, in the Netherlands. At that time, the country was not yet formed, so the city of Amsterdam was the official issuer (Fama 2006).

However, these bonds are considered their predecessor which later merged into Netherlands government bonds. The average interest rate at that time fluctuated around 20% at the time of issuance.

The first national government issuing a bond was England in the late seventeenth century. The war against France, occurring at that time, was the reason for issuing securities in the form of a tontine.

European governments started issuing bonds later, to generally fund public expenditure, mostly in the form of perpetual bonds. They are special bonds with no maturity, but their use was limited in time, with governments issuing only bonds with a limited term to maturity lately.

Another important distinction is between secured and unsecured bonds. The former are backed by a lien on the borrower's assets. If the borrower defaults, those assets can be sold to pay off the bondholders.

Unsecured bonds, also called debentures, are only backed by the general ability of the corporation or other borrower to pay its bills. If the borrower goes bankrupt, debentures can't be paid off until secured bondholders are paid.

Investors in subordinated debentures are a step down in the line for reimbursement and do not get paid until after holders of senior claims get their money. However, they take priority over the equity holders of any type.

Zero-coupon bonds may be secured or unsecured. They are issued at a big discount from face value because they pay all the interest at maturity, with no payments along the way.

Down from the government level, there are bonds that are issued locally by local states or municipalities. They are called municipal bonds and are very popular on US markets. They are issued by state or city governments, or their agencies, and come in two principal varieties.

General obligation bonds are the first type of municipal bonds. It is a type of security that is backed by the full taxing authority of the government. Revenue bonds, the second type, are instead backed by the receipts from a specific source of revenue, from the project that has to be financed by the bond issuance. They are therefore not considered as secure as general obligation bonds.

The interest that is paid to holders of both revenue and general obligation municipal bonds is exempt from federal income taxes and, usually, income taxes of the issuing state as well.

Government bonds in the United States have denominations dependent on the time to maturity. The so-called treasury bills are those that mature in less than 1 year. Bonds with maturity between 1 and 5 years are called treasury notes, and the bonds with maturity longer than 5 years are called treasury bonds. All of them are debentures, backed by the full faith and credit of the federal government.

Another popular type of bonds in the United States are the agency securities, which are issued by various government-sponsored agencies. They are not technically backed by the US government, but still a moral obligation is assumed to exist, which presumably wouldn't let an agency issue fail (Fabozzi 2001).

Callable bonds are a type of bonds that can be called (redeemed) by the issuer before maturity. This is an option that is written on the bond for the benefit of the company or government issuing the bond.

The borrower or buyer of the bond is subject to the issuer's decision of recalling the bond or not at any time. A company might decide to call its bonds if, for instance, interest rates fell so far that it could issue new bonds at a lower rate and thus save money. When bonds are called for an amount higher than paid by the buyer at issuance, the government would charge taxes on the forced sale.

Convertible bonds are the other types of bonds with an option attached. In this case, the option is in favor of the bondholder. Convertible bonds in fact can be swapped for the same company's common stock at a fixed ratio of amount of bonds to number of shares of stock.

The convertible option makes many corporate bonds attractive to the investors. If the price of the stock increases enough after purchase of the bond, the option may be convenient to be exercised, with sometimes a consistent profit on the swap.

Example 5.3   Assume an investor buys ten convertible bonds issued by company A, at 100 € each. Each bond pays a coupon of 5% and can be converted into four shares of the company. At the time of bond purchase, the stock is selling at 20 €. Because break-even conversion price is

$$\frac{100}{4} = 25 \ €$$

It follows that 5 € is the price paid per share for the conversion privilege. If the stock raises above $25, the investor can make a profit by converting the bonds to stock. If the price were to go to 30 €, one could turn the 1000 € bond investment into

$$40 \times 30 = 1200$$

worth of stock.

Because their fate is so closely tied to that of the stock price of the issuing firm, convertible bonds tend to be more closely in sync with the stock market than the bond market.

5.1.3   Knowing the Risks

Both government and corporate bonds represent a good source of income, some-times granting a relatively high rate of return to the investor, while being a safe investment, especially when compared to stocks.

It is still important to know the potential risks and potential downturns associated with holding corporate and/or government bonds. There are several types of risk that are attached to an investment in fixed income.

Credit risk is the risk of loss due to the inability of the borrower to hold its obligations, by failing on its debt. The definition clearly states that whenever someone lends money to someone else, the operation is subject to some degree of credit risk.

Generally, the yield on the government bonds denominated in domestic currency is considered as a proxy of the risk-free rate in the country of issuance. This is because it is assumed that the government can raise taxes or create additional currency in order to redeem the bond at maturity.

The ability to repay its debt is crucial for a company, and it is normally translated into credit ratings by major rating institutions such as Standard & Poor's or Moody's. Ratings range from "AAA" for high credit quality investments to "D" for bonds in default. The investor puts a lot of trust on the judgments expressed by these agencies.

In case a company has a low credit rating, with troubles in repaying its obligations, the banks and other lending institutions may ask for higher interest rates before agreeing to give a loan (Brigo and Mercurio 2001).

This can have an adverse impact on the company's ability to satisfy its debts with current bondholders and will hurt existing bondholders who might have been looking to unload their positions.

The currency risk associated with fixed income investments is the risk that the value of the currency a bond pays out will go down compared to the amount in the holder's domestic currency.

A European investor may invest in US T bonds to get more currency risk exposure than by investing in some European bonds. The same is true for an American investor considering investment in European bonds to have more currency risk than US bonds.

A bond paying in a currency that does not have a history of keeping its value may not be a good deal even if a high interest rate is offered, given that the gain in interest will be most likely offset by the loss on the currency side.

Inflation risk is the risk that the value of the currency a bond pays out will lose purchasing power over time due to an increase in inflation. Investors expect some amount of inflation, so the risk is that the inflation rate will be higher than expected.

Some governments issue inflation-linked types of bonds, protecting investors against upper movements of the inflation rate. These bonds link both the interest payment and the face value to some inflation index.

There is an inverse relationship between interest rates and bond prices, with bond prices raising when the market interest rates go down. It happens because when interest rates are on the decline, investors try to capture or lock in the highest rates they can for as long as they can.

This can be achieved by scooping up the existing bonds that carry an interest rate higher than the market. The surplus of demand implies an increase in bond price. If the prevailing interest rate were on the rise, the reaction of investors would force bond prices down.

Example 5.4 An investor owns a bond that trades at par and has a yield of 5%. If the prevailing market interest rate raises to 6%, the investor will decide to sell the 5% bond to buy the bond at 6%, for a better interest. This drives the price of the 5% bond price down, below par.

A common issue for bond investors is the so-called reinvestment risk, which arises when the proceeds from the sale of some funds have to be reinvested at a lower rate than before, therefore worsening the position of the investor. One of the main ways this risk presents itself is when interest rates fall over time and callable bonds are exercised by the issuers.

The issuer of a callable bond can redeem the bond prior to maturity, by forwarding the principal payment to the bondholder, usually for a value that is slightly above the par value of the bond.

By having the bonds recalled, the bondholder might end up with a lot of cash which is difficult to handle because reinvestment opportunities may present refinancing risk, which can have a major adverse impact on an individual's investment returns over time.

This is why the investors in callable bonds receive a yield that is higher than on a similar noncallable bond. The extra yield compensates for the extra risk in the portfolio.

