10

System Identification

10.1 Purpose

Vibration is defined as an oscillatory movement of some entity about an equilibrium state. It is the means of producing sound in musical instruments, it is the principle underlying the design of loudspeakers, and it describes the response of buildings to earthquakes. The squealing of disc brakes on a car is caused by vibration. The up and down motion of a ship at sea is a low-frequency vibration. Spectral analysis provides the means for understanding and controlling vibration.

Vibration is generally caused by some external force acting on a system, and the relationship between the external force and the system response can be described by a mathematical model of the system dynamics. We can use spectral analysis to estimate the parameters of the mathematical model and then use the model to make predictions of the response of the system under different forces.

10.2 Identifying the gain of a linear system

10.2.1 Linear system

We consider systems that have clearly defined inputs and outputs, and aim to deduce the system from measurements of the inputs and outputs or to predict the output knowing the system and the input. Attempts to understand economies and to control inflation by increasing interest rates provide ambitious examples of applications of these principles.

A mathematical model of a dynamic system is linear if the output to a sum of input variables, x and y, equals the sum of the outputs corresponding to the individual inputs. More formally, a mathematical operator $\mathcal{L}$ is linear if it satisfies

$$\mathcal{L}(ax + by) = a\mathcal{L}(x) + b\mathcal{L}(y)$$

P.S.P. Cowpertwait and A.V. Metcalfe, *Introductory Time Series with R*,
Use R, DOI 10.1007/978-0-387-88698-5_10,

where a and b are constants. For a linear system, the output response to a sine wave input is a sine wave of the same frequency with an amplitude that is proportional to the amplitude of the input. The ratio of the output amplitude to the input amplitude, known as the gain, and the phase lag between input and output depend on the frequency of the input, and this dependence provides a complete description of a linear system.

Many physical systems are well approximated by linear mathematical models, provided the input amplitude is not excessive. In principle, we can identify a linear model by noting the output, commonly referred to as the response, to a range of sine wave inputs. But there are practical limitations to such a procedure. In many cases, while we may be able to measure the input, we certainly cannot specify it. Examples are wave energy devices moored at sea, and the response of structures to wind forcing. Even when we can specify the input, recording the output over a range of frequencies is a slow procedure. In contrast, provided we can measure the input and output, and the input has a sufficiently broad spectrum, we can identify the linear system from spectral analysis. Also, spectral methods have been developed for non-linear systems.

A related application of spectral analysis is that we can determine the spectrum of the response if we know the system and the input spectrum. For example, we can predict the output of a wave energy device if we have a mathematical model for its dynamics and know typical sea spectra at its mooring.

10.2.2 Natural frequencies

If a system is set in motion by an initial displacement or impact, it may oscillate, and this oscillation takes place at the *natural frequency* (or frequencies) of the system. A simple example is the oscillation of a mass suspended by a spring. Linear systems have large gains at natural frequencies and, if large oscillations are undesirable, designers need to ensure that the natural frequencies of the system are far removed from forcing frequencies. Alternatively, in the case of wave energy devices, for example, the designer may aim for the natural frequencies of the device to match predominant frequencies in the sea spectrum. A common example of forcing a system at its natural frequency is pushing a child on a swing.

10.2.3 Estimator of the gain function

If a linear system is forced by a sine wave of amplitude A at frequency f, the response has an amplitude $G(f)A$, where $G(f)$ is the gain at frequency f. The ratio of the variance of the output to the variance of the input, for sine waves at this frequency, is $G(f)^2$. If the input is a stationary random process rather than a single sine wave, its variance is distributed over a range of frequencies, and this distribution is described by the spectrum. It seems intuitively reasonable to estimate the square of the gain function by the ratio of the output spectrum to the input spectrum. Consider a linear system with a single input, x_t, and a single output, y_t. The gain function can be estimated by

$$\hat{G}(f) = \sqrt{\frac{C_{yy}(f)}{C_{uu}(f)}} \tag{10.1}$$

A corollary is that the output spectrum can be estimated if the gain function is known, or has been estimated, and the input spectrum has been estimated by

$$C_{yy} = G^2 C_{uu} \tag{10.2}$$

Equation (10.2) also holds if spectra are expressed in radians rather than cycles, in which case the gain is a function $G(\omega)$ of ω.

