3 The Relationship Between Risk and Return

The relationship between risk and return on the financial market is an issue of primary importance in finance, and it spans all the fields of specialization, including corporate finance.

In fact, one of the most important principles in financial markets states that securities with higher risk are supposed to give a higher expected return in order to be appealing for an investor.

On the other hand, securities with lower risk are those chosen by the investors with lower risk appetite. They therefore offer a lower return and are more indicated for conservative investment strategies.

The risk-return trade-off is at the foundation of modern finance, and finding the right balance between return demand and risk exposure is at the basis of good management of a business.

All the theory of financial risk and return has its roots in the work done in the 1950s about the Modern Portfolio Theory. Nowadays it is then possible to give a mathematical interpretation to the relationship.

After that, the Capital Asset Pricing Model constitutes a further step in identifying the elements of risk and return that optimize an investment in some specific asset and describes the optimal portfolios.

After studying this chapter, you will be able to answer the following questions, among others:

- How can return and risk in financial markets be represented?
- What are the basics of the relationship between risk and return?
- How can Portfolio Theory be used to select the efficient portfolios in the space of return and volatility?
- What are the assumptions of the Capital Asset Pricing Model and how can it be derived?
- How is it possible to determine the right expected return for an investment?

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The first part of the chapter is dedicated to the definition of expected return and volatility as a measure of risk universally used in finance. The second section is about the Modern Portfolio Theory and its implications for investment decisions and asset pricing. The last section deals with the Capital Asset Pricing Model and its application for determining the rate of return demanded by investors for a particular level of risk.

3.1 Expected Return and Volatility

Learning Outcomes

- Understand the concept of portfolio return.
- Understand the concepts of portfolio volatility and correlation.
- Learn how to use Monte Carlo simulation methods.

3.1.1 The Portfolio Return

Financial returns are profits on an investment, including changes in value of the assets held, or dividends coming from owning it, as well as other cash flows, which the investor receives from the investment.

More appropriately, in general, financial returns are referred to as profits on an investment as percentage of the amount invested, for investments made over a specific period.

The period is typically a year, in which case the rate of return is referred to as an annual return, but as mentioned for the interest rates compounding frequencies in Chap. 2, other time periods can be assumed for calculation.

It is a measure of the profitability of an investment, and, by just inverting the calculation, it allows to understand the time it will take to partially or fully recover an amount invested in some project.

Example 3.1 Consider an investor holding 10,000 € and receiving 1200 € in the first year of the investment. The rate of return is then 12%, and the investor will recover the initial 10,000 € in almost 7 years. Different investors have different required rates of return at different levels of risk.

There are several ways financial returns can be measured, which creates a problem of transparency when financial institutions try to market their investment opportunities or try to communicate with clients.

In an ideal world, clients would be updated annually with their portfolio's rate of return, enabling them to compare their performance to the appropriate benchmarks. However, in reality, some advisors may not be providing these results to their clients.

Therefore, there is a question mark on what returns measurement should be used and how the possible measurement methods differ from each other. Moreover, there

Table 3.1 Market values of the portfolio

Date	Market value (€)	Cash flow (€)	Net (€)
December 2014	350,000		
January 2015	357,000		
February 2015	365,000	+30,000	395,000
March 2015	401,000		
April 2015	407,000		
May 2015	412,000	−15,000	397,000
June 2015	403,000		
July 2015	410,000		

are issues related to the contributions and withdrawals from the portfolio, to complicate the picture.

There are four common methodologies for return calculation, and they can sometimes differ substantially from each other, depending mostly on the combination of the timing of cash flows, their size relative to the portfolio value, and the volatility of the portfolio's market value.

Example 3.2 An investor holds a portfolio with market value of 350,000 € in December 2014. The following table shows the market values for the first quarter of 2015, with contributions and withdrawals, as in Table 3.1.

The first methodology for return calculation is called money-weighted rate of return, most commonly known as IRR, already described in Chap. 2. As from the definition, the average return makes the net present value of the sum of all cash flows (during the measurement period) equal to zero.

Assuming that all contributions to the portfolio are positive cash flows, while all the withdrawals from the portfolio are negative cash flows, it is possible to use the method for calculating the rate of return in the presence of relatively small cash flows, compared to the portfolio value (Satchell 2007).

Assuming that all the cash flows earn the same rate of return when they are invested, its return can differ substantially from the true time-weighted rate of return when large cash flows occur during periods of significantly fluctuating portfolio values.

The linked internal rate of return (LIR) is an approximate time-weighted rate of return that can be calculated by performing a geometrical link of monthly money-weighted rates of return.

When the time horizon is 1 year or less, geometric linking involves converting the monthly money-weighted returns to relative form ($1 + r_{\mathrm{MW}i}$), multiplying them together, and subtracting 1 from the result, as from the formula

$$r_{\mathrm{LIR}} = [(1 + r_{\mathrm{MW1}}) \times (1 + r_{\mathrm{MW2}}) \times \ldots \times (1 + r_{\mathrm{MW}n})] - 1$$

where:

r_{MW1} is the monthly money-weighted rate of return at time 1.

In case the external cash flows hitting the portfolio have not been large in combination with volatile portfolio fluctuations during the monthly measurement period, the measured LIR should be very close to the true time-weighted rate of return.

The modified Dietz rate of return (MDR) is a method used by the Canadian company PWL Capital, in order to calculate returns for their customers. The approximation to a true time-weighted rate of return is achieved by weighting.

Each cash flow in fact is weighted by the proportion of the measurement period it is present or absent from the portfolio. Monthly Modified Dietz rates of return are usually calculated and geometrically linked together, similar to the LIR.

The MDR is based on monthly valuation of the portfolio; therefore, the rate it yields can be widely different from the true time-weighted rate of return in the presence of large external cash flows occurring during monthly measurement periods characterized by high volatility.

$$r_{\mathrm{MD}} = \frac{V_{\mathrm{M1}} - V_{\mathrm{M0}} - CF}{V_{\mathrm{M0}} + \sum \mathrm{CF}_i w_i}$$

with

$$CF = \sum CF_i$$

where:

V_{M0} is the full market value of the portfolio at the beginning of the period.
V_{M1} is the full market value of the portfolio at the end of the period.

and

$$w_i = \frac{n_{\mathrm{CD}} - n_{\mathrm{D}}}{n_{\mathrm{CD}}}$$

where:

n_{CD} is the total number of days in the period.
n_{D} is the number of days from the beginning of the period that the CF_i occurs.

All cash flows in the above formulas are assumed to occur at the end of the day.

The final and most relevant portfolio performance measurement tool is the time-weighted rate of return (TRR). It is the most accurate return measure in most cases but requires daily valuations of the portfolio, as soon as a cash flow occurs (Lettau and Van 2008).

Periods of return calculation are divided into subperiods, and for each of them, the total return calculation is performed. These subperiod returns are then geometrically linked together to obtain the TRR over the measurement period.

$$r_{\mathrm{TW}} = [(1 + r_{t,1}) \times (1 + r_{t,1}) \times \ldots \times (1 + r_{t,n})] - 1$$

where

$$r_{t,1} = \frac{V_{\mathrm{M1}} - V_{\mathrm{M0}}}{V_{\mathrm{M0}}}$$

3.1.2 Volatility and Correlation

Many applications in finance require volatility as an input, including corporate finance, for valuation. Knowing how to measure volatility and correlation is therefore crucial, as well as knowing what the economic drivers of volatility in financial markets are.

A financial manager has the aim to understand what the determinants of volatility on financial markets are. Moreover, it is important to highlight the connections between volatility and the risk factors driving it.

Volatility can be measured in several ways, and many approaches are possible. Historical volatility is the most commonly used method, while other methods are implied volatility and econometric modeling.

Historical volatility can be measured as the standard deviation (square root of variance) of the log changes in price, measured at regular time intervals. Estimating volatility from historical data implies starting from the observation of market prices at fixed time intervals (Connor et al. 2010).

The frequency of observation can range from daily to yearly, depending on the nature and purpose of the analysis. There are several methods to scale the volatility conveniently from one frequency to another.

The observation is set on n prices at some point in time, at the end of any time interval i in the time range. First, the returns for each time interval must be calculated, in order to compute the variance afterward.

Each return r_i is calculated from a couple of prices, and therefore $n + 1$ prices are needed in order to calculate n returns. Define

$$\begin{aligned} r_i &= \ln(S_i) - \ln(S_{i-1}) \\ &= \ln\left(\frac{S_i}{S_{i-1}}\right), \quad i = 1, 2, \ldots, n \end{aligned}$$

where:

S_i is the stock price at the end of i-th interval (time i).
S_{i-1} is the stock price at the end of ($i-1$)-th interval (time $i - 1$).

Volatility is measured according to the basic statistical properties of variance and standard deviation. As a convention, it is convenient to define the sum of square deviation from the mean d^2 as

$$d^2 = \sum_{i=1}^{n} \left(r_i - \bar{r}\right)^2$$

where $\bar{r}$ is the mean of the returns over time, calculated as

$$\bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i$$

Once d^2 is defined, the standard deviation estimator can be derived as the square root of the unbiased sample variance.

$$s_\sigma = \sqrt{\frac{\sum_{i=1}^{n} \left(r_i - \bar{r}\right)^2}{n - 1}}$$

Basic knowledge of statistics is sufficient to recall that the estimator s is actually biased, as opposed to s^2 which is an unbiased estimator for the population variance σ^2.

One of the tricky points in the historical volatility estimation is the right choice of n, which is not easy. In fact, choosing a high number of observation increases the accuracy of the estimation but at the cost of including data that go too far back in time which can bias the analysis since the far past may be irrelevant.

Recall that the frequency of observation can be of any kind, and it is possible to scale the volatility to get the annualized value of it from the volatility measured at other frequencies. The yearly estimate of standard deviation therefore is

$$\widehat{\sigma} = s_\sigma \sqrt{m}$$

where m is the number of reference periods in a year, so that $m = 252$ for daily observations, $m = 52$ for weekly observations, $m = 12$ for monthly observations, and so on. The standard error of the estimation is approximately equal to $\widehat{\sigma} / \sqrt{2n}$.

If the volatility measures are high, it means there have been many fluctuations in market price. A low volatility shows the price has been stable over time. It is also possible to compare the volatility of one instrument with other instruments or indices, in order to get a measure of the relative volatility.

In the short run, volatility changes over time, while in the long run, it is possible to identify trends to almost fixed values, well contained in between narrow boundaries.

In order to identify those trends, a technique is to measure volatility over different time windows and average it (Esch et al. 2005).

Comparison between short-term volatility and long-term trend is very useful to determine the trend of volatility over time. For example, a high short-period low volatility combined with a long-period high volatility means the stock has recently calmed down in price fluctuations.

Implied volatility is another method for standard deviation measurement, widely used in finance. It is the volatility implied by an analytical model for pricing financial assets. In particular, it is the volatility implied by the Black-Scholes-Merton (BSM) model for pricing financial derivatives.

The description and analysis of the analytical model are beyond the scope of the chapter and are dealt with in Chap. 9. However, the concept of implied volatility is so important in finance, that it is worthwhile to present it by using appropriate math.

The purpose of this part is to introduce the concept of implied volatility and analyze the use of it in financial analysis. According to BSM model, the price of a European call option written on a stock is given by

$$c = SN(d_1) - K\mathrm{e}^{-rT}N(d_2)$$

with

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}$$
$$d_2 = d_1 - \sigma\sqrt{\Delta t}$$

where:

S is the price of the stock, as observed in the market at the time of calculation.
K is the strike price, as specified in the option contract.
σ is the volatility of the underlying stock.
r is the risk-free rate.
Δt is the time left until maturity of the option.
$N(.)$ is the cumulative normal distribution.

The volatility parameter σ is a constant in the model. By inverting the BSM model, one can achieve a measure of the volatility that is implied by the actual market price of the option.

