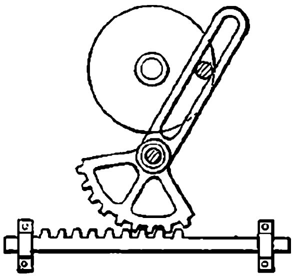
Chapter

7 Robot Arm Kinematics

Take to kinematics. It will repay you.
It is more fecund than geometry; it adds a fourth dimension to space.
Chebyshev



Kinematics is the branch of mechanics that studies the motion of a body, or a system of bodies, without consideration given to its mass or the forces acting on it. A serial-link manipulator comprises a chain of mechanical links and joints. Each joint can move its outward neighbouring link with respect to its inward neighbour. One end of the chain, the base, is generally fixed and the other end is free to move in space and holds the tool or end-effector.

From the Greek word for motion.

Figure 7.1 shows two classical robots that are the precursor to all arm-type robots today. Each robot has six joints and clearly the pose of the end-effector will be a complex function of the state of each joint. Section 7.1 describes a notation for describing the link and joint structure of a robot and Sect. 7.2 discusses how to compute the pose of the end-effector. Section 7.3 discusses the inverse problem, how to compute the position of each joint given the end-effector pose. Section 7.4 describes methods for generating smooth paths for the end-effector. The remainder of the chapter covers advanced topics and two complex applications: writing on a plane surface and a four-legged walking robot.

7.1 Describing a Robot Arm

A serial-link manipulator comprises a set of bodies, called links, in a chain and connected by joints. Each joint has one degree of freedom, either translational (a sliding or prismatic joint) or rotational (a revolute joint). Motion of the joint changes the relative angle or position of its neighbouring links. The joints of most common robot are revolute but the Stanford arm shown in Fig. 7.1b has one prismatic joint.

The joint structure of a robot can be described by a string such as "RRRRRR" for the Puma and "RRPRRR" for the Stanford arm, where the j^{th} character represents the type of joint j, either Revolute or Prismatic. A systematic way of describing the geometry of a serial chain of links and joints was proposed by Denavit and Hartenberg in 1955 and is known today as Denavit-Hartenberg notation.

For a manipulator with N joints numbered from 1 to N, there are $N+1$ links, numbered from 0 to N. Link 0 is the base of the manipulator and link N carries the end-

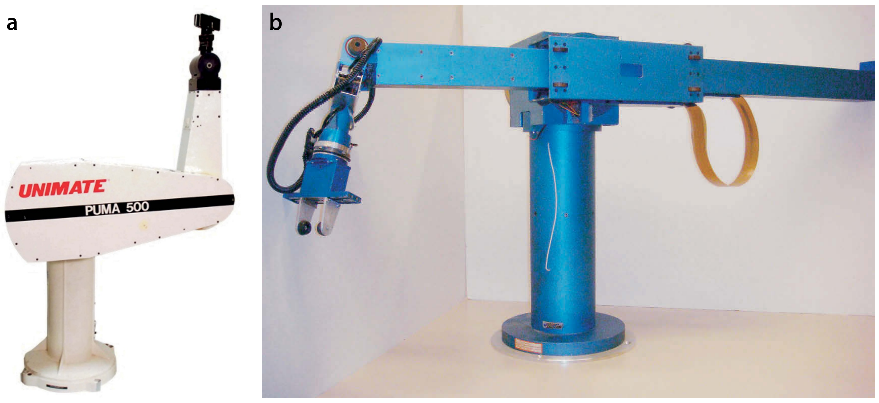


Fig. 7.1. **a** The Puma 560 robot was the first modern industrial robot (courtesy Oussama Khatib). **b** The Stanford arm was an early research arm and is unusual in that it has a prismatic joint (Stanford University AI Lab 1972; courtesy Oussama Khatib). Both arms were designed by robotics pioneer Victor Scheinman and both robots can be seen in the Smithsonian Museum of American History, Washington DC

Jacques Denavit and **Richard Hartenberg** introduced many of the key concepts of kinematics for serial-link manipulators in a 1955 paper (Denavit and Hartenberg 1955) and their later classic text *Kinematic Synthesis of Linkages* (Hartenberg and Denavit 1964).

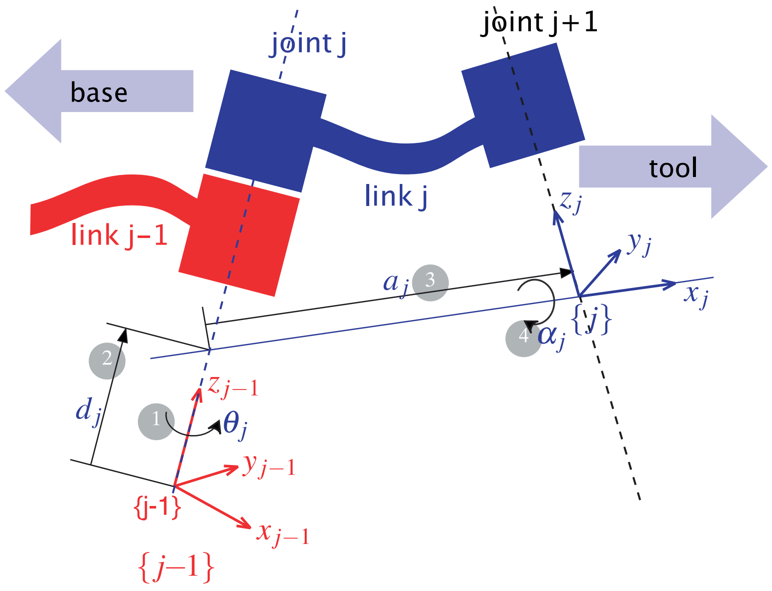


Fig. 7.2. Definition of standard Denavit and Hartenberg link parameters. The colors red and blue denote all things associated with links $j-1$ and j respectively. The numbers in circles represent the order in which the elementary transforms are applied

effector or tool. Joint j connects link $j-1$ to link j and therefore joint j moves link j. A link is considered a rigid body that defines the spatial relationship between two neighbouring joint axes. A link can be specified by two parameters, its length a_j and its twist α_j. Joints are also described by two parameters. The link offset d_j is the distance from one link coordinate frame to the next along the axis of the joint. The joint angle θ_j is the rotation of one link with respect to the next about the joint axis.

Figure 7.2 illustrates this notation. The coordinate frame $\{j\}$ is attached to the far (distal) end of link j. The axis of joint j is aligned with the z-axis. These link and joint parameters are known as Denavit-Hartenberg parameters and are summarized in Table 7.1.

Following this convention the first joint, joint 1, connects link 0 to link 1. Link 0 is the base of the robot. Commonly for the first link $d_1 = \alpha_1 = 0$ but we could set $d_1 > 0$ to represent the height of the shoulder joint above the base. In a manufacturing system the base is usually fixed to the environment but it could be mounted on a mobile base such as a space shuttle, an underwater robot or a truck.

The final joint, joint N connects link $N-1$ to link N. Link N is the tool of the robot and the parameters d_N and a_N specify the length of the tool and its x-axis offset respectively. The tool is generally considered to be pointed along the z-axis as shown in Fig. 2.14.

The transformation from link coordinate frame $\{j-1\}$ to frame $\{j\}$ is defined in terms of elementary rotations▸ and translations as

The 3×3 orthonormal matrix is augmented with a zero translational component to form a 4×4 homogenous transformation.

$$^{j-1}\boldsymbol{A}_j(\theta_j, d_j, a_j, \alpha_j) = \boldsymbol{T}_{Rz}(\theta_j)\boldsymbol{T}_z(d_j)\boldsymbol{T}_x(a_j)\boldsymbol{T}_{Rx}(\alpha_j) \tag{7.1}$$

which can be expanded as

$$^{j-1}\boldsymbol{A}_j = \begin{pmatrix} \cos\theta_j & -\sin\theta_j\cos\alpha_j & \sin\theta_j\sin\alpha_j & a_j\cos\theta_j \\ \sin\theta_j & \cos\theta_j\cos\alpha_j & -\cos\theta_j\sin\alpha_j & a_j\sin\theta_i \\ 0 & \sin\alpha_j & \cos\alpha_j & d_j \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.2}$$

Table 7.1. Denavit-Hartenberg parameters: their physical meaning, symbol and formal definition

Joint angle	θ_j	the angle between the x_{j-1} and x_j axes about the z_{j-1} axis	revolute joint variable
Link offset	d_j	the distance from the origin of frame $j-1$ to the x_j axis along the z_{j-1} axis	prismatic joint variable
Link length	a_j	the distance between the z_{j-1} and z_j axes along the x_j axis; for intersecting axes is parallel to $\hat{z}_{j-1} \times \hat{z}_j$	constant
Link twist	α_j	the angle from the z_{j-1} axis to the z_j axis about the x_j axis	constant
Joint type	σ_j	$\sigma = 0$ for a revolute joint, $\sigma = 1$ for a prismatic joint	constant

Jacques Denavit (1930–) was born in Paris where he studied for his Bachelor degree before pursuing his masters and doctoral degrees in mechanical engineering at Northwestern University, Illinois. In 1958 he joined the Department of Mechanical Engineering and Astronautical Science at Northwestern where the collaboration with Hartenberg was formed. In addition to his interest in dynamics and kinematics Denavit was also interested in plasma physics and kinetics. After the publication of the book he moved to Lawrence Livermore National Lab, Livermore, California, where he undertook research on computer analysis of plasma physics problems. He retired in 1982.

The parameters α_j and a_j are always constant. For a revolute joint θ_j is the joint variable and d_j is constant, while for a prismatic joint d_j is variable, θ_j is constant and $\alpha_j = 0$. In many of the formulations that follow we use generalized joint coordinates

$$q_j = \begin{cases} \theta_j & \sigma_j = 0, \text{ for a revolute joint} \\ d_j & \sigma_j = 1, \text{ for a prismatic joint} \end{cases}$$

For an N-axis robot the generalized joint coordinates $\boldsymbol{q} \in \mathcal{C}$ where $\mathcal{C} \subset \mathbb{R}^N$ is called the joint space or configuration space.◀ For the common case of an all-revolute robot $\mathcal{C} \subset \mathbb{S}^N$ the joint coordinates are referred to as joint angles. The joint coordinates are also referred to as the *pose of the manipulator* which is different to the *pose of the end-effector* which is a Cartesian pose $\xi \in SE(3)$. The term *configuration* is shorthand for *kinematic configuration* which will be discussed in Sect. 7.3.1.

This is the same concept as was introduced for mobile robots on page 65.

Within the Toolbox we represent a robot link with a `Link` object which is created by

```
>> L = Link([0, 0.1, 0.2, pi/2, 0])
L =
 theta=q, d=0.1, a=0.2, alpha=1.571 (R,stdDH)
```

where the elements of the input vector are given in the order θ_j, d_j, a_j, α_j. The optional fifth element σ_j indicates whether the joint is revolute ($\sigma_j = 0$) or prismatic ($\sigma_j = 1$). If not specified a revolute joint is assumed.

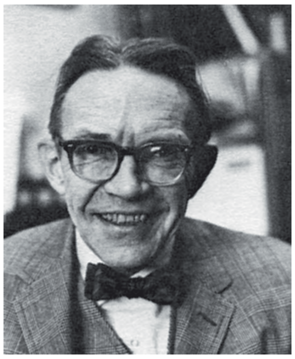
The displayed values of the `Link` object shows its kinematic parameters as well as other status such the fact that it is a revolute joint (the tag `R`) and that the standard Denavit-Hartenberg parameter convention is used (the tag `stdDH`).◀

A slightly different notation, *modified Denavit-Hartenberg* notation is discussed in Sect. 7.5.3.

Although a value was given for θ it is not displayed – that value simply served as a placeholder in the list of kinematic parameters. θ is substituted by the joint variable q and its value in the `Link` object is ignored. The value will be managed by the `Link` object.

A `Link` object has many parameters and methods which are described in the online documentation, but the most common ones are illustrated by the following examples. The link transform Eq. 7.2 for $q = 0.5$ rad is

Richard Hartenberg (1907–1997) was born in Chicago and studied for his degrees at the University of Wisconsin, Madison. He served in the merchant marine and studied aeronautics for two years at the University of Goettingen with space-flight pioneer Theodor von Karman. He was Professor of mechanical engineering at Northwestern University where he taught for 56 years. His research in kinematics led to a revival of interest in this field in the 1960s, and his efforts helped put kinematics on a scientific basis for use in computer applications in the analysis and design of complex mechanisms. His also wrote extensively on the history of mechanical engineering.



```
>> L.A(0.5)
ans =
    0.8776   -0.0000    0.4794    0.1755
    0.4794    0.0000   -0.8776    0.0959
         0    1.0000    0.0000    0.1000
         0         0         0    1.0000
```

is a 4×4 homogeneous transformation. Various link parameters can be read or altered, for example

```
>> L.RP
ans =
     R
```

indicates that the link is revolute and

```
>> L.a
ans =
    0.2000
```

returns the kinematic parameter a. Finally a link can contain an offset

```
>> L.offset = 0.5;
>> L.A(0)
ans =
    0.8776   -0.0000    0.4794    0.1755
    0.4794    0.0000   -0.8776    0.0959
         0    1.0000    0.0000    0.1000
         0         0         0    1.0000
```

which is added to the joint variable before computing the link transform and will be discussed in more detail in Sect. 7.5.1.

7.2 Forward Kinematics

The forward kinematics is often expressed in functional form

$$\xi_E = \mathcal{K}(\boldsymbol{q}) \tag{7.3}$$

with the end-effector pose as a function of joint coordinates. Using homogeneous transformations this is simply the product of the individual link transformation matrices given by Eq. 7.2 which for an N-axis manipulator is

$$\xi_E \sim {}^0\boldsymbol{T}_E = {}^0\boldsymbol{A}_1 \, {}^1\boldsymbol{A}_2 \cdots {}^{N-1}\boldsymbol{A}_N \tag{7.4}$$

The forward kinematic solution can be computed for *any* serial-link manipulator irrespective of the number of joints or the types of joints. Determining the Denavit-Hartenberg parameters for each of the robot's links is described in more detail in Sect. 7.5.2.

The pose of the end-effector $\xi_E \sim \boldsymbol{T}_E \in SE(3)$ has six degrees of freedom – three in translation and three in rotation. Therefore robot manipulators commonly have six joints or degrees of freedom to allow them to achieve an arbitrary end-effector pose. The overall manipulator transform is frequently written as $\boldsymbol{T}_6$ for a 6-axis robot.

