Chapter 20
Convergence of Sequences of Real Numbers

As we saw in the last chapter, when we graph several terms of a sequence, certain behavior may appear. We may become convinced, for whatever reason, that the sequence is unbounded. Or, we may believe that the sequence is bounded and we may even notice the sequence moving toward a particular horizontal line. But how do we check that what we believe is happening really is happening?

Our efforts to explain this require that you fully understand how to measure distance. So we remind you that distance is usually measured using the absolute value function, or $|x|$, and the absolute value of a real number x measures the distance from x to 0. If we want to measure the distance between two real numbers x and a, we would need to look at $|x-a|$.

For an arbitrary positive real number ε, we know what it means to say $|x| = \varepsilon$. What does it mean to say $|x| < \varepsilon$? The answer is, as you can check, that $|x| < \varepsilon$ if and only if $-\varepsilon < x < \varepsilon$. So how do we determine when a sequence of real numbers approaches a real number L? We will use the absolute value function to measure the distance from terms in the sequence to L. We make this precise in the following definition.

We say that a sequence (x_n) **converges** if there exists a real number L such that for all $\varepsilon > 0$ there exists a real number N such that $|x_n - L| < \varepsilon$ for all $n \geq N$. If such an L exists, we call L the **limit** of the sequence (x_n), we say that (x_n) **converges to** L, and we write $x_n \to L$ or $\lim_{n\to\infty} x_n = L$. If no such L exists, we say that the sequence **diverges**. While we allow N to be a real number, we caution you to remember that the indices on the terms of a sequence, x_n, are natural numbers. To really understand this definition, we must understand it visually, and we must also know how to use it to show that a sequence converges. We first turn to the visual aspect of convergence.

Let's think about the definition. If we believe the sequence (x_n) converges, then we need to find a real number L such that the sequence gets really really close to the line $y = L$. Mathematically, we say that we need the sequence arbitrarily close to the horizontal line. This, in turn, implies that the distance from the sequence to the line $y = L$ should be less than every positive real number ε that we can think of; that is what arbitrarily close means. But it may not happen right away; it may only happen eventually (for each ε, there will exist N such that x_n may not satisfy what we want

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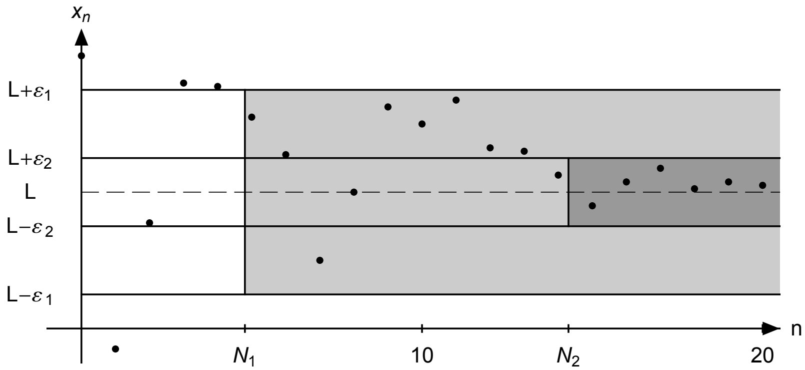

Fig. 20.1 Definition of a convergent sequence; two-dimensional illustration

for $n < N$, but it will satisfy it for $n \geq N$). That's why we defined convergence the way we did.

We illustrate this definition in Figure 20.1. In this picture, we first pick a value $\varepsilon = \varepsilon_1 > 0$. Then we indicate a corresponding real number $N = N_1$ such that for all $n \geq N_1$ we have $|x_n - L| < \varepsilon_1$. Looking at the figure, we see that the strip from $y = L - \varepsilon_1$ to $y = L + \varepsilon_1$ contains all the terms of the sequence from x_5 on. Generally, the smaller the value of ε, the larger the value of N. Let's think about why this is true: Returning to Figure 20.1, we see that $\varepsilon = \varepsilon_2$ is smaller than ε_1, and this, in turn, forces the sequence to be closer to the horizontal line. So we go farther out in the sequence to get closer to the line $y = L$. You also probably noticed that once we find a value of N that works, anything larger than N will work, too.