The problem is that the model is complicated so that it is not possible to be inverted analytically. Therefore, numerical procedures are needed in order to invert the model and get the value for σ.

In the last year, financial mathematicians have worked on advanced financial approximation formulas for the calculation of implied volatility. One of the most popular approximation formulas has been developed by M. Brenner and G. Subrahmanyam, who define implied volatility as

$$\sigma = \sqrt{\frac{2\pi c}{S\Delta t}}$$

The above approximation works for the money options, namely, the options for what the condition $S = Ke^{-rT}$ holds.

Another popular method of approximation was presented by Curtis and Carriker in 1988, and it is called the Direct Implied Volatility Estimate (DIVE), whose formula is given by

$$d_1 = \frac{\sigma\sqrt{\Delta t}}{2}$$
$$d_2 = -\frac{\sigma\sqrt{\Delta t}}{2}$$

The BSM formula then becomes

$$\begin{aligned} c &= S\left[N\left(\frac{\sigma\sqrt{\Delta t}}{2}\right) - N\left(-\frac{\sigma\sqrt{\Delta t}}{2}\right)\right] \\ &= S\left[N\left(\frac{\sigma\sqrt{\Delta t}}{2}\right) - \left(1 - N\left(\frac{\sigma\sqrt{\Delta t}}{2}\right)\right)\right] \\ &= S\left[2N\left(\frac{\sigma\sqrt{\Delta t}}{2}\right) - 1\right] \end{aligned} \tag{3.1}$$

Formula (3.1) can be inverted in order to get the implied volatility (Brenner and Subrahmanyam 1988) as

$$\sigma = \frac{2}{\sqrt{\Delta t}} N^{-1}\left(\frac{c+S}{2S}\right)$$

where:

$N^{-1}(.)$ is the inverse of the cumulative normal distribution.

Implied volatility is a good proxy of the market sentiment about the volatility of a particular asset. Some investors calculate implied volatility on actively traded options and then manipulate it in order to price less actively traded options on the same underlying stock.

Another type of volatility is the one that is estimated through econometric models, in particular the so-called Autoregressive Conditional Heteroscedasticity (ARCH) and General ARCH (GARCH) models (Green 2011).

Econometric models like those allow to model variance in a framework of autoregressive returns, dependent on historical values, and heteroscedasticity, which is the condition of nonconstant conditional variance.

A convenient way to describe the relation between two variables is through covariance, which describes the interdependence among the variables telling if it is direct or inverse, and also to what degree the variables move in the same or opposite direction.

Recall the variance of a random variable x with mean μ is given by

$$\begin{aligned}\sigma^2 &= E\left\{[x-E(x)]^2\right\}\\ &= E\left[(x-\mu)^2\right]\\ &= \int (x-\mu)^2 f(x)dx\\ &= \int x^2 f(x)dx - \left[\int xf(x)dx\right]^2\\ &= E(x^2) - E^2(x)\end{aligned}$$

Using the integration method, it is then possible to describe the covariance between two variables X and Y, which is given by

$$\begin{aligned}\sigma_{x,y} &= E\left[(x-\mu_x)\left(y-\mu_y\right)\right]\\ &= E(xy) - E(x)E(y)\\ &= \int\int xyf(x,y)dxdy - \left[\int\int xf(x,y)dxdy \int\int yf(x,y)dxdy\right]\\ &= \int\int xyf(x,y)dxdy - \left[\int xf_x(x,y)dx \int yf_y(x,y)dy\right]\end{aligned}$$

where:

$f(x, y)$ is the joint density.
$f_x(x, y)$ and $f_y(x, y)$ are the marginal densities.

Correlation is very similar to covariance in that it measures the relation between two variables, but, as opposed to covariance, it also gives information about the degree to which the two variables are related.

The correlation coefficient ranges from -1 to $+1$, making it a standardized measure of the interdependence between two variables. A coefficient of $+1$ corresponds to positive perfect correlation. When the variables are not related at all, the value of correlation is zero, and the perfect negative correlation corresponds to a value of -1.

The correlation coefficient, denoted by $\rho_{x,\ y}$, of random variables x and y is defined to be

$$\rho_{x,y} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

where:

σ_{xy} is the covariance between x and y.
σ_x is the standard deviation of x.
σ_y is the standard deviation of y.

The coefficient is independent of the measurement unit and has the same sign of covariance.

Example The joint density of two random variables x and y is

$$f(x, y) = \begin{cases} y^{-2x} & x \geq 0, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

By integrating, the expected values are given as

$$E(x) = \int_0^{\infty} x f_x(x, y) dx = \ldots = \frac{1}{2}$$
$$E(y) = \int_0^{1} y f_y(x, y) dy = \ldots = \frac{1}{3}$$
$$E(xy) = \int_0^{\infty} \int_0^{1} xy f(x, y) dx dy = \ldots = \frac{1}{12}$$

The covariance is given by the expectation of the product minus the product of the expectations, as defined by the formula

$$\begin{aligned} \sigma_{x,y} &= E(xy) - E(x)E(y) \\ &= \frac{1}{6} - \frac{1}{12} \\ &= \frac{1}{12} \end{aligned}$$

The correlation between x and y is simply the expected product of the corresponding standard scores as can be shown by

$$\begin{aligned} \rho_{x,y} &= \frac{\sigma_{x,y}}{\sigma_x \sigma_y} \\ &= \frac{E\{[x - E(x)][y - E(y)]\}}{\sigma_x \sigma_y} \\ &= E\left\{ \left[\frac{x - E(x)}{\sigma_x} \right] \left[\frac{y - E(y)}{\sigma_y} \right] \right\} \end{aligned}$$

In finance, the measure of correlation between two financial assets is related to the issue of portfolio diversification. Diversification is the practice of investing in many different assets in order to reduce the overall portfolio volatility through correlation effect.

The correlation coefficient can be expanded to give a measure of the correlation between two assets, starting from the historical realized returns based on n observations, as described by

$$\widehat{\rho}_{x,y} = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sqrt{\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]\left[n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2\right]}}$$

which provides the sample correlation coefficient of the two assets rather than the population coefficient.

Example 3.3 Consider the following table with five observations of the returns of two assets x and y.

Stock	$n = 1$	2	3	4	5
x	13.2%	−2.35%	4.53%	10.42%	−3.44%
y	10.3%	3.45%	−1.23%	8.33%	1.21%

Calculation yields

	x_i	y_i	$x_i y_i$	x_i^2	y_i^2
1	13.2%	10.3%	1.36%	1.74%	1.06%
2	−2.35%	3.45%	−0.08%	0.06%	0.12%
3	4.53%	−1.23%	−0.06%	0.21%	0.02%
4	10.42%	8.33%	0.87%	1.09%	0.69%
5	−3.44%	1.21%	−0.04%	0.12%	0.01%

so that

$$\sum_{i=1}^{n} x_i = 22.36\%,\ \sum_{i=1}^{n} y_i = 22.06\%,\ \sum_{i=1}^{n} x_i y_i = 2.05\%,$$
$$\sum_{i=1}^{n} x_i^2 = 3.21\%,\ \left(\sum_{i=1}^{n} x_i\right)^2 = 5.00\%,\ \sum_{i=1}^{n} y_i^2 = 1.90\%,\ \left(\sum_{i=1}^{n} y_i\right)^2 = 4.87\%$$

and the sample correlation is given by

$$\widehat{\rho}_{x,y} = \frac{(5 \times 2.05\%) - (22.36\% \times 22.06\%)}{\sqrt{[(5 \times 3.21\%) - 5.00\%][(5 \times 1.90\%) - 4.87\%]}} = 0.7416$$

3.1.3 Maximum Likelihood Methods

In case of volatility from econometric models, those must be estimated in order to get the parameters that explain the variance. Maximum likelihood estimation (MLE) is a general method for estimating the parameters of an econometric model (McLeish 2005).

As the name suggests the method is based on the fit of the modeled calculated variance to the real one, observable on the market. It involves a random variable with some probability density whose form is known, but not the parameter vector.

Consider a random variable $x_i = x_1, x_2, \ldots, x_n$ with probability density function $f(x_1, x_2, \ldots, x_n; \theta)$. As mentioned above MLE involves choosing the values of the parameters that give the highest probability to match the observed data.

The trick behind MLE is to view the time series under analysis as a series of draws from a probability distribution. Assuming there are n random variables conditional on n parameters, the joint probability density function is given by

$$f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_n) \tag{3.2}$$

The usual interpretation of formula (3.2), as the function of the variable x_t for given parameters θ, in the case of MLE, is reversed. The aim now is to look at f as a function of the parameter given the value of the variable.

The MLE can be used for both time series models and replicated experiments, being its versatility in statistics.

For a variable x with n observations $x_1, x_2 \ldots, x_n$ and a joint probability density given by

$$(x_1, \ldots, x_n) \mapsto p_{n,\theta}(x_1, \ldots, x_n)$$

which depends on a parameter θ, the likelihood function is the stochastic process

$$\theta \mapsto p_{n,\theta}(x_1, \ldots, x_n)$$

The MLE for the parameter θ is the value maximizing the likelihood function. It is now possible to condition repeatedly in order to represent the likelihood appropriately. In fact, the likelihood corresponding to the observations $x_1, \ldots, x_n$ as $\theta \mapsto p_\theta(x_1, \ldots, x_n)$ can be decomposed as

$$\theta \mapsto p_\theta(x_1, \ldots, x_n) = p_\theta(x_1) \times p_\theta(x_2|x_1) \times \ldots \times p_\theta(x_n|x_{n-1}, \ldots, x_1)$$

In case the variable under analysis is continuous, the analysis changes slightly. The probability density function in fact can be written as

$$f(x; \theta_1, \theta_2, \ldots, \theta_k)$$

In addition, it depends on k unknown parameters $\theta_1, \theta_2, \ldots, \theta_k$.

Again, a draw of n independent observations $x_1, x_2, \ldots, x_n$ from an experiment has an associated likelihood function that can be written as

$$L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2, \ldots, \theta_k) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2, \ldots, \theta_k) \tag{3.3}$$

The right-hand side of Eq. (3.3) is clearly very difficult to calculate, being an iterated product of densities. A convenient way to solve the issue is to work with the log of the likelihood, given that it is a monotone transformation that preserves the concavity of the target function, leaving the maximization problem unchanged.

The log likelihood function is given by:

$$\begin{aligned}\ell(x|\theta) &= \ln L(x_1, x_2, \ldots, x_n|\theta_1, \theta_2, \ldots, \theta_k) \\ &= \sum_{i=1}^{n} \ln f(x_i; \theta_1, \theta_2, \ldots, \theta_k)\end{aligned}$$

So, maximizing the log likelihood is much easier than the likelihood L, and the MLEs of θ_1, θ_2, . . ., θ_k are given by the simultaneous solutions of k equations such that

$$\frac{\partial \ell(x|\theta)}{\partial \theta_j} = 0, \quad j = 1, 2, \ldots, k$$

MLE is a very good estimator for big samples, tending to infinite. For finite samples, there are better tools.

In general, the MLE method is used to estimate parameters of well-known distributions, as, for example, the normal distribution. In this case, the MLE distribution for the parameters mean μ and standard deviation σ for the normal distribution can be derived by first introducing its probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Assume the observations x_1, x_2, . . ., x_n are realizations of a normally distributed variable x. The likelihood function is given by

$$\begin{aligned}L(x_1, x_2, \ldots, x_n|\mu,\sigma) &= \prod_{i=1}^{n} \left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X_i-\mu}{\sigma}\right)^2}\right] \\ &= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^N} e^{-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{X_i-\mu}{\sigma}\right)^2}\end{aligned}$$

The log likelihood is

$$\ell(x|\theta) = -\frac{n}{2} \ln(2\pi) - n \ln \sigma - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2$$

The first-order conditions are defined by setting the partial derivatives with respect to mean and standard deviation equal to zero.

$$\begin{aligned}\frac{\partial(\ell)}{\partial \mu} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0 \\ \frac{\partial(\ell)}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 = 0\end{aligned}$$

Solving the two equations simultaneously gives the solution

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\widehat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2}$$

The estimators are correct in the limit, meaning their accuracy increases with the size of the sample. When the sample equals the whole population, the estimators are true values. As for the general MLE, the above estimators have in-the-limit properties, being unbiased, sufficient, consistent, and efficient.

