2 Valuation Tools

A financial market is an aggregation of possible investors, willing to buy or sell financial assets. Financial valuation is a complex discipline, which involves many tools and methods in order to correctly price the assets and the risks on financial markets.

Corporate valuation is based on the correct valuation and time position of the cash flows deriving from daily corporate activities, as well as the correct choice of tools to evaluate the streams of cash flowing in and flowing out, on the other hand, perhaps inflow and outflow.

The knowledge of the rules of time traveling and mastering the timeline are basic skills that must be acquired in order to proceed to valuation at a later stage, as it is most commonly said in the literature: "a dollar today is not equal to a dollar tomorrow."

Whether it is a revenue, or a cost of some type, or an amortization, or any type of operative or non-operative item, basic tools for valuation allow investors to attach the right value to each of the possible corporate balance sheet items.

On the other hand, beside the corporate side, valuation is also important to acquire the price of openly traded assets on the financial market, which are functional to the survival of the corporation.

After studying this chapter, you will be able to answer the following questions, among others:

- What is the timeline, and how can it be used to position cash flows over time?
- What are the rules of time travel in financial valuation, and how do they affect the valuation of cash flows?
- What types of interest rate can be described, and how can they be compounded at different frequencies?
- What are the main drivers of interest rates?
- What are net present value and internal rate of return? How can the present value of specific cash flows like annuities and perpetuities be calculated?

A. Corelli, *Analytical Corporate Finance*, Springer Texts in Business and Economics, https://doi.org/10.1007/978-3-319-95762-3_2

The first section of the chapter is an introduction to the time value of money, with insights on how the rules of time travel can be applied to valuate cash flow streams. The second section deals with the many types of interest rates that can be calculated, with focus on how the rates change at different compound frequencies and what are the drivers of interest rates. The third section is about present value calculation and the concepts of net present value and internal rate of return, at the basis of the theory of financial valuation.

2.1 The Time Value of Money

Learning Outcomes

- Explain the rules of time travel and learn how to use the timeline.
- Learn how to valuate cash flows streams over time.
- Calculate the present value of annuities and perpetuities.

2.1.1 The Rules of Time Travel

A stream of cash flows can be defined as a series of many cash flows happening at several points in time. The timeline is a linear representation of the timing and can be used to graphically place the cash flows at their correct date.

In order to construct a timeline, consider a loan of amount L received today to be repaid in the following 2 years, with payments of C every year. The timeline in this case will look as in Fig. 2.1.

Example 2.1 Assume investor A borrows 1000 € from investor B today, and ignoring any time value of money, the repayment will happen in two equal yearly installments of 500 € in the next 2 years. The timeline is as in Fig. 2.2.

In the timeline, time 0 represents the present date. Date 1 is 1 year later and represents the end of the first year. At time 1, one will receive the 500 € payment, expressed by the cash flow at time 1.

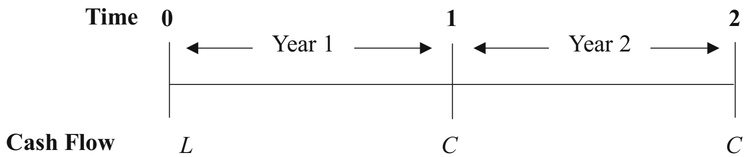

Fig. 2.1 Timeline of a series of cash flows when an amount L is paid at time 0 and an amount C is received at time 1 and 2

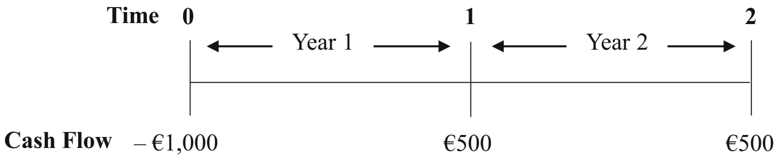

Fig. 2.2 Timeline of a series of cash flows with sample numbers

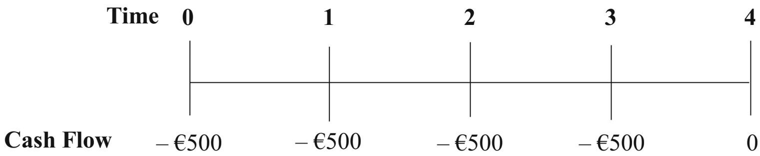

Fig. 2.3 Timeline of a series of a student loan payment

The same will happen at time 2, for the payment covering the period from time 1 to time 2, which generates the cash flow registered at the latter time, adding up another 500 € to the sum of payments over time.

Each point in the timeline can be seen as a date, and the space between two dates is the time between dates. As convention, remember that each point represents the start of the period considered.

To differentiate between the two types of cash flows, we assign a different sign to each. Cash outflows are represented on the timeline with a negative sign, while cash inflows have a positive sign.

For example, if rent is paid each month, one could use a timeline like the one in our first example to represent two rental payments, but the time label would represent months instead of years (Bodie and Merton 2000).

Basic timelines are very simple and just relate to well-structured and easy-to-understand cash flow streams, with a straightforward position on the timeline. However, timelines that are more complicated exist.

However, assigning the right position of cash flows on the timeline can be a difficult task, and it is a crucial step in making right financial decisions. It is then recommended to always draw the timeline when approaching every task.

Example 2.2 A student must pay tuition fees for the next 4 years of his education at some university. In this case, the money must be anticipated at time zero for the first year, and so on, with no payment in the last year. The timeline is as of Fig. 2.3.

In order to make efficient financial decisions, sometimes one must compare cash flows that occur at different points in time. However, the basic principle of comparison is that cash flows at different points in time cannot be compared.

Therefore, in order to make comparisons and sum up or subtract cash flows belonging to a stream, it is necessary to move them backward or forward on the timeline toward a common point. There are three important rules that govern the movement in time of cash flows.

The first rule asserts that only cash flow values at the same point in time can be compared or combined, following the main principle mentioned above. In order to compare or combine cash flows, it is necessary to reduce the analysis to a single time point.

A euro today is not equivalent to a euro in 1 year, and that is why, the first rule is so important. Money today is more valuable, given that it can be invested to earn some interest rate.

It is therefore necessary to discount future cash flows and to account for their equivalent (lower) present value and compound past cash flows, when moving forward in time, in order to get to the higher value the money would get by investing it in time.

The second rule explains how to move a cash flow forward in time and states that in order to do so, the cash flow must be compounded. In order to do so, the interest on the initial amount must be accrued according to the percentage interest rate of reference.

Example 2.3 An investor has 10,000 € to invest for 1 year. In order to calculate the amount he will receive after 1 year if the money is invested at an interest rate of 5%, it is possible to move the cash flow forward in time (compound) by

$$(10,000) \times (1 + 0.05) = 10,500 \ \text{€}$$

In general, if the market interest rate for the year is r, then we multiply by the interest rate factor $(1 + r)$ to move the cash flow from the beginning to the end of the year.

The compounding can be applied repeatedly and iteratively to move the amount forward by more than one time step. There is no limit to how far we can get with the compounding as long as we keep the coherence when accruing for the interest. The value will of course grow more and more as we move further ahead in time.

Example 2.4 Assume the above investor wants to invest the 10,000 € for a 2-year term. Assuming the interest rate is fixed at 5% for both years, calculation shows that the amount due after 2 years is

$$(10,000) \times (1 + 0.05) \times (1 + 0.05) = 11,025 \ \text{€}$$

It is possible to represent the calculation on a timeline as in Fig. 2.4.

The two examples above show that for an interest rate of 5%, the amount available in 1 year will be obtained by multiplying the initial amount by 1.05, while the amount available in 2 years is obtained by multiplying the initial amount by the same factor twice.

The point is that the interest accrued in 1 year also produces interests. Therefore, in the second period, the amount will be augmented of the additional 5% on the initial amount plus the 5% of the interest amount accrued in the first period.