7.2.1 A 2-Link Robot

The first robot that we will discuss is the two-link planar manipulator shown in Fig. 7.3. It has the following Denavit-Hartenberg parameters
which we use to create a vector of `Link` objects

Link	θ_i	d_i	a_i	α_i	σ_i
1	q_1	0	1	0	0
2	q_2	0	1	0	0

```
>> L(1) = Link([0 0 1 0]);
>> L(2) = Link([0 0 1 0]);
>> L
L =
 theta=q1, d=0, a=1, alpha=0 (R,stdDH)
 theta=q2, d=0, a=1, alpha=0 (R,stdDH)
```

which are passed to the constructor `SerialLink`

```
>> two_link = SerialLink(L, 'name', 'two link');
```

which returns a `SerialLink` object that we can display

```
>> two_link
two_link =
two link (2 axis, RR, stdDH)
+---+-----------+-----------+-----------+-----------+
| j |     theta |         d |         a |     alpha |
+---+-----------+-----------+-----------+-----------+
|  1|         q1|          0|          1|          0|
|  2|         q2|          0|          1|          0|
+---+-----------+-----------+-----------+-----------+

grav =    0  base = 1  0  0  0   tool =  1  0  0  0
          0         0  1  0  0           0  1  0  0
       9.81         0  0  1  0           0  0  1  0
                    0  0  0  1           0  0  0  1
```

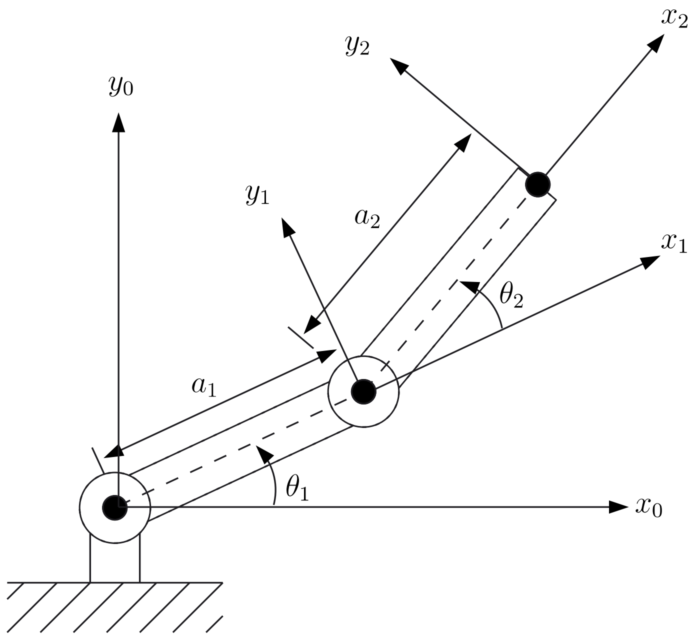


Fig. 7.3. Two-link robot as per Spong et al. (2006, fig. 3.6, p. 84). Reproduced with permission.

This provides a concise description of the robot. We see that it has 2 revolute joints as indicated by the structure string `'RR'`, it is defined in terms of standard Denavit-Hartenberg parameters, that gravity is acting in the default z-direction.▸ The kinematic parameters of the link objects are also listed and the joint variables are shown as variables `q1` and `q2`. We have also assigned a name to the robot which will be shown whenever the robot is displayed graphically. The script

Normal to the plane in which the robot moves.

```
>> mdl_twolink
```

performs the above steps and defines this robot in the MATLAB® workspace, creating a `SerialLink` object named `twolink`.

The `SerialLink` object is key to the operation of the Robotics Toolbox. It has a great many methods and properties which are illustrated in the rest of this part, and described in detail in the online documentation. Some simple examples are

```
>> twolink.n
ans =
     2
```

which returns the number of joints, and

```
>> links = twolink.links
L =
theta=q1, d=0, a=1, alpha=0 (R,stdDH)
theta=q2, d=0, a=1, alpha=0 (R,stdDH)
```

which returns a vector of `Link` objects comprising the robot.▸ We can also make a copy of a `SerialLink` object

`Link` objects are derived from the MATLAB® `handle` class and can be set *in place*. Thus we can write code like `twolink.links(1).a = 2` to change the kinematic parameter a_2.

```
>> clone = SerialLink(twolink, 'name', 'bob')
clone =
bob (2 axis, RR, stdDH)
+---+-----------+-----------+-----------+-----------+
| j |     theta |         d |         a |     alpha |
+---+-----------+-----------+-----------+-----------+
|  1|         q1|          0|          1|          0|
|  2|         q2|          0|          1|          0|
+---+-----------+-----------+-----------+-----------+

grav =    0  base = 1  0  0  0   tool =  1  0  0  0
          0         0  1  0  0           0  1  0  0
       9.81         0  0  1  0           0  0  1  0
                    0  0  0  1           0  0  0  1
```

which has the name `'bob'`.▸

A unique name is required when multiple robots are displayed graphically, see Sect. 7.8 for more details.

Now we can put the robot arm to work. The forward kinematics are computed using the `fkine` method. For the case where $q_1 = q_2 = 0$

```
>> twolink.fkine([0 0])
ans =
     1     0     0     2
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

the method returns the homogenous transform that represents the pose of the second link coordinate frame of the robot, $\boldsymbol{T}_2$. For a different configuration the tool pose is

```
>> twolink.fkine([pi/4 -pi/4])
ans =
    1.0000         0         0    1.7071
         0    1.0000         0    0.7071
         0         0    1.0000         0
         0         0         0    1.0000
```

By convention, joint coordinates with the Toolbox are row vectors.

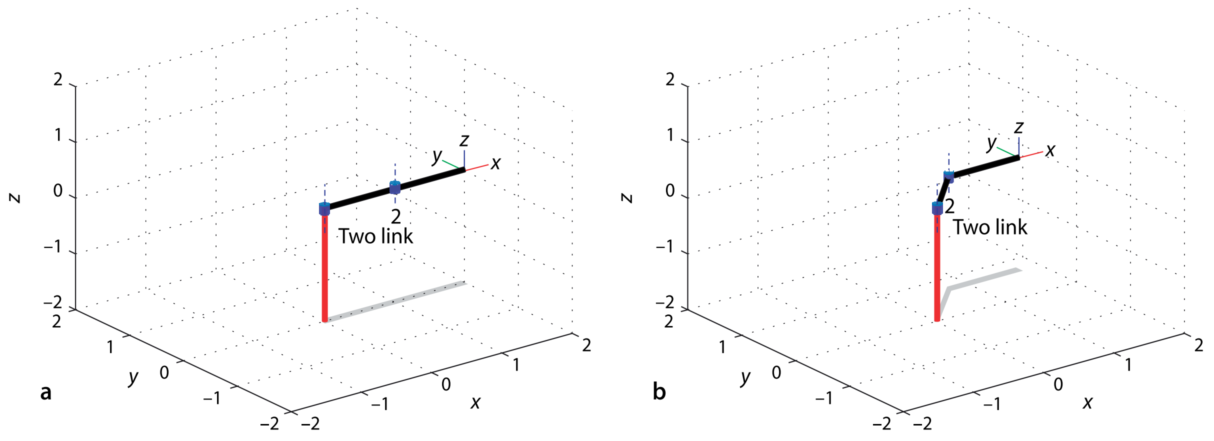


Fig. 7.4. The two-link robot in two different poses, **a** the pose $(0, 0)$; **b** the pose $(\frac{\pi}{4}, -\frac{\pi}{4})$. Note the graphical details. Revolute joints are indicated by cylinders and the joint axes are shown as line segments. The final-link coordinate frame and a shadow on the ground are also shown

The robot can be visualized graphically

```
>> twolink.plot([0 0])
>> twolink.plot([pi/4 -pi/4])
```

as stick figures shown in Fig. 7.4. The graphical representation includes the robot's name, the final-link coordinate frame, $\boldsymbol{T}_2$ in this case, the joints and their axes, and a shadow on the ground plane. Additional features of the plot method such as multiple views and multiple robots are described in Sect. 7.8 with additional details in the online documentation.

The simple two-link robot introduced above is limited in the poses that it can achieve since it operates entirely within the xy-plane, its task space is $\mathcal{T} \subset \mathbb{R}^2$.

7.2.2 A 6-Axis Robot

Truly useful robots have a task space $\mathcal{T} \subset SE(3)$ enabling arbitrary position and attitude of the end-effector – the task space has six spatial degrees of freedom: three translational and three rotational. This requires a robot with a configuration space $\mathcal{C} \subset \mathbb{R}^6$ which can be achieved by a robot with six joints. In this section we will use the Puma 560 as an example of the class of all-revolute six-axis robot manipulators. We define an instance of a Puma 560 robot using the script◂

The Toolbox has scripts to define a number of common industrial robots including the Motoman HP6, Fanuc 10L, ABB S4 2.8 and the Stanford arm.

```
>> mdl_puma560
```

which creates a SerialLink object, p560, in the workspace

```
>> p560
p560 =
Puma 560 (6 axis, RRRRRR, stdDH)
 Unimation; viscous friction; params of 8/95;
+---+-----------+-----------+-----------+-----------+
| j |     theta |         d |         a |     alpha |
+---+-----------+-----------+-----------+-----------+
|  1|         q1|          0|          0|      1.571|
|  2|         q2|          0|     0.4318|          0|
|  3|         q3|       0.15|     0.0203|     -1.571|
|  4|         q4|     0.4318|          0|      1.571|
|  5|         q5|          0|          0|     -1.571|
|  6|         q6|          0|          0|          0|
+---+-----------+-----------+-----------+-----------+

grav =     0  base = 1  0  0  0   tool =  1  0  0  0
           0         0  1  0  0           0  1  0  0
        9.81         0  0  1  0           0  0  1  0
                     0  0  0  1           0  0  0  1
```

The **Puma 560 robot** (Programmable Universal Manipulator for Assembly) shown in Fig. 7.1 was released in 1978 and became enormously popular. It featured an anthropomorphic design, electric motors and a spherical wrist which makes it the first *modern* industrial robot – the archetype of all that followed.

The Puma 560 catalyzed robotics research in the 1980s and it was a very common laboratory robot. Today it is obsolete and rare but in homage to its important role in robotics research we use it here. For our purposes the advantages of this robot are that it has been well studied and its parameters are very well known – it has been described as the "white rat" of robotics research.

Most modern 6-axis industrial robots are very similar in structure and can be accomodated simply by changing the Denavit-Hartenberg parameters. The Toolbox has kinematic models for a number of common industrial robots including the Motoman HP6, Fanuc 10L, and ABB S4 2.8.

Note that a_j and d_j are in SI units which means that the translational part of the forward kinematics will also have SI units.

The script `mdl_puma560` also creates a number of joint coordinate vectors in the workspace which represent the robot in some canonic configurations:

`qz`	$(0, 0, 0, 0, 0, 0)$	*zero angle*
`qr`	$(0, \frac{\pi}{2}, -\frac{\pi}{2}, 0, 0, 0)$	*ready*, the arm is straight and vertical
`qs`	$(0, 0, -\frac{\pi}{2}, 0, 0, 0)$	*stretch*, the arm is straight and horizontal
`qn`	$(0, \frac{\pi}{4}, -\pi, 0, \frac{\pi}{4}, 0)$	*nominal*, the arm is in a dextrous working pose

Well away from singularities, which will be discussed in Sect. 7.4.3.

and these are shown graphically in Fig. 7.5. These plots are generated using the `plot` method, for example

```
>> p560.plot(qz)
```

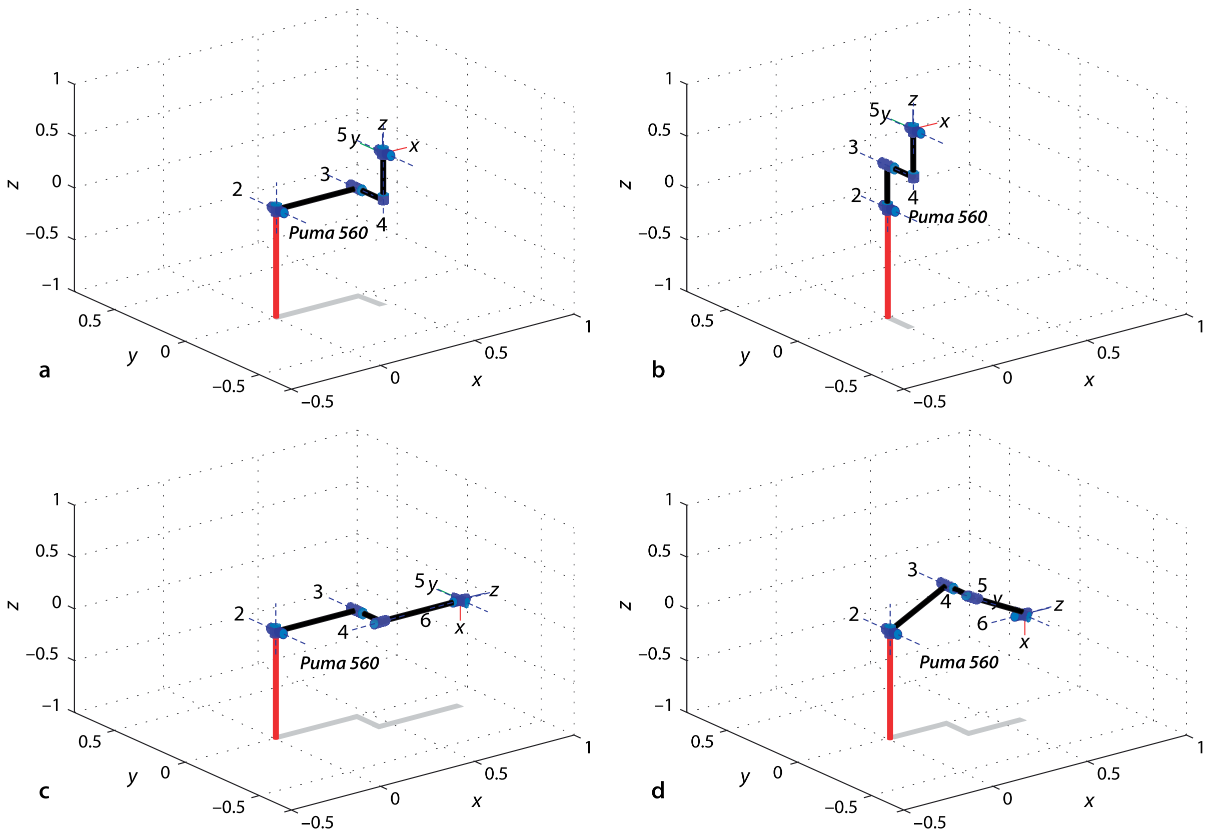


Fig. 7.5. The Puma robot in 4 different poses. **a** *Zero angle*; **b** *ready* pose; **c** *stretch*; **d** *nominal*

Anthropomorphic means having human-like characteristics. The Puma 560 robot was designed to have approximately the dimensions and reach of a human worker. It also had a spherical joint at the wrist just as humans have.

Roboticists also tend to use anthropomorphic terms when describing robots. We use words like waist, shoulder, elbow and wrist when describing serial link manipulators. For the Puma these terms correspond respectively to joint 1, 2, 3 and 4–6.

Forward kinematics can be computed as before

```
>> p560.fkine(qz)
ans =
    1.0000         0         0    0.4521
         0    1.0000         0   -0.1500
         0         0    1.0000    0.4318
         0         0         0    1.0000
```

which returns the homogeneous transformation corresponding to the end-effector pose $\boldsymbol{T}_6$. The origin of this frame, the link 6 coordinate frame {6}, is defined◂ as the point of intersection of the axes of the last 3 joints – physically this point is inside the robot's wrist mechanism. We can define a tool transform, from the $\boldsymbol{T}_6$ frame to the actual tool tip by

By the `mdl_puma560` script.

