

Chapter 13

Consequences of the Completeness of \mathbb{R}

We have now reached the point at which we can give rigorous proofs of two facts you have known for some time. We mention the second first; that is, we will conclude this chapter by showing that there is a nonempty bounded subset of \mathbb{Q} that does not have a supremum in \mathbb{Q} . As an exercise, you will show that there is also a nonempty bounded subset of \mathbb{Q} that does not have an infimum in \mathbb{Q} . In this sense, \mathbb{Q} is not complete. Since \mathbb{R} is complete, there must be numbers in \mathbb{R} that are not in \mathbb{Q} . We know what these numbers are, of course; they are the irrational numbers. Thus far we haven't proven that a particular real number is irrational. But now we can! This will be the first fact that we prove. Here's a rough outline: We will show that if a is a positive real number, then a has a positive square root; that is, there exists $x \in \mathbb{R}^+$ such that $x^2 = a$. But Theorem 5.2 told us that $\sqrt{2}$ is not rational. Thus, we know $\sqrt{2}$ exists and we know it is not rational. Therefore, we have a rigorous proof that $\sqrt{2}$ is irrational.

This might be a good time to put in a plug for a project: It's much harder than you might think to prove that a particular number is irrational. We take for granted that e and π are irrational, but proving this is really not so easy. You can do so if you work Project 29.5.

Before we get to the heart of this chapter, we will get a little more practice with least and greatest lower bound. You should solve the exercise below before proceeding to the next theorem.

Exercise 13.1. Let $a = \sup\{w \in \mathbb{R}^+ : w^2 < 2\}$. Show that if $0 < c < a$, then $c^2 < 2$. ○

Theorem 13.2 (Existence of square roots in \mathbb{R}^+). *There exists a positive real number a such that $a^2 = 2$.*

In order to help you better understand this proof, we'll indicate how we "devised a plan." The basic idea is that the square root should be the supremum of the set $A = \{w \in \mathbb{R}^+ : w^2 < 2\}$. The completeness axiom tells us this supremum exists, so we'll call it a . How do we show that a is the square root of 2? Well, we need to

show that $a^2 = 2$. If $a^2 > 2$, our intuition tells us that we should be able to subtract something off of a , which we'll call x , and come up with something smaller than a that is still an upper bound of the set. Thus $a - x$ would be an upper bound smaller than the supremum, and this would contradict the fact that a is the least upper bound. So we now have to worry about the case in which $a^2 < 2$. In this case, our intuition tells us that we should be able to add just a little bit to a , which we'll call y , and find an element of A , namely, $a + y$, that is bigger than the upper bound a . This now contradicts the fact that a is an upper bound. So, since neither of these two cases can occur, the only other possibility, $a^2 = 2$, must be the one that holds.

Proof. Let $A = \{w \in \mathbb{R}^+ : w^2 < 2\}$. Then A is a nonempty subset of \mathbb{R} (since $1 \in A$), and A is bounded above (by, for example, 2). By the completeness axiom A has a supremum, which we denote by a . What we know so far is that a is a real number, and $1 \leq a \leq 2$. We will show that $a^2 = 2$.

We know that one of the following three cases must occur: $a^2 > 2$, $a^2 < 2$, or $a^2 = 2$. We'll show that the first two cases are impossible.

Case 1. Suppose that $a^2 > 2$. Let $x = (a^2 - 2)/(2a)$. (We'll explain the choice of this x after the proof.) Then, as you can check, $0 < a - x < a$. By Exercise 13.1 we know that $(a - x)^2 < 2$. But

$$(a - x)^2 = a^2 - 2a + x^2 \geq a^2 - 2ax = 2.$$

So $(a - x)^2 < 2$ and $(a - x)^2 \geq 2$. This is impossible and we conclude that Case 1 cannot occur.