Figure 20.2 below is yet another way to illustrate the same situation. Explain this sketch to yourself.

Now let's turn to how we show a sequence converges. First we make a conjecture as to the value of the limit, call it L. The important thing to notice, when we make our conjecture, is that we are interested in the behavior as $n \to \infty$. We don't really care what happens for small n, for example. So if we ask what a sequence converges to, you can guess by ignoring terms that don't really matter in the long run. For example, if we ask you to guess what

$$\left(\frac{2n^2 + 3n + 4}{4n^2 - 3n - 5}\right)$$

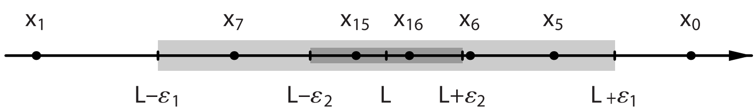

Fig. 20.2 Definition of a convergent sequence; one-dimensional illustration

converges to, you would remember that only behavior near infinity matters, so you would probably guess that $2n^2$ really dominates the quantity in the numerator, while $4n^2$ really dominates the quantity in the denominator. Thus, when you guess this limit, you'll probably guess that it's the same as the limit of the sequence with terms $x_n = 2n^2/(4n^2)$—which it is. You probably also would guess that this limit is $1/2$, and you'd be right again. Before we move to our second step, try practicing your guessing on some of the examples below.

Exercise 20.1. Guess the limits of each of the following sequences:

(a) $\left(\dfrac{3n^2+n+4}{3+n+5n^2}\right)$;

(b) $\left(\dfrac{2n^3+n^2+5}{n^4+6}\right)$;

(c) $\left(\dfrac{\sqrt{n}+n}{\sqrt{n}-n}\right)$. ○

Now, once you have guessed your limit, you must prove that it is correct. To do this, assume that we are given an arbitrarily small number, which we denote by ε. We then try to find a real number N (depending on ε) so that for the remaining terms of the sequence, that is, for $n \geq N$, we have $|x_n - L| < \varepsilon$. This is where things get tricky.

Let's see how we would use this definition, starting with a fairly simple example.

Example 20.2. Show that the sequence (x_n) defined by $x_n = 1/n$ converges to 0.

We'll find this easier if we follow Pólya's list.

"Understanding the problem." We need to show that for every $\varepsilon > 0$ there exists a real number N such that $n \geq N$ implies that $|x_n - 0| < \varepsilon$. We fill in what we can, remembering that ε is arbitrary; that is, it is chosen for us and we have no control over its value. So once someone gives us ε, we are supposed to come up with N so that $|1/n - 0| < \varepsilon$ for $n \geq N$.

"Devising a plan." Now we'll work backwards to see what N has to be. We need $|1/n| < \varepsilon$. So we need $n > 1/\varepsilon$. It appears that if we take $N > 1/\varepsilon$, say $N = 2/\varepsilon$, we'll get exactly what we need when $n \geq N$. Since N depends on ε (there's an ε in our definition of N), many authors write $N = N(\varepsilon)$.

Before we carry out the plan, note that if we knew ε, then this would tell us exactly how big N has to be. If $\varepsilon = 0.1$, then N needs to be bigger than $1/0.1 = 10$; if $\varepsilon = 0.01$, then N needs to be bigger than $1/0.01 = 100$; and if ε just happens to be itself, then N needs to be greater than $1/\varepsilon$. We're now ready to carry out the plan. We have to write this up, so that a reader who has not seen our work will know what we are doing.

Proof. If $\varepsilon > 0$ and we choose $N = 2/\varepsilon$, then for $n \geq N$, we have $|x_n - 0| = |1/n - 0|$ $= 1/n \leq 1/N = \varepsilon/2 < \varepsilon$. Therefore, if $n \geq N$, then $|x_n - 0| < \varepsilon$ as desired. □

Our second example is a bit more challenging, and requires slightly different techniques. You'll see more problems of this type and more challenging limit problems when you take your first analysis course.

Example 20.3. Show that $\lim_{n\to\infty} n/(n+2) = 1$.