```
>> p560.tool = transl(0, 0, 0.2);
```

in this case a 200 mm extension in the $\boldsymbol{T}_6$ z-direction.◂ The pose of the tool tip, often referred to as the tool centre point or TCP, is now

Alternatively we could change the kinematic parameter d_6. The tool transform approach is more general since the final link kinematic parameters only allow setting of d_6, a_6 and α_6 which provide z-axis translation, x-axis translation and x-axis rotation respectively.

```
>> p560.fkine(qz)
ans =
    1.0000         0         0    0.4521
         0    1.0000         0   -0.1500
         0         0    1.0000    0.6318
         0         0         0    1.0000
```

The kinematic definition we have used considers that the base of the robot is the intersection point of the waist and shoulder axes which is a point inside the structure of the robot. The Puma 560 robot includes a 30 inch tall pedestal. We can shift the origin of the robot from the point inside the robot to the base of the pedestal using a base transform

```
>> p560.base = transl(0, 0, 30*0.0254);
```

where for consistency we have converted the pedestal height to SI units. Now, with both base and tool transform, the forward kinematics are

```
>> p560.fkine(qz)
ans =
    1.0000         0         0    0.4521
         0    1.0000         0   -0.1500
         0         0    1.0000    1.3938
         0         0         0    1.0000
```

and we can see that the z-coordinate of the tool is now greater than before.

We can also do more interesting things, for example

```
>> p560.base = transl(0,0,3) * trotx(pi);
>> p560.fkine(qz)
ans =
    1.0000         0         0    0.4521
         0   -1.0000   -0.0000    0.1500
         0    0.0000   -1.0000    2.3682
         0         0         0    1.0000
```

which positions the robot's origin 3 m above the world origin with its coordinate frame rotated by 180° about the x-axis. This robot is now hanging from the ceiling!

The Toolbox supports joint angle time series, or trajectories, such as

```
>> q
q =
         0         0         0         0         0         0
         0    0.0365   -0.0365         0         0         0
         0    0.2273   -0.2273         0         0         0
         0    0.5779   -0.5779         0         0         0
         0    0.9929   -0.9929         0         0         0
         0    1.3435   -1.3435         0         0         0
         0    1.5343   -1.5343         0         0         0
         0    1.5708   -1.5708         0         0         0
```

where each row represents the joint coordinates at a different timestep and the columns represent the joints.▸ In this case the method `fkine`

Generated by the `jtraj` function, which is discussed in Sect. 7.4.1.

```
>> T = p560.fkine(q);
```

returns a 3-dimensional matrix

```
>> about(T)
T [double] : 4x4x8 (1024 bytes)
```

where the first two dimensions are a 4×4 homogeneous transformation and the third dimension is the timestep. The homogeneous transform corresponding to the joint coordinates in the fourth row of `q` is

```
>> T(:,:,4)
ans =
    1.0000    0.0000         0    0.3820
    0.0000    1.0000   -0.0000   -0.1500
         0   -0.0000    1.0000    1.6297
         0         0         0    1.0000
```

Creating a trajectory is the topic of Sect. 7.4.

7.3 Inverse Kinematics

We have shown how to determine the pose of the end-effector given the joint coordinates and optional tool and base transforms. A problem of real practical interest is the inverse problem: given the desired pose of the end-effector ξ_E what are the required joint coordinates? For example, if we know the Cartesian pose of an object, what joint coordinates does the robot need in order to reach it? This is the inverse kinematics problem which is written in functional form as

$$\boldsymbol{q} = \mathcal{K}^{-1}(\xi) \tag{7.5}$$

In general this function is not unique and for some classes of manipulator no closed-form solution exists, necessitating a numerical solution.

7.3.1 Closed-Form Solution

A necessary condition for a closed-form solution of a 6-axis robot is that the three wrist axes intersect at a single point. This means that motion of the wrist joints only changes the end-effector orientation, not its translation. Such a mechanism is known as a spherical wrist and almost all industrial robots have such wrists.

We will explore inverse kinematics using the Puma robot model

```
>> mdl_puma560
```

At the *nominal* joint coordinates

```
>> qn
qn =
         0    0.7854    3.1416         0    0.7854         0
```

the end-effector pose is

```
>> T = p560.fkine(qn)
T =
   -0.0000    0.0000    1.0000    0.5963
   -0.0000    1.0000   -0.0000   -0.1501
   -1.0000   -0.0000   -0.0000   -0.0144
         0         0         0    1.0000
```

Since the Puma 560 is a 6-axis robot arm with a spherical wrist we use the method `ikine6s` to compute the inverse kinematics using a closed-form solution. The required joint coordinates to achieve the pose `T` are

The method `ikine6s` checks the Denavit-Hartenberg parameters to determine if the robot meets these criteria.

```
>> qi = p560.ikine6s(T)
qi =
    2.6486   -3.9270    0.0940    2.5326    0.9743    0.3734
```

Surprisingly, these are quite different to the joint coordinates we started with. However if we investigate a little further

```
>> p560.fkine(qi)
ans =
   -0.0000    0.0000    1.0000    0.5963
    0.0000    1.0000   -0.0000   -0.1500
   -1.0000    0.0000   -0.0000   -0.0144
         0         0         0    1.0000
```

we see that these two different sets of joint coordinates result in the *same* end-effector pose and these are shown in Fig. 7.6. The shoulder of the Puma robot is horizontally offset from the waist, so in one solution the arm is to the left of the waist, in the other it is to the right. These are referred to as the left- and right-handed kinematic configurations respectively. In general there are eight sets of joint coordinates that give the same end-effector pose – as mentioned earlier the inverse solution is not unique.

We can *force* the right-handed solution

```
>> qi = p560.ikine6s(T, 'ru')
qi =
   -0.0000    0.7854    3.1416    0.0000    0.7854   -0.0000
```

which gives the original set of joint angles by specifying a *right handed* configuration with the elbow *up*.

In addition to the left- and right-handed solutions, there are solutions with the elbow either up or down, and with the wrist flipped or not flipped. For the Puma 560 robot the wrist joint, θ_4, has a large rotational range and can adopt one of two angles

More precisely the elbow is above or below the shoulder.

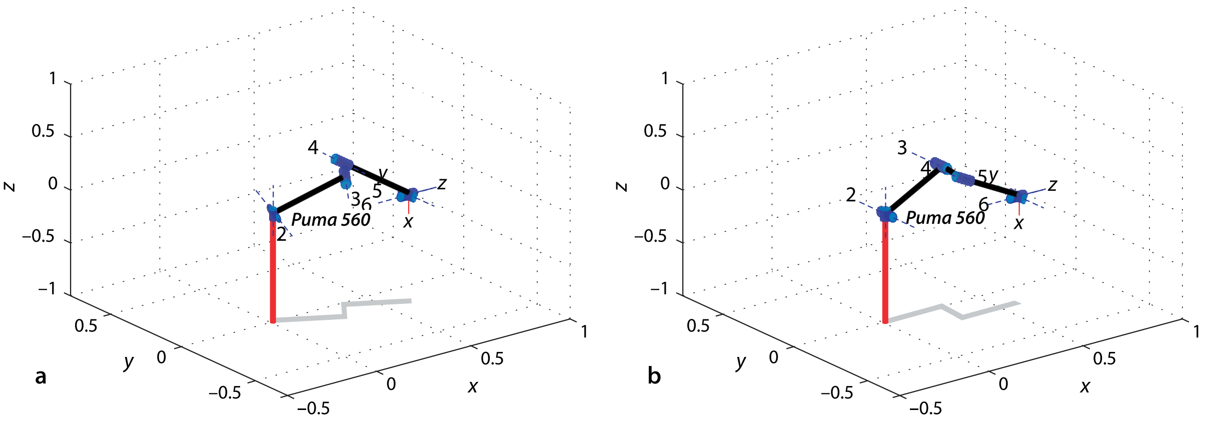


Fig. 7.6. Two solutions to the inverse kinematic problem, left-handed and right-handed solutions. The shadow shows clearly the two different configurations

that differ by π rad. Since most robot grippers have just two fingers, see Fig. 2.14, this makes no difference in its ability to grasp an object.

The various kinematic configurations are shown in Fig. 7.7. The kinematic configuration returned by `ikine6s` is controlled by one or more of the character flags:

left or right handed	`'l'`, `'r'`
elbow up or down	`'u'`, `'d'`
wrist flipped or not flipped	`'f'`, `'n'`

Due to mechanical limits on joint angles and possible collisions between links not all eight solutions are physically achievable. It is also possible that no solution can be achieved, for example

```
>> p560.ikine6s( transl(3, 0, 0) )
Warning: point not reachable
ans =
   NaN   NaN   NaN   NaN   NaN   NaN
```

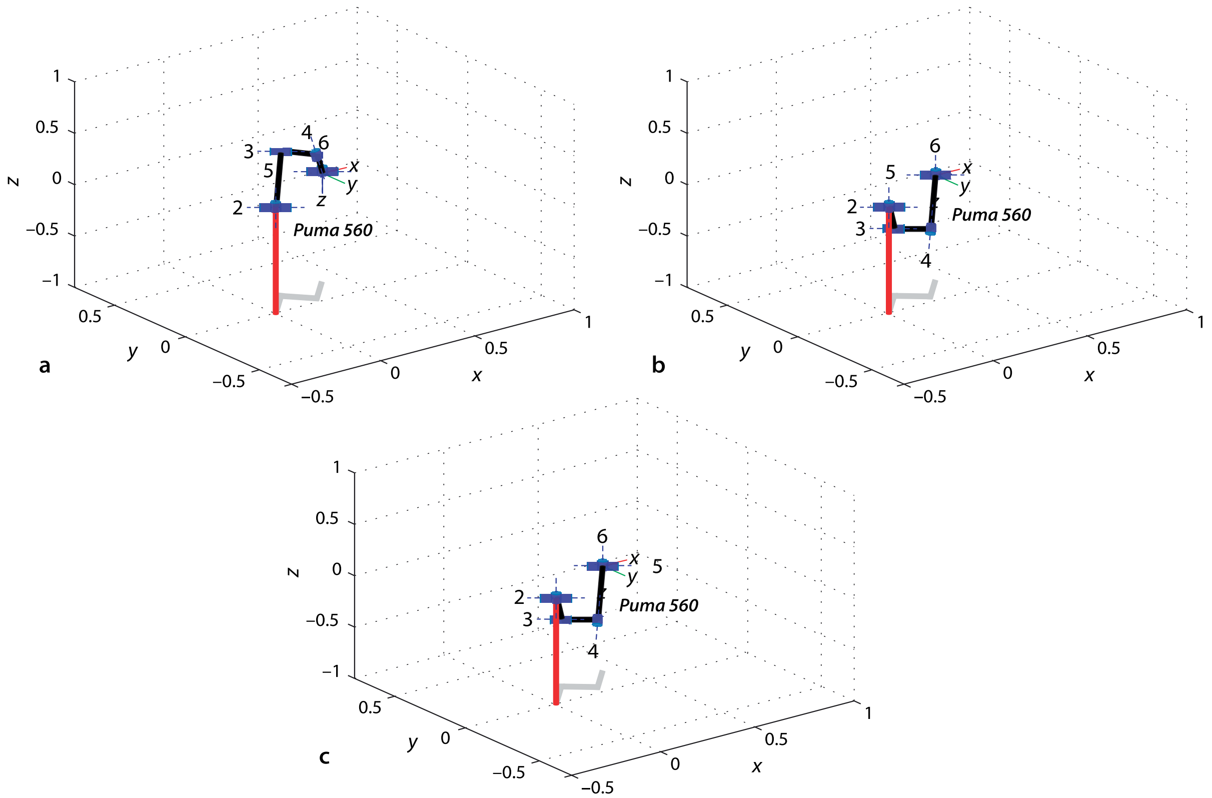
fails because the arm is simply not long enough to reach this pose. A pose may also be unachievable due to singularity where the alignment of axes reduces the effective degrees of freedom (the gimbal lock problem again). The Puma 560 has a singularity when q_5 is equal to zero and the axes of joints 4 and 6 become aligned. In this case the best that `ikine6s` can do is to constrain $q_4 + q_6$ but their individual values are arbitrary. For example consider the configuration

```
>> q = [0 pi/4 pi 0.1 0 0.2];
```

for which $q_4 + q_6 = 0.3$. The inverse kinematic solution is

```
>> p560.ikine6s(p560.fkine(q), 'ru')
ans =
  -0.0000    0.7854    3.1416    2.9956    0.0000   -2.6956
```

Fig. 7.7. Different configurations of the Puma 560 robot. **a** Right-up-noflip; **b** right-down-noflip; **c** right-down-flip



has quite different values for q_4 and q_6 but their sum

```
>> q(4)+q(6)
ans =
    0.3000
```

remains the same.

7.3.2 Numerical Solution

For the case of robots which do not have six joints and a spherical wrist we need to use an iterative numerical solution. Continuing with the example of the previous section we use the method `ikine` to compute the general inverse kinematic solution

```
>> T = p560.fkine(qn)
ans =
   -0.0000    0.0000    1.0000    0.5963
   -0.0000    1.0000   -0.0000   -0.1501
   -1.0000   -0.0000   -0.0000   -0.0144
         0         0         0    1.0000
>> qi = p560.ikine(T)
qi =
    0.0000   -0.8335    0.0940   -0.0000   -0.8312    0.0000
```

which is different to the original value