Snapshot 3.1

Risk and Return in Excel

The calculation of risk and return of some particular assets starts from the time series of the historical market prices of the asset, at any frequency (daily, weekly, monthly, etc.). Let us consider the case of daily prices.

Date	Price
Day_n	P_n
Day_{n-1}	P_{n-1}
...	...
Day_1	P_1

Note that here Day_n is the most recent day in the dataset, while Day_1 is the oldest.

One can then calculate the daily returns by using any the following formulas:

Date	Price	Return	or	Return	or	Return
Day_n	P_n	$r_n = \frac{P_n - P_{n-1}}{P_{n-1}}$		$r_n = \frac{P_n}{P_{n-1}} - 1$		$r_n = \ln\left(\frac{P_n}{P_{n-1}}\right)$
Day_{n-1}	P_{n-1}	$r_{n-1} = \frac{P_{n-1} - P_{n-2}}{P_{n-2}}$		$r_{n-1} = \frac{P_{n-1}}{P_{n-2}} - 1$		$r_n = \ln\left(\frac{P_n}{P_{n-1}}\right)$
...	...	...		...		...
Day_2	P_2	$r_2 = \frac{P_2 - P_1}{P_1}$		$r_2 = \frac{P_2}{P_1} - 1$		$r_n = \ln\left(\frac{P_n}{P_{n-1}}\right)$
Day_1	P_1	...		...		...

It is clear that for n prices in the dataset, a total of $n - 1$ returns can be calculated.

The return of the asset can be calculated as the average of daily returns multiplied by the scaling factor 252 (number of days in a financial year) as

(continued)

Snapshot 3.1 (continued)

$$\boxed{= \text{AVERAGE}\left(\text{`Column of returns'}\right) * 252}$$

The scaling factor depends on the frequency of the dataset and can be adjusted accordingly. For weekly data, the factor is 52, for monthly data is 12, and so on.

The volatility of the returns can be calculated by using the command in Excel for the calculation of the standard deviation of the population of data as

$$\boxed{= \text{STDEV.P}\left(\text{`Column of returns'}\right) * \text{SQRT252}}$$

In this case, the scaling factor is square rooted because the standard deviation is already the square root of the variance, which is scaled as the returns.

3.2 Modern Portfolio Theory

Learning Outcomes

- Understand the trade-off between risk and return on financial markets.
- Learn about market efficiency.
- Understand the main concepts of Modern Portfolio Theory.

3.2.1 The Risk/Return Trade-Off

Financial economist Harry Markowitz developed the Modern Portfolio Theory (MPT) in the 1950s, claiming that the information given on the risk and return of the single assets in a portfolio is not enough to conclude about the overall risk and return.

In practice, MPT tries to give a mathematical description of how diversification affects the portfolio risk and return by modeling the mathematical relationship between risk and return of the assets in a normal market.

Standard deviation is the measure of risk in the MPT framework, and the model aims at defining all the possible combination of risk and return available in the market and then identifying the optimal portfolio.

MPT relies on assumptions on distribution of prices and returns. In particular, they are assumed to follow a normal distribution. So in order to understand the relationship between risk and return, it is important to recall the basic properties of the normal distribution.

- The distribution is symmetric and bell shaped.
- It is continuous for all values of X between $-\infty$ and ∞ so that every interval of real numbers has a non-null probability.
- The distribution is totally defined by two parameters, μ and σ, that determine the shape of the distribution.
- The probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where:
μ is the mean of the distribution.
σ is the standard deviation of the distribution.

- About $\frac{2}{3}$ of all cases fall within one standard deviation from the mean, that is

$$\Pr(\mu - \sigma \leq x \leq \mu + \sigma) \approx 0.68$$

- About 95% of cases lie within two standard deviations from the mean, that is

$$\Pr(\mu - 2\sigma \leq x \leq \mu + 2\sigma) \approx 0.95$$

Recall from basic statistics that the normal distribution (Fig. 3.1) is completely defined by the two parameters mean and standard deviation. In financial terms, if one assumes that returns are normally distributed, the mean corresponds to the expected return, and the standard deviation describes the volatility of returns (Figs. 3.2 and 3.3).

The intuition is that the relationship between the risk and the return of some investment is given by the same corresponding relationship between the mean and standard deviation of the distribution of the returns.

If prices or returns are supposed to be normally distributed, it is possible to infer the properties of the normal distribution to them.

Example 3.4 If we have a time series of normally distributed financial returns, with mean 0.04 and standard deviation 0.22, we can say that the asset associated with that particular series of data has an expected return of 4% (over the time horizon considered) and a volatility (risk) of 22%.

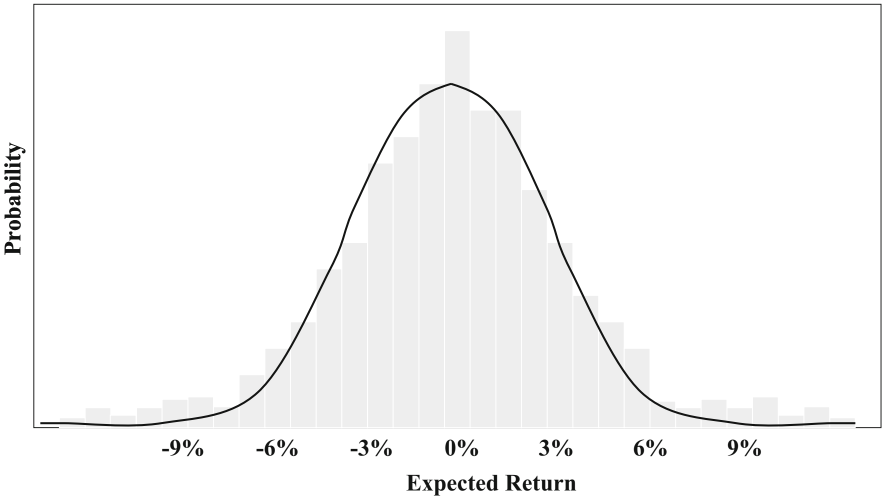

Fig. 3.1 Financial returns of most securities are normally distributed. In this case, the graph shows an average return of almost 3% for that particular security

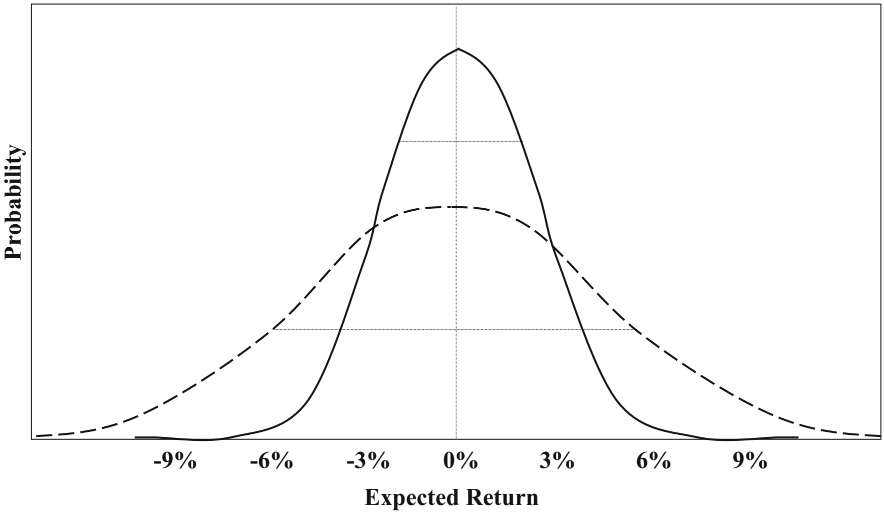

Fig. 3.2 Distribution of possible returns of two different investments. They both have the same expected return, but one of them (dashed line) offers a wider spread of possible outcomes, making other investments safer and more attractive for most of the investors

Given the distributional properties of returns that are normally distributed, it is clear that the relationship between risk and return is direct, in order to get a.

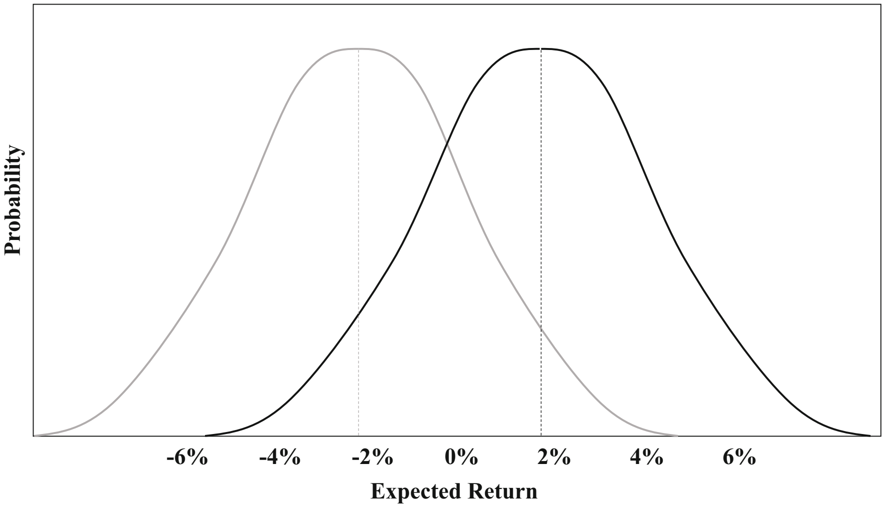

Fig. 3.3 Two Normal distributions, with same standard deviation but different mean (expected return)

Investment returns reflect the degree of risk carried by the investment, and investors should be able to determine what level of return is appropriate for a given level of risk.

Define the portfolio value as

$$V_t = \sum_{i=1}^{n} \frac{\alpha_i}{S_{i,0}} S_{i,t}$$

where:

α_i is the amount invested in asset i.
$S_{i,\,0}$ is the value of asset i at time 0.
$S_{i,\,t}$ is the value of asset i at time t.

The expected return of a portfolio is straightforward to calculate, being the weighted average of the expected returns of the assets composing it. Weights assigned to the calculation reflect the proportion of each asset on the total portfolio value.

$$\begin{aligned} r_{\mathrm{p}} &= \frac{V_t - V_{t-1}}{V_{t-1}} \\ &= \ldots \\ &= \sum_{i=1}^{n} w_i r_i \end{aligned} \tag{3.4}$$

where:

r_i is the expected return on stock i.
n is the number of stocks in the portfolio.
w_i is the weight (proportion) of asset i in the portfolio.

Example 3.5 Suppose stocks A and B have expected returns $E(r_{\mathrm{A}}) = 12.5\%$ and $E(r_{\mathrm{B}}) = 20\%$. The expected return of the portfolio composed of 75% of stock A and 25% of stock B is given by

$$E(r_{\mathrm{p}}) = 0.75 \times 12.5\% + 0.25 \times 20\% = 14.38\%$$

The calculation of the standard deviation of a portfolio is trickier, in that it does not only involve the volatility of the single assets in the portfolio but also their covariance. The co-movements of the asset returns must be taken into account through covariance and correlation measures (Merton 1972).