As above, we'll first understand our problem, and then devise a plan.

"Understanding the problem." We start by writing out the definition: For all $\varepsilon > 0$ there exists N such that $n \geq N$ implies that $|x_n - L| < \varepsilon$.

Next we'll fill in x_n and L: For all $\varepsilon > 0$ there exists N such that $n \geq N$ implies that $|n/(n+2) - 1| < \varepsilon$.

Now simplify the expression $|n/(n+2) - 1| < \varepsilon$ to find out how big n must be in terms of ε: We need to find N so that $n \geq N$ implies that $2/(n+2) < \varepsilon$.

"Devising a plan." You can solve for n, as above, if you wish, but that method only works well in simple cases. So we are going to try to change this problem from the one we have to a simpler problem. How will we do that? Well, we want to make $2/(n+2)$ small. If we can find something bigger and simpler than $2/(n+2)$, and if we can make that less than ε, then we will also know that $2/(n+2)$ is less than ε. What's bigger and simpler? The previous example wasn't too bad. So if we just had a simple fraction, we would be in good shape. A general strategy that often works is this: If the numerator is complicated, we'll try to find something simpler and larger than the numerator. If the denominator is complicated, we'll try to find something simpler and smaller than the denominator. Our simpler expression must still "act the same in the long run" as the original.

For this exercise, the numerator is simple, so we'll leave it alone. For the denominator, we'll try to somehow use the thing that dominates: n. We need $n+2$ greater than or equal to something simple involving n. It's pretty clear that $n+2 \geq n$, so we'll use n. Putting this together, we have found that

$$\frac{2}{n+2} \leq \frac{2}{n}.$$

Thus, if we make $2/n < \varepsilon$, we will also make $2/(n+2) < \varepsilon$. But making $2/n < \varepsilon$ is easy, since $2/n < \varepsilon$ if and only if $n > 2/\varepsilon$. Therefore, it appears that if $N > 2/\varepsilon$, then $2/n < \varepsilon$ and thus $2/(n+2) < \varepsilon$, which is what we need.

"Carrying out the plan." Write out the proof, beginning with "If $\varepsilon > 0$ and ...." The very next phrase should identify N, unless there are things you need to tell the reader in order for the reader to understand your definition of the real number N. Remember that the reader will only see your proof, not your plan.

Proof. If $\varepsilon > 0$ and we choose $N = 3/\varepsilon$, then for $n \geq N$, we have

$$\left|\frac{n}{n+2} - 1\right| = \left|\frac{2}{n+2}\right| \leq \frac{2}{n} \leq \frac{2}{N} = \frac{2\varepsilon}{3} < \varepsilon.$$

Thus, $\lim_{n\to\infty} n/(n+2) = 1$. □

Here's one more exercise on limits.

Exercise 20.4. The sequence $\big((2n+4)/(n^2+n+1)\big)$ converges. Guess the limit and prove that your guess is correct, using the definition of convergence. ○

It's time to think about negating the definition of convergence.

Exercise 20.5. By negating the definition of convergence, explicitly state what it means for a sequence (x_n) to diverge. ○

Exercise 20.6. Using Exercise 20.5, show that the sequence $((-1)^n)$ diverges. ○

We state the most basic properties of limits and convergent sequences here. The first theorem says that there can be one and only one choice for the limit of a convergent sequence.

Theorem 20.7. *If a sequence converges, then the limit is unique.*

We've done uniqueness proofs before and we'll do this one the same way: We suppose to the contrary that there are two different limits L and M, and then we will show that they must be the same. So, we need to show that $L-M=0$. We also use a standard trick: *we add and subtract the same quantity to an object.* Why? Well, since all we know is that L and M get close to the terms of the sequence (x_n), we have to somehow use these terms. But there is no x_n in the equation $L-M=0$. So we will have to insert an x_n where none appears.