```
>> qn
qn =
         0    0.7854    3.1416         0    0.7854         0
```

but does result in the correct tool pose

```
>> p560.fkine(qi)
ans =
   -0.0000    0.0000    1.0000    0.5963
   -0.0000    1.0000   -0.0000   -0.1501
   -1.0000   -0.0000   -0.0000   -0.0144
         0         0         0    1.0000
```

Plotting the pose

```
>> p560.plot(qi)
```

shows clearly that `ikine` has found the elbow-down configuration.

A limitation of this general numeric approach is that it does not provide explicit control over the arm's kinematic configuration as did the analytic approach – the only control is implicit via the initial value of joint coordinates (which defaults to zero). If we specify the initial joint coordinates

```
>> qi = p560.ikine(T, [0 0 3 0 0 0])
qi =
    0.0000    0.7854    3.1416    0.0000    0.7854   -0.0000
```

When solving for a trajectory as on p. 146 the inverse kinematic solution for one point is used to initialize the solution for the next point on the path.

the solution converges on the elbow-up configuration.◂

As would be expected the general numerical approach of `ikine` is considerably slower than the analytic approach of `ikine6s`. However it has the great advantage of being able to work with manipulators at singularities and manipulators with less than six or more than six joints. Details of the principle behind `ikine` is provided in Sect. 8.4.

7.3.3 Under-Actuated Manipulator

An under-actuated manipulator is one that has fewer than six joints, and is therefore limited in the end-effector poses that it can attain. SCARA robots such as shown on page 135 are a common example, they typically have an x-y-z-θ task space, $\mathcal{T} \subset \mathbb{R}^3 \times \mathbb{S}$ and a configuration space $\mathcal{C} \subset \mathbb{S}^3 \times \mathbb{R}$.

We will revisit the two-link manipulator example from Sect. 7.2.1, first defining the robot

```
>> mdl_twolink
```

and then defining the desired end-effector pose

```
>> T = transl(0.4, 0.5, 0.6);
```

However this *pose* is over-constrained for the two-link robot – the tool cannot meet the orientation constraint on the direction of $\hat{x}_2$ and $\hat{y}_2$ nor a z-axis translational value other than zero. Therefore we require the `ikine` method to consider only error in the x- and y-axis translational directions, and ignore all other Cartesian degrees of freedom. We achieve this by specifying a mask vector as the fourth argument

```
>> q = twolink.ikine(T, [0 0], [1 1 0 0 0 0])
q =
   -0.3488    2.4898
```

The elements of the mask vector correspond respectively to the three translations and three orientations: t_x, t_y, t_z, r_x, r_y, r_z in the end-effector coordinate frame. In this example we specified that only errors in x- and y-translation are to be considered (the non-zero elements). The resulting joint angles correspond to an endpoint pose

```
>> twolink.fkine(q)
ans =
ans =
   -0.5398   -0.8418         0    0.4000
    0.8418   -0.5398         0    0.5000
         0         0    1.0000         0
         0         0         0    1.0000
```

which has the desired x- and y-translation, but the orientation and z-translation are incorrect, as *we allowed* them to be.

7.3.4 Redundant Manipulator

A redundant manipulator is a robot with more than six joints. As mentioned previously, six joints is theoretically sufficient to achieve any desired pose in a Cartesian taskspace $\mathcal{T} \subset SE(3)$. However practical issues such as joint limits and singularities mean that not all poses within the robot's reachable space can be achieved. Adding additional joints is one way to overcome this problem.

To illustrate this we will create a redundant manipulator. We place our familiar Puma robot

```
>> mdl_puma560
```

on a platform that moves in the xy-plane, mimicking a robot mounted on a vehicle. This robot has the same task space $\mathcal{T} \subset SE(3)$ as the Puma robot, but a configuration space $\mathcal{C} \subset \mathbb{R}^2 \times \mathbb{S}^6$. The dimension of the configuration space exceeds the dimensions of the task space.

The Denavit-Hartenberg parameters for the base are

Link	θ_i	d_i	a_i	α_i	σ_i
1	0	q_1	0	$-\frac{\pi}{2}$	1
2	$-\frac{\pi}{2}$	q_2	0	$\frac{\pi}{2}$	1

and using a shorthand syntax we create the `SerialLink` object directly from the Denavit-Hartenberg parameters

```
>> platform = SerialLink( [0 0 0 -pi/2 1; -pi/2 0 0 pi/2 1], ...
     'base', troty(pi/2), 'name', 'platform' );
```

The Denavit-Hartenberg notation requires that prismatic joints cause translation in the local z-direction and the base-transform rotates that z-axis into the world x-axis direction. This is a common complexity with Denavit-Hartenberg notation and an alternative will be introduced in Sect. 7.5.2.

The joint coordinates of this robot are its x- and y-position and we test that our Denavit-Hartenberg parameters are correct

```
>> platform.fkine([1, 2])
ans =
     1.0000     0.0000     0.0000     1.0000
    -0.0000     1.0000     0.0000     2.0000
          0    -0.0000     1.0000     0.0000
          0          0          0     1.0000
```

We see that the rotation submatrix is the identity matrix, meaning that the coordinate frame of the platform is parallel with the world coordinate frame, and that the x- and y-displacement is as requested.

We mount the Puma robot on the platform by *connecting* the two robots in series. In the Toolbox we can express this in two different ways, by multiplication

```
>> p8 = platform * p560;
```

or by concatenation

```
p8 = SerialLink( [platform, p560]);
```

which also allows other attributes of the created `SerialLink` object to be specified.

However there is a small complication. We would like the Puma arm to be sitting on its tall pedestal on top of the platform. Previously we added the pedestal as a base transform, but base transforms can only exist at the start of a kinematic chain, and we require the transform between joints 2 and 3 of the 8-axis robot. Instead we implement the pedestal by setting d_1 of the Puma to the pedestal height

```
>> p560.links(1).d = 30 * 0.0254;
```

and now we can compound the two robots

```
>> p8 = SerialLink( [platform, p560], 'name', 'P8')
p8 =

P8 (8 axis, PPRRRRRR, stdDH)
+---+-----------+-----------+-----------+-----------+
| j |     theta |         d |         a |     alpha |
+---+-----------+-----------+-----------+-----------+
|  1|          0|         q1|          0|     -1.571|
|  2|     -1.571|         q2|          0|      1.571|
|  3|         q3|      0.762|          0|      1.571|
|  4|         q4|          0|     0.4318|          0|
|  5|         q5|       0.15|     0.0203|     -1.571|
|  6|         q6|     0.4318|          0|      1.571|
|  7|         q7|          0|          0|     -1.571|
|  8|         q8|          0|          0|          0|
+---+-----------+-----------+-----------+-----------+

grav =     0  base = 0  0  0  0   tool =  1  0  0  0
           0         0  1  0  0           0  1  0  0
        9.81         1  0  0  0           0  0  1  0
                     0  0  0  1           0  0  0  1
```

resulting in a new 8-axis robot which we have named the P8. We note that the new robot has inherited the base transform of the platform, and that the pedestal displacement appears now as d_3.

We can find the inverse kinematics for this new redundant manipulator using the general numerical solution. For example to move the end effector to (0.5, 1.0, 0.7) with the tool pointing downward in the xz-plane the Cartesian pose is

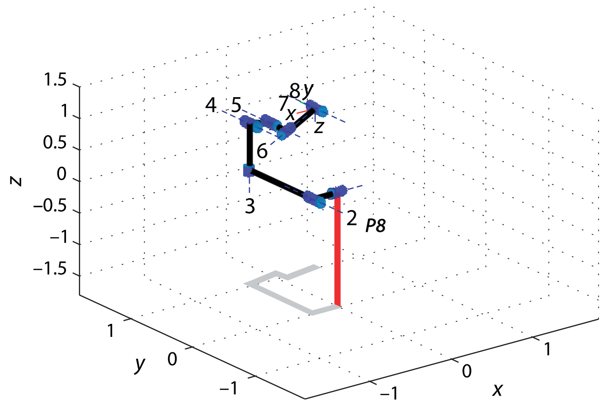


Fig. 7.8.
Redundant manipulator P8

```
>> T = transl(0.5, 1.0, 0.7) * rpy2tr(0, 3*pi/4, 0);
```

The required joint coordinates are

The solution is not unique, an infinite number of joint coordinate vectors will result in the same end effector pose. This is related to the null-space motion discussed in Sect. 8.2.3.

```
>> qi = p8.ikine(T)
qi =
    -0.2854    1.0528    0.1246   -0.5377   -0.6091   -0.0939
         -1.2147    0.1212
```

The first two elements are displacements of the prismatic joints in the base, and the last six are the robot arm's joint angles. We see that the solution has made good use of the second joint, the platform's *y*-axis translation to move the base of the arm close to the desired point. We can visualize the configuration

```
>> p8.plot(qi)
```

which is shown in Fig. 7.8. We can also show that the plain old Puma cannot reach the point we specified

```
>> p560.ikine6s(T)
Warning: point not reachable
```

This robot has 8 degrees of freedom. Intuitively it is more dextrous than its 6-axis counterpart because there is more than one joint that can contribute to every Cartesian degree of freedom. This is more formally captured in the notion of manipulability which we will discuss in Sect. 8.1.4. For now we will consider it as a scalar measure that reflects the ease with which the manipulator's tool can move in different Cartesian directions. The manipulability of the 6-axis robot

```
>> p560.maniplty(qn)
ans =
    0.0786
```

is much less than that for the 8-axis robot

```
>> p8.maniplty([0 0 qn])
ans =
    1.1151
```

which indicates the increased dexterity of the 8-axis robot.

7.4 Trajectories

One of the most common requirements in robotics is to move the end-effector smoothly from pose A to pose B. Building on what we learnt in Sect. 3.1 we will discuss two approaches to generating such trajectories: straight lines in joint space and straight lines in Cartesian space. These are known respectively as joint-space and Cartesian motion.

7.4.1 Joint-Space Motion

We choose the tool z-axis, see Fig. 2.14, downward on these poses as it would be if the robot was reaching down to work on some horizontal surface. For the Puma 560 robot it is physically impossible for the tool z-axis to point upward in such a situation.

Consider the end-effector moving between two Cartesian poses◀

```
>> T1 = transl(0.4, 0.2, 0) * trotx(pi);
>> T2 = transl(0.4, -0.2, 0) * trotx(pi/2);
```

which lie in the xy-plane with the end-effector oriented downward. The initial and final joint coordinate vectors associated with these poses are

```
>> q1 = p560.ikine6s(T1);
>> q2 = p560.ikine6s(T2);
```

and we require the motion to occur over a time period of 2 seconds in 50 ms time steps

```
>> t = [0:0.05:2]';
```

A joint-space trajectory is formed by smoothly interpolating between two configurations `q1` and `q2`. The scalar interpolation functions `tpoly` or `lspb` from Sect. 3.1 can be used in conjunction with the multi-axis *driver* function `mtraj`

```
>> q = mtraj(@tpoly, q1, q2, t);
```

or

```
>> q = mtraj(@lspb, q1, q2, t);
```

which each result in a 50×6 matrix `q` with one row per time step and one column per joint. From here on we will use the `jtraj` convenience function

```
>> q = jtraj(q1, q2, t);
```

which is equivalent to `mtraj` with `tpoly` interpolation but optimized for the multi-axis case and also allowing initial and final velocity to be set using additional arguments. For `mtraj` and `jtraj` the final argument can be a time vector, as here, or an integer specifying the number of time steps.

We obtain the joint velocity and acceleration vectors, as a function of time, through optional output arguments

```
>> [q,qd,qdd] = jtraj(q1, q2, t);
```

An even more concise expression for the above steps is provided by the `jtraj` method of the `SerialLink` class

```
>> q = p560.jtraj(T1, T2, t)
```

The trajectory is best viewed as an animation

```
>> p560.plot(q)
```

but we can also plot the joint angle, for instance q_2, versus time

```
>> plot(t, q(:,2))
```

or all the angles versus time

```
>> qplot(t, q);
```

as shown in Fig. 7.9a. The joint coordinate trajectory is smooth but we do not know how the robot's end-effector will move in Cartesian space. However we can easily determine this by applying forward kinematics to the joint coordinate trajectory

```
>> T = p560.fkine(q);
```

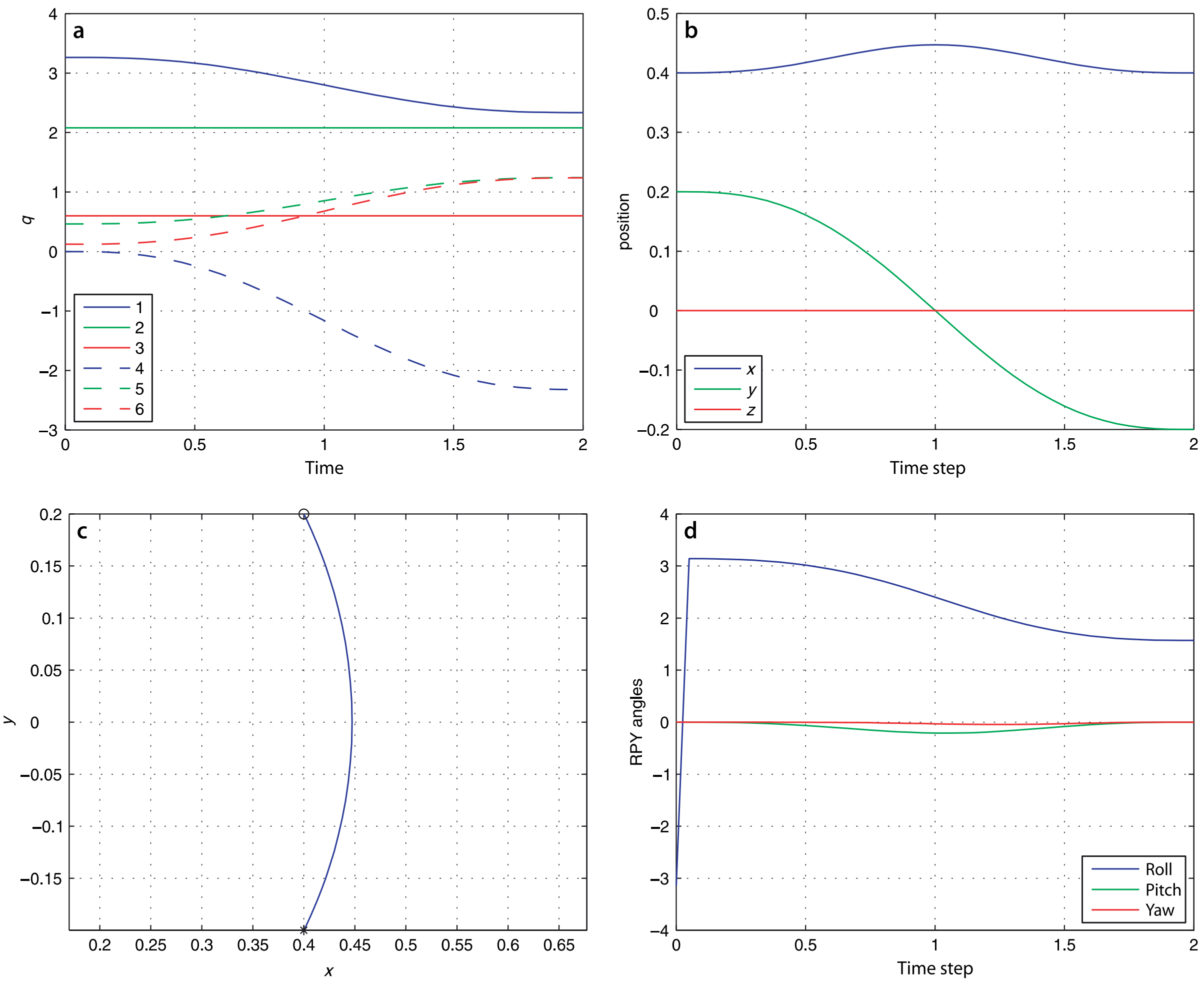


Fig. 7.9. Joint space motion. **a** Joint coordinates versus time; **b** Cartesian position versus time; **c** Cartesian position locus in the *xy*-plane **d** roll-pitch-yaw angles versus time

which results in a 3-dimensional Cartesian trajectory. The translational part of this trajectory is