Active bond investors can attempt to mitigate reinvestment risk in their portfolios by staggering the potential call dates of their differing bonds. This limits the chance that many bonds will be called at once.

Every bond is a certificate of debt, and by buying one, an investor is lending money to the issuer of the bond. This money must be repaid at maturity, with some interest on top. In case of corporate bond, the possibility to cash back the investment relies on the full faith of the issuer.

Among the range of possibilities, the investor must also consider the default of the borrower. If this happens, the debt will not be repaid, and the credit risk involved in the bond purchase will unleash its effects fully.

As one means of analyzing the possibility of default, some analysts and investors determine a company's coverage ratio before initiating an investment. By analyzing the financials of the borrower, they can determine its operating income and cash flow and then weigh that against its debt service expense.

In the case of government bonds, the markets are usually very liquid, with many securities populating the markets and ease of trading. This is not the case of corporate bonds, in some cases.

If there is no sufficient volume of some kind of bonds in the market, investors are at risk of not being able to sell their bonds quickly due to a thin market with few buyers and sellers for the bond.

The low demand for a particular bond issue leads to high volatility of prices, with a potential adverse impact on the return of the bondholder's portfolio. By selling in a thin market, one may be forced to take a much lower price than expected to sell the position in the bond.

There are also minor types of risk that can affect an investment in fixed income. Event risk in particular is the risk that the firm issuing a bond will undertake a process that will change its structure.

Operations like leveraged buyout, merger and acquisitions and debt restructuring can increase the debt ratio, pushing the value of bonds down and increasing the overall riskiness of the firm.

It is possible to talk about event risk also in case of natural and industrial accidents happening to the business, as well as important governmental interventions that produce regulatory changes.

Recall that it is possible to measure the convexity of a bond, as the rate of change of the dollar duration of a bond. Convexity and modified duration combined together improve the estimate of price sensitivity to large changes in interest rates.

By definition, plain bonds with no option attached have positive convexity, while convertible bonds and other bonds with options embedded have negative convexity. It means that the bond value is subject to extension risk when interest rates rise and contraction risk when interest rates fall.

Mortgage-backed bonds present additional risks as well. Prepayment risk is the risk that, in case interest rates go down, the mortgage holders decide to refinance or repay their loans sooner, therefore creating an early return of principal to holders of the loans.

An extension risk on the other hand is the risk that rising interest rates will slow the assumed prepayment speeds of mortgage loans. This will delay the return of principal to the investors, so that they will not be able to reinvest at higher yields.

5.2 The Price/Yield Relationship

Learning Outcomes

- Describe the relationship between bond prices and yields.
- Define fixed income futures and their application to financial risk.
- Explain how yield shifts impact on portfolio immunization.

5.2.1 Bond Prices and Yields

There are several ways to calculate the interest accrued on an investment or loan, with the main distinction being between simple interest and compound interest. The simple interest is just paid at the end of accrual period, without being reinvested.

In this way, there is no extra interest earned on the matured interest, and the equation can be written as

$$I_S = L \times r \times \Delta t$$

where:

I_S is the amount of simple interest generated by the investment.
r is the annual interest rate applied on the nominal amount.
L_0 is the nominal amount on which the interest rate is applied.

In case of compound interest, the accrued interest at the end of compounding period is reinvested together with the capital, increasing the amount earned at the end of later periods. Define the final value of the investment, called the future value, compounded on t years:

$$L_t = L(1 + r)^t \qquad\qquad (5.1)$$

The above equation is specific for annual compounding but can be generalized to allow for different compounding frequencies. Consider an interest rate that is compounded m times per year. In this case formula (5.1) is modified as

$$L_t = L\left(1 + \frac{r}{m}\right)^{tm}$$

For m tending to infinite, the highest compounding frequency is represented by continuous compounding, which can be approximated by

$$L_{\Delta t} = L e^{r\Delta t}$$

Some financial applications require knowledge of the present value of an investment, when knowing its final value and the interest rate applied to the nominal amount. The formulas for compounding can then be inverted to get

$$L_t = \frac{L_t}{(1 + r)^t}$$

$$L = \frac{L_t}{\left(1 + \frac{r}{m}\right)^{tm}}$$

$$L = L_t e^{-r\Delta t}$$

Example 5.5 A consumer wants to buy a car worth 25,900 €, and he is offered two different ways of payment. He can either pay the whole sum cash or go on a zero-interest loan for 1 year. The second choice is obviously the best but does not assume the seller offers a price discount for cash payment. If the discounted price is 23,610 €, this corresponds to a 10% discount on price. Now the convenience of choosing between the two forms of payment depends on the interest rate in the economy. At the 10% rate, it doesn't matter to buy the car from one seller or the other. But other interest rates generate different present values to be compared, therefore making one offer or the other the best one.

Table 5.1 Effective annual rate calculation for different compounding frequencies

Compounding	m	Effective annual rate	$r = 4\%$
Annually	1/1	$r = (1 + r/1)^1 - 1$	4.00%
Semiannually	1/2	$r_S = (1 + r/2)^2 - 1$	4.04%
Monthly	1/12	$r_M = (1 + r/12)^{12} - 1$	4.06%
Weekly	1/52	$r_W = (1 + r/52)^{52} - 1$	4.07%
Daily	1/365	$r_D = (1 + r/365)^{365} - 1$	4.08%
Continuous	$1/\infty$	$r_e = e^r - 1$	4.08%

The last column shows an example of the calculation applied to a nominal rate of 4%

An important concept related to the compounding frequency is the effective annual rate. Based on different frequencies, the same investment at the same rate can give different final values (Merton 1974).

Table 5.1 summarizes the effective rate accrued on some investment for the same nominal rate r compounded at different frequencies m and the calculation for the most common frequencies with an example in terms of numbers.

Clearly, the higher the compounding frequency, the higher the effective return earned on the investment. This is the reason for borrowers to usually prefer paying interest at lower compounding frequencies, while lenders prefer higher compounding frequency.

That is why yield is sometimes referred to as the IRR of an investment in bonds for $n = \frac{T}{\Delta t}$ periods, where T is the maturity.

The IRR is the interest rate r_y that solves the equation

$$B_0 = \sum_{i=1}^{n} \frac{C_i}{\left(1 + r_y\right)^i}$$

where:

C_i are the cash flows paid by the bond at each time i.
$n = t$ in terms of the notation used so far, in case of annual payments.

There is also the continuous compounding version of the IRR which can be expressed as

$$B_0 = \sum_{i=1}^{n} C_i e^{-r_y i} \tag{5.2}$$

For a bond paying periodic coupons to the investor, the payments are the intermediate coupons, plus the sum of last coupon and face value at the end, while for a zero-coupon bond, only the face value at maturity is paid with no intermediate payments.

Example 5.6 Consider now two bond yields above (11%) and below (8%) the average return. For a yield of 11%, the present value of the bond is (continuous compounding

$$B_0 = \sum_{i=1}^{7} 8e^{-0.11i} = 83.25$$

The present value of the bond corresponding to an 8% yield is

$$B_0 = \sum_{i=1}^{7} 8e^{-0.08i} = 98.31$$

The equation has no explicit solution for the yield and therefore cannot be inverted. The way to solve it is computationally or by adopting iterative trial-error procedures in order to identify the yield that gives the market price.

The procedure starts with a guess on the value of r_y as an input in formula (5.2). Then one must check whether the present value from calculation matches the actual price of the bond. The yield is then calibrated to get the match.

