Chapter 23
Countable and Uncountable Sets

I see it but I do not believe it.—Georg Cantor [59, p. 997]

Having mastered finite sets, we now turn to understanding the infinite. We know that $\mathbb{N}$ is infinite, and we know that $\mathbb{Q}$ is infinite (see Problem 22.8). Are they equivalent? In some sense, we can count $\mathbb{N}$ and it may feel as though we cannot count $\mathbb{Q}$—that is, as though we cannot list a first element, second element, third element, and so on. However, we shall see that $\mathbb{Q}$ and $\mathbb{N}$ are, in fact, equivalent.

An infinite set A is said to be **countably infinite** if $A \approx \mathbb{N}$. In Chapter 21 we showed that $\mathbb{Z} \approx \mathbb{N}$ and $2\mathbb{Z} \approx \mathbb{N}$, so these also are countably infinite. It is also easy to show that $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\} \approx \mathbb{N}$. A set is **countable** if it is either finite or countably infinite. A set is said to be **uncountable** if it is not countable. Note that if we only know that a set is countable, we don't necessarily know if it is finite or infinite. If we have an infinite countable set, it automatically is equivalent to $\mathbb{N}$.

Exercise 23.1. Let A and B be two countably infinite sets. Prove that there is a bijection of A onto B.

Theorem 23.2. *Every subset of* $\mathbb{N}$ *is countable.*

The proof of this theorem will be presented as an exercise in reading mathematics.

Exercise 23.3. This is an exercise in reading a proof. We ask that as you read you pretend that the set T (appearing in the proof) is the set of prime numbers. Of course, you are not allowed to pick a particular subset, call it T, and conclude that you have proved the theorem. However, for the purpose of understanding someone else's proof, this might be quite helpful. We will call this set the demo, and denote it by T. Whenever you see (?) you should figure out what happens for this set. If at the end you remain largely unsatisfied, pick another set for T and try again. No matter what, you'll understand more of the proof than if you hadn't tried anything at all. So read the proof, think about the question marks, and then answer the set of questions provided below.

U. Daepp and P. Gorkin, *Reading, Writing, and Proving: A Closer Look at Mathematics*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4419-9479-0_23,

(a) In the statement of Theorem 18.6 a function f appears. What is this function in the proof (when the demo set T is used)?
(b) Find $g(0), g(1), \ldots, g(5)$ in the demo. What is the subset of T that is used to define $g(6)$ in our demo? What is $g(6)$ (when you use the demo set T to define $g(6)$)?
(c) For one-to-one, we thought we had to show that if $g(j) = g(\ell)$, then $j = \ell$. What's going on in this proof?
(d) Say $t = 19$. What element is mapped to t when the demo set is used? ○

Before proceeding with our proof, we give a sense of the idea behind it: If T is an infinite subset of $\mathbb{N}$, we will construct a function $g : \mathbb{N} \to T$ recursively, listing the elements of T in increasing order. Since the list is strictly increasing, g will be one-to-one. And, since the list is constructed to look at what has been chosen and then move on to the next largest number in T, the function g will also be onto. This part of the proof will require some work, however, and we will proceed as follows: We will suppose that g is not onto. The well-ordering principle will allow us to conclude that there is a smallest element in T that is not in the range of g. However, g will be constructed in a way that will not allow us to "skip over" elements of T and so we will obtain a contradiction. We now make these ideas precise.

Note that the (?)'s in this proof refer to Exercise 23.3. Once you have worked through the exercise, you should be able to read through the proof, ignoring the symbol (?) as you read.

Proof. Let T denote a subset of $\mathbb{N}$. If T is finite, it is countable and we are done. (?) We suppose, then, that T is infinite and show that it is countably infinite. To this end we construct a bijection $g : \mathbb{N} \to T$ recursively.

Since T is a nonempty subset of $\mathbb{N}$, the well-ordering principle implies that T has a least element. We set $g(0) = \min T$. (?) Note that for any $t \in T$ we can write $T = \{x \in T : x \leq t\} \cup \{x \in T : x > t\}$. (?) Since $\{x \in T : x \leq t\} \subseteq \{0, \ldots, t\}$, we may use Theorem 21.9 to conclude that $\{x \in T : x \leq t\}$ is finite. Since T is infinite, Theorem 21.11 implies that $\{x \in T : x > t\}$ is infinite for every $t \in T$. (?) In particular, the set $\{x \in T : x > t\}$ is nonempty for every $t \in T$ and contains a minimum by the well-ordering principle. Thus, we can define $g(n+1) = \min\{x \in T : x > g(n)\}$. (?) By the recursion theorem, Theorem 18.6, the function $g : \mathbb{N} \to T$ is well-defined.