Proof. Let (x_n) be a convergent sequence. Suppose to the contrary that $x_n \to L$ and $x_n \to M$, where $L \neq M$. Let $\varepsilon = (1/4)|L-M|$. Then $\varepsilon > 0$. By the definition of convergence, since $\varepsilon > 0$, there exists N_1 such that $|x_n - L| < \varepsilon$ for $n \geq N_1$ and there exists N_2 such that $|x_n - M| < \varepsilon$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$ we have

$$\begin{aligned} |L-M| &= |L-x_n+x_n-M| \\ &\leq |L-x_n|+|x_n-M| && \text{(by the triangle inequality)} \\ &< \varepsilon+\varepsilon && \text{(since } n \geq N_1 \text{ and } n \geq N_2\text{)} \\ &= \tfrac{1}{2}|L-M|. \end{aligned}$$

But this is silly, since no positive real number is smaller than half of itself. This contradiction establishes the result that limits of sequences are unique. □

Here's another important theorem. It uses Exercise 19.3, which says that a sequence (x_n) is bounded if and only if there exists a real number M such that $|x_n| \leq M$ for all n.

Theorem 20.8. *Every convergent sequence is bounded.*

Proof. Suppose that the sequence $(x_n)_{n=1}^{\infty}$ converges to the real number L. Let $\varepsilon = 1$. Then there exists N such that $|x_n - L| < 1$ for all $n \geq N$. Let K be the smallest integer satisfying $K \geq N$. Thus $|x_n| = |x_n - L + L| \leq |x_n - L| + |L| < 1 + |L|$

for all $n \geq K$. Consider the real numbers $|x_1|, |x_2|, \ldots, |x_{K-1}|$ and $1 + |L|$. Since there are finitely many such numbers, we may choose the maximum of these. Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{K-1}|, 1 + |L|\}$. Then $|x_n| \leq M$ for all n, and we conclude that the sequence (x_n) is bounded. □

Part (i) of the next theorem says that if we sum two convergent sequences, the new sequence converges, too. It also says "the limit of the sum is the sum of the limits." What do the other parts say?

Theorem 20.9. *Let (x_n) and (y_n) be two sequences that converge. Let L and M be real numbers such that $x_n \to L$ and $y_n \to M$. Then*

(i) $x_n + y_n \to L + M$,
(ii) $\alpha x_n \to \alpha L$, *for every real number* α,
(iii) $x_n y_n \to LM$, *and*
(iv) *if* $M \neq 0$ *and* $y_n \neq 0$ *for all* n, *then* $1/y_n \to 1/M$.

We prove part (i) here. All the proofs are similar, and this part will illustrate the most important idea, which is that we need to choose things carefully to make everything work out. Here's what we mean: For every $\varepsilon > 0$, we need to find N such that for $n \geq N$ we have $|(x_n + y_n) - (L + M)| < \varepsilon$. We can make $|x_n - L| < \varepsilon$ for n large enough, and we can make $|y_n - M| < \varepsilon$ for n large enough, but if we add these together we get 2ε. You'll now see how we handle this problem.

Proof. [Proof of (i)] Let $\varepsilon > 0$. Since $\varepsilon/2 > 0$ and $x_n \to L$, there exists N_1 such that $|x_n - L| < \varepsilon/2$ for $n \geq N_1$. Again, since $\varepsilon/2 > 0$ and $y_n \to M$, there exists N_2 such that $|y_n - M| < \varepsilon/2$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we know that $n \geq N_1$ and $n \geq N_2$, so

$$\begin{aligned}
&|(x_n + y_n) - (L + M)| \\
&\qquad = |(x_n - L) + (y_n - M)| \\
&\qquad \leq |x_n - L| + |y_n - M| && \text{(by the triangle inequality)} \\
&\qquad < \varepsilon/2 + \varepsilon/2 && \text{(since } n \geq N_1 \text{ and } n \geq N_2\text{).}
\end{aligned}$$

We conclude that for all $\varepsilon > 0$ there exists N such that for all $n \geq N$ we have $|(x_n + y_n) - (L + M)| < \varepsilon$, as desired. □

This theorem can be used to find the limit of a recursively defined function rather easily, if we know that the sequence converges.