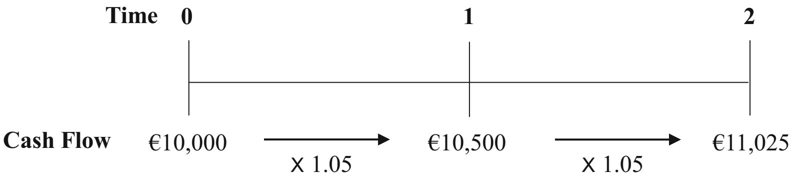

Fig. 2.4 Timeline for an interest-earning investment

The value of a cash flow that is moved forward in time is known as its future value, and depending on whether only the notional amount is compounded or the interest also matured in the previous period, the interest is defined as simple or compounded, respectively.

Consider now a movement forward of more steps than just one or two. Using the same approach, we compound the cash flow a third time, fourth time, and so on. The approach can be extended too many consecutive periods or jumps of many periods.

Example 2.5 Assume the investor wants to invest the 10,000 € for a 3-year term. Assuming the interest rate is fixed at 5% for all years, calculation shows that the amount due after 3 years is

$$(10,000) \times (1 + 0.05) \times (1 + 0.05) \times (1 + 0.05) = 11,576.25 \ €$$

In general, if we have a cash flow now C_0 to compute its future value FV_n for n periods into the future, we must compound it by the n intervening interest rate factors. If the interest rate r is constant, this calculation yields

$$\mathrm{FV}_n = C_0 \times (1 + r) \times (1 + r) \times \ldots \times (1 + r) = C_0(1 + r)^n$$

The third rule of time travel states that in order to move a cash flow backward in time, it must be discounted at the relevant interest rate.

The third rule of time travel shows how to push the cash flows backward in time to get a present value of some amount supposed to be paid or received at some point in the future.

Example 2.6 Suppose one wants to calculate the value today of 10,000 € you anticipate receiving in 1 year. If the current market interest rate is 5%, it is possible to invert the compound relationship shown above to obtain that

$$\frac{10,000}{1.05} = 9523.81$$

To move the cash flow backward in time, one must divide it by the interest rate factor (1 + r) where r is the interest rate—this is the same as multiplying by the discount factor $\frac{1}{(1+r)}$.

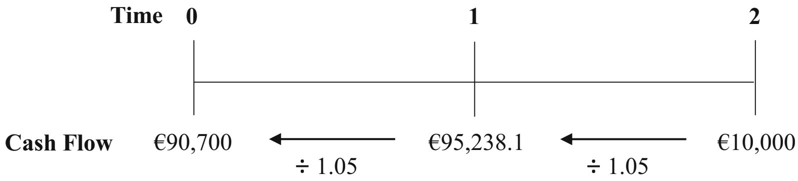

Fig. 2.5 Timeline for the present value of a future amount

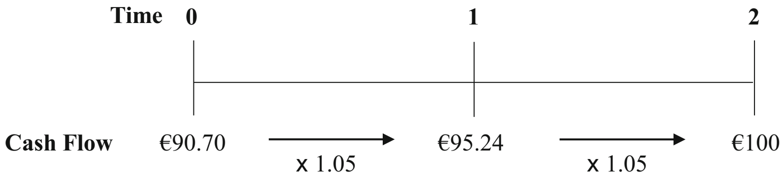

Fig. 2.6 Timeline for the future value of a sum at present value

The process of moving a value or cash flow backward in time—finding the equivalent value today of a future cash flow—is known as discounting.

Example 2.7 An investor knows he will receive an amount of 10,000 € in 2 years from today rather than in 1 year. If the interest rate for both years is 5%, it is possible to draw the timeline as in Fig. 2.5.

An interest rate of 5% makes 10,000 € in 2 years being equivalent to 95,238.1 € in 1 year and 90,700 € today. The amounts represent the same value in different units (different points in time). The arrow points to the left to indicate that the value is being moved backward in time or discounted. Note that the value decreases as we moved the cash flow further back (Kaminsky 2003).

The same example can be again applied to moving forward cash flows and compounding them. One just has to invert the timeline to obtain the one in Fig. 2.6.

An amount of 90,700 € invested today is equivalent to an amount of 95,238.1 € in 1 year and 10,000 € in 2 years. Again, further extension of the timeline is possible, and for a 3-year period, it can be drawn as in Fig. 2.7.

That is, the present value today of a cash flow of 100 € in 3 years discounted at an interest rate of 5% is given by 86,380 €.

In general, to compute the present value today (date 0) of a cash flow C_n that comes n periods from now, we must discount it by the n intervening interest rate factors. If the interest rate r is constant, this yields

$$\mathrm{PV}_0 = C_n \times \frac{1}{(1+r)} \times \frac{1}{(1+r)} \times \ldots \times \frac{1}{(1+r)} = \frac{C_n}{(1+r)^n}$$

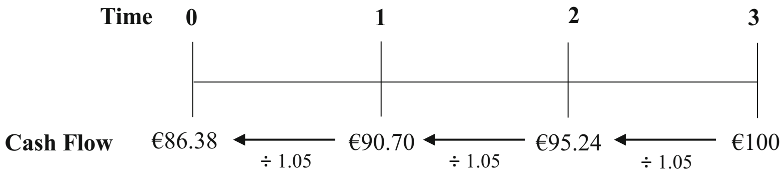

Fig. 2.7 Timeline for the present value of a future amount

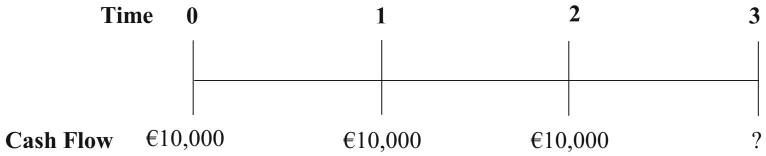

Fig. 2.8 Undefined timeline for a saving plan

2.1.2 Valuation of Cash Flow Streams

Proper application of the rules of time travel allows the investor to compare and combine cash flows occurring at different times. Based on the analysis of the cash flows, it becomes straightforward to give valuations in terms of future value or present value.

Example 2.8 A saving plan consists in saving 10,000 € for three times, including current time and 2 further years. The investor wants to calculate how much money will be available after a further year (therefore 3 years from today), if the interest rate is fixed at 5% for all years.

In this case, the timeline is given in Fig. 2.8.

The rules of time travel again allow to calculate the value of every cash flow depending on the time they belong to. Every single cash flow in this case will have to move for the right amount of time steps, which is one in this case, and be summed up to the correspondent cash flow for the time of arrival of the compounded cash flow.

First, the deposit at date 0 is moved forward to date 1. Because it is then in the same time period as the date 1 deposit, the two amounts can be combined to find out the total in the bank on date 1.

The new aggregate amount can then be moved forward to time 2 in the same way, and compounded first, to be then summed up to the deposit at that time. The last step is to push the overall amount to time 3 by a last compounding. The overall result is shown by the timeline in Fig. 2.9.

There is another possible approach to the problem, consisting in calculating the future value at year 3 of each cash flow separately. Once all the three amounts are in year 3, it is possible to combine them (see Fig. 2.10).