Case 2. Suppose that $a^2 < 2$. Let $y = (2 - a^2)/(3a)$. (You'll explain the choice of this y in Problem 13.10, after you have read our explanation for the choice of x for Case 1.) Then $y > 0$, and therefore $a + y > a$. Since a is an upper bound of A , we see that $a + y \notin A$. Thus $(a + y)^2 \geq 2$. We will show that $y < a$, and then use this to obtain our contradiction. To this end, note that because $1 \leq a$ we have

$$a - y = (4a^2 - 2)/(3a) > 0.$$

Thus, $y < a$, as claimed. Since $y < a$, we also know that $y^2 < ay$. Thus,

$$(a + y)^2 = a^2 + 2ay + y^2 < a^2 + 2ay + ay = a^2 + 3ay = 2.$$

So, $(a + y)^2 \geq 2$ and $(a + y)^2 < 2$. This is impossible, and we conclude that Case 2 cannot occur.

Thus we conclude that the only remaining possibility holds, and therefore $a^2 = 2$. \square

We promised an explanation for our choice of x . Here it is: We wanted x to satisfy two conditions, $0 < x < a$ and $(a - x)^2 \geq 2$. The second condition is difficult to solve (without resorting to $\sqrt{2}$). Let's concentrate on $(a - x)^2 \geq 2$. Once we know which x satisfy this inequality, we'll worry about satisfying $0 < x < a$. So we want to solve $(a - x)^2 \geq 2$, and we will do so by following Pólya's advice about simplifying the problem. We note that $(a - x)^2 = a^2 - 2ax + x^2 \geq a^2 - 2ax$. Now, if we make

$a^2 - 2ax = 2$, then we have also made $(a - x)^2 \geq 2$. So, now we have arrived at an easier problem: Find x such that $a^2 - 2ax = 2$. Solving this we get $x = (a^2 - 2)/2a$. This will be our first guess for x . If it doesn't satisfy $0 < x < a$, we'll need to adjust it. But we know that $1 \leq a \leq 2$ so $x < 2$, and we suppose that $a^2 > 2$ so $0 < x$. Thus, $0 < x < 2$, as desired. The proof shows that, in fact, this choice does everything it needs to do.

A similar proof can be used to show that for every $x > 0$ and every positive integer n , there exists a unique real number $a > 0$ such that $a^n = x$. Uniqueness follows easily from the fact that if $a, b > 0$ and $a < b$, then $a^n < b^n$. (See [92, p. 10] for a proof of the existence and uniqueness of n th roots.) This real number is sometimes called the principal n th root of x . From here on, we will assume the existence of n th roots and we will use the usual rules of exponentiation.

The completeness axiom of \mathbb{R} says that if we start with a nonempty set that is bounded above, we can find its supremum. In \mathbb{Q} , this is not the case: We can find a nonempty subset of \mathbb{Q} that is bounded above but has no supremum in \mathbb{Q} . In other words, there is no completeness axiom for \mathbb{Q} . There are simpler examples of sets that are not complete and we provide one such example below.

Exercise 13.3. Show that if $S = [0, 1) \cup (1, 2]$, there is a nonempty bounded subset T of S for which there is no $b \in S$ that satisfies both

- (i) $x \leq b$ for all $x \in T$, and
- (ii) if $c \in S$ and $x \leq c$ for all $x \in T$, then $b \leq c$. ○

There are many subsets of \mathbb{R} that are complete and there are many that are not complete, but it is not our goal in this chapter to discuss the completeness properties of all subsets of \mathbb{R} . Dealing with subsets can be a bit tricky, and we prefer not to get into the details in this text. Instead, we will focus on one of the most important subsets of \mathbb{R} —the rationals.

Example 13.4. Show that the set \mathbb{Q} is not complete; that is, show that there is a nonempty set B of rational numbers that is bounded above, but no rational number b satisfies both

- (i) $x \leq b$ for all $x \in B$, and
- (ii) if $c \in \mathbb{Q}$ and $x \leq c$ for all $x \in B$, then $b \leq c$.