Exercise 20.10. Define the sequence (x_n) by $x_1 = 1$ and $x_{n+1} = 0.02x_n^2 + 8$ for all $n \in \mathbb{Z}^+$. You may assume that the sequence (x_n) converges and is bounded above by 20. Find $\lim_{n\to\infty} x_n$. ○

Definitions

Definition 20.1. We say that a sequence (x_n) **converges** if there exists a real number L such that for all $\varepsilon > 0$ there exists a real number N such that $|x_n - L| < \varepsilon$ for all $n \geq N$. If such an L exists, we call L the **limit** of the sequence (x_n), we say that (x_n) **converges to** L, and we write $x_n \to L$ or $\lim_{n\to\infty} x_n = L$.

Definition 20.2. A sequence **diverges** if it does not converge.

Definition 20.3 (for Problem 20.20). Let (x_n) be a bounded sequence. For each positive integer n, let $s_n = \sup\{x_m : m \geq n\}$. Then (s_n) converges and its limit is called the **limit superior** of (x_n), written $\limsup(x_n)$.

Definition 20.4 (for Problem 20.20). Let (x_n) be a bounded sequence. For each positive integer n, let $t_n = \inf\{x_m : m \geq n\}$. Then (t_n) converges and its limit is called the **limit inferior** of (x_n), written $\liminf(x_n)$.

Definition 20.5 (for Problem 20.21). A sequence (x_n) of real numbers **diverges to infinity** (written $x_n \to \infty$) if for every $M \in \mathbb{R}$ there exists a real number N such that $n \geq N$ implies $x_n > M$.

Definition 20.6 (for Problem 20.21). A sequence is **monotone** if it is an increasing sequence or a decreasing sequence.

Definition 20.7 (for Problem 20.21). A sequence (x_n) is a **Cauchy sequence** if for all $\varepsilon > 0$ there exists a real number N such that $n, m \geq N$ implies that $|x_n - x_m| < \varepsilon$.

Definition 20.8 (for Problem 20.22). Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and let (n_k) be a strictly increasing sequence of positive integers. The sequence (x_{n_k}) is called a **subsequence** of (x_n).

Solutions to Exercises

Solution (20.1). The answers are (in this order): $3/5$, 0, and -1.

Solution (20.4). Normally when we write up our solution we will include the proof and not our work on devising a plan. But one more careful example here will certainly be useful. So here's our plan: We guess that this converges to 0, so we need to show that for all $\varepsilon > 0$, there exists N such that $|(2n+4)/(n^2+n+1)| < \varepsilon$ for all $n \geq N$. Now both numerator and denominator are a bit complicated. For the denominator, we need to find something smaller and simpler than n^2+n+1 involving the highest-order term n^2. So for this part, we note that $n^2+n+1 > n^2$. Now for the numerator, we need to find something larger and simpler than $2n+4$ involving the highest-order term n. For $n \geq 1$, since $4 \leq 4n$, we see that $2n+4 \leq 2n+4n = 6n$. Putting this together, for $n \geq 1$, we have

$$\left|\frac{2n+4}{n^2+n+1}\right| \leq \frac{6n}{n^2} = \frac{6}{n}.$$

So, if we make $6/n < \varepsilon$, we should be able to complete the proof. Thus, we'll choose $N = 7/\varepsilon$.

Proof. If $\varepsilon > 0$, we choose $N = 7/\varepsilon$. Then for $n \geq N$, we have

$$\left|\frac{2n+4}{n^2+n+1} - 0\right| \leq \frac{6n}{n^2} = \frac{6}{n} \leq \frac{6}{N} = \frac{6\varepsilon}{7} < \varepsilon,$$

where the first inequality follows since $n \geq 1$ and, consequently, $2n+4 \leq 6n$. Thus $\big((2n+4)/(n^2+n+1)\big)$ converges to 0. □

Solution (20.5). A sequence (x_n) diverges if for every real number L there exists $\varepsilon > 0$ such that for all $N \in \mathbb{R}$ there exists $n \geq N$ with $|x_n - L| \geq \varepsilon$.

Solution (20.6).