The combined analysis of the single volatilities and covariance among couples of assets leads to the formula for the variance of the portfolio, which is given by

$$\begin{aligned} \sigma_{\mathrm{p}}^2 &= E\left[\left(r_{\mathrm{p}} - E\left(r_{\mathrm{p}}\right)\right)\right]^2 \\ &= \ldots \\ &= \sum_{i=1}^{n} \sum_{j \neq i=1}^{n} w_i w_j \sigma_{ij} \end{aligned} \tag{3.5}$$

where:

σ_i^2 is the variance of the i-th asset.
w_i is the weight of an asset i in the portfolio.
w_j is the weight of an asset j in the portfolio.
σ_{ij} is the covariance between asset i and j.

Equation (3.5) can be written in matrix notation (Best 2010) as

$$\sigma_{\mathrm{p}}^2 = \mathbf{w}^{\mathrm{T}} \mathbf{\Sigma} \mathbf{w}$$

where:

$\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is the vector of weights of assets in the portfolio.

$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1,n} \\ \vdots & \iddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_n^2 \end{pmatrix}$ is the covariance matrix of the portfolio.

The covariance between any couple i,j of asset returns is given by

$$\begin{aligned} \sigma_{ij} &= E\{[r_i - E(r_i)][r_j - E(r_j)]\} \\ &= \ldots \\ &= E(r_i r_j) - E(r_i)E(r_j) \end{aligned}$$

The calculation of the correlation coefficient between the returns on two stocks is given by

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

where:

σ_i is the standard deviation of asset i.
σ_j is the standard deviation of asset j.

Example 3.6 Suppose stock A and stock B have expected returns $E(r_\mathrm{A}) = 12.5\%$ and $E(r_\mathrm{B}) = 20\%$ respectively, and standard deviations $\sigma_\mathrm{A} = 5.12\%$ and $\sigma_\mathrm{B} = 20.49\%$, correlation coefficient is $\rho_\mathrm{AB} = -1$. First, we compute the covariance to be

$$\sigma_\mathrm{AB} = \frac{-1}{0.0512 \times 0.2049} = -0.0105$$

The variance of a portfolio of 75% stock A and 25% stock B is

$$\begin{aligned} \sigma_\mathrm{p}^2 &= (0.75)^2(0.0512)^2 + (0.25)^2(0.2049)^2 + 2 \times 0.75 \times 0.25 \times -0.0105 \\ &= 0.00016 \end{aligned}$$

3.2.2 Optimal Portfolios

The aim of a financial investor is to maximize the expected return on their investment, given an accepted level of risk. The MPT defines the analytics of the relationship between risk and return in a complete market.

MPT says that the investors minimize the risk of the portfolio, hence its volatility, for a given level of return. This is done by choosing the right amount of each security to include in the portfolio, so that the total portfolio variance is minimized.

One must take into account not only how the single assets price changes, according to the volatility, but also how the changes for all asset compare, so to use covariance to quantify the diversification effect.

The approach of the theory is mathematical, and it is based on the construction of the ideal portfolio that fits the needs of an investor minimizing the risk given some fixed return. The relationship between risk and return is taken under consideration for the analysis.

Each security carries its own risk, and mixing many securities in the same portfolio should reduce the risk through diversification effect. The emphasis is in fact on the power of covariance to reduce the overall risk.

The MPT resembles a series of mathematical procedures to identify the optimal portfolio in the set of all possible portfolios and choose it in the context of wealth-maximizing, risk-adverse investors.

The model can be represented on a graph in the risk-return space. Different investors have different risk tolerance and return appetite. Taking into consideration the utility function of the investors, the model identifies a set of feasible risk/return combinations.

Securities can be combined in portfolios (and portfolios of portfolios) in a way to minimize the risk for some level of return. The set of portfolios with minimum variance for a given return is called efficient frontier.

Linear algebra is not sufficient to handle calculation when portfolios are very large. It is much more convenient to use matrix algebra, in order to simplify the problem. Matrix algebra formulas are faster to process and much easier to implement on the computer (Fig. 3.4).

An optimization problem, for minimizing the variance, leads to the analytical derivation of the efficient frontier. As mentioned above, it is better to show the calculation in matrix algebra, for the sake of simplicity.

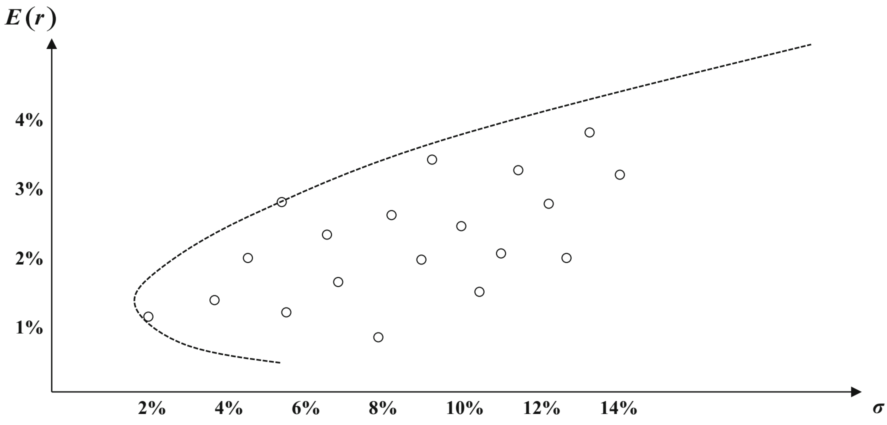

Fig. 3.4 Efficient frontier for portfolio or risky assets. Various combinations of available assets generate different portfolios, with different combinations of risk and return (dots). The portfolios lying on the upper side of the curve are efficient

The minimum variance portfolio with expected return μ is the solution of the following minimization program

$$\min\left(\frac{1}{2}\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}\right)$$

subject to

$$\begin{aligned}\mathbf{w}^{\mathrm{T}}\mathbf{1} &= 1\\ \mathbf{w}^{\mathrm{T}}\mathbf{r} &= \mu_{\mathrm{P}}\end{aligned}$$

where:

$\mathbf{w}$ is the vector of weights assigned to each asset.
$\boldsymbol{\Sigma}$ is the variance-covariance matrix of the asset returns.
$\mathbf{r}$ is the vector of the expected returns of the assets.
$\mathbf{1}$ is a vector of ones.
μ_{P} is the expected return of the portfolio.

It is possible to simplify the problem, for example, by reducing the amount of constraints. Allowing for short selling in fact, weights can take any value, even negative, removing one of the constraints.

The Lagrangian function of the problem is

$$L \equiv \frac{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}{2} + \lambda_1\left(\mu_{\mathrm{P}} - \mathbf{w}^{\mathrm{T}}\mathbf{r}\right) + \lambda_2\left(1 - \mathbf{w}^{\mathrm{T}}\mathbf{1}\right)$$

where:

λ_1 and λ_2 are the Lagrange multipliers.

with first-order conditions given by the partial derivatives of the Lagrangian function with respect to the weights vector and the Lagrangian multipliers, which can be written as

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{w}} &= \boldsymbol{\Sigma}\mathbf{w} - \lambda_1\mathbf{r} - \lambda_2\mathbf{1} = 0\\ \frac{\partial L}{\partial \lambda_1} &= \mu_{\mathrm{P}} - \mathbf{w}^{\mathrm{T}}\mathbf{r} = 0\\ \frac{\partial L}{\partial \lambda_2} &= 1 - \mathbf{w}^{\mathrm{T}}\mathbf{1} = 0\end{aligned}$$

Some straightforward calculation leads to the solution for the optimal weights, which minimize the variance of the portfolio, as defined by

$$\mathbf{w}^* = \lambda_1\left(\boldsymbol{\Sigma}^{-1}\mathbf{r}\right) + \lambda_2\left(\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) \tag{3.6}$$

It is very common in the literature to now simplify the notation by indicating the following:

$$\begin{aligned} A &= \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{r} > 0 \\ B &= \mathbf{r}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{r} > 0 \\ C &= \mathbf{1}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1} > 0 \\ \Delta &= BC - A^2 > 0 \end{aligned} \tag{3.7}$$

The popular Cauchy-Schwarz inequality shows that $\Delta > 0$, due to the condition that the covariance matrix is non-singular and not all assets have the same mean. It follows that

$$\mathbf{r} \neq k\mathbf{1}$$

where:

k is some constant.

Following the result in Eq. (3.7) and given the constraint of the optimization problem, the Lagrange multipliers are given by

$$\begin{aligned} \lambda_1 &= \frac{C\mu_{\mathrm{P}} - A}{\Delta} \\ \lambda_2 &= \frac{B - A\mu_{\mathrm{P}}}{\Delta} \end{aligned}$$

The financial economist Merton (1972) shows that the equation of the minimum variance portfolio, as a consequence of the optimization algorithm, is given by

$$\sigma^2 = \frac{C}{\Delta}\left(\mu_{\mathrm{P}} - \frac{A}{C}\right)^2 + \frac{1}{C}$$

This is clearly a quadratic equation describing a parabola in the space of return and volatility. The program can be implemented as an interesting application with software and real data, in order to draw the efficient frontier of the market analyzed and conclude about efficiency of single portfolios.

The last step is to substitute the explicit values of the Lagrange multipliers into Eq. (3.6) in order to get the explicit extended formulation of the vector of portfolio weights as solution of the optimization problem, defined as

$$\begin{aligned} \mathbf{w}^* &= \left(\frac{C\mu_{\mathrm{P}} - A}{\Delta}\right)\left(\boldsymbol{\Sigma}^{-1}\mathbf{r}\right) + \left(\frac{B - A\mu_{\mathrm{P}}}{\Delta}\right)\left(\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) \\ &= \frac{1}{\Delta}\left[\mu_{\mathrm{P}}\left(C\boldsymbol{\Sigma}^{-1}\mathbf{r} - A\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) + \left(B\boldsymbol{\Sigma}^{-1}\mathbf{1} - A\boldsymbol{\Sigma}^{-1}\mathbf{r}\right)\right] \end{aligned}$$

The representation shows a portfolio whose variance is minimized for some specific value of expected return. It is possible to show that the minimum variance portfolio is obtained for a value μ_P corresponding to

$$\mu_P = \frac{A}{C}$$

so that the optimal portfolio is given by

$$\mathbf{w}^*_{\mathrm{OPT}} = \frac{1}{\Delta}\left[\frac{A}{C}\left(C\boldsymbol{\Sigma}^{-1}\mathbf{r} - A\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) + \left(B\boldsymbol{\Sigma}^{-1}\mathbf{1} - A\boldsymbol{\Sigma}^{-1}\mathbf{r}\right)\right]$$

The derivation of the efficient frontier for a market with only risky assets can be extended to the case when a risk-free asset is also present. Such an asset is like a constant in the model, so that the following condition holds:

$$\sigma_f^2 = \sigma_{if} = 0$$

where:

σ_f^2 is the variance of the risk-free asset.
σ_{if} is the covariance between the risk-free and the risky asset.

First, consider a convex combination of two portfolios Π_1 and Π_2 that lie on the efficient frontier, which is given by

$$\alpha\Pi_1 + (1-\alpha)\Pi_2, \quad \forall\alpha, \quad -\infty < \alpha < \infty$$

Now consider the minimum variance portfolio (MVP) from the above derivation in the space of return and risk, in a market with only risky assets, having expected return r_p and variance σ_p^2.

In case a proportion α of the investor's wealth is invested in the MVP and the remaining $(1-\alpha)$ is invested in the risk-free rate, the new combined portfolio Π has new expected return and variance.

The expected return of the portfolio is defined as

$$E(r_\Pi) = \alpha E(r_i) + (1-\alpha)r_f$$

where:

$E(r_i)$ is the expected return on the risky asset.

In addition, the variance can be written as

$$\begin{aligned}\sigma_\Pi^2 &= \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_f^2 + 2\alpha(1-\alpha)\sigma_{A,f}\\ &= \alpha^2\sigma_A^2\end{aligned}$$

The new equation leads to another frontier, for different values of α. The tangent portfolio r_m covers all the portfolios with expected return in the range between zero and $E(r_m)$, for $0 < \alpha < 1$.