There is also an approximation equation that allows for a fair approximation of the yield to maturity of a coupon bond, as closed form solution. The equation is

$$r_y = \frac{C_i + \frac{B_{FV} - B}{t}}{\frac{B_{FV} + B}{2}}$$

where:

B_{FV} is the face value of the bond at maturity.
B is the market price of the bond.

As a note to avoid confusion, just have in mind that the notation B_{FV} is used to identify the value at maturity (face value) of the bond, as well as B_T, which is by convention 100 €.

Example 5.7 Consider a 7-year bond pays 8% coupon rate, face value 100 €, currently selling at 90 €. The average return per year is

$$r_y = \frac{8 + \frac{100 - 90}{7}}{\frac{100 + 90}{2}} = 9.92\%$$

Back to trial-and-error procedures for yield calculation, this is sometimes performed by bounding the right yield between two extreme values, one higher and the other lower than the actual yield.

The aim is to get closer to market price, just changing the guess on the yield appropriately, from a lower rate r_l, corresponding to a positive NPV_+, and a higher rate r_h, corresponding to a negative NPV_-. Then the yield to maturity is given by interpolation:

$$r_y = r_1 + \frac{NPV_+}{NPV_+ - NPV_-}(r_h - r_1)$$

where:

NPV_+ is the positive net present value, from a low guess on the yield.
NPV_- is the negative net present value, from a high guess on the yield.

Example 5.8 Given the two bonds in Example 5.6, the NPV at 11% is

$$-90 + 83.25 = -6.75$$

NPV at 8% is

$$-90 + 98.31 = 8.31$$

The yield to maturity is then

$$r_y = 0.08 + \left(\frac{8.31}{8.31 - (-6.75)}\right)(0.11 - 0.08) = 9.66\%$$

which is a fair approximation of the initial calculation. Computing the yield numerically gives a value of $r_y = 10.06\%$.

Standard calculations for bond price and yield to maturity, as presented till now, imply the yield to maturity is the same for all time periods considered during the life of the bond.

But if one accounts for the fact that the payments happen at different points in time, it can be assumed that they should be discounted at different yields, meaning that each cash flow is discounted by a rate appropriate to its maturity. The present value of a bond in discrete compounding is given by

$$B_0 = \sum_{i=1}^{n} \frac{C_i}{(1 + r_i)^i}$$

and continuous compounding can be written as

$$B_0 = \sum_{i=1}^{n} C_i e^{-r_i i}$$

where:

r_i is the yield to maturity corresponding to time i.

The pricing done by choosing different rates for different maturities is more accurate, without making necessary assumptions on reinvestment rates. The term structure rates can be any current spot rates, expectations of future spot rates, expected inflation, liquidity, and risk premium.

When pricing between two maturities both forward in time, it is appropriate to work with forward yields, which indicate the expected spot yield at some date in the future and can be derived directly from spot rates.

By indicating the spot rate at time 1 by r_1 available for investing for 1 year, and the spot rate r_2 as the rate available now for investing for two periods, the forward rate between periods 1 and 2 is the rate implied for investing for a 1-year period in 1 year's time.

This can be generalized to any forward rate between time t_1 and t_2, which can be written as

$$r_{1,2} = \left(\frac{(1+r_2)^{t_2}}{(1+r_1)^{t_1}} \right)^{\frac{1}{t_1-t_2}} - 1$$

where:

r_1 is the spot rate at time t_1.
r_2 is the spot rate at time t_2.

Finally, it is worthwhile to mention the par yield, the theoretical coupon rate r_{FV} that makes the bond calculated price equal to par value. The calculation is made by reworking the bond pricing equation as

$$B_T = \sum_{i=1}^{t} \frac{r_P B_T}{(1+r_i)^i} + \frac{r_P B_T + B_T}{(1+r_t)^t}$$
$$= \sum_{i=1}^{t} \frac{C_P}{(1+r_i)^i} + \frac{C_P + B_T}{(1+r_t)^t}$$

where:

B_T is the bond face value.
r_P is the par yield.
C_P is the coupon calculated on the par yield.

Example 5.9 Assume the spot rates in the term structure are

Year	Spot rate
1	6.00%
2	6.75%
3	7.00%

The par yield on a bond such that will be priced at par therefore is

$$100 = \frac{C_\mathrm{P}}{(1.06)} + \frac{C_\mathrm{P}}{(1.0675)^2} + \frac{C_\mathrm{P} + 100}{(1.07)^3}$$

The solution to the equation is $C_\mathrm{P} = 7.21$. Therefore, the par yield is $r_\mathrm{P} = 7.21\%$.

Once the par yield curve is obtained, it can be used for calculation on the coupon to be set on new bond issues and the relative value assessment.

5.2.2 The Yield Curve

Many theoretical models in financial economics are based on the assumption of constant interest rates. But in reality this is not true and interest rates may vary through time. This occurs primarily because inflation rates are expected to differ through time.

One way to find the variability of rates over time is to compare bonds with the same risk and plot on a graph their yields for different maturities. The relationship between interest rates (yields) and maturities is called term structure of interest rates, and the graph plotting it is called yield curve.

The curve can take different shapes according to the structure of the stochastic model for the short rate that is used to generate it. In general, it can take four different shapes, so as to be upward sloping, downward sloping, flat, or humped (see Fig. 5.1).

The upward shape is the most common and most observed on the market. It implies that short-term rates are lower than long-term rates, due to expectations that macroeconomic factors will push future rates to be higher (Litterman and Scheinkman 1991).
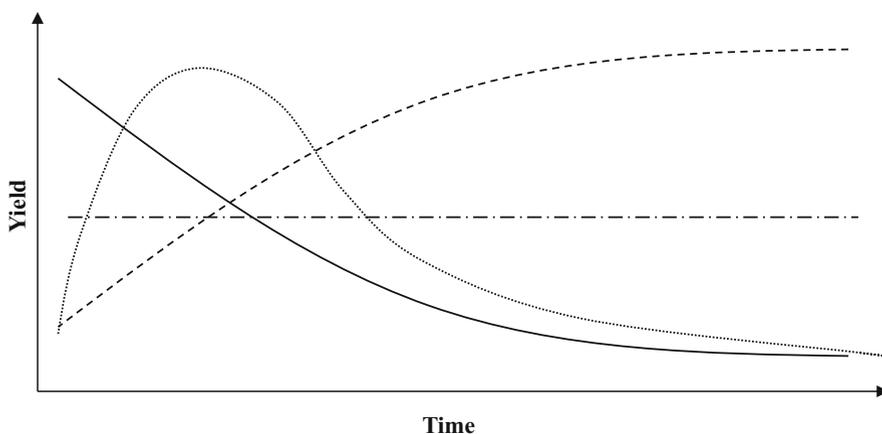

Fig. 5.1 The yield curves, as resulting from most common models of the interest rates, and observed empirically, can take different forms

The mechanism underlying such a result is that the investor has expectations of higher inflation and a tighter monetary policy to compensate that. As a consequence, investors demand a higher-risk premium. Such a risk is priced in the yield curve by demanding higher yields for longer maturities.

In case of a downward-sloped curve, interest rates imply expectations of future downturn in the economy. A flat yield curve indicates expectations on the economy have a negligible effect on the risk premium demand of investors.

A humped curve shows the demand of capital on intermediate maturities, a phenomenon observed in reality, where in recent years, on some markets, investors have demanded high volumes of long maturity instruments, leaving the intermediate maturities at lower demand, and consequently higher yields.