10.3 Spectrum of an AR(p) process

Consider the deterministic part of an AR(p) model with a complex sinusoid input,

$$x_t - \alpha_1 x_{t-1} - \ldots - \alpha_p x_{t-p} = e^{i\omega t} \tag{10.3}$$

Assume a solution for x_t of the form $A\,e^{i\theta}e^{i\omega t}$, where A is a complex number, and substitute this into Equation (10.3) to obtain

$$A = \left(1 - \alpha_1 e^{-i\omega} - \ldots - \alpha_p e^{-i\omega p}\right)^{-1} \tag{10.4}$$

The gain function, expressed as a function of ω, is the absolute value of A. Now consider a discrete white noise input, w_t, in place of the complex sinusoid. The system is now an AR(p) process. Applying Equation (10.2), with population spectra rather than sample spectra, and noting that the spectrum of white noise with unit variance is $1/\pi$ (§9.8.1), gives

$$\Gamma_{xx}(\omega) = |A|^2\,\Gamma_{ww} = \frac{1}{\pi}\left(1 - \alpha_1 e^{-i\omega} - \ldots - \alpha_p e^{-i\omega p}\right)^{-2} \qquad 0 \le \omega < \pi \tag{10.5}$$

The deterministic part of an AR(p) model is a *linear difference* equation of order p.

10.4 Simulated single mode of vibration system

The simplest linear model (SI units in parentheses) for a vibrating system is that of a mass m (N) on a spring of stiffness k (Nm^{-1}) with a damper characterised by a damping coefficient c (Nsm^{-1}). Denote the displacement of the mass by y, differentiation with respect to time by a dot placed above it, and the forcing term by x. If we apply Newton's second law of motion, equating the product of mass and acceleration with the forces acting on the mass, we obtain:

$$m\ddot{y} + c\dot{y} + ky = x \tag{10.6}$$

Equation (10.6) has the same form as that given in Exercise 1: the undamped natural frequency is $\sqrt{(k/m)}$ and the damping coefficient is $c/(2\sqrt{\{km\}})$. The gain is given by

$$G(\omega) = \left[(k - m\omega^2)^2 + c^2\omega^2\right]^{-\frac{1}{2}} \tag{10.7}$$

and the damped natural frequency, which corresponds to the maximum gain, is given by

$$\sqrt{\frac{k}{m}\left(1 - \frac{c^2}{4km}\right)} \tag{10.8}$$

These results can be derived by substituting $x = \sin(\omega t)$ and $y = G\sin(\omega t - \psi)$ into Equation (10.6) (also see Exercise 1).

Equation (10.6) represents a single mode of vibration because there is a single mass that is constrained to move in a straight line without rotating. The equation is a good model for a pendulum making small oscillations. It might be a reasonable model for vibration of a street lamp on a metal pole in gusts of wind from the same direction. It would be a poor model for vibration of a violin string because it could only describe the fundamental mode shape and would miss all of the harmonics and overtones. Nevertheless, Equation (10.6) is a widely used approximation in vibration analysis. Thomson (1993) is a nice introduction to the theory of vibration.

If we take a time step of Δ (that is, a small fraction of the unit of time) we can approximate derivatives by backward differences. Thus Equation (10.6) can be approximated by the difference equation

$$a_0 y_t + a_1 y_{t-1} + a_2 y_{t-2} = x_t \tag{10.9}$$

where

$$a_0 = \frac{m}{\Delta^2} + \frac{c}{\Delta} + k; \quad a_1 = -\frac{2m}{\Delta^2} - \frac{c}{\Delta}; \quad a_2 = \frac{m}{\Delta^2}$$