When $\alpha > 1$, the frontier includes portfolio with an expected return higher than $E(r_m)$. It corresponds to leveraged portfolios, which are created by borrowing money at r_f and investing it into the tangent portfolio.

Now let us change notation and define a portfolio composed of a proportion w_f of the risk-free asset and a proportion $w_i = (1 - w_f)$ of the risky asset. Since the two assets are uncorrelated, the properties of variance hold, as defined by

$$\sigma_p^2 = (1 - w_f)^2 \sigma_i^2$$

where σ_p^2 is the portfolio variance. Straightforward mathematics shows that the portfolio weights can be expressed as

$$w_f = 1 - \frac{\sigma_p}{\sigma_i}$$
$$w_i = \frac{\sigma_p}{\sigma_i}$$

Therefore, we can calculate the portfolio return as

$$E(r_p) = \left(1 - \frac{\sigma_p}{\sigma_i}\right) r_f + \left(\frac{\sigma_p}{\sigma_i}\right) E(r_i)$$

The Capital Market Line takes the form

$$E(r_p) = r_f + \left(\frac{\sigma_p}{\sigma_i}\right) [E(r_i) - r_f]$$

The risk-free asset r_f adds up in the previous optimization as a zero-risk element in the market. Therefore, the efficient frontier expands, increasing the range of investment opportunities (Fig. 3.5).

The model then turns from a curve (the efficient frontier) to a line, which is called Capital Market Line (CML). It represents the capital allocation between the risk-free security and the risky asset. The tangent point between the new linear frontier and previous curve is the tangency portfolio, also known as the market portfolio.

The CML is the line that can be drawn between the risk-free rate, on the vertical axis, and the tangency point. The new line is considered dominant to the efficient frontier because the inclusion of a risk-free asset in the economy allows better portfolios to be formed.

The line is the new frontier for efficient portfolios so that the investor can move on it to choose the desired mix of return and risk. Introducing the risk-free rate allows for borrowing and lending funds in order to leverage or deleverage the investment portfolio.

The presence of the risk-free rate allows the investor to borrow funds at that rate and invest them in the risky asset, in order to increase the leverage. On the other

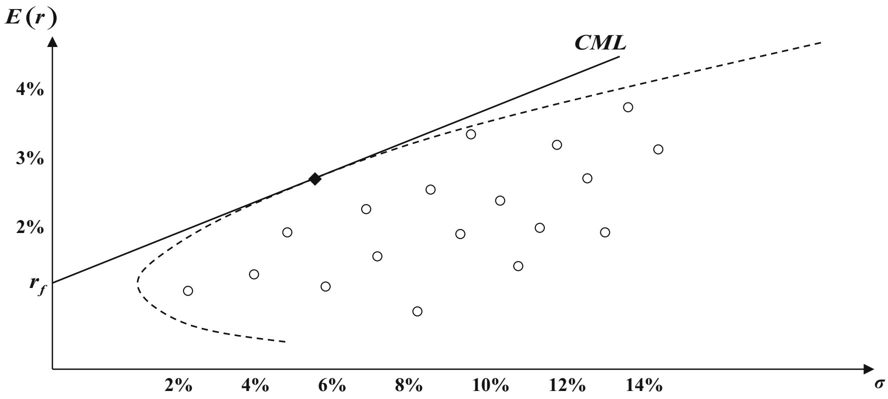

Fig. 3.5 Introducing a risk-free asset in the market allows choosing efficient portfolios on the straight line. Combinations of risk-free assets and risky assets lead to different points on the line

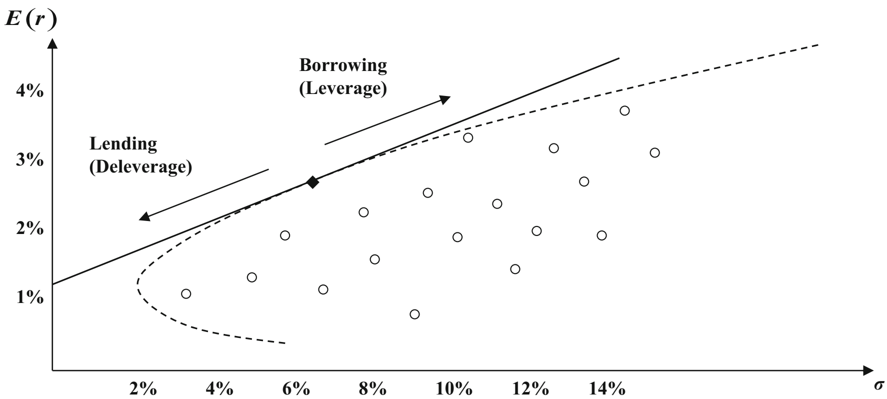

Fig. 3.6 Lending and borrowing money are possible to leverage or deleverage the position over the optimal portfolio

hand, it is also possible to save the proceedings from sale of risky asset at the risk-free rate, so to reduce the riskiness.

The theory has important implications. For example, risk-averse investors want to hold portfolios that are combinations of the risk-free asset and the tangency risky portfolio, at appropriate balance.

The fact that all investors hold the same risky portfolio must be unique, and it identifies like the market portfolio. The only choice left to risk-averse investors is the proportion to put in the risk-free asset (Fig. 3.6).

3.2.3 The Market Price of Risk

The market price of risk can be defined as the extra return on the risk-free rate, required by an investor to compensate the risk bore on the market by investing in some specific asset.

In derivatives, theory quantities are modeled as stochastic, and randomness leads to risk. The purpose of financial analysis is to determine how much extra return an investor should expect for taking risk (Renneborg 2006).

The abovementioned BSM derivative pricing model does not express the market price of risk directly, since the risk in an option position, for example, can be hedged away by taking a position on the underlying asset. This concept will be clearer after Chap. 9, but this information is important for the continuation of this section.

The derivation of the market price of risk goes through considering two different claims written on the same underlying variables. Consider the properties of derivatives dependent on the value of a single variable θ, following the stochastic process

$$d\theta = \mu_\theta \theta dt + \sigma_\theta \theta dW$$

where μ_θ and σ_θ are the expected growth rate and volatility of θ, respectively. It is assumed that the parameters depend only on the variable itself and the time.

According to the Ito's lemma, it is possible to define the process of any derivative v_i written on θ, as

$$\begin{aligned} dv_i &= \left(\frac{\partial v_i}{\partial \theta} \mu_\theta \theta + \frac{\partial v_i}{\partial t} + \frac{1}{2} \frac{\partial^2 v_i}{\partial \theta^2} \sigma_\theta^2 \theta^2 \right) dt + \left(\frac{\partial v_i}{\partial \theta} \sigma_\theta \theta \right) dW \\ &= m_i dt + s_i dW \end{aligned}$$

Consider then the process for the variable transformation $\varsigma_i = \ln v_i$. The application again of the Ito's lemma leads to

$$\begin{aligned} d\varsigma_i &= \left(\frac{1}{v_i} m_i - \frac{1}{2v_i^2} s_i^2 \right) dt + \left(\frac{1}{v_i} m_i \right) dW \\ &= \mu_i dt + \sigma_i dW \end{aligned}$$

The next step is to consider the above process for two different derivatives

$$\begin{aligned} d\varsigma_1 &= \mu_1 dt + \sigma_1 dW \\ d\varsigma_2 &= \mu_2 dt + \sigma_2 dW \end{aligned}$$

and then form a portfolio V of the two, by including them with weights w_1 and w_2 (summing up to unity), respectively. Letting Π denote the natural log of the portfolio value, the process followed by $d\Pi$

$$d\Pi = (w_1 \mu_1 + w_2 \mu_2) dt + (w_1 \sigma_1 + w_2 \sigma_2) dW$$

In order to investigate the risk premium, it is crucial to know under what circumstances the above portfolio is riskless. Choosing values

$$w_1 = -\frac{\sigma_2}{\sigma_1 - \sigma_2}$$
$$w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}$$

The portfolio process becomes

$$d\Pi = \left(\frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2 - \frac{\sigma_2}{\sigma_1 - \sigma_2}\mu_1\right)dt + \left(\frac{\sigma_1}{\sigma_1 - \sigma_2}\sigma_2 - \frac{\sigma_2}{\sigma_1 - \sigma_2}\sigma_1\right)dW$$

It follows that the term in dW vanishes, meaning that the volatility of the process is zero. Only the drift of the process is left, and in order for the portfolio to be riskless, the drift must be equal to the risk-free rate, to comply with no arbitrage conditions. It follows that

$$\frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2 - \frac{\sigma_2}{\sigma_1 - \sigma_2}\mu_1 = r$$

which implies

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda \tag{3.8}$$

The parameter λ can be defined as the market price of risk of the underlying variable θ, and it depends on the single derivatives. Moreover, it must be the same for all the claims written on the same underlying variable.

The market price of risk is a measure of the trade-off between risk and return associated to some variable. The product of the quantity of risk, σ, and its price, λ, gives the amount of corresponding return.

In investment finance, it is called Sharpe ratio, and it is mostly used to assess the level of compensation obtained by the investor for the risk taken. It is especially useful to compare two assets with different returns and volatility.

The higher the Sharpe ratio, the higher the expected return from an asset in relation to the risk it carries. Therefore, good investment strategies consist in picking the investment with the highest Sharpe ratio.

Based on the formula (3.8), it is now possible to explicitly define the relationship between the risk premium demanded by investors, in terms of excess return, for holding derivative i and the price of risk, as

$$\mu_i - r = \lambda\sigma_i$$

The risk premium is therefore the market price of risk λ, multiplied by the amount of risk derivative i holds, σ_i.

From basic stochastic calculus, the process followed by any derivative v can be written as

$$dv = \mu v dt + \sigma v dW$$

In the traditional risk-neutral world, the leading assumption is that investors do not demand any compensation for the risk taken, leaving the market price of risk equal to zero. As a consequence $\mu = r$, and the process followed by r can be written as

$$dv = rvdt + \sigma vdW$$

Other worlds can now be defined, based on the specific level of market risk characterizing them. Generalizing the framework, the growth rate of the process drift can be written as

$$\mu = r + \lambda\sigma$$

so that

$$dv = (r + \lambda\sigma)vdt + \sigma vdW$$

The market price of risk of a variable determines the growth rates of all the securities dependent on that variable.

Snapshot 3.2

Portfolio Optimization in Excel

In order to apply portfolio optimization in Excel, first it is necessary to create a matrix of the n observed returns for the chosen m assets in the market and the covariance matrix, as in the following example:

E(R)	*0.178*	*0.216*	...	*0.049*	**Var(R)**	*0.178*	*0.216*	...	*0.049*

Using the formulas from Markowitz Portfolio Theory, the following step is to calculate, from the expected return and variances of the m single assets, the portfolio expected return and portfolio variance

R(P)	*0.178*	**Var(P)**	*0.178*

In order to set for the optimization, a line of weights of the assets in the portfolio must be filled.

	1	2	...	m
Weights	0.111	0.223	...	0.015

It is then possible to start the program, using the Solver Tool in Excel and implementing the following steps:

1. Set your desired portfolio return.
2. Start the Excel Solver.

(continued)

Snapshot 3.2 (continued)

3. Minimize the portfolio variance.
4. Change “By changing variable cells” to the range of cells containing the portfolio allocation.
5. Set the constraints (the total portfolio allocation must add up to one, and the portfolio return should be you desired value).
6. Click Solve.

Once the set of optimal weights are obtained, they can be used to build efficient portfolios for different combinations of expected return and variance. In order to get the efficient frontier, it is sufficient to plot the line crossing these portfolios, using the graph function in Excel.

3.3 The Capital Asset Pricing Model

Learning Outcomes

- Understand the Capital Asset Pricing Model.
- Draw and describe the Capital Market Line and Security Market Line.
- Acquire introductory knowledge of advanced factor models for asset pricing.