A complete term structure of interest rates must cover all the maturities between the shortest and the longest represented on the yield curve. Therefore, it is necessary to complete the structure by inferring the rates for the other maturities.

In general, fixed income markets are very liquid, and instruments are available to cover all maturities. However, in order to complete for missing maturities, a bootstrapping approach can be used.

The method consists in constructing some longer-term zero-coupon bonds by forming portfolios of traded coupon bonds. Zero-coupon yields can be derived from the prices of these bonds (Longstaff and Schwartz 1995).

Consider a simplified market with T bonds, one for each available maturity. As another simplification, let's assume that all bonds make payments at each maturity with identical payment dates. It is then possible to construct zero-coupon bonds for each maturity $1, 2, \ldots T$.

The bootstrapping of new rates over the maturity T goes through construction of appropriate discount factors $v_{t+1}, v_{t+2}, \ldots, v_{t+T}$ for all maturities considered. The discount factors are defined as the present value of a bond with face value 1, expiring at each maturity involved.

The next step is to derive yields from the discount factors, by building the yield curve up to time $t + T$. Recall the price equation for a coupon-paying bond:

$$B = \sum_{i=1}^{T} C_i e^{-y_i t_i} \simeq \sum_{i=1}^{T} \frac{C_i}{(1 + y_i)^i}$$

where:

C_i is the payment made by the bond at time i.
y_i is the yield corresponding to maturity i.
t_i is the time corresponding to maturity i. For annual compounding $t_i = i$.

Indicating the payment of bond $i = 1, \ldots, T$ at time $t + j$ ($j = 1, \ldots, T$) with $C_{i,j}$ and the price of bond i with $B_{i,t}$, the discount factors must then satisfy the pricing kernel, as above, given by equations

$$\begin{pmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,T} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ C_{T,1} & C_{T,2} & \cdots & C_{T,T} \end{pmatrix} \begin{pmatrix} v_{t+1} \\ v_{t+2} \\ \vdots \\ v_{t+T} \end{pmatrix} = \begin{pmatrix} B_{1,t} \\ B_{2,t} \\ \vdots \\ B_{T,t} \end{pmatrix} \tag{5.3}$$

The nature of the bonds ensures that the matrix on the left-hand side of Eq. (5.3) is non-singular, with only one solution possible.

The next step consists in forming a portfolio of the T bonds, with elements zero-coupon bonds paying 1 at $t + j$. Denoting by $\alpha_{i,j}$ the units of bond i needed to replicate the zero-coupon bond maturing at $t + j$, it must hold

$$\begin{pmatrix} C_{1,1} & C_{2,1} & \cdots & \cdots & \cdots & C_{T,1} \\ C_{1,2} & C_{2,2} & \cdots & \cdots & \cdots & C_{T,2} \\ \vdots & \vdots & \ddots & & & \vdots \\ C_{1,j} & C_{2,j} & \cdots & \cdots & \cdots & C_{T,j} \\ \vdots & \vdots & & & \ddots & \vdots \\ C_{1,T} & C_{2,T} & \cdots & \cdots & \cdots & C_{T,T} \end{pmatrix} \begin{pmatrix} \alpha_{1,j} \\ \alpha_{2,j} \\ \vdots \\ \alpha_{j,j} \\ \vdots \\ \alpha_{T,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{5.4}$$

Equation (5.4) gives a unique solution associated with the maturity $t + j$. By comparing it to Eq. (5.3), the equality

$$v_{t+j} = \begin{pmatrix} \alpha_{1,j} & \alpha_{2,j} & \cdots & \alpha_{T,j} \end{pmatrix} \begin{pmatrix} B_{1,t} \\ B_{2,t} \\ \vdots \\ B_{T,t} \end{pmatrix} = \sum_{i=1}^{T} \alpha_{i,j} B_{i,t} \tag{5.5}$$

can be derived, yielding the discount factor corresponding to the j-th entry.

By running Eq. (5.4) for all $j = 1, \ldots, T$, the values for all the zero-coupon bonds are obtained. Equation (5.5) can then be applied to get the corresponding discount factors.

Example 5.10 Two bonds are traded on the market. Bond A is a 2-year 4% coupon bond; 8% provides payments of 4 € in the first year and 104 € in the second year. Bond B is a 1-year zero-coupon bond providing a payment of 104 €. Assume the price of bond A is 97 €, while the price of bond B is 90 €:

$$\begin{pmatrix} 91 \\ 96 \end{pmatrix} = \begin{pmatrix} 5 & 105 \\ 104 & 0 \end{pmatrix} \begin{pmatrix} v_{t+1} \\ v_{t+2} \end{pmatrix}$$

The solution is $v_{t+1} = 0.96$ and $v_{t+2} = 0.91$.

5.2.3   Duration and Convexity

The cash flows of a bond follow a pattern that depends on the sensitivity of bond prices to changes in bond yields. There is an inverse relationship since more risky bonds, with higher yield, are cheaper to buy.

Another factor is the maturity of the asset. In general, the longer the time to maturity, the greater the sensitivity to changing interest rates, and the greater the yield to maturity, the lower the sensitivity to changing interest rates.

The sensitivity of the bond price to the yield is generally implemented into the duration measure. Duration measures the average time length for the cash flows of the bonds, and it is a good estimate of the sensitivity.

It is a time-weighted measure of the length of life of a bond, and a high duration implies a high volatility with respect to changes in the yield to maturity. Duration can be measured in some ways for different purposes: Macaulay duration, modified duration, and effective duration.

In continuous compounding, the Macaulay duration is defined as

$$D = -\frac{dB_y}{dr_y} \times \frac{e^{r_y}}{B_y}$$

where:

r_y is the yield to maturity of the bond.
B_y is the price of the bond corresponding to the yield r_y.

The first-order derivative can be expressed as

$$\frac{dB_y}{dr_y} = -e^{-r_y} \sum_{t=1}^{T} tC_t e^{-r_y \Delta t}$$

where:

C_t is the payment made by the bond at time t.
Δt is the time interval between payments.

The equation for the duration follows as

$$D = \frac{1}{B_y} \sum_{t=1}^{T} tC_t e^{-r_y \Delta t}$$

which, for annual coupon, in discrete notation becomes

$$D = \frac{1}{B_y} \sum_{t=1}^{T} \frac{tC_t}{\left(1 + r_y\right)^t}$$

If price normalization is taken out of the formula, the result is the so-called dollar duration, defined as

$$D_\$ = \sum_{t=1}^{T} tC_t e^{-r_y \Delta t}$$

The discounted payments divided by the bond price sum up to one as defined by

$$\sum_{t=1}^{T} \frac{C_t e^{-r_y \Delta t}}{B_y} = 1$$

Therefore, each payment can be seen as a weight w_t, and one can write

$$D = \sum_{t=1}^{T} tw_t$$

Example 5.11 A 5-year bond with face value 100 €, selling at par, pays semiannual coupon rate 5%. If the yield of the bond is 5% as well, the Macaulay duration of the bond (discrete compounding) is given by

$$D = \frac{1}{100} \times \left(\frac{5}{1.05} + \frac{10}{1.05^2} + \frac{15}{1.05^3} + \frac{20}{1.05^4} + \frac{525}{1.05^5} \right) = 4.55$$

The duration of the bond is 4.55 years.