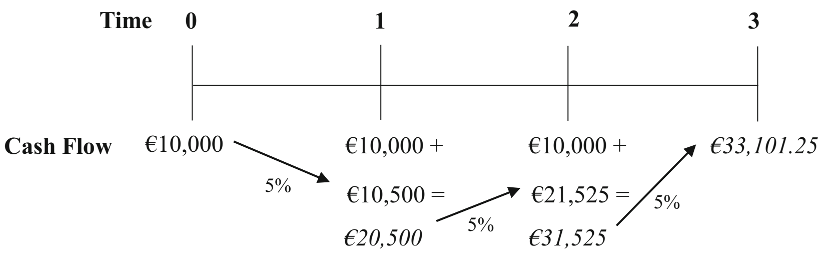

Fig. 2.9 Timeline for a saving plan

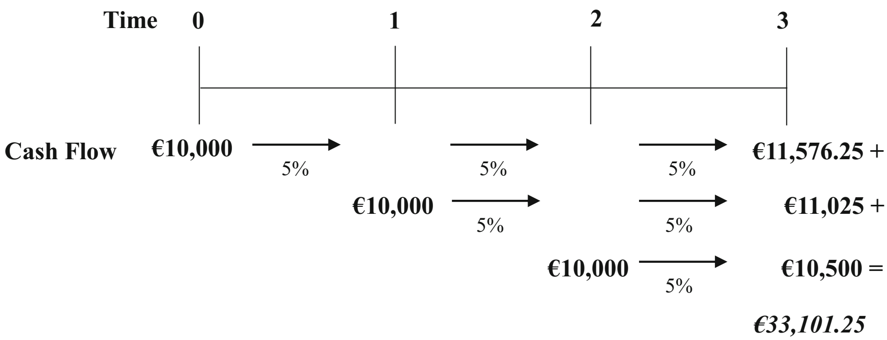

Fig. 2.10 Timeline for a saving plan with separate cash flow calculation

Obviously, both calculations lead to the same result. As long as the computation is consistent with the rules of time travel, the amount obtained is the same in both ways. The order of application of the rules has no role, and the chosen calculation depends on which is more convenient for the problem at hand.

2.1.3 Annuities and Perpetuities

Some types of cash flows have very well-defined properties, so that they get specific names and become easily identifiable. In particular, cash flows that involve equal payments over regular time spans can be defined as annuities and perpetuities.

A regular perpetuity is a stream of equal cash flows that occur at constant time intervals with infinite maturity (never stop). The attribute regular is used to distinguish it from the growing perpetuity, to be discussed later on. The timeline for a perpetuity can be drawn as in Fig. 2.11.

Normally the first cash flow of a perpetuity arrives at the end of the first period (time 1 on the usual timeline). This timing is sometimes referred to as payment in arrears and is a standard convention that is adopted in literature and in this book (Shapiro and Streiff 2001).

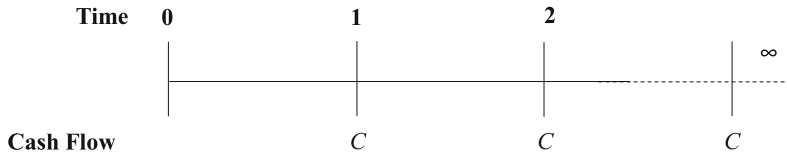

Fig. 2.11 Timeline for a perpetuity

The equation for the present value of a perpetuity is derived from the standard present value calculation, where all cash flows and rates are constant and time steps increase toward infinity, so that it can be simplified as

$$\mathrm{PV}_0 = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots = \sum_{t=1}^{n} \frac{C}{(1+r)^t} \tag{2.1}$$

Notice that $C_t = C$ in the present value equation because the cash flow for a perpetuity is constant. Also, because the first cash flow is in one period $C_0 = 0$.

The trick from the perpetuity present value calculation equation is that even the sum of an infinite number of positive terms becomes finite. This is because the cash flows in the future are discounted for an ever-increasing number of periods, so their contribution to the sum eventually becomes negligible.

Equation (2.1) can therefore be further simplified in order to get a shortcut formula, which simplifies the calculation. The shortcut is derived by calculating the value of a perpetuity by creating our own perpetuity.

By the law of one price, once the calculated and the created perpetuity are taken back at present values, those must be equal given that they come from the same cash flow structure.

Example 2.9 An investor can invest 10,000 € in a bank account paying a 5% interest forever. Every year you can then withdraw the money from the bank, keep the interest part for yourself, and reinvest the initial capital again, at the same conditions.

The situation can be then represented on the timeline in Fig. 2.12.

The strategy creates a perpetuity paying 500 € per year, on the assumption that the bank will remain solvent and the interest rate will not change.

Recall that the law of one price tells us that the same good must have the same price in every market. The bank allows creating the perpetuity for the initial cost of 10,000 €. The present value of the 500 € per year in perpetuity is the self-made perpetuity, which costs 10,000 €.

Generalizing the argument, assume an investment in a bank of an amount P. It is possible to withdraw the interest $C = rP$ every year as per the example above and leave the principal for another round of compounding (Taylor 1986).

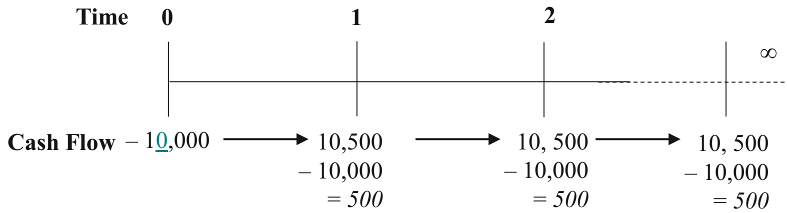

Fig. 2.12 Timeline for a perpetuity with calculations

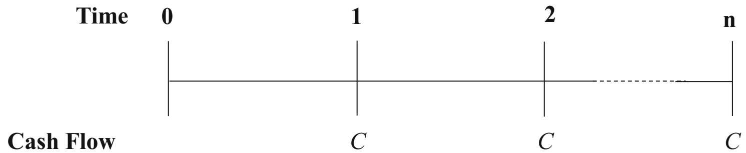

Fig. 2.13 Timeline for an annuity

The condition of present value of a perpetuity with discount rate r and constant cash flows C, starting in one period, can be expressed mathematically as

$$\mathrm{PV_P} = \frac{C}{r}$$

By depositing the amount $\frac{C}{r}$ today, we can withdraw an interest of

$$\frac{C}{r} \times r = C$$

each period in perpetuity.

Similar to perpetuities, a regular annuity is a stream of a finite number of equal cash flows coming at constant time intervals, for a period, which is not infinite. As for the perpetuities, growing annuities are a special case, introduced later on.

The main difference between an annuity and a perpetuity is the absence of an infinite time horizon and infinite cash flows paid. There are n payments that are spanned over n time steps on the timeline. As for the perpetuity, the first cash flow conventionally corresponds to time 1.

The timeline for representing an annuity looks like in Fig. 2.13.

The same convention of the first payment happening at time one, as for the perpetuity, is adopted. The present value of an n-period annuity with payment C and interest rate r is

$$\mathrm{PV_A} = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^n} = \sum_{t=1}^{n} \frac{C}{(1+r)^t}$$

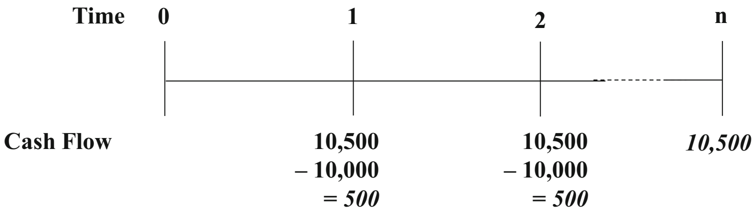

Fig. 2.14 Timeline for an annuity derived from an investment

The derivation of the present value of an annuity is based on the same principles adopted for the perpetuity. Obviously, the change to a finite maturity and finite number of cash flows cuts some of the assumptions made in the previous case.

This is especially true in the second step of the derivation, when deriving the shortcut formula, which in the case of an annuity is more complicated in that it cannot benefit from the assumption of negligible cash flows made for the perpetuity.

The approach to be used to find the shortcut equation is nevertheless the same as before and goes through the creation of an annuity and comparison of present value with an equivalent one.

The process to artificially create an annuity is similar to the one used for building up a perpetuity. Starting with an investment 10,000 € in a bank account paying a 5% interest, after one year there will be 10,500 € in the bank.

By withholding the interest and reinvesting the principal further, one builds up the cash flow system that generates an annuity. The only difference with the case of the perpetuity is that the time is limited to *n* years and does not go to infinite. In that case, the cash flows will look like in Fig. 2.14.