The following short `R` script investigates a difference equation approximation to a lightly damped system represented by Equation (10.6) with $m = 1$, $c = 1$, and $k = 16.25$. The undamped natural frequency is 4.03 and the damping coefficient is 0.124, so the damped natural frequency is 4 radians per second, assuming time is measured in seconds. The maximum gain is obtained by substituting the damped natural frequency into Equation (10.7) and is 0.250. Equation (10.6) is approximated with Equation (10.9). The input x_t is an AR(2) process with α_1 and α_2 set at 1 and -0.5, respectively, driven by Gaussian white noise with unit variance. The sampling rate is 100 per second, so the spectrum is defined from 0 to 50 Hz. The record length, n, is 100,000 and `R` calculates the spectrum at 50,000 points. The natural frequency is around 0.64 Hz, so the gain function is only plotted up to a frequency of 5 Hz. The arrays Freq, FreH, Omeg, and OmegH contain the discretized frequencies in cycles per sampling interval, Hz, radians per sampling interval, and radians per second, respectively. Gth is the theoretical gain of the linear system. Gemp is the empirical estimate of the gain calculated as the square root of the ratio of the output spectrum to the input spectrum. Gar is the theoretical gain of the difference equation approximation, and it is indistinguishable from the empirical estimate (Fig. 10.1). As the signals are noise-free this is not surprising. You are asked to investigate the effects of adding noise to the input and output signals in Exercise 2. The empirical estimate of the gain identifies the natural frequency accurately but slightly underestimates the maximum gain, and you are asked to investigate possible reasons for this in Exercise 3.

```
> m <- 1; c <- 1; k <- 16.25; Delta <- 0.01
> a0 <- m / Delta^2 + c / Delta + k
> a1 <- -2 * m / Delta^2 - c / Delta; a2 <- m / Delta^2
> n <- 100000
> y <- c(0, 0); x <- c(0, 0)
> set.seed(1)
> for (i in 3:n) {
     x[i] <- x[i-1] - 0.5 * x[i-2] + rnorm(1)
     y[i] <- (-a1 * y[i-1] - a2 * y[i-2]) / a0 + x[i] / a0
  }
> Sxx <- spectrum(x, span = 31)
> Syy <- spectrum(y, span = 31)
> Gemp <- sqrt( Syy$spec[1:5000] / Sxx$spec[1:5000] )
> Freq <- Syy$freq[1:5000]
> FreH <- Freq / Delta
> Omeg  <- 2 * pi * Freq
> OmegH <- 2 * pi * FreH
> Gth <- sqrt( 1/( (k-m*OmegH^2)^2 + c^2*OmegH^2 ))
> Gar <- 1 / abs( 1 + a1/a0 * exp(-Omeg*1i) + a2/a0 * exp(-Omeg*2i) )
> plot(FreH, Gth, xlab = "Frequency (Hz)", ylab = "Gain", type="l")
> lines(FreH, Gemp, lty = "dashed")
> lines(FreH, Gar, lty = "dotted")
```

10.5 Ocean-going tugboat

The motion of ships and aircraft is described by displacements along the orthogonal x, y, and z axes and rotations about these axes. The displacements are surge, sway, and heave along the x, y, and z axes, respectively. The rotations about the x, y, and z axes are roll, pitch, and yaw, respectively (Fig. 10.2). So, there are six degrees of freedom for a ship's motion in the ocean, and there are six natural frequencies. However, the natural frequencies will not usually correspond precisely to the displacements and rotations, as there is a coupling between displacements and rotations. This is typically most pronounced between heave and pitch. There will be a natural frequency with

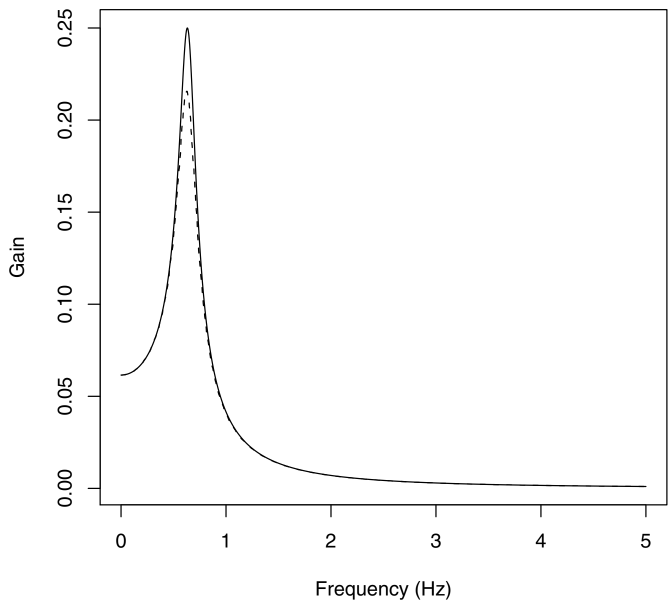

Fig. 10.1. Gain of single-mode linear system. The theoretical gain is shown by a solid line and the estimate made from the spectra obtained from the difference equation is shown by a broken line. The theoretical gain of the difference equation is plotted as a dotted line and coincides exactly with the estimate.

a corresponding mode that is predominantly heave, with a slight pitch, and another natural frequency that is predominantly pitch, with a slight heave.