3.3.1 Model Assumptions

The previous sections have been dedicated to developing the CML. Once the line is generated, a further step consists in finding a way to measure systematic risk, given that diversification can eliminate only a part of it.

Modern Portfolio Theory describes systematic risk and how it should be compensated. After diversification takes out all asset-specific risk in fact, the systematic risk is the only one to be compensated.

A solution to the issue of risk measurement and compensation is given by the Capital Asset Pricing Model (CAPM), a model describing and solving the issue through mathematical modeling of systematic risk.

The CAPM framework starts with the risk-free rate as a benchmark for measuring a risk premium that determines the expected return on some specific asset. The premium is a compensation for the risk borne by the investor.

After the premium is calculated, the following step is to multiply it by a beta coefficient, which is a measure of the riskiness of the asset in proportion to the market risk. This is the link between expected return on the asset and the risk premium in the market.

Modern Portfolio Theory is then meant at determining the right expected return on a portfolio of assets. One of the main defining points of the method is diversification, toward elimination of all non-systematic risk.

Moreover, the investors hold combinations of just the risk-free asset and the tangency portfolio. Finally, systematic risk of the asset is a proportion of the market risk, and the risk premium of an asset is proportional to its systematic risk.

Recall that the CML describes the relationship between the excess return of an asset and the excess return of the market, as a line in the risk-return space.

$$r_i - r_{\mathrm{f}} = \beta(r_{\mathrm{m}} - r_{\mathrm{f}})$$

where:

r_{m} is the return of the market.
β is the dependence factor between market and asset.

The CAPM identifies the tangency portfolio, lying on both the efficient frontier and CML, as the market portfolio, and relating the price of any asset on the market to the market itself.

Consider a model where there are n assets in the market. Each asset $i = 1, \ldots, n$ has a market capitalization

$$\mathrm{Mcp}_i = V_i n_i$$

where:

V_i is the price per share of asset i.
n_i is the number of shares outstanding of asset i.

It follows that the total capitalization of the assets in the market is

$$\mathrm{Mcp}_{\mathrm{m}} = \sum_{i=1}^{n} \mathrm{Mcp}_i$$

The weights for each asset are given by

$$\begin{aligned} w_i &= \frac{\mathrm{Mcp}_i}{\sum_{i=1}^{n} \mathrm{Mcp}_i} \\ &= \frac{\mathrm{Mcp}_i}{\mathrm{Mcp}_{\mathrm{m}}} \end{aligned}$$

The CAPM result is based on the assumption that investors invest their money between the risk-free and the tangency portfolio, which is held in same proportions by all the investors.

The main assumptions underlying the derivation of CAPM are:

- Markets are complete with price-taking investors.
- Markets are frictionless and transactions are free of costs.
- The holding period of the assets is the same for all investors.
- Investors optimize the investment in the risk-return space, as by MPT.
- Funds are lent and borrowed at the same risk-free rate.
- The market is characterized by homogeneous information for all investors.

The CAPM states that if an investor holds some amount of the market portfolio, the risk compensation for single assets should be a consequence of how the single asset behaves compared to the market (Sharpe 1964).

If the volatility of an asset is the same as another asset (or the market), then also the expected return must be equal. By indicating the variance of the asset by σ^2, the risk premium per unit of risk λ can be written as

$$\lambda_{\mathrm{P}} = \frac{[E(r_{\mathrm{m}}) - r_{\mathrm{f}}]}{\sigma_{\mathrm{m}}^2}$$

There is a direct relation between the risk premium and the excess return on the market. The relation is instead inverse with the market risk, indicated as variance. The right compensation for each asset is calculated according to the covariance between the asset and the market.

The expected premium on asset i should be equal to the risk premium per each unit of risk multiplied by the relationship of the asset with the market expressed in the form of covariance.

$$E(r_i) - r_{\mathrm{f}} = \sigma_{i\mathrm{m}} \frac{[E(r_{\mathrm{m}}) - r_{\mathrm{f}}]}{\sigma_{\mathrm{m}}^2}$$

In case of a null covariance with the market, the asset has no risk attached and can only earn the risk-free rate, in normal market conditions. For a value of the covariance equal to 1, the asset is perfectly related to the market, earning the market return.

The formula of the CAPM can be simplified as

$$E(r_i) = r_{\mathrm{f}} + \frac{\sigma_{i\mathrm{m}}}{\sigma_{\mathrm{m}}^2}[E(r_{\mathrm{m}}) - r_{\mathrm{f}}]$$

The factor $\frac{\sigma_{i\mathrm{m}}}{\sigma_{\mathrm{m}}^2}$ called the beta β (beta) of the asset shows the proportionality of the asset risk premium to the market risk premium. The resulting model is the CAPM and is typically written as:

$$E(r_i) = r_{\mathrm{f}} + \beta_i[E(r_{\mathrm{m}}) - r_{\mathrm{f}}]$$

Example 3.7 Suppose the covariance of some stock A with the market is 15% and the variance of A is 12%. The market expected return is 11% and the risk-free rate is 5%. We can calculate then the expected return of A applying the CAPM

$$E(r_{\mathrm{A}}) = 0.05 + \frac{0.15}{0.12}[0.11 - 0.05] = 0.125$$

The expected return for the stock is 12.5%.

The implementation of CAPM involves some practical issues related to the input factors and their estimation. For example, the market portfolio is hard to proxy because a portfolio of all assets in the market would be needed.

It is therefore necessary to find a good proxy for the market return, which is usually represented by index funds. It is a sort of smaller portfolio representative of the market, including all the most dominant assets, to capture the essence of it.

One of the most famous stock indices in the world is the Standard & Poor's (S&P) 500 which resembles 500 representative stocks as a proxy of the market. In this specific case, CAPM implementation goes through an elaborated model that applies to reality. Consider estimating the beta coefficient for asset $i = 1, 2, \ldots, m$ over a discrete time horizon of n time points. For every $k = 1, 2, \ldots, n$, the variable average (expected) return for the asset i is defined by

$$\hat{\bar{r}}_i = \frac{1}{n}\sum_{k=1}^{m} r_{i,k}$$

where:

$\hat{\bar{r}}_i$ is the average return of asset i.
r_{ik} is the k-th sampled value of asset return.
$r_{\mathrm{SP},\,k}$ is the k-th sampled value of the S&P 500 index return.

For the index is

$$\hat{\bar{r}}_{\mathrm{SP}} = \frac{1}{n}\sum_{k=1}^{m} r_{\mathrm{SP},k}$$

The estimation of the index variance σ^2_{SP} is

$$\widehat{\sigma}^2_{\mathrm{SP}} = \frac{1}{n-1}\sum_{k=1}^{m}\left(r_{\mathrm{SP},k} - \hat{\bar{r}}_{\mathrm{SP}}\right)^2$$

And the covariance $\sigma_{\mathrm{SP}i}$ between asset i and the index is estimated by

$$\widehat{\sigma}_{\mathrm{SP}i} = \frac{1}{n-1} \sum_{k=1}^{m} \left(r_{\mathrm{SP},k} - \hat{\bar{r}}_{\mathrm{SP}} \right) \left(r_{ik} - \hat{\bar{r}}_{i} \right)$$

The beta value estimation is finally given by

$$\widehat{\beta} = \frac{\widehat{\sigma}^2_{\mathrm{SP}}}{\widehat{\sigma}_{\mathrm{SP}}}$$

3.3.2 The Security Market Line

The CAPM model applied to an asset gives the right discount rate for the future cash flows generated by it. Such a rate depends on the riskiness of the asset compared to the market, as defined by the beta.

The beta is in fact proportional to the riskiness of the asset. The higher the beta, the higher the discount rate, and the lower the present value of the future cash flows, with a resulting lower present value of the asset.

The opposite works for low-beta stocks, which are less sensitive to market changes and less risky, yielding lower discount rate and higher present value for the stock. Recall that one of the main findings of CAPM is the identification of the market portfolio as the tangent portfolio (Schneeweis et al. 2010).

Moreover, the risk associated to an asset is dependent on its covariance with the market portfolio. The market affects the asset in that the systematic risk is not diversifiable and must be compensated, according to the relationship between excess return of the asset and excess return of the market.

It is possible to derive the CAPM from decomposing a portfolio between an asset and the market. Consider forming a portfolio p by investing an amount w_i in a risky asset I and an amount $w_{\mathrm{m}} = (1 - w_i)$ in the market portfolio m. The expected return of the portfolio is then given by

$$E\left(r_{\mathrm{p}}\right) = w_{\mathrm{m}} E(r_{\mathrm{m}}) + w_i E(r_i)$$

where:

$E(r_{\mathrm{m}})$ is the expected return of the market.
w_{m} is the weight of the market in the portfolio.

Recall that the risk can be represented through the volatility (standard deviation), which can be defined as

$$\sigma_{\mathrm{p}} = \sqrt{{w_{\mathrm{m}}}^2 \sigma^2_{\mathrm{m}} + {w_i}^2 \sigma^2_i + 2 w_i w_{\mathrm{m}} \sigma_{i\mathrm{m}}}$$

where:

σ_{m}^2 is the variance of the market.
$\sigma_{i\mathrm{m}}$ is the covariance between the market and the asset.

The analysis then deals with the slope of the efficient frontier and the slope of the CML, with a focus on their tangency point. At that point, the slope of the CML is given by

$$\text{Slope}_{\text{CML}} = \frac{[E(r_{\mathrm{m}}) - r_{\mathrm{f}}]}{\sigma_{\mathrm{m}}}$$

First of all, the slope of the efficient frontier (curve) at the tangency point can be obtained by differentiating the expected portfolio return $E(r_{\mathrm{P}})$ with respect to portfolio volatility σ_{P}. The chain rule applies as

$$\text{Slope}_{\text{EFR}} = \frac{\partial E(r_{\mathrm{p}})}{\partial \sigma_{\mathrm{p}}} = \frac{\dfrac{\partial E(r_{\mathrm{p}})}{\partial w_i}}{\dfrac{\partial \sigma_{\mathrm{p}}}{\partial w_i}}$$

The chain is solved piecewise, starting from the numerator of the expression at the extreme right-hand side that can be written as

$$\frac{\partial E(r_{\mathrm{p}})}{\partial w_i} = E(r_i) - E(r_{\mathrm{m}})$$

The denominator is the sensitivity of the portfolio volatility with respect to the weight of asset i

$$\begin{aligned}\frac{\partial \sigma_{\mathrm{p}}}{\partial w_i} &= \frac{1}{2}\left[{w_{\mathrm{m}}}^2\sigma_{\mathrm{m}}^2 + w_i^2\sigma_i^2 + 2w_i w_{\mathrm{m}}\sigma_{i\mathrm{m}}\right]^{-\frac{1}{2}} \times \left[-2w_{\mathrm{m}}\sigma_{\mathrm{m}}^2 + 2w_i\sigma_i^2 + 2(1-2w_i)\sigma_{i\mathrm{m}}\right] \\ &= \frac{\left[-w_{\mathrm{m}}\sigma_{\mathrm{m}}^2 + w_i\sigma_i^2 + (1-2w_i)\sigma_{i\mathrm{m}}\right]}{\sigma_{\mathrm{p}}}\end{aligned}$$

The calculation in theory is very difficult. But the properties of the model at the tangency point can be used to simplify it. At tangency point all investors choose the market portfolio.