The artificially created annuity generates a positive payoff of 500 € per year for 20 years, and it is possible to get the present value of the annuity by applying the law of one price, as in the case of the perpetuity.

Since it took an initial investment of 10,000 € to create the cash flows on the timeline, the present value of these cash flows is 10,000 €, which is the sum of the present value PV_{CF} of the *n* cash flows of 10,500 € and the present value PV_A of the *n* annuity cash flows of 500 €. It follows that

$$PV_{CF} + PV_A = 10,000 \ €$$

The present value of the annuity can be therefore derived as

$$PV_A = 10,000 \ € - PV_{CF} = 10,000 - \frac{10,000}{(1.05)^{20}} = 10,000 - 3768.89 = 6231.10$$

It follows that the present value of 500 € for 20 years is 6231.10 €. Intuitively, the value of the annuity corresponds to the initial investment in the bank account minus the present value of the principal still in the account after 20 years.

As for the perpetuity, the numerical example can be generalized in order to obtain the general equation for the present value of a regular annuity.

Generalizing the argument, assume an investment in a bank of an amount P. It is possible to withdraw the interest $C = rP$ every year as per the example above and leave the principal for another round of compounding (Watson 1936).

For an initial investment of P, we will receive an n-period annuity of C per period, plus we will get back our original P at the end. P is the total present value of the two sets of cash flows. The present value of receiving C in annuity is therefore given by

$$\mathrm{PV_A} = P - \frac{P}{(1+r)^n} = P\left(1 - \frac{P}{(1+r)^n}\right)$$

By recalling that

$$P = \frac{C}{r}$$

it follows that

$$\mathrm{PV_A} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

As opposed as for the perpetuity, an annuity also has a future timeFV_n, which starts after the last payment made. It is then possible to calculate the future value, through a simple equation for the future value.

In case one is interested in the future, value of an annuity in n periods forward on the timeline, it is necessary to compound the present value for n periods at the relevant interest rate r.

$$\mathrm{FV}_n = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)(1+r)^n = \frac{C}{r}\left[(1+r)^n - 1\right]$$

2.2 Interest Rates

Learning Outcomes

- Define the various types of interest rates.
- Learn how to use various compound frequencies.
- Explain what are the drivers of interest rates.

2.2.1 Types of Interest Rates

There are several types of interest rates that can be defined depending on the way they are calculated. The easiest type of rate involves a simple loan, where a specific amount of funds is borrowed, which must be repaid at maturity to the lender, along with some additional amount, represented by the interest.

Commercial loans are usually structured like that. As an example, consider a loan of 10,000 € at 5% rate, implying that the borrower will have to repay 10,500 € in 1 year.

A fixed-payment loan is fully amortized and consists in borrowing an amount of funds that will be repaid with the interest on top, through several fixed regular payments, until maturity is reached (Brealey et al. 2006).

As an example consider the above loan of 10,000 € but given out as a mortgage. This implies it will be repaid in yearly installments of 1260 € for 25 years, implying an interest rate of 12%.

The yield to maturity (YTM) can be defined as the interest rate equating the present value of payments received from a debt instrument to the price or value of that debt instrument today (market value).

The YTM is sometimes also called the internal rate of return (IRR) of an investment, and it is the most accurate and widely applicable measure of interest rates, given that it allows to understand how much interest an investment is yielding to the investor.

The above definitions lead to several types of calculations that can be performed on the loan, according to how it is considered. For a simple loan, for example, the yield to maturity associated to the loan, given the data above, must satisfy the condition that

$$PV = \frac{FV}{1+r}$$

so that

$PV(1+r) = FV \quad \Rightarrow \quad r = \frac{FV-PV}{PV}$ where:

PV is the present value of the loan.
FV is the future value of the loan at maturity.

Example 2.10 Consider calculating the yield on a payment of 10,500 € in 1 year, whose present value today is 10,000 €. Calculation shows that

$$r = \frac{10,500 - 10,000}{10,000} = 0.05 = 5\%$$

Thus, for simple loans, the yield to maturity equals the simple interest rate.

Example 2.11 In the case of a fixed-payment loan instead, recall the borrower pays 1260 € per year, for 25 years, to repay an initial loan of 10,000 €.

The yield to maturity r must satisfy

$$10,000 = \frac{1260}{1+r} + \frac{1260}{(1+r)^2} + \frac{1260}{(1+r)^3} + \ldots + \frac{1260}{(1+r)^{25}}$$

The calculation for r cannot be performed analytically, but it requires a numerical approach or the use of a financial calculator or spreadsheet. Calculation done via computer or scientific calculator shows that the yield to maturity is equal to 0.1183 or 11.83%. More generally, for any fixed-payment loan, if

$$\text{PV} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^n}$$

where:

C is the amount of fixed annual payments.
n are the years to maturity.

In finance and corporate finance, the distinction between the nominal interest rates and the actual return on a bond is very important, in that it affects the perception of the profitability of the investment.

The rate of return on a bond in fact is a complex calculation that takes into account not only any interest payments made by the bond but also any changes in the market price of the bond itself, which also adds up.

Example 2.12 A coupon bond has face value of 100 € and a coupon of 5%, and it is bought at par. If the interest rate falls over time, the bond price will increase. In case the investor receives one coupon payment during the time span of the investment and is then able to sell the bond after 1 year for 115 €, the total investment return includes both capital gain and coupon. The total return on the investment will therefore be given by

$$\text{PV} = \frac{115 - 100 + 5}{100} = 20\%$$

Sometimes the strategy can lead to a loss, with the process working the other way round. If the interest rates rise over the next year, so that the bond price falls to 80 €, the gain from the coupon is completely offset by the capital loss. In this case, the total return would be

$$\text{PV} = \frac{80 - 100 + 5}{100} = -15\%$$

2.2.2 Compounding Frequencies

When investing money in a deposit account in a bank or other financial institution, some interest is paid on the sum, which is deposited. The same works when borrowing money, in which case some interest is owed to the institution lending the money.

In fact, the interest paid on deposited money is usually much lower than the interest an individual has to pay to a bank when borrowing money. This is how banks make profits from circulating the money in the financial system and that is how the bank pays the salaries for all their employers, the cost of buying and maintaining their premises, and the dividends paid to shareholders (Brigham 1992).

As mentioned in previous sections, a very important distinction when calculating an interest is between simple and compound interest. Simple interest is paid every term (year), but the interest is not reinvested, so it doesn't attract extra interest if you leave it in the bank.

Compound interest on the other hand implies that the interest payments are reinvested in the same account, at the same interest rate, for the following periods, generating new interest themselves, to add up to the interest generated by the capital amount.

When shifting to compound interest, the issues that determine the right amount to be repaid is how often the interest is calculated and added to the account. The frequency of compounding determines the final amount of the loan plus interest. Common practice for interest rate compounding is to have, for example, weekly, monthly, yearly, etc.

The terminal value of a current amount L, invested for n years, at a rate R per annum, is given by

$$L_n = L(1 + R)^n$$

If the interest rate is compounded m times per year, the terminal value of the investment can be written as

$$L_{n,m} = L\left(1 + \frac{R_m}{m}\right)^{nm}$$

As the compounding frequency tends to infinity, the computation shifts in continuous compounding, which gives the highest value for the terminal value and can be written as

$$L_{\mathrm{c}} = L\mathrm{e}^{R_{\mathrm{c}}n}$$

The relation between discrete and continuous compounding is given by

Table 2.1 Example of various compounding frequencies

Compounding frequencies	Interest rate
Annual ($m = 1$)	10.0000%
Semiannual ($m = 2$)	10.2500%
Quarterly ($m = 4$)	10.3813%
Monthly ($m = 12$)	10.4713%
Weekly ($m = 52$)	10.5065%
Daily ($m = 365$)	10.5156%

$$
\begin{aligned}
&Le^{R_c n} = L\left(1 + \frac{R_m}{m}\right)^{nm} \\
&\Rightarrow e^{R_c n} = \left(1 + \frac{R_m}{m}\right)^{nm} \\
&\Rightarrow R_m n = nm\left(e^{\frac{R_c}{m}} - 1\right) \\
&\Rightarrow R_m = m\left(e^{\frac{R_c}{m}} - 1\right)
\end{aligned}
$$

To show the effect of compounding frequency on an interest rate of 10% per annum, it is possible to apply the above formulas, and the effective rates at different compound frequencies are shown in Table 2.1.