Naval architects will start with computer designs and then proceed to model testing in a wave tank before building a prototype. They will have a good idea of the frequency response of the ship from the models, but this will have to be validated against sea trials. Here, we analyse some of the data from the sea trials of an ocean-going tugboat. The ship sailed over an octagonal course, and data were collected on each leg. There was an impressive array of electronic instruments and, after processing analog signals through anti-aliasing filters, data were recorded at 0.5s intervals for roll (degrees), pitch (degrees), heave (m), surge (m), sway (m), yaw (degrees), wave height (m), and wind speed (knots).

```
> www <- "http://www.massey.ac.nz/~pscowper/ts/leg4.dat"
> tug.dat <- read.table(www, header = T)
> attach(tug.dat)
> Heave.spec <- spectrum( Heave, span = sqrt( length(Heave) ),
                                 log = c("no"), main = "" )
```

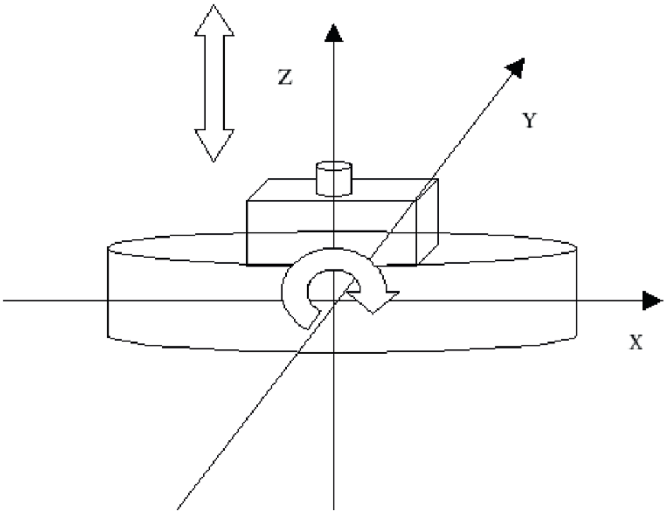

Fig. 10.2. Orthogonal axes for describing motion of a ship. Heave and pitch are shown by block arrows.

```
> Wave.spec  <- spectrum( Wave,  span = sqrt( length(Heave) ),
                                  log = c("no"), main = "" )
> G <- sqrt(Heave.spec$spec/Wave.spec$spec)
> par(mfcol = c(2, 2))
> plot( as.ts(Wave) )
> acf(Wave)
> spectrum(Wave, span = sqrt(length(Heave)), log = c("no"), main = "")
> plot(Heave.spec$freq, G, xlab="frequency Hz", ylab="Gain", type="l")
```

Figure 10.3 shows the estimated wave spectrum and the estimated gain from wave height to heave. The natural frequencies associated with the heave/pitch modes are estimated as 0.075 Hz and 0.119 Hz, and the corresponding gains from wave to heave are 0.15179 and 0.1323. In theory, the gain will approach 1 as the frequency approaches 0, but the sea spectrum has negligible components very close to 0, and no sensible estimate can be made. Also, the displacements were obtained by integrating accelerometer signals, and this is not an ideal procedure at very low frequencies.