Proof. Let L be a real number, and let $\varepsilon = 1/2$. We break this into two cases. First suppose that $L < 0$. Let $N \in \mathbb{R}$ and choose n to be an even integer satisfying $n \geq N$. Then $|x_n - L| = |1 - L| > 1 > \varepsilon$. Now if $L \geq 0$, then for $N \in \mathbb{R}$ choose an odd integer with $n \geq N$. It follows that $|x_n - L| = |-1-L| = |1+L| \geq 1 > \varepsilon$. Therefore (x_n) diverges. □

Solution (20.10). First note that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n$. This is, as you should check, an immediate consequence of the definition of the limit of a sequence. Since $L = \lim_{n\to\infty} x_n$ exists, Theorem 20.9 implies that $\lim_{n\to\infty}(x_n)^2$ exists and is L^2. Further, and again using Theorem 20.9, we get

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(0.02x_n^2 + 8) = 0.02L^2 + 8.$$

We solve this quadratic equation and get $L = 10$ or $L = 40$. Since (x_n) is bounded above by 20, we conclude that $\lim_{n\to\infty} x_n = 10$.

Using induction, you can prove that (x_n) is strictly increasing and bounded above by 20. Prove it! Once you've done this, Theorem 20.11 of Problem 20.17 below implies that (x_n) converges. This justifies the hypothesis that we told you to assume.

Problems

Problem 20.1. For each of the following, give an example that satisfies all requirements or prove that no such example exists:

(a) a divergent sequence that is bounded;
(b) an increasing sequence (x_n) with $\lim_{n\to\infty} x_n = 9$ and $\lim_{n\to\infty}(-2x_n) = -8$;

(c) a bounded increasing sequence (x_n) such that $x_n \neq 0$ for all n and such that $(1/x_n)$ diverges.

Problem# 20.2. We used the following several times in this chapter: Let $x, y, z \in \mathbb{R}$. Then $|x-y| \leq |x-z| + |y-z|$. Prove this statement.

Problem 20.3. Let a and δ be real numbers with $\delta > 0$. Show that for all real numbers x, we have $|x-a| < \delta$ if and only if $a - \delta < x < a + \delta$.

Problem 20.4. For each of the following, guess the limit and then prove (using the definition of convergence) that your guess is correct:

(a) $\lim_{n\to\infty} \frac{1}{3n}$;
(b) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$;
(c) $\lim_{n\to\infty} \frac{1}{\sqrt{n+7}}$;
(d) $\lim_{n\to\infty} \frac{n^2+4}{n^2}$;
(e) $\lim_{n\to\infty} \frac{2n+1}{n+2}$;
(f) $\lim_{n\to\infty} \frac{3}{n!}$;
(g) $\lim_{n\to\infty} \frac{1}{(n+7)!}$;
(h) $\lim_{n\to\infty} \frac{3n^2+1}{4n^2+n+2}$.

Problem 20.5. Let (x_n) and (y_n) be convergent sequences. Use the definition of convergence (no limit theorems!) to prove that the sequence $(3x_n - 2y_n)$ converges.

Problem 20.6. This is a continuation of Problem 19.5. We will use all the notation that was introduced there.

(a) Find an explicit formula for the terms S_n of the sequence (S_n). (You may need the formula for a geometric sum, see Problem 19.4.)
(b) Guess $\lim_{n\to\infty} S_n$ and then prove that your guess is correct.
(c) Find $\lim_{n\to\infty} s_n$.
(d) Find the minimum and maximum amount of phenytoin in the patient's blood after one week and after one month (60 administrations of the drug).
(e) Find the minimum and maximum amount of phenytoin in the patient's blood in the long run (that is, as $n \to \infty$).

Problem 20.7. Prove Theorem 20.9 part (ii).

Problem 20.8. (a) Suppose that (x_n) and (y_n) are sequences and $0 \leq x_n \leq y_n$ for all positive integers n. Show that if $y_n \to 0$, then $x_n \to 0$.

(b) Suppose that (x_n) and (y_n) are sequences and $-y_n \leq x_n \leq y_n$ for all positive integers n. Show that if $y_n \to 0$, then $x_n \to 0$.
(c) Find $\lim_{n\to\infty} \dfrac{\sin^2 n}{n}$ and $\lim_{n\to\infty} \dfrac{(-1)^n}{n^2+1}$, and prove that your answers are correct.