Putting it in words, each weight in the duration equation indicates the relative importance of that specific payment, with larger payments having a higher weight and smaller payments having a lower weight.

Through the Macaulay duration, it is possible to measure the percentage change in the price of a bond, for a minimal change in its yield. It is therefore a measure of elasticity.

A further step as a measure of riskiness of a bond is the modified duration. It can be directly derived by the Macaulay duration and is defined as

$$MD = -\frac{dB_y}{dr_y} \times \frac{1}{B_y}$$

The equation describes the negative of the first-order derivative of the price, normalized by the bond price. As for the duration, modified duration can be expressed analytically in terms of cash flows of the bonds modified duration, as

$$MD = \frac{e^{-r_y}}{B_y} \sum_{t=1}^{T} tC_t e^{-r_y \Delta t}$$

which can be discretized as

$$MD = \frac{D}{(1 + r_y)}$$

Modified duration informs about the percentage change in prices due to any specific change in percentage of the yield.

Example 5.12 Using the data in Example 5.11, the modified duration of the bond is given by

$$MD = \frac{4.55}{1.05} = 4.33$$

The modified duration of the bond is 4.33 years.

Duration measures are first-order approximations, therefore more accurate for small changes in the yields, and less accurate for large changes, due to the curvature of the bond price-yield relationship.

Both duration and modified duration work for changes of the yields in the context of flat yield curves. When the curve is not flat, or does not shift parallely, they are not efficient measures.

For more complex bonds, for example, those embedding an option, other measures must be used. Effective duration allows to analyze a bond with embedded option, given that Macaulay and modified duration are not applicable because of nonlinearity of the price changes:

$$ED = \frac{B_l - B_h}{2B_y \Delta r_y}$$

where:

Δr_y is change in yield.
B_h is price of the bond corresponding to the yield going up by Δr_y.
B_l is price of the bond corresponding to the yield going down by Δr_y.

One of the main properties of a zero-coupon bond is that its duration coincides with maturity, given there are no intermediate payments. It follows that the modified duration instead will be slightly lower than the zero-coupon bond's maturity.

In case of coupon bonds, the coupons act as partial intermediate repayments, so to decrease the overall risk and shorten the effective maturity of the cash flows. It follows that the greater the coupon, the lower the duration, because the bond is repaying back the investment faster than a similar bond with lower coupons.

Following the same logic, it is clear that the duration is lower for greater yield, given that payments are discounted at higher rate, with consequently a lower present value, that impacts on duration.

As a first-order derivative measure, the duration can capture a linear relationship between yield and bond price. But the relationship turns out to be nonlinear in practice. Therefore, in order to capture the sensitivity of bond prices to larger changes in yield, a second-order approximation is needed.

Convexity is a measure of the curvature of the value of a security or portfolio as a function of interest rates. Convexity is related to the second-order derivative of the price function.

By combining duration and convexity when measuring the relationship between bond prices and yields, it is possible to reduce substantially the approximation error in presence of the curvature.

The expression for the second-order derivative is

$$\frac{d^2 B_y}{dr_y^2} = \sum_{t=1}^{T} t^2 C_t e^{-r_y \Delta t}$$

Convexity is defined as

$$CV = \frac{d^2 B_y}{dr_y^2} \times \frac{1}{B_y}$$

which can be discretized as

$$CV = \frac{1}{(1+r_y)^2 B_y} \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+r_y)^t}$$

Example 5.13 Using the data in Examples 5.10 and 5.11, the convexity of the bond is given by

$$CV = \frac{1}{100 \times (1.05)^2} \times \left(\frac{5 \times 2}{1.05} + \frac{10 \times 3}{1.05^2} + \frac{15 \times 4}{1.05^3} + \frac{20 \times 5}{1.05^4} + \frac{525 \times 6}{1.05^5} \right) = 23.93$$

The convexity of the bond is 23.93.

By performing a Taylor series expansion on the second-order derivative, the approximation of the change in yield is given by

$$\frac{\Delta B_y}{B_y} = -D \Delta r_y + \frac{1}{2} CV (\Delta r_y)^2$$

Effective convexity is the equivalent of effective duration for duration and can be calculated with the formula

$$ECV = \frac{B_l + B_h - 2B_y}{2B_y (\Delta r_y)^2}$$

5.3 The Term Structure of Interest Rates

Learning Outcomes

- Describe the term structure of interest rates and the yield curve.
- Define and comment single-factor models of the short rate.
- Define and comment multifactor models of the short rate.

5.3.1 The Brownian Motion

A stochastic process can be defined to be Markovian when its value for next time step only depends on the current value, without taking into account the history of the process.

The past is not relevant to the future realizations of the process. The condition can be mathematically formulated as

$$\Pr(X_n = x_n | X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0) = \Pr(X_n = x_n | X_{n-1} = x_{n-1})$$

Example 5.14 There is an urn containing two black balls and one white ball. A ball is drawn in each of the three consecutive trials. There is no information about the color of the ball in the first trial, and the second trial yielded a black ball. It follows that there is a 50% chance to withdraw a black ball in the third trial, given that one black ball and one white ball are left. If there is information provided about the color of the balls in both the first and second trial, then it would be a 100% surety to guess the color of the ball in the third trial. It is clear that the probability distribution for tomorrow-ball's color is affected by the information about the past. This stochastic process of observed colors does not have the Markov property.

The stochastic movement of particles in a system was first observed by R. Brown in the nineteenth century. He let grains of pollen move suspended in the water, and, using a microscope, he observed their strange but regular zigzag path.

After the first formulation of the theory of Brownian motion, developed by Bachelier in the early 1900s, Einstein used a probabilistic model to explain the basic properties of the process.

He could in fact observe the behavior of water molecules concluding that the kinetic energy of fluids was acting on it, so that any molecule receives a random number of impacts from random directions, from other molecules, in any short period of time.

The Brownian motion, or Wiener process, can be conveniently seen as a limit of the random walks with infinitesimal steps (Musiela and Rutkowski 1997).

Recall that standard random walk is a sum process in which each step is a Bernoulli variable, getting values +1 or −1, with equal probability:

$$S_T = X_1 + X_2 + \ldots + X_T$$

where

$$X_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

with expected value and variance given by

$$E(S_T) = 0$$
$$\text{Var}(S_T) = T$$

It is possible to show that, in the limit for $n \to \infty$, the Brownian motion can be defined as the stochastic process W_t, such that

$$\Delta W = \varepsilon \sqrt{\Delta t}, \quad \varepsilon \sim N(0, 1)$$

where

$$E(\Delta W) = 0$$
$$\text{Var}(\Delta W) = \sqrt{\Delta t}$$

The basic properties of the Brownian motion are

$$W_0 = 0$$

$$W_t \sim N(0, t)$$

- The paths of the process are continuous function of time $t \in [0, \infty)$.
- For any set of time points s and t, such that $0 \leq s \leq t$, it holds that

$$E\left(W_t^2\right) = \text{Var}(W_t) + [E(W_t)]^2 = t + 0 = t$$
$$E(W_t - W_s) = E(W_t) - E(W_s) = 0 - 0 = 0$$
$$E\left[(W_t - W_s)^2\right] = E\left(W_t^2\right) + E\left(W_s^2\right) - 2\text{Cov}(W_t, W_s)$$
$$= t + s - 2\min(W_t, W_s) = |t - s|$$

- The covariance between the process at two different points in time is $\text{Cov}(W_t, W_s) = \min(s, t)$
- Process increments on nonoverlapping intervals are independent, meaning:
  - The distribution of $W_t - W_s$ depends on $t - s$ only.
  - If $0 \leq u \leq s \leq t$, the random variables $W_t - W_s$ and W_u are independent.