Therefore the proportion of asset i is zero, and the variance of the portfolio turns into the variance of the market, the only variable left. Therefore, the derivative of the variance wr to the weight w_i becomes

$$\left.\frac{\partial \sigma_{\mathrm{p}}}{\partial w_i}\right|_{\substack{w_i = 0 \\ \sigma_{\mathrm{p}} = \sigma_{\mathrm{m}}}} = \frac{\sigma_{i\mathrm{m}} - \sigma_{\mathrm{m}}^2}{\sigma_{\mathrm{m}}}$$

The chain rule leads to the expression for the slope of the efficient frontier as

$$\text{Slope}_{\text{EFR}} = \frac{\dfrac{\partial E(r_{\text{p}})}{\partial w_i}}{\dfrac{\partial \sigma_{\text{p}}}{\partial w_i}}$$
$$= \frac{E(r_i) - E(r_{\text{m}})}{\dfrac{\sigma_{i\text{m}} - \sigma_{\text{m}}^2}{\sigma_{\text{m}}}}$$
$$= \frac{[E(r_i) - E(r_{\text{m}})]\sigma_{\text{m}}}{\sigma_{i\text{m}} - \sigma_{\text{m}}^2}$$

The equilibrium is defined by the equality of the two slopes at tangency point. The market portfolio is the most efficient portfolio, and the equality is such that

$$\frac{[E(r_i) - E(r_{\text{m}})]\sigma_{\text{m}}}{\sigma_{i\text{m}} - \sigma_{\text{m}}^2} = \frac{[E(r_{\text{m}}) - r_{\text{f}}]}{\sigma_{\text{m}}}$$

so that

$$E(r_i) = r_{\text{f}} + \beta_i[E(r_{\text{m}}) - r_{\text{f}}]$$

where

$$\beta_i = \frac{\sigma_{i\text{m}}}{\sigma_{\text{m}}^2}$$

The graph of the beta against the volatility is called Security Market Line (SML). It is the line where all efficient portfolios lie.

If the utility function of the investors is concave, the findings of CAPM theory are consistent with the direct relationship between the risk and the expected return of an investment (Fig. 3.7).

A simple regression allows estimating the beta of an asset starting from the time series of risk-free rate and market returns. The regression takes the form

$$r_{i,t} = \alpha_{i,t} + \beta_{i,t} r_{\text{m}} + \varepsilon_{i,t}$$
$$\varepsilon_i \sim N(0, \sigma^2)$$

where:

$\beta_{i,\ t}$ is the beta of asset i at time t.

The volatility of the return can then be expressed as

$$\sigma_i = \beta_i^2 \sigma_{\text{m}} + \sigma_\varepsilon \tag{3.9}$$

where:

σ_ε is the volatility of the error term.

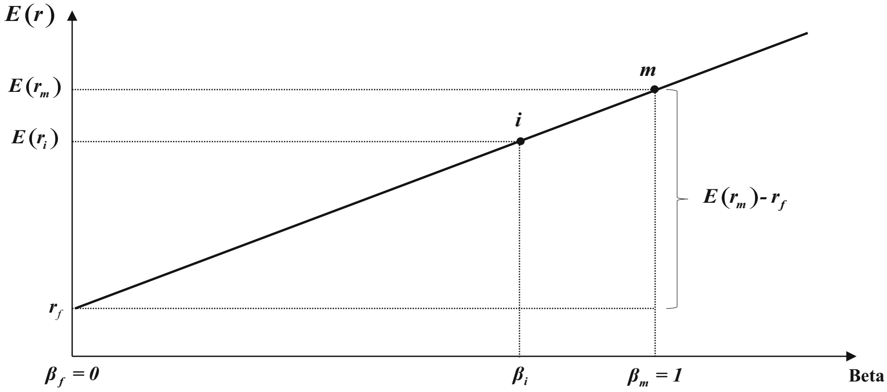

Fig. 3.7 The CAPM relation identifies the Security Market Line (SML) on which all the efficient combinations of expected return and asset betas lie. The slope of the curve identifies the relation between asset and market excess return

The layout of formula (3.9) separates the variance of the market return from the variance of the error term, and it is known as the variance decomposition of returns. It highlights the systematic risk, related to the market, from the diversifiable risk, related to the error term.

When the error term is close to zero, the portfolio is well diversified, and no more action is needed in order to eliminate risk. Only systematic risk is left, and it is rewarded by the market, according to the beta of the asset.

Equation (3.8) can be extended to an equally weighted portfolio of n assets as defined by

$$\sigma_{\mathrm{p}} = \beta_{\mathrm{p}}^2 \sigma_{\mathrm{m}} + \frac{1}{n}\sigma_{\mathrm{p}}^2$$

where:

β_{p} is the beta of the portfolio.

The term on the extreme right-hand side shows the diversification effect; with portfolio, variance reduced when n is particularly high. The covariance between any couple of assets i and j can be expressed as

$$\begin{aligned}\sigma_{i,j} &= \mathrm{cov}\left[\alpha_i + \beta_i r_{\mathrm{m}} + \varepsilon_i, \alpha_j + \beta_j r_{\mathrm{m}} + \varepsilon_j\right] \\ &= \ldots \\ &= \beta_i \beta_j \sigma_{\mathrm{m}}^2\end{aligned}$$

3.3.3 Beyond CAPM

After the development of CAPM as an asset-pricing model, its potential has been increased by the development of other models that take inspiration from it and extend the features of the original framework.

One example is the consumption-based CAPM (CCAPM), a model that expands the analysis introducing a consumption factor for calculating the expected return on an investment.

The CCAPM structure implies that the (expected) risk premium on the asset in this case is proportional to the covariance between the asset return and the consumption calculated for the analyzed period.

Consider a multi-period model with an infinitely lived representative household and an expected lifetime utility function defined as

$$E(u_t) = u(c_t) + E\left[\sum_{i=1}^{\infty} d^i u(c_{t+i})\right], \quad 0 < d < 1$$

where:

c_t is the consumption at time t.
$u(c_t)$ is the utility associated to consuming at time t.
d is a discount function.
$u(c_{t+i})$ is the utility associated to consuming at time $t + i$.

The level of consumption c_t, subject to a budget constraint, is chosen at time t. If the price for a financial security is p_t, the household can buy the security and redeem later at $t + 1$, to finance consumption at that time.

The first-order condition on the expected utility shows the optimal quantity of financial asset demanded by the investor as

$$dE[c_{t+1}u'(c_{t+1})] - u'(c_t)p_t = 0 \tag{3.10}$$

The result is the utility from consuming at $t + 1$ when financing through settlement of a marginal unit of the asset, net of the marginal utility lost from not consuming when purchasing the security.

Given that the one-period return on the asset can be written as

$$r_{t+1} = \frac{c_{t+1}}{p_t} - 1$$

Equation (3.10) changes to

$$\begin{aligned}
&dE[c_{t+1}u'(c_{t+1})] - u'(c_t)p_t = 0\\
&\Rightarrow dE\left[\frac{c_{t+1}}{p_t}u'(c_{t+1})\right] - u'(c_t) = 0\\
&\Rightarrow dE[(1 + r_{t+1})u'(c_{t+1})] - u'(c_t) = 0\\
&\Rightarrow dE\left[(1 + r_{t+1})\frac{u'(c_{t+1})}{u'(c_t)}\right] = 1
\end{aligned}$$

This is the CCAPM standard equation, showing that in equilibrium, the product of 1 plus the expected value of asset return and the marginal rate of substitution of the consuming utilities is equal to 1.

The model for all the securities, including the risk-free asset, in what case the formula becomes

$$d(1 + r_{\mathrm{f}})E\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] = 1$$

It is then possible to express CCAPM in terms of excess returns, by subtraction of the risk-free rate as described by

$$\begin{aligned}
&dE\left[(1 + r_{t+1})\frac{u'(c_{t+1})}{u'(c_t)}\right] - d(1 + r_{\mathrm{f}})E\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] = 0\\
&\Rightarrow \ldots\\
&\Rightarrow E\left[(r_{t+1} - r_{\mathrm{f}})\frac{u'(c_{t+1})}{u'(c_t)}\right] = 0
\end{aligned}$$

The product of the expected excess rate and the marginal rate of substitution of consumption is equal to zero.

Another popular variation of the CAPM was proposed by Eugene Fama and Kenneth French in 1992. After years of empirical analysis of financial returns, they found out that two classes of stock have the tendency of performing better than average.

Small capitalization stocks and high book-to-value ratio stocks showed returns higher than average of all classes of stocks. They therefore modified the standard CAPM formula in order to capture the effect of three factors.

The Fama-French formula is exposed to factors related to the returns of small minus big (SMB) capitalization stocks and high minus low (HML) book-to-value stocks. The resulting model is a three-factor formula in the form

$$E(r_i) = r_{\mathrm{f}} + \beta_i[E(r_{\mathrm{m}}) - r_{\mathrm{f}}] + \beta_{i,\mathrm{SMB}}\mathrm{SMB} + \beta_{i,\mathrm{HML}}\mathrm{HML} \tag{3.11}$$

where:

SMB is the expected excess return on SMB factor.
HML is the expected excess return on HML factor.

$\beta_{i,\ \mathrm{SMB}}$ is the beta of small capitalization stock factor, for asset i.
$\beta_{i,\ \mathrm{HML}}$ is the beta of high-minus-low book-to-value factor, for asset i.

The model improves the CAPM by adding new information, and the dynamics of the regression is pretty interesting, given that when returns increase with book-to-value, stocks with high ratio must be more risky than average.

The strong points of the model are the great diversity of the factors included in formula (3.11), based thus on heterogeneous variables. The indices weigh stocks according to capitalization, meaning that factor actually defines the market portfolio of reference.

Finally, it is interesting to have a look at another extension of the CAPM developed by Stephen Ross in 1976, called the Arbitrage Pricing Theory (APT). It is based on the idea that asset returns can be predicted by relating it to many common risk factors.

ATP does not require homogeneous behavior among investors, nor does it claim that the tangency portfolio is the only risky asset that will be held by the investors. It assumes that expected returns on a security should be related to the security's covariance with the common factors.

Suppose that returns are driven by a set of factors F_1, F_2, . . ., F_m. The model in case of m factors can be expressed in the form

$$E(r_i) = r_{\mathrm{f}} + \beta_{i,F_1}[E(r_{F_1}) - r_{\mathrm{f}}] + \beta_{i,F_2}[E(r_{F_2}) - r_{\mathrm{f}}] + \ldots + \beta_{i,F_m}[E(r_{F_m}) - r_{\mathrm{f}}]$$

where:

$E\left(r_{F_j}\right)$ is the expected return on factor j.
$\beta_i^{F_1}$ is the beta of factor j, for asset i.

The parameters can be estimated by regressing the model, as for the CAPM, in the form

$$r_i = \alpha_i + \beta_{i,F_1} r_{F_1 1} + \beta_{i,F_1} r_{F_2} + \ldots + \beta_{i,F_1} r_{F_m} + \varepsilon_i$$

ATP estimation is based on multivariate regression analysis.

3.4 Summary

The relationship between risk and return is of primary importance in financial analysis and corporate finance. Through the interaction of profitability and risk of all assets on the market, it is possible to determine the overall risk of portfolios.

Good understanding of the risk-return paradigm goes through the analysis of variables like portfolio return, volatility, and correlation and the understanding of techniques like maximum likelihood estimation.

Modern Portfolio Theory is the result of the formalization of the risk-return relationship in a framework of efficient markets and rational investors. The risk-return trade-off is formalized through a parabola in the risk-return space, with the upper part being the curve of the efficient portfolios in the market.

Optimal portfolios are those who lie on the upper side of the curve and represent all the portfolios giving the maximum return for a given level of risk (volatility). They therefore represent the optimal choices for each level of risk.

The formalization of the relationship also allows giving a representation of the market price of risk, which can be defined as the unit price for each unit of risk an investor is willing to bear.

The Capital Asset Pricing Model is the result of establishing a relationship between the risk of some asset and the risk of the market. By introducing a risk-free asset in the space of possible investment opportunities, it is possible to derive the line of the optimal portfolio to invest in, tangent to the curve previously derived.

The beta measures the relationship between excess return of an asset and the excess return of the market portfolio, giving the idea of the specific riskiness of each asset and a price to the market risk taken by an investor.