Proof. Let $B = \{x \in \mathbb{Q}^+ : x^2 < 2\}$. Then B is nonempty ($1 \in B$) and bounded above (2 is an upper bound). Suppose to the contrary that the rational number b satisfies conditions (i) and (ii) above. If such a rational number b exists, it must satisfy one of the following three things: $b = \sqrt{2}$, $b > \sqrt{2}$, or $b < \sqrt{2}$. We'll show that no rational number can satisfy one of these.

We know from Theorem 5.2 that $\sqrt{2}$ is not rational, and we know from our assumptions that b is rational. So $b \neq \sqrt{2}$.

Suppose that $b > \sqrt{2}$. By Theorem 12.12, there is a rational number c such that $\sqrt{2} < c < b$. Now the supremum of the set $A = \{x \in \mathbb{R}^+ : x^2 < 2\}$ is, as we have just seen in Theorem 13.2, $\sqrt{2}$. For every $x \in B$ we know that $x \in A$ and consequently

$x \leq \sqrt{2} < c$. Thus c satisfies the hypotheses of (ii). But $c < b$, and therefore c does not satisfy the conclusion of (ii). This implies that this case cannot occur.

Now suppose that $b < \sqrt{2}$. By Theorem 12.12, there is a rational number c with $b < c < \sqrt{2}$. The right side of this inequality (together with the fact that $c > 0$) tells us that $c^2 < 2$ and therefore $c \in B$. But $c > b$, and this contradicts the fact that b satisfies condition (i) above. This implies that this case cannot occur.

Thus we conclude that there is no rational number satisfying conditions (i) and (ii) above. \square

Exercise 13.5. We showed that \mathbb{Q} is not complete by showing that when we consider the set $B = \{x \in \mathbb{Q}^+ : x^2 < 2\}$, there is no $b \in \mathbb{Q}$ satisfying (i) and (ii) of Example 13.4. Using that result, show that there is a nonempty bounded set C in \mathbb{Q} for which there is no $u \in \mathbb{Q}$ satisfying

- (i) $w \geq u$ for all $w \in C$, and
- (ii) if $z \in \mathbb{Q}$ and $w \geq z$ for all $w \in C$, then $u \geq z$. \circ

By the way, there is still something “missing” in \mathbb{R} —the square root of -1 . So you might decide to look at complex numbers ... but that’s another story. If it’s a story you are interested in learning more about, you can begin by looking at Project 29.6 and the references that appear there.

Definitions

Definition 13.1 (for Problems 13.13 through 13.15). The relation \preceq on a nonempty set S is called a **partial order** if the following three conditions are satisfied:

- (i) (Reflexive property) For all $x \in S$, we have $x \preceq x$.
- (ii) (Transitive property) For all $x, y, z \in S$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.
- (iii) (Antisymmetric property) For all $x, y \in S$, if $x \preceq y$ and $y \preceq x$, then $x = y$.

Definition 13.2 (for Problems 13.13 through 13.15). The relation \preceq on a nonempty set S is called a **total order** if it is a partial order with the following additional property:

- (iv) For all $x, y \in S$, we have $x \preceq y$ or $y \preceq x$.

Solutions to Exercises

Solution (13.1). Since $c < a$, we know that $c \neq \sup\{w \in \mathbb{R}^+ : w^2 < 2\}$. Therefore, there exists $w \in \mathbb{R}^+$ such that $w^2 < 2$ and $c < w$. Since $c > 0$, we have $c^2 < cw$. Since $w > 0$, we have $cw < w^2$. Putting this together, we have $c^2 < w^2 < 2$, as desired.

Solution (13.3). Claim: If $T = [0, 1)$, then there is no element $b \in S = [0, 1) \cup (1, 2]$ satisfying the stated conditions for this T : Suppose there were such an element. We cannot have $b = 1$, because $b \in S$ and $1 \notin S$. Thus, $b < 1$ or $b > 1$.