Problem 20.9. Redo Problem 20.4, parts (a), (c), (d), (e), (g), and (h) using theorems in this chapter or Problem 20.8.

Problem 20.10. Let (x_n) be a convergent sequence defined recursively by $x_1 = 50$ and $x_{n+1} = \frac{1}{2}x_n + 5$ for $n \geq 1$. Without finding an explicit form of x_n, calculate $\lim_{n\to\infty} x_n$. Justify all steps in your calculation.

Problem 20.11. Claim: Let (y_n) be a sequence defined recursively by $y_1 = 5$ and $y_{n+1} = 3y_n - 6$ for $n \geq 1$. Then $\lim_{n\to\infty} y_n = 3$.

Proof? Applying Theorem 20.9,

$$\lim_{n\to\infty} y_{n+1} = 3 \lim_{n\to\infty} y_n - 6.$$

Hence $2\lim_{n\to\infty} y_n = 6$. This implies that $\lim_{n\to\infty} y_n = 3$. ⍰

If this proof is correct, fill in the details, citing the theorems that are applied. If this proof is incorrect, show exactly where an incorrect conclusion is drawn. Is the claim correct? Explain.

Problem 20.12. (a) Show that for every sequence (x_n) we have

$$0 \leq |x_n| + x_n \leq 2|x_n|.$$

(b) Prove that if $|x_n| \to 0$, then $x_n \to 0$. (See Problem 20.8.)
(c) If (x_n) is a sequence such that $|x_n| \to 1$, must $x_n \to 1$?

Problem 20.13. Prove Theorem 20.9 part (iii). (Hint: You may want to use Theorem 20.8.)

Problem 20.14. The proof of Theorem 20.9 part (iv) is outlined below.

(a) Prove that for real numbers y and M, if $|y - M| \leq |M|/2$, then $|y| \geq |M|/2$. (You might wish to use the lower triangle inequality to establish this implication.)
(b) Let (y_n) be a sequence of nonzero real numbers, and suppose that $y_n \to M$, where $M \neq 0$. Prove that if $0 < \varepsilon < |M|/2$, then there exists N such that for $n \geq N$ if $|y_n - M| < \varepsilon$, then $|(M - y_n)/(My_n)| \leq (2/M^2)|M - y_n|$.
(c) Prove Theorem 20.9 part (iv).

Problem 20.15. Let a be a real number satisfying $0 < a < 1$.

(a) Show that there exists a real number x such that $x > 0$ and $a = 1/(1+x)$.
(b) Show that $a^n \leq 1/(1+nx)$ for all $n \in \mathbb{N}$. (You will need to do Problem 18.6 if you have not already done it.)

(c) Show carefully that $1/(1+nx) \to 0$.
(d) Show that $a^n \to 0$. (You will want to do Problem 20.8 if you have not already done it.)
(e) Show that if $x_n = \underbrace{0.999...9}_{n\,9's}$, then $x_n \to 1$.

Problem 20.16. Let $x_n = F_{n+1}/F_n$, where F_n denotes the nth Fibonacci number. In Problem 19.19 we showed that $x_n = 1 + 1/x_{n-1}$. Assume further that there exists a nonzero real number L such that $x_n \to L$. Explain why $1/x_n \to 1/L$ and use these facts to compute L.

The number L is the golden ratio, which appears frequently in architecture and in nature. The Greeks, and others, felt (and still feel) that rectangles with sides in golden ratio are the most beautiful.

Problem 20.17. (An exercise in reading and writing.) The combination of the two theorems in this problem is usually called the **monotone convergence theorem for sequences**.

(a) Read the proof below until you understand it. Mathematicians often read proofs many times, and you may have to do so with this one.

Theorem 20.11. *Every increasing bounded sequence converges to its supremum.*

Proof. Let $l = \sup(x_n)$, and let $\varepsilon > 0$. Since $l - \varepsilon$ is not an upper bound, there exists N such that $x_N > l - \varepsilon$. We have assumed that (x_n) is an increasing sequence. Therefore, if $n \geq N$, we know that $x_n \geq x_N > l - \varepsilon$. Since $x_n \leq l$ for all n, for $n \geq N$ we have $l - \varepsilon < x_n < l + \varepsilon$, and thus $|x_n - l| < \varepsilon$. Therefore, the sequence (x_n) converges to l. □

(b) Use the ideas in the proof above to prove Theorem 20.12.