Models that are more sophisticated expand the idea underlying the CAPM by identifying specific factors that affect the relationship between the asset and the market, therefore making the relationship more factor-specific.

Problems

1. Briefly explain the risk-return relationship, as from modern portfolio theory.
2. Why do some investors put a large portion of their portfolios into risky assets, while others invest largely in the risk-free asset?
3. Explain why the following assumptions in particular are required for the Capital Asset Pricing Model to hold:
   (a) A single investor cannot impact the share price.
   (b) Time horizon is the same for all investors.
   (c) All assets are marketable and infinitely divisible.
4. Is it entirely correct, as an analyst recently commented, that market crashes are never advertised in advance?
5. Recall that criticisms of EMH claim that the rate of return on small-cap stocks tend to be higher than for large-cap.
   (a) Is this compatible with CAPM?
   (b) How could such a claim be tested?
6. Assume beta coefficients are estimated for a large number of assets, finding that average rates of return of assets and their beta coefficients are not significantly correlated.
   (a) What does the result imply in terms of validity of the CAPM?
   (b) Would it make sense to completely reject the CAPM on the basis of this evidence?

7. What happens to the riskiness of a portfolio if assets with very low or negative correlations are combined?
8. Comparing diversifiable and non-diversifiable risk, which one do you think is more important to financial managers in corporations?
9. Discuss financial risk from the perspective of the CAPM.
10. What features define the points on the efficient frontier? Do portfolios exist above the frontier?
11. Explain the concept of beta, its calculation, and meaning.
12. What does the SML indicate? What are the main differences with the CML?
13. Derive the intermediate passages to solve Eq. (2.1) in the chapter.
14. Consider the following investments

Investment	Expected return	Standard deviation
A	6.2%	10.5%
B	8.1%	11.2%
C	7.2%	12.3%
D	6.5%	10.8%

Which investment would you prefer between the following pairs?
(a) A and D?
(b) B and C?
(c) C and D?

15. Derive the intermediate passages to solve Eq. (3.4) in the chapter.
16. Assume two assets A and B for which the following estimates have been derived:

$$E(r_{\mathrm{D}}) = 12\%, \sigma_{\mathrm{D}} = 19\%$$

$$E(r_{\mathrm{E}}) = 15\%, \sigma_{\mathrm{E}} = 22\%$$

Consider the portfolios that can be formed investing 50% of total wealth in A and 50% in B. What is the portfolio's standard deviation if the assets are perfectly positively correlated?

17. A portfolio that consists of the following assets:

Stock	Investment (€)	Beta
A	250,000	1.15
B	330,000	−0.65
C	570,000	1.18
D	1,150,000	0.78

The market return is 13% and the risk-free rate is 2%.
(a) Calculate the beta of the portfolio.
(b) Calculate the required rate of return of this portfolio.

18. Assume the market portfolio has an expected return of 12% and a volatility of 28%. There is a risky asset i on which limited information is available. It is known that the expected return of the asset is 9%, the volatility is bounded between 18% and 32%, and the covariance between the asset and the market is bounded between 0.014 and 0.026.
    (a) Find the interval of possible values of $\rho_{i,\,\mathrm{M}} = \mathrm{Corr}(r_i, r_\mathrm{M})$ and of β_i.
    (b) What range of values for the risk-free rate do you find realistic? Give an interval in which the risk-free rate should be.
    (c) Find the interval of risk-free rates implied by the CAPM in the above system.
    (d) How would you price an asset with the same risk characteristics as asset i?
19. Denote by X a stock and by M the market. The correlation coefficient $\rho_{\mathrm{P,\,M}}$ between the stock and the market is 0.80. The volatility of stock X is 25% and the volatility of the market is 12%.
    (a) Calculate the systematic variance, the unsystematic variance, and β_P.
    (b) Show that the beta of a portfolio equals the weighted average of the assets' betas.
20. When estimating CAPM from data, a regression error must be taken into account and the formula modified accordingly. The error is normally distributed. Suppose there are three risky assets with the following betas and values of $\sigma^2_{\varepsilon_j}$ in the CAPM.

j	β_j	$\sigma^2_{\epsilon_j}$
A	1.3	0.008
F	1.1	0.012
E	0.6	0.010

Suppose also that the market excess returns have a variance of 0.016.
(a) What is the beta of an equally weighted portfolio of these three assets?
(b) What is the variance of the excess return on the equally weighted portfolio?
(c) What proportion of the total risk of asset A is due to market risk?

21. Consider the following data for a one-factor economy. All portfolios are well diversified.

Portfolio	$E(R_i)$	Beta
A	10%	1.0
B	4%	0
C	9%	2/3

(a) Is there any arbitrage opportunity in the market?
(b) If so, what would the arbitrage strategy be?

22. The weight measurement of individuals picked in two different areas of the world X and Y is given in the following table:

i	X_i	Y_i
1	73	67
2	58	59
3	56	62
4	69	53
5	73	75
6	75	52
7	63	61
8	67	58
9	72	64
10	69	59
11	66	62

(a) Calculate the sample correlation of the weight in the two areas of the world.

Appendix: Liquidity CAPM

A version of CAPM based on the liquidity effect was developed in 2000 by Jacoby et al. The model can be described by the formula

$$E(r_i - c_i) = r_\mathrm{f} + \beta_i^* [E(r_\mathrm{M} - c_\mathrm{M}) - r_\mathrm{f}]$$

where

$$\beta_i^* = \frac{\mathrm{cov}(r_i - c_i, r_\mathrm{M} - c_\mathrm{M})}{\mathrm{var}(r_\mathrm{M} - c_\mathrm{M})}$$

The advantage of the revised model is to capture the impact of liquidity costs on the systematic risk, quantified by the liquidity-adjusted beta β_i^*.

By manipulating the beta equation, it is possible to isolate and decompose the covariance as shown by

$$\begin{aligned}\beta_i^* &= \frac{\mathrm{cov}(r_i - c_i, r_\mathrm{M} - c_\mathrm{M})}{\mathrm{var}(r_\mathrm{M} - c_\mathrm{M})}\\ &= \frac{\mathrm{cov}(r_i, r_\mathrm{M})}{\mathrm{var}(r_\mathrm{M} - c_\mathrm{M})} + \frac{\mathrm{cov}(c_i, c_\mathrm{M})}{\mathrm{var}(r_\mathrm{M} - c_\mathrm{M})} - \frac{\mathrm{cov}(r_i, c_\mathrm{M})}{\mathrm{var}(r_\mathrm{M} - c_\mathrm{M})} - \frac{\mathrm{cov}(c_i, r_\mathrm{M})}{\mathrm{var}(r_\mathrm{M} - c_\mathrm{M})}\end{aligned}$$

The liquidity-CAPM equation can be then reformulated as

$$E(r_i) = r_{\mathrm{f}} + E(c_i) + \left[E(r_{\mathrm{M}} - c_{\mathrm{M}} - r_{\mathrm{f}})\left(\beta_i + \beta_i^{L1} - \beta_i^{L2} - \beta_i^{L3}\right)\right]$$

where

$$\beta_i = \frac{\operatorname{cov}(r_i, r_{\mathrm{M}})}{\operatorname{var}(r_{\mathrm{M}} - c_{\mathrm{M}})}$$

$$\beta_i^{L1} = \frac{\operatorname{cov}(c_i, c_{\mathrm{M}})}{\operatorname{var}(r_{\mathrm{M}} - c_{\mathrm{M}})}$$

$$\beta_i^{L2} = \frac{\operatorname{cov}(r_i, c_{\mathrm{M}})}{\operatorname{var}(r_{\mathrm{M}} - c_{\mathrm{M}})}$$

$$\beta_i^{L3} = \frac{\operatorname{cov}(c_i, r_{\mathrm{M}})}{\operatorname{var}(r_{\mathrm{M}} - c_{\mathrm{M}})}$$

The four beta coefficients resulting from the decomposition can be commented as follows:

- β_i is the classic beta of the CAPM formula, when liquidity issues are not considered.
- β_i^{L1}, or $\operatorname{cov}(c_i, c_{\mathrm{M}})$, represents the *commonality in liquidity*. Expected return increases with the covariance between the asset's illiquidity and the market illiquidity, because investors demand a premium for holding a security that becomes illiquid when their portfolio (market) becomes illiquid.
- β_i^{L2}, or $\operatorname{cov}(r_i, c_{\mathrm{M}})$, measures the sensitivity of asset return to market liquidity. This beta loads *negatively* with expected returns, because investors are willing to give up return on an asset with a high return in times of market illiquidity.
- β_i^{L3}, or $\operatorname{cov}(c_i, r_{\mathrm{M}})$, measures the sensitivity of asset liquidity to market return. This beta also loads *negatively* with expected returns, because investors are willing to give up return on a security that is liquid in a down market. When the market declines, investors are poor, and the ability to easily sell becomes valuable.

Case Study: Risk and Return

Financo Ltd

The Case

Financo Ltd is a primary financial institution, managing the wealth of few selected high-profile investors, with big amounts to invest. It is therefore crucial for the company to manage the clients' portfolios with the highest possible attention to return and risk.

Led by Mr. Edward, CEO and certified financial analyst, the company is currently managing a total of \$12 billion of assets, with a team of 50 skilled portfolio managers.

Financo Ltd has a very good reputation among investors in the region, and it is able to acquire new customers every semester. Most of their portfolios have survived the financial crisis years with minimum losses while recovering immediately afterward.

Mr. William is a young but wealthy investor who is seeking for new opportunities to make his money grow and achieve good diversification. He is willing to invest an initial amount of \$25 million.

He therefore reaches the Financo headquarters and Mr. Edward welcomes him. After a brief chat with the client and some introduction of the company, the CEO decides to take care of this particular customer himself.

In fact, it is now few weeks that Mr. Edward is analyzing the markets, trying to understand how the company can improve his services to the customers by extending the range of assets to invest in.

In particular, Mr. William has a low-medium risk profile, with preference for income-paying and noncomplex assets. He has been very clear in telling Mr. Edward that he is not willing to consider investments in credit derivatives, as an example.

Even though the range of possibilities allowed by the customer is not extremely wide, Mr. Edward knows he can offer to his customer a wide choice of financial products to be included in the ideal portfolio. Given his recent time spent on learning from the market, there are few types of assets that are interesting.

One possibility is to invest in the equity. Even though the equity market has been subject to drawbacks in the last few months, this newly discovered company seems to be appealing, in that it has been recently made public through an interesting IPO.

Since not much public financial information is available for such a company, the analysis of the investment opportunity must be carried out by comparing the company to the comparable firms in the high-tech sector.

Actually, the average correlation of the industry with the market return is 21.50%. The volatility of the stock is 23.50%, and the volatility of the market is 21.50%. The market return calculated on a 52-week basis is 10.50%, and the risk-free rate in the economy is set at 2.21%, which corresponds to the yield to maturity of government long-term bonds.

Another stock is available from the same market, representing equity of a major well-established corporation. In this case, it is known that the correlation of the stock return with the market return is 19.50%, while the volatility of the stock is 20.50%.

Mr. Edward recommends to focus on the two stocks and combine them in different ways. He then asks Financo to start the analysis of such an investment opportunity, in order to identify the best combination of the available stocks.

Questions

1. If you were to follow the Portfolio Theory like Mr. Edward, what would you do first in order to measure the profitability of the idea?

2. Use CAPM to calculate the expected return on each single asset.
3. What profitability measure can be used in order to assess the relative validity of each single asset and combinations of them?
4. Calculate the return and risk associated with an equally weighted portfolio of the two stocks.
5. Now repeat point 4 for a portfolio made of 80% of the first stock and 20% of the second stock and another portfolio made of 20% of the first stock and 80% of the second stock.
6. Concluding, what is the best combination of the two stocks in terms of relationship between return and risk?

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