The process path over a long interval T is such that $W_T - W_0$ is the sum of n small changes in Δt of the process, and it is equal to

$$W_T - W_0 = \sum_{i=1}^{n} \varepsilon_i \sqrt{\Delta t}$$

so that

$$E(W_T - W_0) = 0$$
$$\text{Var}(W_T - W_0) = n\Delta t = T$$

From the basic properties above described, it is then possible to derive other properties allowing for calculus simplifications, when handling the Brownian motion. Consider the model expressed in continuous time, dW, and recall that, for $\Delta t \to 0$, the time interval becomes very small and equal to dt.

Also recall from basic properties that

$$E(dW) = 0$$
$$E(dWdt) = E(dW)dt = 0$$
$$E(dW^2) = dt$$

It follows that

$$\text{Var}(dW^2) = E(dW^4) - [E(dW^2)]^2 = 3(dt)^2 - (dt)^2 = 0$$
$$E[(dWdt)^2] = E[(dW)^2](dt)^2 = 0$$
$$\text{Var}(dWdt) = E[(dWdt)^2] - [E(dWdt)]^2 = 0$$

The argument to derive the final important properties of the Brownian motion is that, if the above equations hold, and all the elements have a zero second-order moment, their expected and actual values coincide.

Moreover, since the square of dt is supposed to converge to zero even faster than dt, it is possible to conclude that

$$dW^2 = dt$$
$$dWdt = 0$$
$$dt^2 = 0$$

Now consider introducing a drift term in Δt, so to obtain a new Δx process, that can be defined as

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}$$

where a and b are constants, and

$$E(\Delta x) = a\Delta t$$
$$\text{Var}(\Delta x) = b^2 \Delta t$$

The model can obviously be expressed in continuous time and takes the form

$$dx = adt + bdW_t$$

which is called generalized Brownian motion.

An Ito process is defined as a generalized Wiener process in which a and b are not constant but dependent on x and t, respectively:

$$dx = a(x, t)dt + b(x, t)dW_t$$

The generalized Wiener process is the leading process in defining stock price movements in the markets but can also be used to model the behavior of the short rate of interest, as clarified in the next section.

5.3.2   Single-Factor Models

The classical interest rate models start from a process describing the behavior of the short rate, at infinitesimal time steps. The state variable is the instantaneous forward rate, and the short rate is modeled using a Brownian motion (Duffie and Singleton 1999).

The stochastic differential equation describing the dynamics of short rate follows an Ito process:

$$dr_t = m_{r,t}dt + s_{r,t}dW_t$$

where:

$m_{r,\,t}$ is a drift parameter dependent on both r_t and t.
$s_{r,\,t}$ is a diffusion parameter dependent on both r_t and t.

The single-factor models are characterized by having a single driving factor which is modeled in multiple possible forms, according to the form chosen for the drift and the diffusion. In fact, by varying the choice of the coefficients $m_{r,\,t}$ and $s_{r,\,t}$, the process can lead to different dynamics of the instantaneous rate.

Equilibrium models are those where the short rate is fully endogenous, and generated by the model, through equilibrium conditions of no-arbitrage, and without any external input.

The Vasicek model is a popular equilibrium model which is founded on a short rate modeled as

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

As from the formula, the process is characterized by mean reversion, meaning that the rate aims in the long run at a fixed constant level, b, floating around it at a reverting speed a, which is constant as well. All parameters are nonnegative constants.

The main advantage of such an equilibrium model is the explicit form taken by the short rate, which is normally distributed. One downturn of the model is that in its form, it allows for negative rates as outcome.

In order to derive the analytical expression of the short rate, it is necessary to introduce the variable change:

$$x_t = r_t - b$$

The process in dx_t becomes

$$dx_t = -ax_t dt + \sigma dW_t$$

Consider another change of variable:

$$z_t = e^{at} x_t$$

The process becomes

$$\begin{aligned} dz_t &= ae^{at} x_t dt + e^{at} dx_t \\ &= ae^{at} x_t dt + e^{at}(-ax_t dt + \sigma dW_t) \\ &= e^{at} \sigma dW_t \end{aligned}$$

Integration on both sides gives

$$z_t = z_0 + \int_0^t e^{as} \sigma dW_s$$

$$\rightarrow e^{at} x_t = e^{a0} x_0 + \int_0^t e^{as} \sigma dW_s$$

$$= x_0 + \int_0^t e^{as} \sigma dW_s$$

so that

$$x_t = e^{-at}\left(x_0 + \int_0^t e^{as}\sigma dW_s\right)$$

Recall that the equality $x_t = r_t - b$ holds, so that the short rate can be calculated as

$$r_t - b = e^{at}(r_0 - b) + e^{-at}\sigma\int_0^t e^{as}dW_s$$

$$\to r_t = e^{at}(r_0 - b) + \sigma\int_0^t e^{-a(t-s)}dW_s + b$$

$$= e^{at}r_0 + b(1 - e^{-at}) + \sigma\int_0^t e^{-a(t-s)}dW_s$$

Generalizing the solution to any time interval $\Delta t = t - s$ gives

$$r_t = e^{a\Delta t}r_s + b\left(1 - e^{-a\Delta t}\right) + \sigma\int_s^t e^{-a\Delta t}dW_u$$

Through some complicated calculus, it is possible to show that every model for a short rate implies a specific solution for the price of the bond and term structure.

For example, in the case of the Vasicek model, the solution for the price at time t of a bond with face value equal to one, maturing at time T ($\Delta t = T - t$), is given by the formula

$$P_{t,T} = \Lambda_{t,T}e^{-\Theta_{t,T}r_t}$$

where

$$\Theta_{t,T} = \frac{1 - e^{-a\Delta t}}{a}$$

and

$$\Lambda_{t,T} = e^{\frac{\left(\Theta_{t,T-\Delta t}\right)\left(a^2 b - \frac{\sigma^2}{2}\right)}{a^2} - \frac{\sigma^2\Theta_{t,T}^2}{4a}}$$

Starting from the bond price is then possible to derive the term structure. In particular, the continuously compounded interest rate at time t for a term of $\Delta t = T - t$ can be extracted from the formula

$$P_{t,T} = e^{-r_{t,T}\Delta t}$$

So that the point $r_{t,\,T}$ which determines the term structure in that specific time interval is

$$r_{t,T} = \frac{1}{\Delta t}\ln P_{t,T}$$

In order to overcome the possibility of negative rates implied by using the Vasicek model, Cox, Ingersoll, and Ross (CIR) proposed a model incorporating a different form for the diffusion part, with the short rate defined as

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$$

The CIR equation allows only for rates that are bounded in the region of nonnegative numbers, by multiplying the diffusion term by the square root of the short rate itself. Bond prices in the CIR model have the same general form as in Vasicek's model, but the functions $\Lambda_{t,\,T}$ and $\Theta_{t,\,T}$ are given as

$$\Theta_{t,T} = \frac{2(e^{\gamma\Delta t} - 1)}{(\gamma + a)(e^{\gamma\Delta t} - 1) + 2\gamma}$$

and

$$\Lambda_{t,T} = \left[\frac{2\gamma e^{\frac{(\gamma+a)\Delta t}{2}}}{(\gamma + a)(e^{\gamma\Delta t} - 1) + 2\gamma}\right]^{\frac{2ab}{\sigma^2}}$$

where

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

The bond price and interest rate are again dependent on the short rate at time t. The value of the latter determines the level of the term structure but does not define the shape, which is only dependent on time.