In order to prove that our function g is one-to-one we first show that if $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, then $g(k+n) > g(k)$. To show this we use induction on n. By definition, $g(k+1) = \min\{x \in T : x > g(k)\} > g(k)$. (?) This concludes the base step. For the induction step, we let $n \in \mathbb{Z}^+$ and suppose that $g(k+n) > g(k)$. Then $g(k+n+1) = \min\{x \in T : x > g(k+n)\} > g(k+n)$. (?) By the induction hypothesis we get $g(k+n+1) > g(k)$. The result follows by induction. It is now immediate that if $j \neq \ell$, then $g(j) \neq g(\ell)$ (in Exercise 23.3 part (c) you are asked to provide the details for this claim). Hence g is one-to-one.

Finally, we show that the function g is onto. Suppose to the contrary that it is not. Then the set $S = \{x \in T : x \notin \operatorname{ran}(g)\}$ is nonempty. By the well-ordering principle, since S is a subset of $\mathbb{N}$ it contains a least element. Call this element a(?), and note that a is simply the smallest element in T that is not in the range of g, and therefore everything strictly smaller than a (and in T) is in the range of g. If $a = \min T$, then $a = g(0)$, which is not possible. Hence the set $\{x \in T : x < a\} \neq \emptyset$. This set is also bounded above (by a), and by Problem 12.18 it has a maximum, which we call b.(?) Since $b = \max\{x \in T : x < a\}$, we know that b is strictly smaller than a and so b is in the range of g. Therefore, there exists $n \in \mathbb{N}$ with $g(n) = b$. Now there can be nothing in T between a and b, so $\{x \in T : x > b\} = \{x \in T : x \geq a\}$, and we have

$$\begin{aligned} g(n+1) &= \min\{x \in T : x > g(n)\} \\ &= \min\{x \in T : x > b\} = \min\{x \in T : x \geq a\}.(?) \end{aligned}$$

Therefore $g(n+1) = a$, contradicting our choice of $a \in S$, so g is onto.

Since g is a bijection, T is also countable if it is infinite. □

As you read the theorems and corollaries below, think about whether or not you know a corresponding result for finite sets. If so, what was the proof? Do the ideas from those proofs help you here? Why or why not? We leave the corollaries for you to prove in Problems 23.9 and 23.10.

Corollary 23.4. *Every subset of a countable set is countable.*

Remember that when we say that a set is countable, we mean that it is finite or countably infinite. The next exercise will often allow you to handle both cases at once.

Exercise 23.5. Prove that a nonempty set A is countable if and only if there exists a one-to-one function $f : A \to \mathbb{N}$. ○

Theorem 23.6. *Suppose that A and B are countable. Then $A \cup B$ is countable.*

Our proof begins with something we have used several times before.

Proof. If $A \subseteq B$ or $B \subseteq A$, the result is clear. So suppose that $A \setminus B$ and $B \setminus A$ are both nonempty. Now note that $A \cup B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \emptyset$. Since $B \setminus A \subseteq B$, Corollary 23.4 implies that $B \setminus A$ is countable. Further, A and $B \setminus A$ are countable, so by Exercise 23.5 there exist one-to-one functions f and g such that $f : A \to \mathbb{N}$ and $g : B \setminus A \to \mathbb{N}$. Define $H : A \cup B \to \mathbb{N}$ by

$$H(x) = \begin{cases} 2f(x) & \text{if } x \in A \\ 2g(x)+1 & \text{if } x \in B \setminus A \end{cases}.$$

You can check (as you have many times before) that H is well-defined and one-to-one. Using Exercise 23.5 once again, we conclude that $A \cup B$ is countable. □

Corollary 23.7. *The union of finitely many countable sets is countable.*

In the next theorem, we want to show that $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$. It is oh, so tempting to go to the definition and try to define a function that is a bijection from our set onto $\mathbb{N}$; after all, that is the definition of equivalence. But if we do everything using definitions, we will not be taking advantage of the useful body of mathematics we have proved thus far, and we will have to re-prove everything we have done. Some of it was quite difficult to prove! Life will be much easier if we think about the theorems we have proved already and see when and how we can use them.

Theorem 23.8. *The set* $\mathbb{N} \times \mathbb{N}$ *is countable.*

Proof. We show that $\mathbb{N} \times \mathbb{N}$ is countable by defining a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ explicitly. So, let $f(n,m) = 2^n 3^m$ for all $(n,m) \in \mathbb{N} \times \mathbb{N}$. This is clearly a well-defined function. (Note that this function is not onto, since a number like 7 is not in the range. Therefore, we will not try to show that f is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.) Since the prime factorization of a natural number is unique, the function is one-to-one. Thus we have a one-to-one mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. By Exercise 23.5, we conclude that $\mathbb{N} \times \mathbb{N}$ is countable. □

Though we did not say so explicitly, $\mathbb{N} \times \mathbb{N}$ is infinite. Therefore, what we have shown is that $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$.