Suppose $b < 1$, then $(b+1)/2 < 1$ and, since $b \in S$, we have $(b+1)/2 > 0$. So $(b+1)/2 \in T$ and $(b+1)/2 > b$, which contradicts (i).

Therefore, we must have $b > 1$. In this case, we consider $(b+1)/2 > 1$. Then $1 < (b+1)/2 < b \leq 2$. But then $c = (b+1)/2 \in S$ and $t \leq c$ for all $t \in T$. Therefore c satisfies the hypothesis of (ii), but not the conclusion since $c < b$. This completes the proof.

Solution (13.5). We define the set $C = \{w \in \mathbb{Q}^- : w^2 < 2\}$. Suppose to the contrary that there exists $u \in \mathbb{Q}$ that satisfies conditions (i) and (ii) stated in this exercise. Let $b = -u$ and note that $b \in \mathbb{Q}$. We will show that b satisfies conditions (i) and (ii) of Example 13.4.

If $x \in B$, we let $w = -x$. Then $w^2 = (-x)^2 = x^2 < 2$. Since $x \in \mathbb{Q}^+$, we have $w \in \mathbb{Q}^-$. Hence $w \in C$. Condition (i) shows that $u \leq w$. Thus $x = -w \leq -u = b$. Hence for all $x \in B$, we have $x \leq b$. Therefore, condition (i) of Example 13.4 is satisfied.

Now suppose $c \in \mathbb{Q}$ and $x \leq c$ for all $x \in B$. If $w \in C$, let $x = -w$. Note that $x^2 = (-w)^2 = w^2 < 2$ and $x \in \mathbb{Q}^+$. Thus, $x \in B$ and $x \leq c$. So $w = -x \geq -c$. Taking $z = -c$ in condition (ii), we have shown that $w \geq z$ for all $w \in C$. We conclude that $u \geq z$. Thus $b = -u \leq -z = c$; that is, $b \leq c$ and condition (ii) of Example 13.4 is satisfied.

The existence of the rational number b contradicts the result of Example 13.4. Hence there can be no rational number u corresponding to the set C that satisfies the two conditions of this exercise.

Problems

Problem 13.1. Find the supremum and infimum (in \mathbb{R}) of the subset $T = [0, \pi) \cap \mathbb{Q}$ of \mathbb{R} . Justify your answers! You may assume that π is irrational, if that is helpful.

Problem 13.2. Let T be a nonempty subset of $[0, 1]$. Then T has an infimum m and a supremum M in \mathbb{R} . Show that m and M belong to $[0, 1]$.

Problem 13.3. Consider the subset $S = \mathbb{R} \setminus \mathbb{Z}$ of \mathbb{R} . Give an example of a subset T_1 of S that has an infimum and supremum in S and a bounded subset T_2 of S that has neither an infimum nor supremum in S .

Problem 13.4. Suppose S is a nonempty bounded subset of \mathbb{R} and u is the infimum of S . Let $v = \inf\{x \in S : x > u\}$. Show that $v \geq u$.

Problem 13.5. Show that the set $S = [0, 1] \cap \mathbb{Q}$ is not complete by showing that there is a nonempty bounded subset T of S for which there is no element $b \in S$ satisfying

- (i) $x \leq b$ for all $x \in T$, and
- (ii) if $c \in S$ and $x \leq c$ for all $x \in T$, then $b \leq c$.

Problem 13.6. Let x and y be two real numbers and let $S = \{x, y\}$.

- (a) Prove that

$$\max\{x, y\} = \frac{|x - y| + x + y}{2}.$$

- (b) Find a similar formula for the minimum.
- (c) Explain why these formulas also yield formulas for $\inf S$ and $\sup S$.

Problem 13.7. If A and B are subsets of \mathbb{R} for which $\sup A = \sup B$ and $\inf A = \inf B$, must $A = B$? Either prove this or give a counterexample.