The main disadvantage of equilibrium models is that there is no link between the output structure and the actual situation of the rates in the real market. The model parameters can of course be chosen and calibrated, but they can never fit reality very closely.

In the fixed income market, when large amounts of capital are employed, very small differences in the rates can generate very high differences in profits or losses. Therefore, for advanced investor requiring more accurate estimates, equilibrium models are not good.

The problem of fit can be overcome by letting some parameters vary deterministically with time. No-arbitrage models are designed to be exactly consistent with today's term structure of interest rates.

Ho and Lee (1986) described the simplest form of no-arbitrage model and the first to be proposed. The model uses the current term structure of rates as an input to the model, making a step further than equilibrium models.

The Ho-Lee process for the short rate is given by (Hull and White 1993)

$$dr_t = \theta_t dt + \sigma dW_t$$

where θ is nonrandom function of t and σ is, as usual, a nonnegative constant.

The analytical solution for the rate is

$$r_t = r_s + \sigma(W_t - W_s) + \int_s^t \theta_u ds$$

and the expression for the price of the bond is

$$P_{t,T} = \Lambda_{t,T} e^{-r_t \Delta t}$$

where

$$\ln \Lambda_{t,T} = \ln \left(\frac{P_{0,T}}{P_{0,t}} \right) + \Delta t F_{0,t} - \frac{1}{2} \sigma^2 t \Delta t^2$$

The model is interesting because of its intrinsic simplicity. However, the simplicity is accompanied by some disadvantages. The main drawback of the model is that it does not take into account for mean reversion of rates.

Mean reversion is defined as the tendency of the interest rates to turn around some average in the long term, at a speed and frequency that is variable. It is an empirically verified important feature of interest rates.

Therefore, for a model to be accurate, it must incorporate it. The solution has been given by the Hull-White model. It is also known as the extended Vasicek model, in that it is basically the no-arbitrage version of it. The process is

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t$$

In this case, the analytical solution to the model is

$$r_t = e^{-a(t-s)} r_s + \int_s^t e^{-a(t-u)} \theta_u du + \int_s^t e^{-a(t-u)} \sigma dW_u$$

and the price of the bond for the given short rate becomes

$$P_{t,T} = \Lambda_{t,T} e^{-\Theta_{t,T} r_t}$$

where

$$\Theta_{t,T} = \frac{1 - e^{-a \Delta t}}{a}$$

and

$$\ln \Lambda_{t,T} = \ln\left(\frac{P_{0,T}}{P_{0,t}}\right) + \Theta_{t,T}\frac{\partial f_{0,t}}{\partial t} - \frac{1}{4a^3}\sigma^2\left(e^{-aT} - e^{-at}\right)^2\left(e^{2at} - 1\right)$$

5.3.3   Multifactor Models

One important limitation of one factor models is that, given their nature, they are accurate only for parallel shifts of the yield curve. The reality is that curves like the LIBOR, for example, often move steepening, so that more complicated models are required.

The use of a parallel shifting setting is, for example, not adequate to price fixed income derivatives maturing in a long term. The same works for derivatives written on long-term assets.

It is often observed that yields at opposite ends of the term structure move in opposite directions. This is a behavior that single-factor models cannot explain, and more factors are needed for the analysis.

The solution to the above problems is given by introducing multifactor models, as pioneered by Merton (1974) and Vasicek (1977), among others. The efficiency of these models is given by allowing for instantaneous correlation between bond returns to be not perfect, leading to higher flexibility of the term structure.

Using a multifactor model allows to establish the dependence between short rate and a vector of factors **x**, as given by

$$r_t = \sum_{i=0}^{n} x_{i,t} \tag{5.6}$$

where

$$dx_t = \mathbf{A}(b - x_t)dt + \mathbf{\Sigma}dW_t, \quad x_t, b \in \mathbb{R}^n \tag{5.7}$$

where:

A and **Σ** are $n \times n$ matrices.

The solution to Eq. (5.6) is a Gaussian process in the form

$$x_t = e^{-At}x_0 + \int_0^t e^{-A(t-s)}\mu ds + \int_0^t e^{-A(t-s)}\mathbf{\Sigma}dW_s$$

As for the single-factor models, the bond prices are expressed in affine form, according to the functional form of the parameters. The high tractability makes Gaussian multifactor models a very good choice to capture different empirical aspects of the data.

A popular multifactor model is the multifactor CIR model, which is based on the independency among factors. The short rate satisfies Eq. (5.7), and the dynamics of each $x_{i,\,t}$ are derived from the single-factor CIR model and are given by

$$dx_{i,t} = a_i(b_i - X_{i,t})dt + c_i\sqrt{x_{i,t}}dW_{i,t}, \quad a_i, b_i, c_i > 0$$

There exist again functional forms for parametrized $A_{i,\,t}$ and $B_{i,\,t}$ that help to express the expectation of the discounting factor in an affine form, as defined by

$$E\left(e^{-\int_t^T X_{i,u}du}\right) = e^{A_{i,\Delta t} + B_{i,\Delta t}X_{i,t}} \tag{5.8}$$

It is then possible to derive the term structure, recalling that the short rate is given by the sum of the single factors, so that

$$E\left(e^{-\int_t^T r_u du}\right) = E\left(e^{-\int_t^T \sum_i X_{i,u}du}\right)$$
$$= \ldots$$
$$= \prod_{i=1}^n E\left(e^{-\int_t^T X_{i,u}du}\right)$$

Substituting for Eq. (5.8) yields the ZCB price as defined by

$$P_{t,T} = \prod_{i=1}^n e^{A_{i,\Delta t} + B_{i,\Delta t}X_{i,t}} = e^{\sum_i e^{A_{i,\Delta t} + B_{i,\Delta t}X_{i,t}}}$$

Another popular model is the Hull and White (2001) which represents the instantaneous forward rate as a linear combination of the stochastic variables x_t and y_t, each driving a specific feature in the shape of the yield curve.

The two stochastic variables follow the processes

$$dx_t = -\lambda_1 x_t dt + \sigma_1 dW_1$$
$$dy_t = -\lambda_2 x_t dt + \sigma_2 dW_2$$

where:

σ_1 and σ_2 are the instantaneous volatilities.

The Brownian motions are correlated by some typically largely negative coefficient (close to -1), so that parallel moves of the yield curve are inversely correlated with the steepening of the curve.

5.4   Summary

Bond valuation is an important task for the calculation of the cost of debt of the firm, given that a large part of the debt of big corporations is nowadays issued in the form of bond issuance.

The fundamentals of bond valuation rely on the proper identification and understanding of the features that make a bond a unique type of debt security. Features like maturity, face value, and coupon payment are components that identify a bond.

There are several types of bonds, namely, government (treasury) bonds and corporate bonds. Government bonds can be sovereign bonds, issued by a country or federation, or municipal bonds, issued by local municipalities and states.

When investing in bonds, it is very important to be fully aware about the risk attached to them. Such a risk reflects the creditworthiness of the issuing party, and it is related to the return the bonds offer to the investor.