```
>> p = transl(T);
```

which is in matrix form

```
>> about(p)
p [double] : 41x3 (984 bytes)
```

and has one row per time step, and each row is the end-effector position vector. This is plotted against time in Fig. 7.9b. The path of the end-effector in the *xy*-plane

```
>> plot(p(:,1), p(:,2))
```

is shown in Fig. 7.9c and it is clear that the path is not a straight line. This is to be expected since we only specified the Cartesian coordinates of the end-points. As the robot rotates about its waist joint during the motion the end-effector will naturally follow a circular arc. In practice this could lead to collisions between the robot and nearby objects even if they do not lie on the path between poses A and B. The orientation of the end-effector, in roll-pitch-yaw angles form, can be plotted against time

```
>> plot(t, tr2rpy(T))
```

as shown in Fig. 7.9d. Note that the roll angle▸ varies from π to $\frac{\pi}{2}$ as we specified. However while the pitch angle has met its boundary conditions it has varied along

Rotation about *x*-axis from Sect. 2.2.1.2.

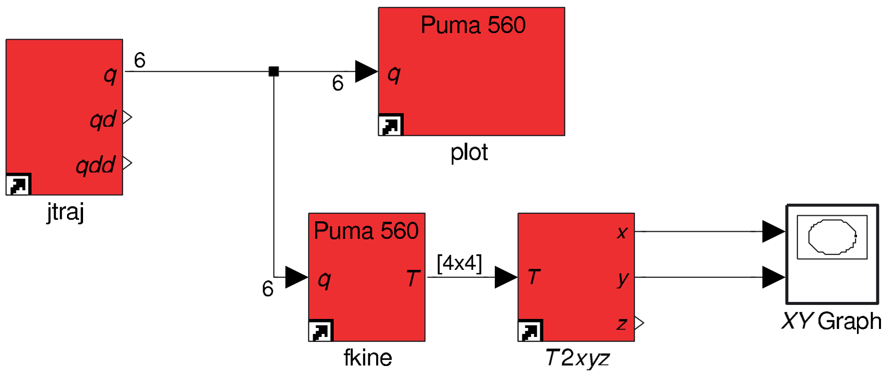


Fig. 7.10. Simulink® model `sl_jspace` for joint-space motion

the path. Also note that the initial roll angle is shown as $-\pi$ but at the next time step it is π. This is an artifact of finite precision arithmetic and both angles are equivalent on the circle.

We can also implement this example in Simulink®

```
>> sl_jspace
```

and the block diagram model is shown in Fig. 7.10. The parameters of the `jtraj` block are the initial and final values for the joint coordinates and the time duration of the motion segment.◂ The smoothly varying joint angles are wired to a `plot` block which will animate a robot in a separate window, and to an `fkine` block to compute the forward kinematics. Both the `plot` and `fkine` blocks have a parameter which is a `SerialLink` object, in this case `p560`. The Cartesian position of the end-effector pose is extracted using the `T2xyz` block which is analogous to the Toolbox function `transl`. The `XY Graph` block plots `y` against `x`.

The Simulink® integration time needs to be set equal to the motion segment time, through the `Simulation` menu or from the MATLAB® command line `>> sim('sl_jspace', 10);`.

7.4.2 Cartesian Motion

For many applications we require straight-line motion in Cartesian space which is known as Cartesian motion. This is implemented using the Toolbox function `ctraj` which was introduced in Chap. 3. Its usage is very similar to `jtraj`

```
>> Ts = ctraj(T1, T2, length(t));
```

where the arguments are the initial and final pose and the *number of* time steps and it returns the trajectory as a 3-dimensional matrix.

As for the previous joint-space example we will extract and plot the translation

```
>> plot(t, transl(Ts));
```

and orientation components

```
>> plot(t, tr2rpy(Ts));
```

of this motion which is shown in Fig. 7.11 along with the path of the end-effector in the *xy*-plane. Compared to Fig. 7.9b and c we note some important differences. Firstly the position and orientation, Fig. 7.11b and d vary linearly with time. For orientation the pitch angle is constant at zero and does not vary along the path. Secondly the end-effector follows a straight line in the *xy*-plane as shown in Fig. 7.9c.

The corresponding joint-space trajectory is obtained by applying the inverse kinematics

```
>> qc = p560.ikine6s(Ts);
```

and is shown in Fig. 7.11a. While broadly similar to Fig. 7.9a the minor differences are what result in the straight line Cartesian motion.

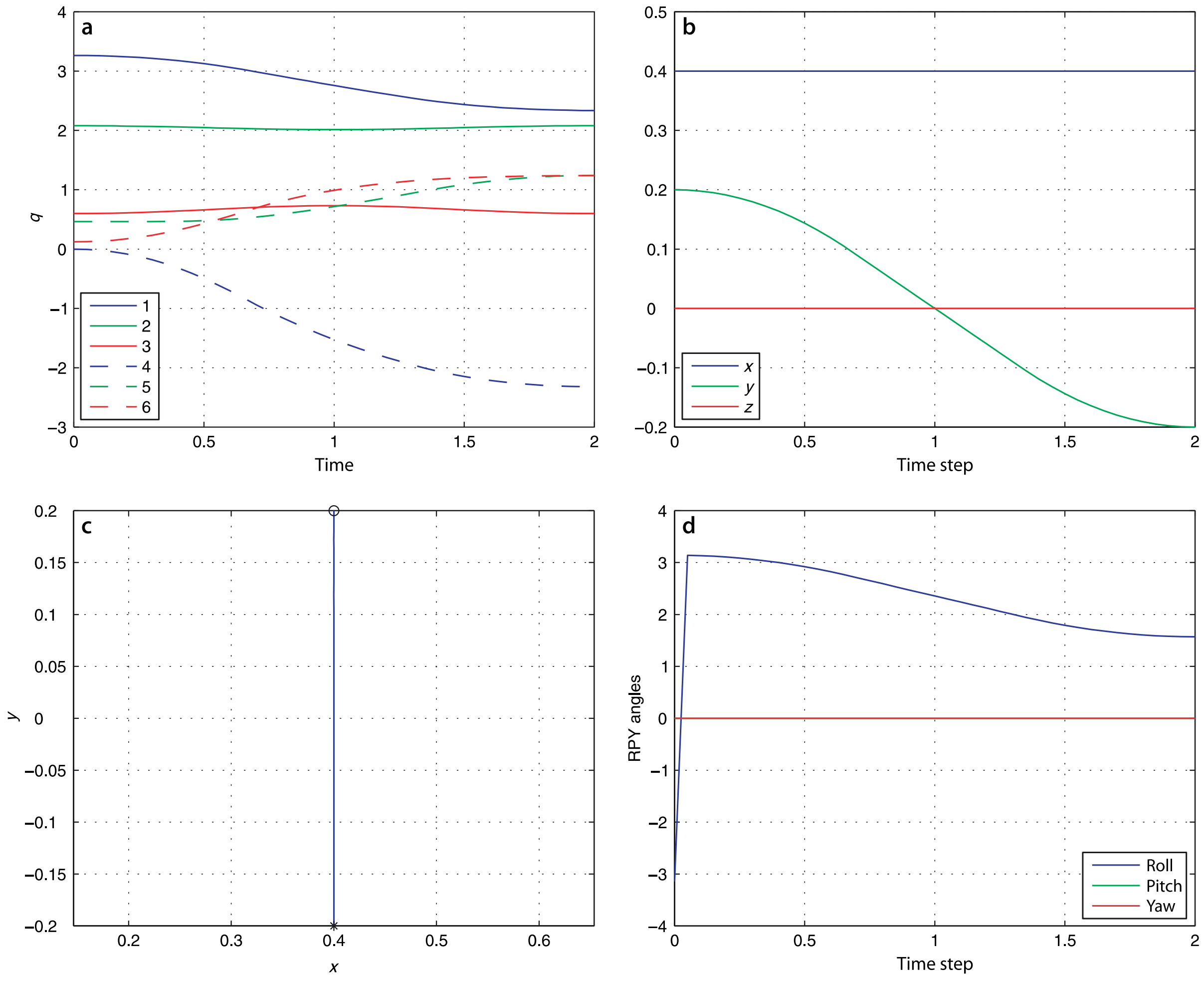


Fig. 7.11. Cartesian motion. **a** Joint coordinates versus time; **b** Cartesian position versus time; **c** Cartesian position locus in the *xy*-plane; **d** roll-pitch-yaw angles versus time

7.4.3 Motion through a Singularity

We have already briefly touched on the topic of singularities (page 148) and we will revisit it again in the next chapter. In the next example we deliberately choose a trajectory that moves through a robot singularity. We change the Cartesian endpoints of the previous example to

```
>> T1 = transl(0.5, 0.3, 0.44) * troty(pi/2);
>> T2 = transl(0.5, -0.3, 0.44) * troty(pi/2);
```

which results in motion in the y-direction with the end-effector z-axis pointing in the x-direction. The Cartesian path is

```
>> Ts = ctraj(T1, T2, length(t));
```

which we convert to joint coordinates

```
>> qc = p560.ikine6s(Ts)
```

and is shown in Fig. 7.12a. At time $t \approx 0.7$ s we observe that the rate of change of the wrist joint angles q_4 and q_6 has become very high.▸ The cause is that q_5 has become almost zero which means that the q_4 and q_6 rotational axes are almost aligned – another gimbal lock situation or singularity.

q_6 has increased rapidly, while q_4 has decreased rapidly and wrapped around from $-\pi$ to π.

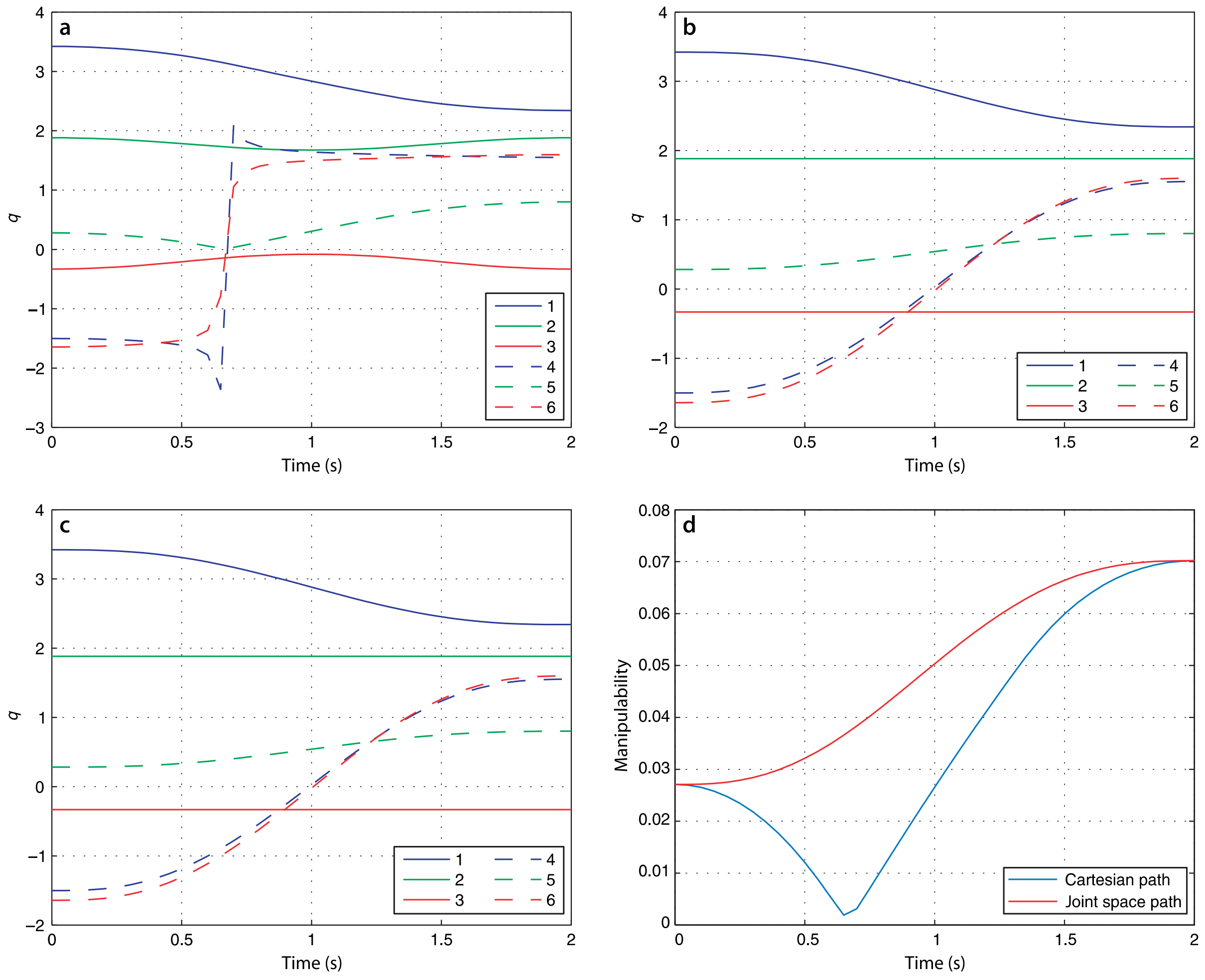


Fig. 7.12. Cartesian motion through a wrist singularity. **a** Joint coordinates computed by inverse kinematics (`ikine6s`); **b** joint coordinates computed by generalized inverse kinematics (`ikine`); **c** joint coordinates for joint-space motion; **d** manipulability

The joint alignment means that the robot has lost one degree of freedom and is now effectively a 5-axis robot. Kinematically we can only solve for the sum $q_4 + q_6$ and there are an infinite number of solutions for q_4 and q_6 that would have the same sum. The joint-space motion between the two poses, Fig. 7.12b, is immune to this problem since it is does not require the inverse kinematics. However it will not maintain the pose of the tool in the *x*-direction for the whole path – only at the two end points. From Fig. 7.12c we observe that the generalized inverse kinematics method `ikine` handles the singularity with ease.

The manipulability measure for this path