10.6 Non-linearity

There are several reasons why the hydrodynamic response of a ship will not be precisely linear. In particular, the varying cross-section of the hull accounts

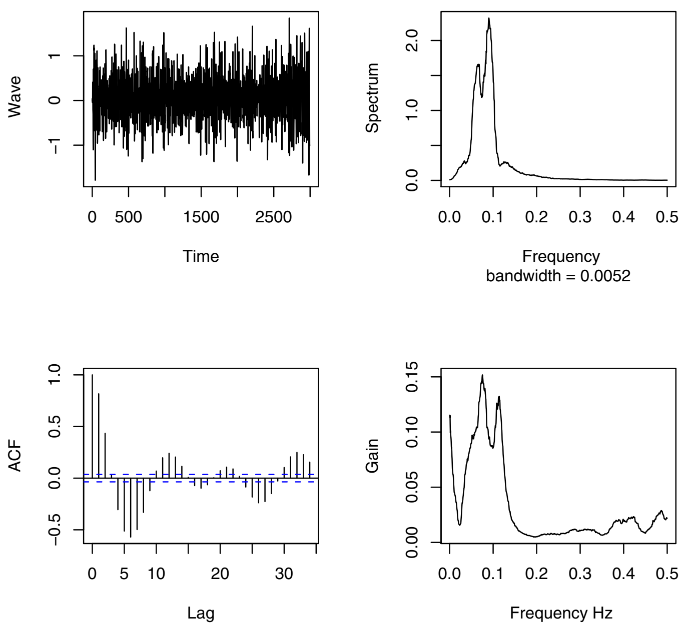

Fig. 10.3. Gain of heave from wave.

for non-linear buoyancy forces. Metcalfe et al. (2007) investigate this by fitting a regression of the heave response on lagged values of the response, squares, and cross-products of these lagged values, wave height, and wind speed. The probing method looks at the response of the fitted model to the sum of two complex sinusoids at frequencies ω_1 and ω_2. The non-linear response can be shown as a three-dimensional plot of the gain surface against frequency ω_1 and ω_2 or by a contour diagram. However, in this particular application the gain associated with the non-linear terms was small compared with the gain of the linear terms (Metcalfe et al., 2007). This is partly because the model was fitted to data taken when the ship was in typical weather conditions – under extreme conditions, when capsizing is likely, linear models are inadequate.

10.7 Exercises

1. The differential equation that describes the motion of a linear system with a single mode of vibration, such as a mass on a spring, has the general form

$$\ddot{y} + 2\zeta\Omega\dot{y} + \Omega^2 y = x$$

The parameter Ω is the undamped natural frequency, and the parameter ζ is the damping coefficient. The response is oscillatory if $\zeta < 1$.

a) Refer to Equation 10.7 and express ζ and Ω in terms of m, c, and k.
b) Suppose there is no forcing term ($x = 0$), assume that $y = e^{mt}$, and substitute into the general form of the differential equation. Show that $m = -\zeta\Omega \pm i\sqrt{[\Omega^2(1-\zeta^2)]}$. The damped natural frequency is $\Omega\sqrt{(1-\zeta^2)}$.
c) Take the initial condition of the unforced system as $y = 1$ when $t = 0$. Find the solution for y, and explain why this is referred to as the transient response.
d) Now consider a periodic forcing term $x = e^{i\omega t}$. Write the steady state response, y, as $y = Ae^{i(\omega t+\phi)}$. Substitute into the general form of the differential equation and show that

$$A = \left(\Omega^2 - \omega^2 + 4\zeta^2\omega^2\Omega^2\right)^{-1/2}$$

$$\tan(\Omega) = \frac{2\zeta\omega\Omega}{\Omega^2 - \omega^2}$$

2. Refer to the R script in §10.4, which compares a difference equation approximation to a model of a mass vibrating on a spring with the theoretical results. Insert another loop after that in lines 6–10 to simulate measurement noise added to the input x and y:

```
for (i in 1:n) {
  x[i] <- x[i] + nax * rnorm(1)
  y[i] <- y[i] + nay * rnorm(1)
}
```

Note that you also need to specify numerical values for the noise amplitudes, nax and nay, earlier in your script.

a) Why does the addition of noise need to be put in a separate loop?
b) How does the addition of white measurement noise to the output, but not the input, affect the estimate of the spectrum?
c) How does the addition of independent white measurement noise to both input and output affect the estimate of the spectrum?

3. The difference equation approximation used in §10.4 underestimates the maximum gain.

a) Investigate the effect of the span parameter.
b) Investigate the effect of increasing the sampling rate to 1000 per second.
c) Investigate the effect of using a centred difference approximation to the derivatives.

$$\dot{y} \approx \frac{y_{t+1} - y_{t-1}}{2\Delta}$$