2.2.3 The Drivers of Interest Rates

Interest rates, like any other financial variable, take their value from the balance of supply and demand of credit. As the forces of supply and demand change, so do interest rates to bring the credit markets back to equilibrium.

It is therefore interesting to analyze what are the drivers of interest rates, for shorter and longer maturities, as well as for the demand and supply side. First, consider the factors that affect long-term interest rate cycles on the demand side.

The risks involved in holding the asset of reference are at the basis of the driving factors from the demand side. The risk perception of the investor may change overtime and so does the valuation of the asset.

The debt holder holds the risk of insufficient funds available to repay for the obligation, to make it difficult to repay the principal and interest when the maturity of the loan approaches.

In case the debt is riskless, like investing in government securities of very advanced countries, there is no risk related to the possibility of repayment, which is guaranteed. For a treasury bond holder, the major risk to the investment is the erosion of value of the coupon payments and the principal through price increases.

Inflation plays a role in valuation of future cash flows, in that an increase in prices of goods and services in the domestic economy results in the decrease of the value of future cash flows.

Given that coupon and nominal principal of a bond are fixed in case of fixed-rate instrument, this translates into a decrease in the value of bond future cash flows, with consequences on the overall valuation of the bond.

The consequence in terms of risk and value is that investors will demand, in the presence of a high inflation, higher interest rates to compensate for the possible loss of value. Oppositely, in case deflation takes hold, the value of the future cash flows actually increases, making the government bond more valuable.

In general, for a fixed income instrument like a bond with fixed rate, the risk is mostly concentrated in the value of future cash flows, given that current coupon payment is a small percentage of the total value (Copeland et al. 2003).

Therefore, it is the expectation of future inflation that is most relevant as a factor, rather than the current rate. It follows that expectations of investors on inflation are a driver of the demand for debt.

The inflation expectations are themselves driven by several factors, which an investor may consider in order to assess the inflation risk in a fixed income investment. There is a strong connection between the economic cycle and the investor perception of inflation.

The actual dynamics behind that relationship are complex, and factors can interact in different ways to form expectations. However, the basic logic entails a few basic steps related to the state of the economy.

More precisely, when the economy starts growing, the companies increase production to meet increasing demand. The increase in production corresponds to a heavier utilization of current capacity.

Labor becomes a scarce resource and salaries increase in order to attract working force. As a consequence, consumer prices increase as well, leading to a vicious circle of salary/prices increase.

The chain reaction makes salaries increase further, and economic agents create expectations of future rise in prices, which leads to building expectations of high inflation in the near future, as well.

The opposite obviously works in times of economic downturn, when both salaries and prices tend to decrease because of economic stagnation, and the expectations of investors and economic agents are then formed based on deflation expectations.

Government expenditure is another important factor in expectation formation for investors. When structural government spending and cyclical spending rise, the economy tends to heat up.

It has the same effect of overproduction of corporation, with a powerful impact on the private sector as well. As government spending rises, the increasing amount of debt the government needs to take on also crowds out private sector debt, leading to interest rate increases.

The aggregation of the overall demand for long-term funds by economic entities can be referred to as the leverage cycle, where leverage is the term usually employed in describing the amount of debt versus assets of either an individual or the economy as an aggregate.

The most direct effect of leverage cycle on the long-term interest rates is direct, with interest rates increasing as long as the demand for debt increases. However, it is not that simple, and the process is mostly discontinuous, so that increasing amounts of leverage do not necessarily lead to increases in interest rates as a steady line.

An important role as a driving factor for the level of domestic interest rates is played by the regulatory framework and the structure of banking system in an economy. In fact, strict regulations preventing inflows and outflows of capital can have a significant impact on domestic interest rates (Webster's Dictionary 1992).

When foreign capitals are blocked to enter the domestic economy, interest rates will tend to be higher than with an open economy, while the opposite occurs in a framework of impossibility of outflows to abroad, with capital held captive domestically.

Central banks worldwide control the cycles of interest rate movements, by employing policy actions that interact with the economic data. Depending on the country or regional area, the central banks take direct control of different types of rates.

For example, in the United States, the Fed directly controls the overnight rate in the United States, but its influence goes far beyond the short rate. This is because the short end of the yield curve is not independent of longer maturities.

Economic data are an important factor driving changes and expectations on future interest rates. Macroeconomic variables are crucial to follow for any financial market but in particular for interest rates.

Every single piece of economic data can be analyzed in order to get important information about the trades in the economy. Mostly, the evaluation of macroeconomic data consists in looking at gross domestic production (GDP), employment data, manufacturing data, and more.

Another important factor to consider when forming expectations on fixed income securities is that such a class of instruments is exposed to interest rate risk. It stems from fixed cash flows being received or paid, with these fixed cash flows made more or less attractive by changes in the broader market interest rates. This change in attractiveness translates into change in the price of the security.

Example 2.13 A bond pays a 5% coupon, and immediately after its purchase by an investor, the average yield on similar investments raises to 7%. The investor can then try to sell the 5% bond, which represents a poor investment, but he will have to do it at a discount. This is because other investors can buy other bonds with higher interest payment. The price of the bond will therefore go down therefore to the increase in interest rates. An opposite argument explains why the 5% bond would gain in value if interest rates on similar instruments fell to 3%.

The reference rate, to what the bond is sensitive to, I changed the too but do you mean the bond is "also" sensitive to the reference rate. If so, then you can ignore my correction. I was not sure about the rate offered in the market on similar assets. Interest rates in the markets are highly correlated, making it crucial to track global flows and developments to effectively manage the interest rate risk.

To summarize, there is a wide range of factors affecting interest rates, and all of them are constantly in a state of flux, with some of them moving up or down in importance according to the state of the economy at some specific time.

More factors could emerge in time, due to the dynamic nature of interest rate markets. In all cases, a logical approach is needed in order to take into account those drivers whose horizons range from decades to days.

When approaching the interest rate markets, it is important to know the major drivers on long and short time scales. At the same time, also being aware of new factors that may arise from time to time, it is an important step in understanding interest rate markets.

2.3 Present Value Calculation

Learning Outcomes

- Explain the concept of net present value.
- Learn how to calculate the internal rate of return.
- Apply the knowledge of time value to growing cash flows.

2.3.1 Net Present Value

The market of interest rates and credit market are characterized by various instruments that give or require payment at different times. From simple loans to coupon bonds, payments can happen only at maturity as well as at multiple times until maturity.

The goal of this section is to describe a unified methodology to approach measurement of interest rates on the various types of instruments available on the market. Given such a framework, valuation of financial assets, projects, and investment opportunities becomes possible and straightforward.

The concept underlying the valuation of credit and interest rate instruments is the present value that captures the idea that a dollar received in the future is less valuable than a dollar received today.

Example 2.14 A bank issues a loan of 100,000 € with a 5000 € interest payment over 1 year. This simple framework makes the interest payment a measure of the interest rate as from

$$r = \frac{I}{L} = \frac{5000}{100,000} = 0.05 = 5\%$$

where:

r is the simple interest rate.
I is the amount of interest paid.
L is the principal amount.