```
>> m = p560.maniplty(qc);
```

is plotted in Fig. 7.12d and shows that manipulability was almost zero around the time of the rapid wrist joint motion. Manipulability and the generalized inverse kinematics function are based on the manipulator's Jacobian matrix which is the topic of the next chapter.

7.4.4 Configuration Change

Earlier (page 147) we discussed the kinematic configuration of the manipulator arm and how it can work in a left- or right-handed manner and with the elbow up or down. Consider the problem of a robot that is working for a while left-handed at one work

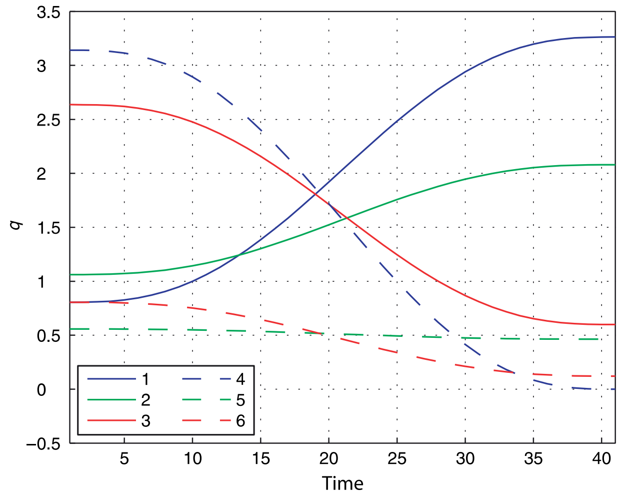


Fig. 7.13.
Joint space motions for configuration change from right-handed to left-handed

station, then working right-handed at another. Movement from one configuration to another ultimately results in no change in the end-effector pose since both configuration have the same kinematic solution – therefore we *cannot* create a trajectory in Cartesian space. Instead we must use joint-space motion.

For example to move the robot arm from the right- to left-handed configuration we first define some end-effector pose

```
>> T = transl(0.4, 0.2, 0) * trotx(pi);
```

and then determine the joint coordinates for the right- and left-handed elbow-up configurations

```
>> qr = p560.ikine6s(T, 'ru');
>> ql = p560.ikine6s(T, 'lu');
```

and then create a path between these two joint coordinates

```
>> q = jtraj(qr, ql, t);
```

Although the initial and final end-effector pose is the same, the robot makes some quite significant joint space motion as shown in Fig. 7.13 – in the real world you need to be careful the robot doesn't hit something. Once again, the best way to visualize this is in animation

```
>> p560.plot(q)
```

7.5 Advanced Topics

7.5.1 Joint Angle Offsets

The pose of the robot with zero joint angles is frequently some rather unusual (or even mechanically unachievable) pose. For the Puma robot the zero-angle pose is a non-obvious *L-shape* with the upper arm horizontal and the lower arm vertically upward as shown in Fig. 7.5a. This is a consequence of constraints imposed by the Denavit-Hartenberg formalism. A robot control designer may choose the zero-angle configuration to be something more obvious such as that shown in Fig. 7.5b or c.

The joint coordinate offset provides a mechanism to set an arbitrary configuration for the zero joint coordinate case. The offset vector, $\boldsymbol{q}_0$, is added to the user specified joint angles before any kinematic or dynamic function is invoked,▸ for example

It is actually implemented within the `Link` object.

$$\xi = \mathcal{K}(\boldsymbol{q} + \boldsymbol{q}_0) \tag{7.6}$$

Similarly it is subtracted after an operation such as inverse kinematics

$$\boldsymbol{q} = \mathcal{K}^{-1}(\xi) - \boldsymbol{q}_0 \tag{7.7}$$

The offset is set by assigning the `offset` property of the `Link` object, for example

```
>> L = Link([0 0 1 0]);
>> L.offset = pi/4
```

or

```
>> p560.links(2).offset = pi/2;
```

7.5.2 Determining Denavit-Hartenberg Parameters

The classical method of determining Denavit-Hartenberg parameters is to systematically assign a coordinate frame to each link. The link frames for the Puma robot using the standard Denavit-Hartenberg formalism are shown in Fig. 7.14a. However there are strong constraints on placing each frame since joint rotation must be about the *z*-axis and the link displacement must be in the *x*-direction. This in turn imposes constraints on the placement of the coordinate frames for the base and the end-effector, and ultimately dictates the zero-angle pose just discussed. Determining the Denavit-Hartenberg parameters and link coordinate frames for a completely new mechanism is therefore more difficult than it should be – even for an experienced roboticist.

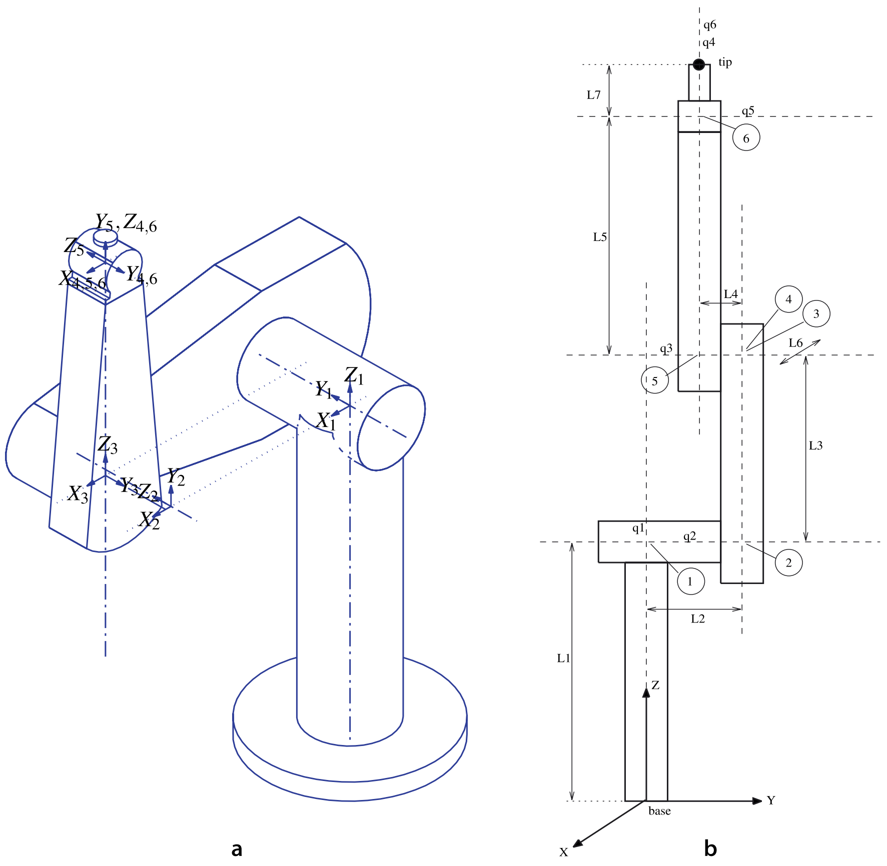


Fig. 7.14. Puma 560 robot coordinate frames. **a** Standard Denavit-Hartenberg link coordinate frames for Puma in the zeroangle pose (Corke 1996b); **b** alternative approach showing the sequence of elementary transforms from base to tip. Rotations are about the axes shown as dashed lines (Corke 2007)

An alternative approach, supported by the Toolbox, is to simply describe the manipulator as a series of elementary translations and rotations from the base to the tip of the end-effector. Some of the the elementary operations are constants such as translations that represent link lengths or offsets, and some are functions of the generalized joint coordinates q_i. Unlike the conventional approach we impose no constraints on the axes about which these rotations or translations can occur.

An example for the Puma robot is shown in Fig. 7.14b. We first define a convenient coordinate frame at the base and then write down the sequence of translations and rotations, from "toe to tip", in a string

```
>> s = 'Tz(L1) Rz(q1) Ry(q2) Ty(L2) Tz(L3) Ry(q3) Tx(L6) Ty(L4)
              Tz(L5) Rz(q4) Ry(q5) Rz(q6)'
```

Note that we have described the second joint as `Ry(q2)`, a rotation about the y-axis, which is not possible using the Denavit-Hartenberg formalism.

This string is input to a symbolic algebra function▸

Written in Java and part of the Robotics Toolbox, the MATLAB® Symbolic Toolbox is not required.

```
>> dh = DHFactor(s);
```

which returns a `DHFactor` object▸ that holds the kinematic structure of the robot that has been factorized into Denavit-Hartenberg parameters. We can display this in a human-readable form

Actually a Java object.

```
>> dh.display
DH(q1, L1, 0, -90).DH(q2+90, 0, -L3, 0).DH(q3-90, L2+L4, L6, 90).
DH(q4, L5, 0, -90).DH(q5, 0, 0, 90).DH(q6, 0, 0, 0)
```

which shows the Denavit-Hartenberg parameters for each joint in the order θ, d, a and α. Joint angle offsets (the constants added to or subtracted from joint angle variables such as `q2` and `q3`) are generated automatically, as are base and tool transformations. The object can generate a string that is a complete Toolbox command to create the robot named "puma"

```
>> cmd = dh.command('puma')
cmd =
SerialLink([-pi/2, 0, 0, L1; 0, -L3, 0, 0; pi/2, L6, 0, L2+L4;
     -pi/2, 0, 0, L5; pi/2, 0, 0, 0; 0, 0, 0, 0; ], ...
 'name', 'puma', ...
 'base', eye(4,4), 'tool', eye(4,4), ...
 'offset', [0 pi/2 -pi/2 0 0 0 ])
```

which can be executed

```
>> robot = eval(cmd)
```

to create a workspace variable called `robot` that is a `SerialLink` object.▸

The length parameters `L1` to `L6` must be defined in the workspace first.

7.5.3 Modified Denavit-Hartenberg Notation

The Denavit-Hartenberg parameters introduced in this chapter is commonly used and described in many robotics textbooks. Craig (1986) first introduced the modified Denavit-Hartenberg parameters where the link coordinate frames shown in Fig. 7.15 are attached to the near (proximal), rather than the far end of each link. This modified notation is in some ways clearer and tidier and is also now commonly used. However its introduction has increased the scope for confusion, particularly for those who are new to robot kinematics. The root of the problem is that the algorithms for kinematics, Jacobians and dynamics depend on the kinematic conventions used. According to Craig's convention the link transform matrix is

$$^{j-1}\boldsymbol{A}_j = \boldsymbol{R}_x(\alpha_{j-1})\boldsymbol{T}_x(a_{j-1})\boldsymbol{R}_z(\theta_j)\boldsymbol{T}_z(d_j) \qquad (7.8)$$

denoted by Craig as $^{j-1}_{\ \ j}\boldsymbol{A}$. This has the same terms as Eq. 7.1 but in a different order – remember rotations are not commutative – and this is the nub of the problem. a_j is

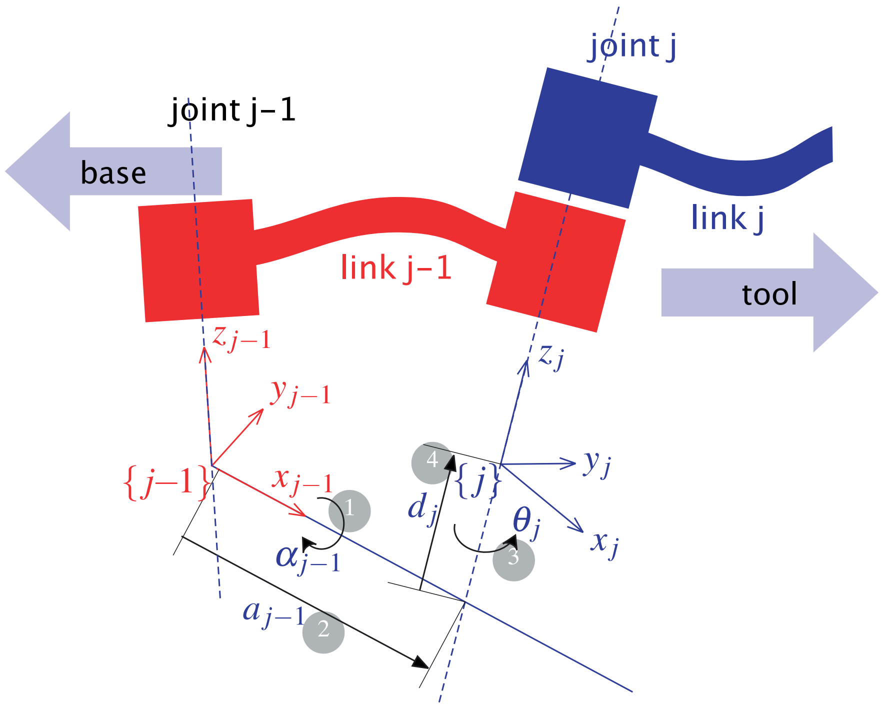


Fig. 7.15. Definition of modified Denavit and Hartenberg link parameters. The colors red and blue denote all things associated with links $j-1$ and j respectively. The numbers in circles represent the order in which the elementary transforms are applied

always the length of link j, but it is the displacement between the origins of frame $\{j\}$ and frame $\{j+1\}$ in one convention, and frame $\{j-1\}$ and frame $\{j\}$ in the other.

If you intend to build a Toolbox robot model from a table of kinematic parameters provided in a paper it is really important to know which convention the author of the table used. Too often this important fact is not mentioned. An important clue lies in the column headings. If they all have the same subscript, i.e. θ_j, d_j, a_j and α_j then this is standard Denavit-Hartenberg notation. If half the subscripts are different, i.e. θ_j, d_j, a_{j-1} and α_{j-1} then you are dealing with modified Denavit-Hartenberg notation. In short, you must know which kinematic convention your Denavit-Hartenberg parameters conform to.

You can also help the cause when publishing by stating clearly which kinematic convention is used for your parameters.

The Toolbox can handle either form, it only needs to be specified, and this is achieved via an optional argument when creating a link object