Problem 13.8. Suppose that A and B are disjoint bounded intervals in \mathbb{R} . Prove that $\inf A \geq \sup B$ or $\inf B \geq \sup A$. You may use the fact that if $a_1, a_2 \in A$ with $a_1 < a_2$, then the interval $[a_1, a_2] \subseteq A$. (Of course, you may use the corresponding fact for B .)

Problem 13.9. For each of the following, either prove the statement or provide a counterexample.

- (a) Let S be a subset of \mathbb{R} consisting of 20 positive integers. Then S has a supremum U and an infimum u and both u and U belong to S .
- (b) Suppose that S is a nonempty subset of \mathbb{R} and S has a supremum U . Let $T = \{x \in S : x \leq U\}$. Then $T = S$.
- (c) Suppose S is a nonempty bounded subset of \mathbb{R} and $U = \sup S$. Suppose further that there exists an $x \in S$ with $x < U$. Let $v = \sup\{x \in S : x < U\}$. Then $v < U$.

Problem 13.10. In the proof of Theorem 13.2, we let $y = (2 - a^2)/(3a)$ for the case $a^2 < 2$.

Explain this choice of y along the lines of the explanation for the choice of x following the proof of Theorem 13.2. (We need a real number y satisfying $0 < y < a$ and $(a + y)^2 \leq 2$. Solve a simpler problem.)

Problem 13.11. Prove that if a is a rational number, then there is an irrational number b such that $a < b$.

Problem 13.12. Prove that for two arbitrary real numbers a and b with $a < b$, there is an irrational number c such that $a < c < b$. (Hint: Consider $a/\sqrt{2}$ and $b/\sqrt{2}$.)

Problem 13.13. Let \preceq denote a relation on a nonempty set S . The relation \preceq is called a **partial order** if the following three conditions are satisfied.

- (i) (Reflexive property) For all $x \in S$, we have $x \preceq x$.
- (ii) (Transitive property) For all $x, y, z \in S$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.
- (iii) (Antisymmetric property) For all $x, y \in S$, if $x \preceq y$ and $y \preceq x$, then $x = y$.

The relation \preceq is a **total order** on the nonempty set S if, in addition, (iv) below is satisfied.

(iv) For all $x, y \in S$, we have $x \preceq y$ or $y \preceq x$.

The last condition says that we can always compare two elements x and y in S using the relation. (A partially ordered set is sometimes called a poset. It's much less common to call a totally ordered set a toset.)

Show that the usual \leq relation defines a total order on \mathbb{R} .

Problem 13.14. This problem uses the definitions introduced in Problem 13.13.

Let A be a set containing at least two elements. We define an order on $\mathcal{P}(A)$ using set inclusion \subseteq . Show that \subseteq is a partial order, but not a total order on $\mathcal{P}(A)$.

Problem 13.15. This problem uses the definitions introduced in Problem 13.13. Consider the relation $<$ on \mathbb{R} . Show that this is not a total order by exhibiting counterexamples for each property ((i)–(iv)) violated in the definition of total order.

Problem 13.16. Prove that there exists $x \in \mathbb{R}^+$ with $x^2 = 3$.

Problem 13.17. Prove that there is no $x \in \mathbb{Q}$ with $x^2 = 12$, but there is an $x \in \mathbb{R}^+$ such that $x^2 = 12$. (You might find Problem 13.16 helpful.)

Problem 13.18. In Problem 13.14 you showed that $(\mathcal{P}(\mathbb{Z}), \subseteq)$ is a partial order. For every nonempty subset \mathcal{A} of $\mathcal{P}(\mathbb{Z})$ we say that $U \in \mathcal{P}(\mathbb{Z})$ is an upper set of \mathcal{A} , if $X \subseteq U$ for all $X \in \mathcal{A}$. A nonempty set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Z})$ will be called an upper bounded set if there is an upper set of \mathcal{A} in $\mathcal{P}(\mathbb{Z})$. We say $U_0 \in \mathcal{P}(\mathbb{Z})$ is a least upper set if (i) U_0 is an upper set of \mathcal{A} and (ii) if U is another upper set of \mathcal{A} , then $U_0 \subseteq U$.