Example 2.15 It is possible to reverse the previous example and calculate the total amount available to the investor at the end of the investing period, given the interest rate of 5% on the initial investment of 100,000 €. Calculation shows that

$$100,000 + (100,000 \times r) = 100,000(1 + r) = 100,000(1.05) = 105,000 \text{ €}$$

It is then possible to extend the investment for a further year by lending the 105,000 € out again for another year at the same simple interest rate of 5%, and you get

$$105,000 + (105,000 \times r) = 105,000(1.05) = 110,250 \text{ €}$$

at the end of the second year. Equivalently, we can write

$$100,000 \times (1 + r) \times (1 + r) = 100,000(1 + r)^2 = 100,000(1.05)^2 = 110,250 \text{ €}$$

and so on for more years, if one wants.

It is possible to generalize the above examples into an equation for the compounded value V of an amount L at an interest rate i for n years (periods).

$$V = L(1 + r)^n$$

By reworking the formula, it is also possible to express the present value L of a future payment V, by the formula

$$L = \frac{V}{(1 + r)^n}$$

2.3.2 Internal Rate of Return

The internal rate of return (IRR) in capital budgeting is a measure of profitability of investments by comparison. According to the context of use, it is also called rate of return (ROR) or effective interest rate.

The term internal refers to the fact that its calculation does not incorporate environmental factors (e.g., the interest rate or inflation), but it only takes into account the cash flows that are generated internally by the investment.

The IRR of an investment measures the annualized effective compounded return rate that makes the NPV of any investment (sum of discounted positive and negative cash flows) equal to zero.

Put another way, the IRR of an investment is the discount rate that makes the net present value of costs (negative cash flows) of the investment equal to the net present value of the benefits (positive cash flows) of the investment.

The budgeting IRR rule states that a higher a project's internal rate of return makes a project or investment to be more desirable. When comparing projects with same up-front investment, the one with highest IRR should be prioritized.

The rule of thumb for a firm should be to undertake all available projects where the IRR exceeds the cost of capital, given the capital constraints and feasibility. Investment in fact may be limited by availability of funds to the firm by the firm's ability to manage multiple projects.

Among other things, the IRR is an indicator of efficiency, quality, and yield of investment, as oppose as NPV, which refers more to the value and magnitude of an investment.

In a scenario where an investment is considered by a firm that has equity holders, this threshold minimum rate corresponds to the cost of capital of the investment (which in turn is determined by the risk-adjusted cost of capital of alternative investments).

The above condition guarantees that equity holders support the investment, given that for an IRR that exceeds the cost of capital, the corresponding investment is profitable and adds value to the company.

The IRR can be calculated from pairs of time and cash flow involved in the project. The NPV is set to zero, and the model is solved for the discount rate, which in turn is the IRR, as an output.

Given pairs of period and cash flow, where n is a positive integer, the total number of periods, and the NPV, the IRR is given by

$$\mathrm{NPV} = \sum_{t=0}^{N} \frac{C_t}{(1+r)^t} \tag{2.2}$$

The common standard is to calculate IRR in yearly term, but the calculation can be made simpler if one considers the actual timing of the cash flows, which is usually monthly, converting the result to a yearly period thereafter.

Example 2.16 Consider an investment represented by the cash flows in the below table

Year	Cash flow
0	−4000
1	1200
2	1410
3	1875
4	1050

The IRR is given by setting the NPV equal to zero, as

$$0 = -4000 + \frac{1200}{(1+\mathrm{IRR})^1} + \frac{1410}{(1+\mathrm{IRR})^2} + \frac{1875}{(1+\mathrm{IRR})^3} + \frac{1050}{(1+\mathrm{IRR})^4}$$

corresponding to

$$\text{IRR} = 14.3\%$$

Instead of the present time, it is also possible to perform the calculation based on any other time, like an end of interval of an annuity, for example. In this case, the value obtained is zero if the NPV is zero.

If cash flows are random, the expected values go in the Eq. (2.2) in place of the actual nonrandom ones. The value of the IRR cannot be found analytically, and numerical methods or graphical methods must be used.

There are numerical methods to solve the equation for the IRR. The secant method, for example, gives the IRR of a period $t + 1$ and can be written as

$$\text{IRR}_{t+1} = \text{IRR}_t - \text{NPV}_t \left(\frac{\text{IRR}_t - \text{IRR}_{t-1}}{\text{NPV}_t - \text{NPV}_{t-1}} \right)$$

where:

IRR_t is considered the nth approximation of the IRR.

The IRR in the above way can be obtained with an arbitrary degree of accuracy.

An interesting case is where the stream of payments consists of a single outflow, followed by multiple inflows occurring at equal periods. It can be written as

$$C_0 < 0, \quad C_0 \geq 0 \ \text{ for } \ n \geq 1$$

In this case, the NPV of the payment stream is a function of interest rate. There is always a single unique solution for IRR given the convexity of the function. A more accurate interpolation equation is given by

$$\begin{aligned} \text{IRR}_{t+1} = \text{IRR}_t - \text{NPV}_t \left(\frac{\text{IRR}_t - \text{IRR}_{t-1}}{\text{NPV}_t - \text{NPV}_{t-1}} \right) \\ \times \left[1 - 1.4 \times \left(\frac{\text{NPV}_{t-1}}{\text{NPV}_t - 3\text{NPV}_t + 2C_0} \right) \right] \end{aligned}$$

This has been shown to be ten times more accurate than the standard secant equation or a wide range of interest rates and initial guesses.

2.3.3 Growing Cash Flows

Regular cash flows are those which are not growing over time but stay constant. As described above, such cash flows are the basis for the calculation of regular annuities and perpetuities.

There is also the case where the cash flows are expected to grow at some (constant) rate in each period. In this case, it is again possible to derive formulas in order to calculate the present value of the associated perpetuity and annuity.

A growing perpetuity is then a stream of cash flows occurring at regular intervals over an infinite period, with an infinite number of payments. Payments grow at a constant rate over time.

The usual convention of the first payment occurring at time 1 is adopted, so that from the first payment to the last (occurring at time t), there are only $t-1$ periods of growth between every t periods, until infinite.

Following the same approach and substituting the so obtained timeline into the general equation for present value of a stream of cash flows yields a summation that simplifies the calculation

$$\mathrm{PV} = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots = \sum_{t=1}^{\infty} \frac{C(1+g)^{t-1}}{(1+r)^t}$$

It is important to note that when $g > r$ cash flows grow faster than they are discounted, the summation diverges, getting larger over time. The sum is then infinite, meaning that it is impossible to reproduce the cash flows associated to the growing perpetuity.

In fact, recall that the approach used to build the formulas for the regular perpetuity is based on replication of cash flows, at a cost equal to the present value of the perpetuity itself.

An infinite present value means that no matter how much money you start with, it is impossible to reproduce those cash flows on your own. This type of perpetuities cannot exist in practice because no one would be willing to offer one at any finite price.

Nobody would ever offer a payment of an amount that grows faster than the interest rate on it. Therefore, the only growing perpetuities that make sense are those with a growing rate lower than the interest rate, so that each successive term in the sum is less than the previous term and the overall sum is finite. It makes sense therefore to assume that $g < r$ for a growing perpetuity.

The equation for a growing perpetuity can be derived by following the same logic applied to a regular perpetuity. Therefore, the amount needed to be deposited today in order to create the perpetuity must be computed.

Recall that for the regular perpetuity, the trick was to create a constant payment forever by withdrawing the interest earned each year and reinvesting the principal. In the case of growing perpetuity, the amount that can be withdrawn each year must be increased, and the principal reinvested each year must grow.

The method to implement such a strategy is to withdraw less than the full amount of interest accrued between each two periods and use the remaining interest to increase the principal amount.