```
>> L1 = link([0 1 0 0 0], 'modified')
L1 =
         q1           0           1           0 (modDH)
```

rather than

```
>> L1 = link([0 1 0 0 0])
L1 =
         q1           0           1           0 (stdDH)
```

Everything else from here on, creating the robot object, kinematic and dynamic functions works as previously described.

The two forms can be interchanged by considering the link transform as a string of elementary rotations and translations as in Eq. 7.1 or Eq. 7.8. Consider the transformation chain for standard Denavit-Hartenberg notation

$$\underbrace{\boldsymbol{T}_{Rz}(\theta_1)\boldsymbol{T}_z(d_1)\boldsymbol{T}_x(a_1)\boldsymbol{T}_{Rx}(\alpha_1)}_{\mathrm{DH}_1} \cdot \underbrace{\boldsymbol{T}_{Rz}(\theta_2)\boldsymbol{T}_z(d_2)\boldsymbol{T}_x(a_2)\boldsymbol{T}_{Rx}(\alpha_2)}_{\mathrm{DH}_2} \cdots$$

which we can regroup as

$$\underbrace{\boldsymbol{T}_{Rz}(\theta_1)\boldsymbol{T}_z(d_1)}_{\text{base}} \cdot \underbrace{\boldsymbol{T}_x(a_1)\boldsymbol{T}_{Rx}(\alpha_1)\boldsymbol{T}_{Rz}(\theta_2)\boldsymbol{T}_z(d_2)}_{\text{MDH}_1} \cdot \underbrace{\boldsymbol{T}_x(a_2)\boldsymbol{T}_{Rx}(\alpha_2)\cdots}_{\text{MDH}_2}$$

where the terms marked as MDH_j have the form of Eq. 7.8 taking into account that translation along, and rotation about the same axis *is* commutative, that is, $\boldsymbol{T}_{Rk}(\theta)\boldsymbol{T}_k(d) = \boldsymbol{T}_k(d)\boldsymbol{T}_{Rk}(\theta)$ for $k \in \{x, y, z\}$.

7.6 Application: Drawing

Our goal is to create a trajectory that will allow a robot to draw the letter 'E'. Firstly we define a number of via points that define the strokes of the letter

```
>> path = [ 1 0 1; 1 0 0; 0 0 0; 0 2 0; 1 2 0;
      1 2 1; 0 1 1; 0 1 0; 1 1 0; 1 1 1];
```

which is defined in the xy-plane in arbitrary units. The pen is down when $z = 0$ and up when $z > 0$. The path segments can be plotted

```
>> plot3(path(:,1), path(:,2), path(:,3), 'color', 'k', 'LineWidth', 2)
```

as shown in Fig. 7.16. We convert this to a continuous path

```
>> p = mstraj(path, [0.5 0.5 0.3], [], [2 2 2], 0.02, 0.2);
```

where the second argument is the maximum speed in the x-, y- and z-directions, the fourth argument is the initial coordinate followed by the sample interval and the acceleration time. The number of steps in the interpolated path is

```
>> about(p)
p [double] : 1563x3 (37512 bytes)
```

and will take

```
>> numrows(p) * 0.02
ans =
   31.2600
```

seconds to execute at the 20 ms sample interval.

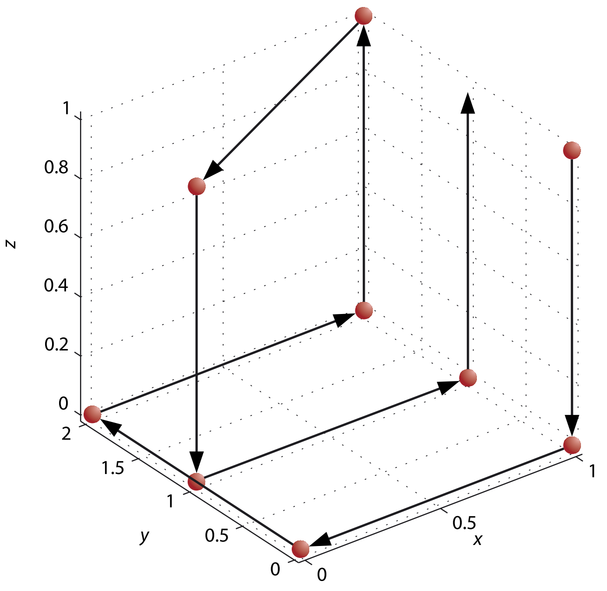


Fig. 7.16. The letter 'E' drawn with a 10-point path. Markers show the via points and solid lines the motion segments

p is a sequence of x-y-z-coordinates which we must convert into a sequence of Cartesian poses. The path we have defined draws a letter that is two units tall and one unit wide so the coordinates will be scaled by a factor of 0.1 (making the letter 200 mm tall by 100 mm wide)

```
>> Tp = transl(0.1 * p);
```

which is a sequence of homogeneous transformations describing the pose at every point along the path. The origin of the letter will be placed at (0.4, 0, 0) in the workplace

```
>> Tp = homtrans( transl(0.4, 0, 0), Tp);
```

which premultiplies each pose in Tp by the first argument.

Finally we need to consider orientation. Each of the coordinate frames defined in Tp assumes its z-axis is vertically upward. However the Puma robot is working on a horizontal surface with its elbow up and writing just like a person at a desk. The orientation of its tool, its z-axis, must therefore be downward. One way to fix this mismatch is by setting the robot's tool transform to make the tool axis point upwards

```
>> p560.tool = trotx(pi);
```

Now we can apply inverse kinematics

```
>> q = p560.ikine6s(Tp);
```

to determine the joint coordinates and then animate it.

```
>> p560.plot(q)
```

The Puma is drawing the letter 'E', and lifting its pen in between strokes! The approach is quite general and we could easily change the size of the letter, draw it on an arbitrary plane or use a robot with different kinematics.

7.7 Application: a Simple Walking Robot

Four legs good, two legs bad!
Snowball the pig, Animal Farm by George Orwell

Our goal is to create a four-legged walking robot. We start by creating a 3-axis robot *arm* that we use as a leg, plan a trajectory for the leg that is suitable for walking, and then instantiate four instances of the leg to create the walking robot.

7.7.1 Kinematics

Kinematically a robot leg is much like a robot arm, and for this application a three joint serial-link manipulator is sufficient. Determining the Denavit-Hartenberg parameters, even for a simple robot like this, is an involved procedure and the zero-angle offsets need to be determined in a separate step. Therefore we will use the procedure introduced in Sect. 7.5.2.

As always we start by defining our coordinate frame. This is shown in Fig. 7.17 along with the robot leg in its zero-angle pose. We have chosen the aerospace coordinate convention which has the x-axis forward and the z-axis downward, constraining the y-axis to point to the right-hand side. The first joint will be hip motion, forward and backward, which is rotation about the z-axis or $\boldsymbol{R}_z(q1)$. The second joint is hip motion up and down, which is rotation about the x-axis, $\boldsymbol{R}_x(q_2)$. These form a spherical hip joint since the axes of rotation intersect. The knee is translated by L_1 in the y-direction or $\boldsymbol{T}_y(L_1)$. The third joint is knee motion, toward and away from the body,

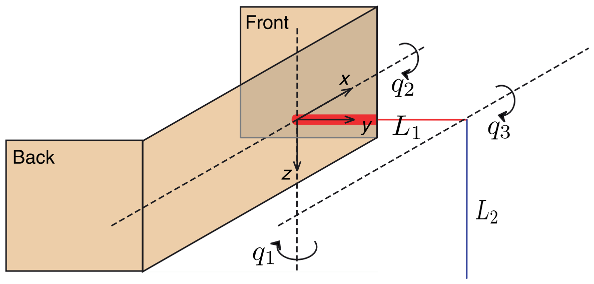


Fig. 7.17. The coordinate frame and axis rotations for the simple leg. The leg is shown in its zero angle pose

which is $\boldsymbol{R}_x(q_3)$. The foot is translated by L_2 in the z-direction or $\boldsymbol{T}_z(L_2)$. The transform sequence of this robot, from hip to toe, is therefore $\boldsymbol{R}_z(q1)\boldsymbol{R}_x(q_2)\boldsymbol{T}_y(L_1)\boldsymbol{R}_x(q_3)\boldsymbol{T}_z(L_2)$.

Using the technique of Sect. 7.5.2 we write this sequence as the string

```
>> s = 'Rz(q1).Rx(q2).Ty(L1).Rx(q3).Tz(L2)';
```

Note that length constants must start with L. The string is automatically manipulated into Denavit-Hartenberg factors

```
>> dh = DHFactor(s)
DH(q1+90, 0, 0, +90).DH(q2, L1, 0, 0).
DH(q3-90, L2, 0, 0).Rz(+90).Rx(-90).Rz(-90)
```

The last three terms in this factorized sequence is a tool transform

```
 >> dh.tool
ans =
trotz(pi/2)*trotx(-pi/2)*trotz(-pi/2)
```

that changes the orientation of the frame at the foot. However for this problem the foot is simply a point that contacts the ground so we are not concerned about its orientation. The method `dh.command` generates a string that is the Toolbox command to create a `SerialLink` object

```
>> dh.command('leg')
ans =
SerialLink([0, 0, 0, pi/2; 0, 0, L1, 0; 0, 0, -L1, 0; ],
  'name', 'leg', 'base', eye(4,4),
  'tool', trotz(pi/2)*trotx(-pi/2)*trotz(-pi/2),
  'offset', [pi/2 0 -pi/2 ])
```

which is input to the MATLAB® `eval` command

```
>> L1 = 0.1; L2 = 0.1;
>> leg = eval( dh.command('leg') )
>> leg
leg =
leg (3 axis, RRR, stdDH)
+---+-----------+-----------+-----------+-----------+
| j |     theta |         d |         a |     alpha |
+---+-----------+-----------+-----------+-----------+
|  1|         q1|          0|          0|      1.571|
|  2|         q2|          0|        0.1|          0|
|  3|         q3|          0|       -0.1|          0|
+---+-----------+-----------+-----------+-----------+

grav =     0  base = 1  0  0  0   tool =  0  0 -1  0
           0         0  1  0  0           0  1  0  0
        9.81         0  0  1  0           1  0  0  0
                     0  0  0  1           0  0  0  1
```

after first setting the the length of each leg segment to 100 mm in the MATLAB® workspace.

As usual we perform a quick sanity check of our robot. For zero joint angles the foot is at

```
>> transl( leg.fkine([0,0,0]) )'
ans =
         0    0.1000    0.1000
```

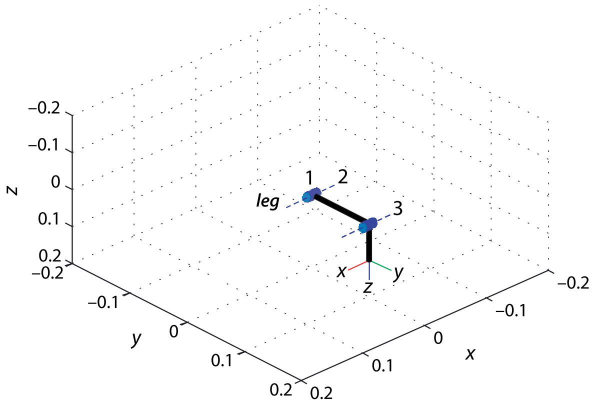


Fig. 7.18. Robot leg in its zero angle pose

as we designed it. We can visualize the zero-angle pose

```
>> leg.plot([0,0,0], 'nobase', 'noshadow')
>> set(gca, 'Zdir', 'reverse'); view(137,48);
```

which is shown in Fig. 7.18. Note that we tell MATLAB® that our *z*-axis points downward. Now we should test that the other joints result in the expected motion. Increasing q_1

```
>> transl( leg.fkine([0.2,0,0]) )'
ans =
   -0.0199    0.0980    0.1000
```

results in motion in the *xy*-plane, and increasing q_2

```
>> transl( leg.fkine([0,0.2,0]) )'
ans =
   -0.0000    0.0781    0.1179
```

results in motion in the *yz*-plane, as does increasing q_3

```
>> transl( leg.fkine([0,0,0.2]) )'
ans =
   -0.0000    0.0801    0.0980
```

We have now created and verified a simple robot leg.

7.7.2 Motion of One Leg

The next step is to define the path that the end-effector of the leg, its foot, will follow. The first consideration is that the end-effector of all feet move backwards at the same speed in the ground plane – propelling the robot's body forward without its feet slipping. Each leg has a limited range of movement so it cannot move backward for very long. At some point we must reset the leg – lift the foot, move it forward and place it on the ground again. The second consideration comes from static stability – the robot must have at least three feet on the ground at all times so each leg must take its turn to reset. This requires that any leg is in contact with the ground for ¾ of the cycle and is resetting for ¼ of the cycle. A consequence of this is that the leg has to move much faster during reset since it has a longer path and less time to do it in.

The required trajectory is defined by the via points