There is an inverse relationship between a bond's price and yield, since the latter is used to discount the cash flows of the bond at present value so to obtain the former. Furthermore, the relationship is convex, defined by a curve.

The yield curve describes the relationship existing on the real market between the maturity of the traded bonds and the yield they offer. The term structure of interest rates is the mathematical description of such a relationship.

Duration and convexity are risk measures that define the intrinsic riskiness of a bond, due to its maturity and the intermediate payments it offers, when present. The convexity adjustment allows for a better approximation of the bond price given the yield.

The term structure of interest rates is mathematically defined as the relationship between the bond yield and the maturity. It is therefore a complex relationship that depends on the assumptions made on the short rate (continuous).

Several models use the above assumptions to derive a closed form solution for the yield and the price of the bond associated to some level of the short rate. Such a derivation can be obtained through either a single-factor or a multifactor model.

Problems

1. Explain what is meant by basis risk when futures contracts are used for hedging.
2. Which bond's price is more affected by a change in interest rates, a short-term bond or a longer-term bond, being all the other features fixed? Why?
3. Provide the definitions of a discount bond and a premium bond. Give examples.
4. All else equal, which bond's price is more effected by a change in interest rates, a bond with a large coupon or a small coupon? Why?
5. What is the difference between the forward price and the value of a forward contract?
6. Someone argued that airlines have no point in using oil futures given that the chance of oil price being lower than futures price in the future is the same as the chance of it being lower. Discuss this.

7. A futures price can be assimilated to a stock paying a dividend yield. What is the dividend yield in the futures case?
8. The annual effective yield on a bond is 7%. A 5-year bond pays coupons of 5% per year in semiannual payments. Calculate the duration.
9. Calculate the modified duration and convexity of the bond in Exercise 8.
10. Prove that the duration of a portfolio of many assets is the weighted average of all durations of the single assets.
11. Consider the following portfolio:

Bond	Coupon	Maturity	Par value	Price value	YTM
1	7.0	5	10,000,000	9,209,000	9.0%
2	10.5	7	20,000,000	20,000,000	10.5%
3	6.0	3	30,000,000	28,050,000	8.5%

Determine the yield to maturity of the portfolio.

12. Consider the two bonds in the following table:

	Bond A	Bond B
Coupon	8%	9%
Yield to maturity	8%	8%
Maturity	2 years	5 years
Par	100.00 €	100.00 €
Price	100.00 €	104.055 €

(a) Compute the duration and modified duration for the two bonds.
(b) Compute the convexity for the two bonds.

13. Recall the two bonds in Exercise 4.

(a) Repeat the calculations of duration, modified duration, and convexity, using shortcut formula, by changing the yields by 0.2%.
(b) Compare the results with those found in Exercise 4 and comment.
14. An investor holds 100,000 units of a bond whose features are summarized in the following table. He wishes to be hedged against a rise in interest rates.

Maturity	Coupon rate	Yield	Duration	Price
18 years	9.5%	8%	9.14	114.18 €

Characteristics of the hedging instrument, which is here a bond are as follows:

Maturity	Coupon rate	Yield	Duration	Price
20 years	10%	8%	9.49	119.79 €

Coupon frequency and compounding frequency are assumed to be semiannual. YTM stands for yield to maturity. The YTM curve is flat at an 8% level.

(a) What is the quantity of hedging instrument that the investor has to trade? What type of position should the investor take on the hedging instrument?
(b) Suppose that the YTM curve increases instantaneously by 0.1%. Calculate the corresponding new price for the two bonds.

15. Consider the two bonds in Exercise 7.
    (a) When YTM curve increases instantaneously by 0.1%, what happens to the portfolio in terms of profits or losses when the portfolio is not hedged? What if it is hedged?
    (b) If the curves shift by 2% instead, how does the answer to point a changes?

16. A bank is required to pay 1,100,000 € in 1 year. There are two investment options available with respect to how funds can be invested now in order to provide for the 1,100,000 € payback. First asset is a noninterest-bearing cash fund, in which an amount x will be invested, and the second is a 2-year zero-coupon bond earning the 10% risk-free rate in the economy, in which an amount y will be invested.
    (a) Develop an asset portfolio that minimizes the risk that liability cash flows will exceed asset cash flows.

17. What position is equivalent to a long forward contract to buy an asset at K on a certain date and a put option to sell it for K on that date?

18. How can a forward contract on a stock with a particular delivery price and delivery date be created from options?

Appendix: Principal Component Analysis of the Term Structure

The term structure can be alternatively described by using a principal component analysis (PCA). The changes in the term structure (ΔTS), by means of principal components x_i, can then be defined by

$$\Delta TS = (\Delta x_1, \Delta x_2, \ldots, \Delta x_n)$$

Knowledge of matrix calculus is needed in order to apply the method. There is a unique change in the key rates for each realization of the principal components, where the latter are linear combinations of changes in the interest rates, given as

$$\Delta x_i = \sum_{i=1}^{n} \eta_{ij} \Delta y_j, j = 1, 2, \ldots n$$

where:

η_{ij} are the principal component coefficients.
y_j is the yield corresponding to maturity j.

Each component explains the maximum percentage of the total residual variance not explained by previous components. The matrix of zero-coupon rates is

symmetric with m independent eigenvectors, corresponding to m nonnegative eigenvalues. Looking at eigenvalues in order of size, the highest eigenvalue corresponds to a specific eigenvector, whose elements are identified the coefficients of the first principal component.

The second highest eigenvalue corresponds to another specific eigenvector, whose elements are identified as the coefficients of the second principal component. And so on, for all eigenvalues.

So the variance of each component is given by the size of the corresponding eigenvalue, and the proportion of total variance of the interest changes explained by the i-th principal component is

$$\sigma_{y,i}^2 = \frac{\lambda_i}{\sum\limits_{i=1}^{m} \lambda_i} \tag{5.9}$$

From the condition of independency of eigenvectors, it follows that the matrix of coefficients η_{ij} is orthogonal, so that its inverse corresponds to the transpose. Equation (5.9) can then be inverted, to get the interest rates, as

$$\Delta y_j = \sum_{i=1}^{n} \eta_{ij} \Delta x_i, j = 1, 2, \ldots n$$

From how the model is built, it is clear that lowest eigenvalues play a very little role in determining the changes in interest rates. Therefore, it is possible to reduce the dimensionality of the model to the m highest eigenvalues, as given by

$$\Delta y_j = \sum_{i=1}^{m} \eta_{ij} \Delta x_i + \varepsilon_i$$

where:

ε_i is an error term due to the approximation from reduced dimensionality.

The first k components are then able to give a sufficiently accurate approximation of the changes in interest rates. The portfolio sensitivity to these components can be used to define the IRR profile.

Difference variances for each principal component are implied by the model, with corresponding even (i.e., unitary) shift in all components making them not equally likely.

A further step involves giving to each factor a unit variance, so to make changes in each factor comparable. Again from matricial calculus, the unit variance is obtained by multiplying each eigenvector by the square root of the corresponding eigenvalue, so that the model gets the form

$$\Delta y_j = \sum_{i=1}^{m} \left(\eta_{ij} \sqrt{\lambda_i} \right) \frac{\Delta x_i}{\sqrt{\lambda_i}} + \varepsilon_i$$

so that in an equivalent equation, the product of eigenvalue and eigenvector is isolated. The new factor loading in parenthesis measures the impact of one standard deviation move in each principal component.

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