Example 2.17 Consider creating a perpetuity growing at a rate of 3%, by investing 1000 € into a bank account, at an interest rate of 5%. At the end of 1 year, the deposit will be worth 1050 €. By withdrawing only 20 €, it is possible to reinvest 1030 €. This amount will then grow in the following year to

$$1030 \times 1.05 = 1081.5 \text{ €}$$

It is now possible to withdraw an amount

$$20 \times 1.03 = 20.6 \text{ €}$$

leaving a principal of

$$1081.5 - 20.6 = 1060.9 \text{ €}$$

By noting that

$$1030(1.03) = 1060.9 \text{ €}$$

It is clear that both the principal and the withdrawals grow at the same rate. The repeated withdrawals after initial investment is equivalent to creating a perpetuity starting at 20 € and growing at a rate of 3% per year. This growing perpetuity must have a present value equal to the cost of 1000 €.

As in previous cases, the argument can be generalized by recalling that in case of a regular perpetuity, an amount C was deposited in an account and the corresponding accrued interest was withdrawn each year.

In order to increase the withdrawn amount by the growth rate g to implement a growing perpetuity, the principal deposited must grow at the same rate. It means that instead of reinvesting C in the second year, we should reinvest

$$C(1+g) = C + gC$$

In order to increase our principal by gC, we can only withdraw

$$C_{\text{w}} = rC - gC = C(r-g) \tag{2.3}$$

It is possible to withdraw an amount $C(r - g)$ after one period, keeping the account balance and cash flow growing at a rate of g forever. Solving Eq. (2.3) gives the present value PV of a growing perpetuity as

$$\text{PV}_{\text{GP}} = \frac{C}{r-g}$$

The intuition behind the equation of a growing perpetuity starts from the regular one. In a regular perpetuity, enough money had to be put in an account in order to ensure that the interest earned matched the cash flows of the regular perpetuity.

In a growing perpetuity, the amount to be put in the account must be larger in order to finance the growth of cash flows. The additional amount is calculated by difference between the interest rate and growth.

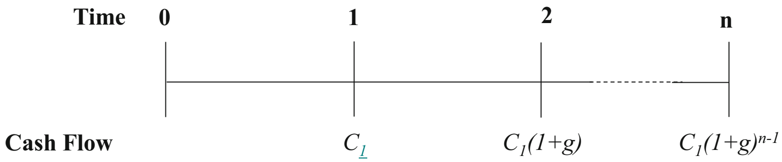

Fig. 2.15 Timeline for a growing annuity

A growing annuity can be represented on a timeline, with initial cash flow, which grows at a rate g in every period considered, until the maturity n of payments, which is finite, as the number n of payments, as shown in Fig. 2.15.

When representing the growing annuity on a timeline, this looks equivalent to a growing perpetuity where the payments from time n+1 onward are removed. The removed cash flows can then be seen themselves as a growing perpetuity starting at n +one.

It follows that in order to determine the present value of the growing annuity, the present value of a growing perpetuity starting at time n+1 can be calculated and subtracted to the value of a corresponding perpetuity starting at time 1.

The method translates into the following formulas:

$$\mathrm{PV_{GA}} = \frac{C}{r-g} - \frac{C_{n+1}}{r-g} \times \frac{1}{(1+r)^n}$$

where the first addendum identifies the present value of a growing perpetuity and the second addendum identifies the present value of a growing perpetuity starting at time n + 1. By substituting in

$$C \times (1+g)^n$$

for

$$C_{n+1}$$

we get

$$\mathrm{PV_{GA}} = \frac{C}{r-g} - \frac{C \times (1+g)^n}{r-g} \times \frac{1}{(1+r)^n}$$

By collecting the terms and simplifying the calculation, the present value of a growing annuity, with interest rate r and cash flows growing at a rate g, on a time interval of n periods, is given by

$$\mathrm{PV_{GA}} = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right] \tag{2.4}$$

Recall that an annuity involves a finite number of payments, so that Eq. (2.4) works for $g > r$.

The equation for the present value of a growing annuity is a general solution, and all other formulas for calculation in this section can be directly derived from it. Consider the case of a growing perpetuity, which is a special case for $n = \infty$. If $g < r$, then

$$\frac{1+g}{1+r} < 1$$

and

$$\lim_{n \to \infty} \left(\frac{1+g}{1+r}\right)^n = 0$$

It follows that the equation for a growing perpetuity is

$$\mathrm{PV_{GP}} = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right] = \frac{C}{r-g}$$

which is exactly the equation for a growing perpetuity. By letting the growth rate equal to zero, it is finally possible to derive the present values of a regular annuity and a regular perpetuity.

Starting then from a regular annuity, it is straightforward to derive a simple equation for the future value of a growing annuity. The value n years in the future is given by moving the present value n periods forward on the timeline, thus compounding the present value for n periods at interest rate r as defined as

$$\mathrm{PV_{GA}} = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right]$$

2.4 Summary

The valuation of cash flows implies the ability to move them backward and forward in time in order to make them compatible to each other. This is why the rules of time travel allow to discount and compound cash flows making them homogeneous in time.

Cash flows streams can then be valued according to the amount they show once the time travel is completed and appropriate compounding or discounting has been applied. The overall value of the stream is the sum of the transformed value of each cash flow.

Interest rates can be measured in different forms, and various types of rate are classified, depending on the definition and the framework where the rate is applied. The bottom line is that they all represent the percentage of gain-loss on an investment.

It is possible to compound and discount rates at different frequencies. Depending on the frequency of recalculation of the interest on the interest, the overall amount of interest and the effective rate applied vary according to the frequency.

The dynamics of interest rates is influenced by several drivers that affect the rates and determine their value. Macroeconomic variables are the most important factors that influence the level and shape of the interest rates.

The net present value is an important measure of the profitability of an investment, given by the sum of all its cash flows, appropriately moved in time. The internal rate of return is the rate that makes the NPV of a cash flow stream equal to zero.

Annuities and perpetuities are specific types of cash flow streams with well-defined features that make them unique. An annuity is a stream of non-perpetual cash flows, while the perpetuity has an infinite time horizon.

Both types of cash flows can also be calculated in the form of growing cash flows. By adding a growth term, the formulas for the present value of both annuity and perpetuity change considerably, allowing measuring the present value of growing cash flows.

Problems

1. Arianna deposits money today in an account that pays 6.5% annual interest. How long will it take to double her money?
2. Sonia has 42,180.53 € in a brokerage account and plans to deposit an additional 5000 € at the end of every future year until the account totals 250,000 €. The account is expected to earn 12% annually on the account. How many years will it take to reach the goal?
3. What is the future value of a 7%, 5-year ordinary annuity that pays 300 € each year? If this were an annuity due, what would its future value be?
4. An investment will pay 100 € at the end of each of the next 3 years, 200 € at the end of year 4, 300 € at the end of year 5, and 500 € at the end of year 6. If other investments of equal risk earn 8% annually, what is its present value? Its future value?
5. Cristina is paying 20,000 € of tuition fees to the university every year. Upon successful completion of a certain amount of credits with high grade, she can get a reimbursement of 20% of the fees every 2 years. Assuming Cristina is always having a good performance, how does the timeline look like?
6. A mortgage of 100,000 € is issued to a company, with yearly repayment of 10,000 € including 7000 € capital and 3000 € interest, for 13 years. The company decides to pay all the mortgage in one solution after 4 years of payments. What is the timeline in this case?
7. Consider an investor depositing an amount of 20,000 € in a bank, for 5 years, with an interest to be earned of 3% every year. How much will the investor have at the end of the period?