Theorem 20.12. *Every decreasing bounded sequence converges to its infimum.*

(c) Can you find another proof of Theorem 20.12, this time using the statement of Theorem 20.11 rather than its proof?

Problem 20.18. Use Problems 20.15 and 20.17 to prove your guess of Exercise 19.9, namely, that if $x_n = \underbrace{0.999...9}_{n\,9's}$, then $\sup(x_n) = 1$.

Problem 20.19. This problem takes up the situation and notation of Problem 19.5.

(a) Use your work of parts (b), (c), and (d) of Problem 19.5 and the theorems of Problem 20.17 to find $\sup(S_n)$ and $\sup(s_n)$.
(b) Use part (a) to find the range of the amount of phenytoin in the patient's blood in the long run (that is, as $n \to \infty$). (Note that though this is the same question as in Problem 20.6 part (e), the solution does not require the explicit form of the sequence (S_n)!)

Problem 20.20. Use the theorems of Problem 20.17 for the following.

(a) Show that $(n!/n^n)$ converges.
(b) Let (x_n) be a bounded sequence. For $n \in \mathbb{Z}^+$, let $s_n = \sup\{x_m : m \geq n\}$. Prove that (s_n) converges. The limit of (s_n) is called the **limit superior** of (x_n), and is usually denoted by $\limsup(x_n)$. What is the $\limsup((-1)^n)$?
(c) Let (x_n) be a bounded sequence. For $n \in \mathbb{Z}^+$, let $t_n = \inf\{x_m : m \geq n\}$. Prove that (t_n) converges. The limit of (t_n) is called the **limit inferior** of (x_n), and is usually denoted by $\liminf(x_n)$.

Problem 20.21. Sequences afford an excellent opportunity to practice everything you have learned. That's what you'll do in this problem: For each of the definitions below, do the following.

1. Read the definition.
2. Try to find an example of something that illustrates the definition.
3. Try to find an example of something that does not satisfy the defining conditions.
4. Write the definition in symbols.
5. Negate the definition.

(a) A sequence (x_n) of real numbers **diverges to infinity** (written $x_n \to \infty$) if for every $M \in \mathbb{R}$ there exists a real number N such that $n \geq N$ implies $x_n > M$.
(b) A sequence is **monotone** if it is an increasing sequence or a decreasing sequence.
(c) A sequence (x_n) is a **Cauchy sequence** if for all $\varepsilon > 0$ there exists a real number N such that $n, m \geq N$ implies that $|x_n - x_m| < \varepsilon$.
In addition, for this one, pretend you were talking to a high school student who loves mathematics and just *has* to know what a Cauchy sequence is but has never heard of ε and N. What would you tell him or her?

Problem 20.22. Consider the following definition.

Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and let (n_k) be a strictly increasing sequence of positive integers. The sequence (x_{n_k}) is called a **subsequence** of (x_n). For this problem, do all the things you did in Problem 20.21, plus (a), (b), and (c) below.

(a) This definition says that when you choose a subsequence, you must do two things: you need to list all the x_n in order of appearance, and then you obtain the x_{n_k} by choosing one element after the other from your list, making sure that the term you choose comes after the one you just chose. How does the definition tell you that you must choose from this list? How does it tell you that you must choose in order of increasing appearance?
(b) Let $x_n = 1/n$. Is (x_n) a subsequence of itself? If $y_n = x_{2n}$, give a formula for y_n in terms of n. If $z_n = x_{n+4}$, give a formula for z_n in terms of n.
(c) How can you tell that the sequence (w_n) given by $w_n = (-1)^n/n$ is not a subsequence of (x_n) as defined in (b)?

Problem 20.23. In Problem 20.21 part (c) we defined a Cauchy sequence. Show that every convergent sequence is a Cauchy sequence.