```
>> xf = 50; xb = -xf;  y = 50; zu = 20; zd = 50;
>> path = [xf y zd; xb y zd; xb y zu; xf y zu; xf y zd] * 1e-3;
```

where `xf` and `xb` are the forward and backward limits of leg motion in the *x*-direction (in units of mm), `y` is the distance of the foot from the body in the *y*-direction, and `zu` and `zd` are respectively the height of the foot in the *z*-direction for foot up and foot down. In this case the foot moves from 50 mm forward of the hip to 50 mm behind.

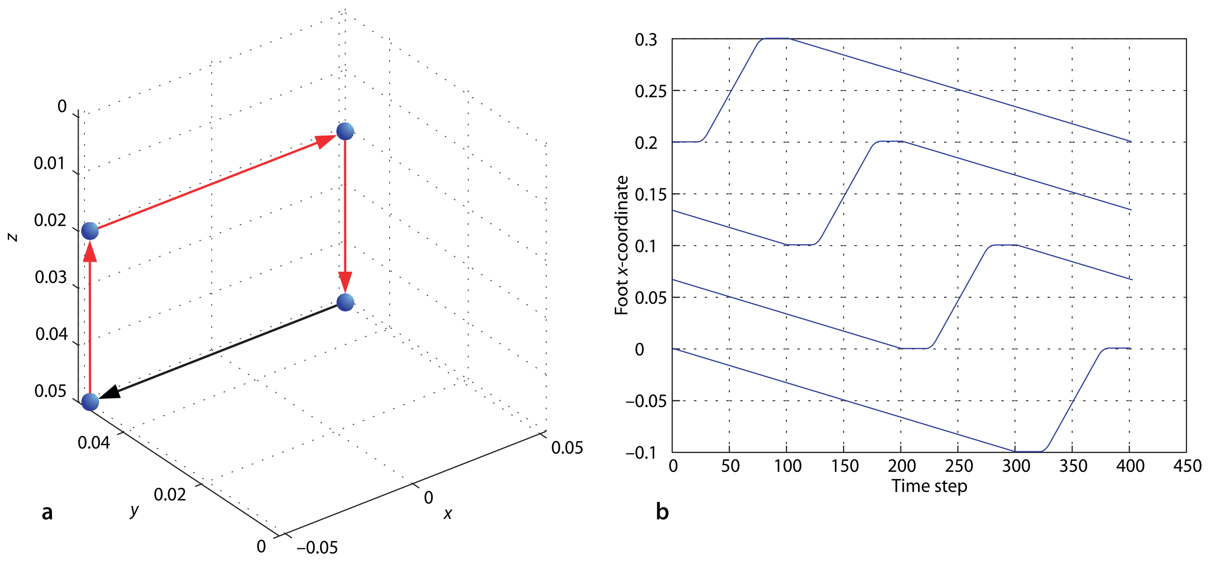


Fig. 7.19. a Trajectory taken by a single leg. Recall from Fig. 7.17 that the *z*-axis is downward. The red segments are the leg reset. **b** The *x*-direction motion of each leg (offset vertically) to show the gait. The leg reset is the period of high *x*-direction velocity

When the foot is down it is 50 mm below the hip and it is raised to 20 mm below the hip during reset. The points in `path` comprise a complete cycle correspond to the start of the stance phase, the end of stance, top of the leg lift, top of the leg return and the start of stance. This is shown in Fig. 7.19a.

Next we sample the multi-segment path at 100 Hz

```
>> p = mstraj(path, [], [0, 3, 0.25, 0.5, 0.25]', path(1,:), 0.01, 0);
```

In this case we have specified a vector of desired segment times rather than maximum joint velocities. The final three arguments are the initial position, the sample interval and the acceleration time. This trajectory has a total time of 4 s and therefore comprises 400 points.

This way we can ensure that the reset takes exactly one quarter of the cycle.

We apply inverse kinematics to determine the joint angle trajectories required for the foot to follow the path. This robot is underactuated so we use the generalized inverse kinematics `ikine` and set the mask so as to solve only for end-effector translation

```
>> qcycle = leg.ikine( transl(p), [], [1 1 1 0 0 0] );
```

We can view the motion of the leg in animation

```
>> leg.plot(qcycle, 'loop')
```

to verify that it does what we expect: slow motion along the ground, then a rapid lift, forward motion and foot placement. The `'loop'` option displays the trajectory in an endless loop and you need to type control-C to stop it.

7.7.3 Motion of Four Legs

Our robot has width and length

```
>> W = 0.1; L = 0.2;
```

We create multiple instances of the leg by cloning the `leg` object we created earlier, and providing different base transforms so as to attach the legs to different points on the body

```
>> legs(1) = SerialLink(leg, 'name', 'leg1');
>> legs(2) = SerialLink(leg, 'name', 'leg2', 'base', transl(-L, 0, 0));
>> legs(3) = SerialLink(leg, 'name', 'leg3', 'base', transl(-L, -W, 0)*trotz(pi));
>> legs(4) = SerialLink(leg, 'name', 'leg4', 'base', transl(0, -W, 0)*trotz(pi));
```

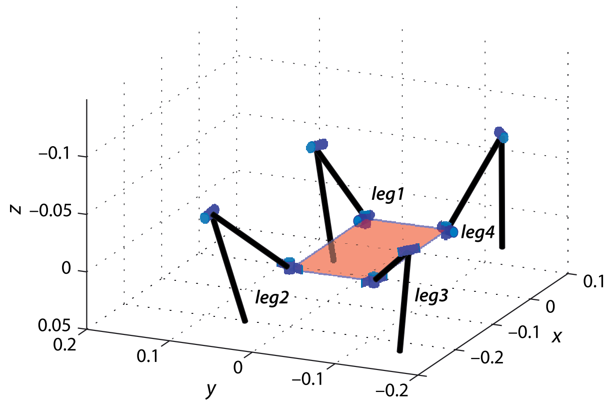


Fig. 7.20. The walking robot

The result is a vector of `SerialLink` objects. Note that legs 3 and 4, on the left-hand side of the body have been rotated about the *z*-axis so that they point away from the body.

As mentioned earlier each leg must take its turn to reset. Since the trajectory is a cycle, we achieve this by having each leg run the trajectory with a phase shift equal to one quarter of the total cycle time. Since the total cycle has 400 points, each leg's trajectory is offset by 100, and we use modulo arithmetic to index into the cyclic gait for each leg. The result is the gait pattern shown in Fig. 7.19b.

The core of the walking program is

```
k = 1;
while 1
    q = qleg(p,:);
    legs(1).plot( gait(qcycle, k, 0,   0) );
    legs(2).plot( gait(qcycle, k, 100, 0) );
    legs(3).plot( gait(qcycle, k, 200, 1) );
    legs(4).plot( gait(qcycle, k, 300, 1) );
    drawnow
    k = k+1;
end
```

where the function

```
gait(q, k, ph, flip)
```

returns the `k+ph`th element of `q` with modulo arithmetic that considers `q` as a cycle. The argument `flip` reverses the sign of the joint 1 motion for the legs on the left-hand side of the robot. A snapshot from the simulation is shown in Fig. 7.20. The entire implementation, with some additional refinement, is in the file `examples/walking.m` and detailed explanation is provided by the comments.

7.8 Wrapping Up

In this chapter we have learnt how to describe a serial-link manipulator in terms of a set of Denavit-Hartenberg parameters for each link. We can compute the relative pose of each link as a function of its joint variable and compose these into the pose of the robot's end-effector relative to its base. For robots with six joints and a spherical wrist we can compute the inverse kinematics which is the set of joint angles required to achieve a particular end-effector pose. This inverse is not unique and the robot may have several joint configurations that result in the same end-effector pose.

For robots which do not have six joints and a spherical wrist we can use an iterative numerical approach to solving the inverse kinematics. We showed how this could be applied to an under-actuated 2-link robot and a redundant 8-link robot. We also touched briefly on the topic of singularities which are due to the the alignment of joints.

We also learnt about creating paths to move the end-effector smoothly between poses. Joint-space paths are simple to compute but do not result in straight line paths in Cartesian space which may be problematic for some applications. Straight line paths in Cartesian space can be generated but singularities in the workspace may lead to very high joint rates.

Further Reading

Serial-link manipulator kinematics are covered in all the standard robotics textbooks such as by Paul (1981), Spong et al. (2006), Siciliano et al. (2008). Craig's text (2004) is also an excellent introduction to robot kinematics and uses the modified Denavit-Hartenberg notation, and the examples in the third edition are based on an older version of the Robotics Toolbox. Most of these books derive the inverse kinematics for a two-link arm and a six-link arm with a spherical wrist. The first full description of the latter was by Paul and Zhang (1986).

Closed-form inverse kinematic solution can be derived symbolically by writing down a number of kinematic relationships and solving for the joint angles, as described in Paul (1981). Software packages to automatically generate the forward and inverse kinematics for a given robot have been developed and these include Robotica (Nethery and Spong 1994) and SYMORO+ (Khalil and Creusot 1997).

The original work by Denanvit and Hartenberg was their 1955 paper (Denavit and Hartenberg 1955) and their textbook (Hartenberg and Denavit 1964). The book has an introduction to the field of kinematics and its history but is currently out of print, although a version is available online. Siciliano et al. (2008) provide a very clear description of the process of assigning Denavit-Harteberg parameters to an arbitrary robot. The alternative approach based on symbolic factorization is described in detail by Corke (2007). This has some similarities to the product of exponential (POE) form proposed by Park (1994). The definitive values for the parameters of the Puma 560 robot are described in the paper by Corke and Armstrong-Hélouvry (1995).

Robotic walking is a huge field in its own right and the example given here is very simplistic. Machines have been demonstrated with complex gaits such as running and galloping that rely on dynamic rather than static balance. A good introduction to legged robots is given in the Robotics Handbook (Siciliano and Khatib 2008, § 16).

Parallel-link manipulators were mentioned briefly on page 136 and have advantages such as increased actuation force and stiffness (since the actuators form a truss-like structure). For this class of mechanism the inverse kinematics is usually closed-form and forward kinematics often requiring numerical solution. Useful starting points for this class of robots are a brief section in Siciliano et al. (2008), the handbook (Siciliano and Khatib 2008, § 12) and in Merlet (2006).

The `plot` Method

The `plot` method was introduced in Sect. 7.2.1 to display the pose of a robot. Many aspects of the created figure such as scaling, shadows, joint axes and labels, tool coordinate frame labels and so on can be customized. The `plot` method also supports multiple views of the same robot, and figures can contain multiple robots. This section provides examples of some of these features and, as always, the full description is available in the online documentation.

The `hold` method works in an analogous way to normal data plotting and allows multiple robots to be drawn into the one set of axes. First create a clone of our standard Puma robot, with its base at a different location

```
>> p560_2 = SerialLink(p560, 'name', 'puma #2', ...
 'base', transl(0.5,-0.5,0));
```

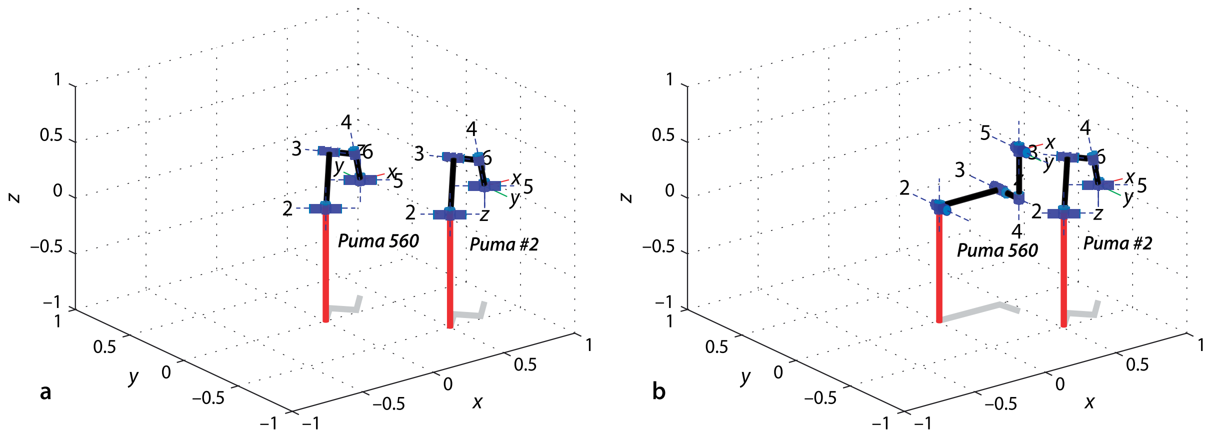


Fig. 7.21. `plot` displaying multiple robots per axis. **a** Both robots have the same joint coordinates; **b** the robots have different joint coordinates

Draw the first robot

```
>> p560.plot(qr)
```

then add the second robot

```
>> hold on
>> p560_2.plot(qr)
```

which gives the result shown in Fig. 7.21a. The graphical robots can be separately animated, for instance

```
>> p560.plot(qz)
```

changes the pose of only the first robot as shown in Fig. 7.21b.

We can also create multiple views of the same robot or robots. Following on from the example above, we create a new figure

```
>> figure
>> p560.plot(qz)
```

and plot the *same* robot. You could alter the viewpoint in this new figure using the MATLAB® figure toolbar options. Now

```
>> p560.plot(qr)
```

causes the robot to change in *both* figures. The key to this is the robot's name which is used to identify the robot irrespective of which figure it appears in. The name of each robot must be unique for this feature to work.

The algorithm adopted by `plot` with respect to the usage of existing figures is:

1. If a robot with the specified name exists in the current figure, then redraw it with the specified joint angles. Also update the same named robot in all other figures.
2. If no such named robot exists, then if hold is on the robot will be added to the current plot else a new graphical robot of the specified name will be created in the current window.
3. If there are no figures, then a new figure will be created.

The robot can be *driven* manually using a graphical teach-pendant interface

```
>> p560.teach()
```

which displays a GUI with one slider per joint. As the sliders are moved the specified graphical robot joint moves, in all the figures in which it has been drawn.

Exercises

1. Experiment with the `teach` method.
2. Derive the forward kinematics for the 2-link robot from first principles. What is end-effector position (x, y) given q_1 and q_2?
3. Derive the inverse kinematics for the 2-link robot (page 141) from first principles. What are the joint angles (q_1, q_2) given the end-effector coordinates (x, y)?
4. Compare the solutions generated by `ikine6s` and `ikine` for the Puma 560 robot at different poses. Is there any difference in accuracy? How much slower is `ikine`?
5. For the Puma 560 at configuration `qn` demonstrate a configuration change from elbow up to elbow down.
6. Drawing an 'E' (page 162)
   a) Change the size of the letter.
   b) Construct the path for the letter 'C'.
   c) Write the letter on a vertical plane.
   d) Write the letter on an inclined plane.
   e) Change the robot from a Puma 560 to the Fanuc 10L.
   f) Write the letter on a sphere.
   g) This writing task does not require 6DOF since the rotation of the pen about its axis is not important. Remove the final link from the Puma 560 robot model and repeat the exercise.
7. Walking robot (page 163)
   a) Shorten the reset trajectory by reducing the leg lift during reset.
   b) Increase the stride of the legs.
   c) Figure out how to steer the robot by changing the stride length on one side of the body.
   d) Change the gait so the robot moves sideways like a crab.
   e) Add another pair of legs. Change the gait to reset two legs or three legs at a time.
   f) Currently in the simulation the legs move but the body does not move forward. Modify the simulation so the body moves.