8. Now assume the interest rate in exercise 7 starts at 3% and grows every year by 0.2% on top of the previous year. Repeat the calculation.
9. Consider an investor who will receive an amount of 50,000 € in 5 years from now. If the interest rate earned on the 5-year time is going to be 2.5% every year, how much is the present value of the sum?
10. Now assume the interest rate in exercise 9 starts at 2.5% and grows every year by 0.15% on top of the previous year. Repeat the calculation.
11. Mahsa can invest 15,000 € in a bank account paying a 4% interest forever. Every year she withdraws an amount equal to the 20% of the deposited money from the bank and reinvests the remaining capital again, at the same conditions. How can the situation be represented on the timeline?
12. Calculate the yield on a payment of 10,000 € in 3 years from now, whose present value today is 9500 €.
13. A fixed-payment loan is paid in yearly instalments of 2450 €, for 15 years, to repay an initial loan of 27,500 €. What is the yield to maturity of the loan?
14. A coupon bond has face value of 100 € and a coupon of 5%, and it is bought at par. In case the investor receives four coupon payments during the time span of the investment and is then able to sell the bond after 2 years for 118 €, what is the total investment return? Can this be solved analytically?
15. Vittorio, manager of Furjan LTD, wants to sell on credit, giving customers 3 months payment deadline. However, the customer Clelia LTD will have to borrow from her bank to carry the accounts payable. The bank will charge a nominal 15% but with monthly compounding. The customer wants to quote a nominal rate to customers that will exactly cover financing costs. What nominal annual rate should Clelia LTD credit customers?
16. Fred wishes to accumulate 1,000,000 € by his retirement date, which is 25 years from now. He will make 25 deposits in your bank, with the first occurring today. The bank pays 8% interest, compounded annually. He expects to get an annual raise of 3%, so he will let the amount he deposit each year also grow by 3%. How much must your first deposit be to meet your goal?
17. Your parents make you the following offer. They will give you 10,000 € at the end of every year for the next 5 years if you agree to pay them back 10,000 € at the end of every year for the following 10 years (i.e., from year 6 on). Should you accept this offer if your discount rate is 12% a year?
18. A rich entrepreneur would like to set up a foundation that will pay a scholarship to one deserving student every year. The first such scholarship will pay 8000 € and is to be awarded in 5 years from now. Then others will follow in perpetuity every year after, and the amount will be indexed at 1.2% per year. How much money should the entrepreneur put in the foundation's account today if that account earns 10% per year?
19. Elisa runs a construction firm. She has just won a contract to build a government office building. Building it will take 1 year and requires an investment of 12,000,000 € today and 6,000,000 € in 1 year. The government will pay her 25,000,000 € upon the building's completion. Suppose the cash flows and their times of payment are certain and the risk-free rate is 5%.

(a) What is the NPV of this opportunity?
(b) How can Elisa turn this NPV into cash today?

20. A firm is considering a project that will require an up-front investment of 10,000,000 € today and will produce 11,000,000 € in cash flow for the firm in 1 year without risk. One option is to pay the 9,000,000 € all cash, while another option is to issue a security that will pay investors 5,600,000 € in 1 year. Risk-free rate is 4%.
    (a) Is the project a good investment paying all cash?
    (b) Is the project a good investment issuing the new security?
    (c) What can be concluded?
21. Suppose Guzeliya wishes to retire 40 years from today. She determines that she needs 50,000 € per year once she retires, with the first retirement funds withdrawn 1 year from the day of retirement. She estimates to earn 6% per year on the retirement funds and that she will need funds up to including her 22nd birthday after retirement.
    (a) How much must she deposit in an account today so that there are enough funds for retirement?
    (b) How much must she deposit each year in an account, starting 1 year from today, so that there are enough funds for retirement?

Case Study: Time Value of Money

Lottery in the United States

The Case

In the United States, lotteries are subject to the laws of and operated independently by each jurisdiction. Even though there is no national lottery, organization consortiums of state lotteries organize games spanning nationwide and carrying larger jackpots. That is the case of Powerball, which is in turn one of the major national lotteries.

For those who are not familiar with gambles, the Powerball is one the most popular and richest in prizes lotteries in the country. As any other lottery, it can be played by picking random numbers and a small monetary investment.

The winning of the jackpot is indeed a very unlikely task to accomplish given that the chances of scoring the winning combination are approximately 1 in 300 million, equivalent to looking for a microscopic needle in a gigantic haystack.

Clelia is a student of English Literature who recently won a jackpot of $250,000,000 at Powerball, an event that is now changing her life and turning her life upside down in some sense.

Jackpot winners have the option of receiving their prize in cash (lump sum) or as an annuity paid in 30 yearly instalments, starting from year 1 (no payments at current time). Each annuity payment is 2% higher than in the previous year to adjust for inflation. The lump sum means taking the entire cash value at once, but there is a

Table 2.2 Schedule of pretax payments, over 30 years, of $250,000,000 growing annuity

Year	Annuity payment
1	4,059,095
2	4,140,277
3	4,223,082
4	4,307,544
5	4,393,695
6	4,481,568
7	4,571,200
8	4,662,624
9	4,755,876
10	4,850,994
11	4,948,014
12	5,046,974
13	5,147,913
14	5,250,872
15	5,355,889
16	5,463,007
17	5,572,267
18	5,683,712
19	5,797,387
20	5,913,334
21	6,031,601
22	6,152,233
23	6,275,278
24	6,400,783
25	6,528,799
26	6,659,375
27	6,792,563
28	6,928,414
29	7,066,982
30	7,208,322

Payments increase by 2% per year, and interest rate is constant at 3% per year

catch: the lump sum is less than the value of the total jackpot, namely, 56% of the announced nominal amount.

Assume that federal taxes on lottery winnings are 25% of the nominal amount, followed by a further 14.6% tax deduction during season (topping up to the maximum US federal tax rate of 39.6%) which applies to both a lump sum and a yearly annuity payment.

Clelia is currently challenged with making the decision of whether to cash the lump-sum payment or to accept the annuity payments every year for the next 30 years, without loss in the nominal amount, besides taxes.

She is not much expert on financial matters, so she decides to ask her friend Allison (a financial analyst in an important consulting firm) to help her with the decision. Winners of the Powerball have 60 days to decide which form of payment they prefer.

Allison accepts to help her friend and decides to start a thorough analysis of both the payment options, given some reasonable assumptions about the evolution of the financial variables involved.

In particular, she wants to compare the future value of the sum obtained by accepting the lump-sum payment to that obtained by accepting the annual annuity payments to be reinvested at the reference interest rate (yield).

Assume that in case the annuity is chosen, the yearly payments can be reinvested at the available interest rate in the economy, which in turn depends on the state of the economy and is supposed to be constant at 3% for the next 30 years. Also, assume returns to be taxed at the same 39.6% as for the lump-sum payment.

Questions

1. If you were Allison, which type of payment would you recommend to Clelia, in case the before-tax return is 3%? Use Table 2.2 to start your analysis.
2. How does your answer to question 1 change in case the reinvestment rate is 6%?
3. Build up a scenario analysis and a graph to describe the relationship between the future value of the lump-sum payment and the future value of the annuity, for possible interest rates ranging from 1% to 5%.
4. Under which conditions is it, in general, convenient to accept the lump sum rather than the annuity payments?
5. Beside the mere numbers and rates, which are other factor(s) to consider when choosing a lump-sum payment instead of an annuity over the years?
6. What considerations would you put forward to the fact that most categories of winners do prefer a lower lump-sum payment?

References

Bodie Z, Merton R (2000) Finance. Prentice Hall, Upper Saddle River, NJ
Brealey R, Myers S, Allen F (2006) Principles of corporate finance. McGraw-Hill Irwin, New York
Brigham EF (1992) Fundamentals of financial management, 6th edn. Harcourt Brace Jovanovich, Orlando, FL
Copeland T, Weston F, Shastri K (2003) Financial theory and corporate policy, 4th edn. Addison-Wesley, Reading, MA
Kaminsky K (2003) Financial literacy: introduction to the mathematics of interest, annuities, and insurance. University Press of America, Lankham, MD
Shapiro D, Streiff T (2001) Annuities. Dearborn Financial Publishing, Chicago, IL
Taylor RW (1986) Future value of a growing annuity: a note. J Financ Educ Fall:17–21
Watson JD (1936) Annuities illustrated by diagrams. Account Rev 11(2):192–195
Webster's Dictionary (1992) G&C Merriam Company. Springfield, MA