- Let $\mathcal{B} = \{\{1, 2, 5, 7\}, \{2, 8, 10\}, \{2, 5, 8\}\}$. Show that \mathcal{B} is an upper bounded set and find a least upper set of \mathcal{B} , if there is one.
- Prove that every nonempty subset of $\mathcal{P}(\mathbb{Z})$ is upper bounded.
- Define “lower set,” “lower bounded set,” and “greatest lower set.”
- Let \mathcal{A} be a nonempty subset of $\mathcal{P}(\mathbb{Z})$. Using union and intersection, find an expression for least upper set of \mathcal{A} and greatest lower set of \mathcal{A} .
- Prove that $(\mathcal{P}(\mathbb{Z}), \subseteq)$ has the “least upper set property” (in other words, show every upper bounded set has a least upper set).

Problem 13.19. Prove that (\mathbb{Z}, \leq) is complete in the following sense: If A is a nonempty set of integers that is bounded above, then there is an integer a such that $a = \sup A$.

Problem 13.20. Let S be a nonempty subset of \mathbb{R} , and $x \in \mathbb{R}$. Suppose that there exists $y \in S$ with $y < x$. Let $y_0 = \sup\{y \in S : y < x\}$.

- Give an example of such a set S and real number x .
- Show that $y_0 \leq x$.
- Give an example to show that y_0 may equal x .
- Give an example to show that y_0 may be strictly less than x .

Problem 13.21. Prove the following statement: For every positive irrational real number a with $\sqrt{a} < 10000$, there is a positive integer n such that

$$\frac{10000}{n} < \sqrt{a} < \frac{10000}{n-1}.$$

Problem# 13.22. To practice some of the more challenging theorems in this and the previous chapter, we use some of the techniques we have discussed to establish the division algorithm in \mathbb{Z} . We wish to prove:

Theorem 13.6 (Division algorithm). *Let m and n be integers with $n \neq 0$. Then there exist $q, r \in \mathbb{Z}$ satisfying $m = nq + r$, where $0 \leq r < |n|$ and q and r are unique.*

We outline the proof below. You should complete the details.

- Let $A = \{m - nx : x \in \mathbb{Z} \text{ and } m - nx \geq 0\}$. Show that A is nonempty.
- Explain why there exists $r \in \mathbb{Z}$ such that $r = \min A$.
- Explain why $0 \leq r < |n|$.
- Use your work above to prove that there are $q, r \in \mathbb{Z}$ such that $m = nq + r$ and $0 \leq r < |n|$.
- We now need to show uniqueness. Recall that to show q and r are unique, we will assume that there exist q_1 and r_1 satisfying several properties. Write out the properties.
- Show that $|n| \mid (r - r_1)$ and that $-|n| < r - r_1 < |n|$.
- Finish the proof.

Tips: You Solved It. Now What?

Let's say we are now at the point where you solved the given problem and wrote up your first draft.

- Look over your solution. Does it use everything you are given?
- Is the answer reasonable?
- If there were places where you were unsure of your argument, check over those arguments carefully. You might find it helpful to write the solution, take a break, and then check the solution. (We say that a lot, don't we?)
- Is your argument clear? Did you choose your notation well? Is your notation clear and unambiguous? Did you introduce all notation before you used it?
- Is there a shorter or more intuitive argument?
- Do you fully understand what you did? Spend some time thinking about the method and what you proved. Could you have gotten a better result?
- When do these methods work? What are the restrictions? Where have you seen them before?
- If the problem was hard for you to solve, what made it hard? What were the important ideas that you were missing?

- This is a good opportunity to learn about yourself, too. Which problems do you like best? Why?