Exercise 23.9. Let A be a finite set and let B be a countable set. Prove that $A \times B$ is countable. ○

Corollary 23.10. *If A and B are countable sets, then $A \times B$ is countable.*

Assuming you were paying attention to all the previous results, this will not be hard to prove (see Problem 23.11).

We are now ready for the two main theorems of this chapter. After all this work, we can finally show that the set of rational numbers is countably infinite.

Theorem 23.11. *The set of rational numbers,* $\mathbb{Q}$, *is countably infinite.*

What follows is a natural way to attempt to prove this. It is, unfortunately, incorrect. But it's worthwhile to present it, see what goes wrong, and fix it.

Not a proof. The rationals can be thought of as p/q where p and q are integers with $q \neq 0$. Thus, we can define a map from $\mathbb{Q}$ to $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by $f(p/q) = (p,q)$. Then f is bijective, so $\mathbb{Q} \approx \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. By Corollary 23.10, the latter set is countable. Thus we conclude that $\mathbb{Q}$ is countable. ⍰

As we mentioned (though not quite this dramatically) there's a *HUGE* error in this proof. Find it, fix it, and then read on and see if you really figured it out.

The problem above was that the function was not well-defined. So let's try again. In the proof below, we begin by considering the positive rationals so that we don't have to worry about whether to put the minus sign in the numerator or denominator.

Proof. [Proof; the real thing] We will begin by showing that $\mathbb{Q}^+$ is countable. We define $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ as follows. Write each member of $\mathbb{Q}^+$ as p/q where $p, q > 0$ and p/q is in reduced form; that is, p and q have no positive common factor other than 1. Now define $f(p/q) = (p,q)$. Because p/q is in reduced form, f is well-defined and one-to-one. Since $\mathbb{N} \times \mathbb{N}$ is countable (Theorem 23.8), and $f(\mathbb{Q}^+)$ is a subset of it, we know from Corollary 23.4 that $f(\mathbb{Q}^+)$ is countable. We conclude that $\mathbb{Q}^+$ is countable. Now the set of negative rationals, $\mathbb{Q}^-$, is equivalent to $\mathbb{Q}^+$. Since $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$, and we have a finite union of countable sets, we use Corollary 23.7 to conclude that $\mathbb{Q}$ is countable. Since $\mathbb{Q}$ is infinite we know that it is countably infinite. □

Looks like we live in a countable world! Not quite—it's time to give an example of an uncountable set. The next theorem will show that the set of real numbers is uncountable.

There's one sticky point in our proof that the reals are uncountable. We will use the decimal representation of real numbers in $(0,1)$. Thus, a word about decimal expansions is in order here. Each element of $(0,1)$ has a decimal representation; that is, for $x \in (0,1)$, there exist integers $a_1, a_2, \ldots, a_n, \ldots$ with $0 \le a_n \le 9$ such that $x = 0.a_1 a_2 \ldots a_n \ldots$. In Problem 20.15, we showed that $0.999\ldots = 1.000\ldots$. This means that the number 1 has two representations. In fact, many numbers in $(0,1)$ have two decimal representations. It can be shown, however, that the only time this can happen is when the representations are of the form $0.a_1 a_2 a_3 \ldots a_n 999\ldots$ for some $n \in \mathbb{Z}^+$, or $0.a_1 a_2 a_3 \ldots a_m 1000\ldots$ for some $m \in \mathbb{Z}^+$. When given a choice, we will always choose the representation ending with repeated 9's. (See also Problem 20.18.)

Theorem 23.12. *The set of real numbers, $\mathbb{R}$, is uncountable.*

The idea of this proof is due to Georg Cantor and is called Cantor's diagonal argument.

Proof. We will suppose, to the contrary, that $\mathbb{R}$ is countable and see what happens. Since every subset of a countable set is countable, the open interval $(0,1)$ must be countable, too. Since $(0,1)$ is clearly infinite, and we have shown that $\mathbb{Z}^+$ is countably infinite, there exists a bijective function $f : \mathbb{Z}^+ \to (0,1)$. We will list the values of f using the decimal expansion of each element of $(0,1)$. So,

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}\ldots \\ f(2) &= 0.a_{21}a_{22}a_{23}\ldots \\ f(3) &= 0.a_{31}a_{32}a_{33}\ldots \\ &\vdots \end{aligned}$$

where each a_{ij} represents an integer between 0 and 9. Since f is onto, each number in $(0,1)$ appears in this list.

The odd thing is this: we can construct a number $b = 0.b_1 b_2 \ldots \in (0,1)$ not in this list (hence showing that our function cannot possibly be onto) by describing its decimal representation as follows. Look at $f(1)$. If $a_{11} = 2$, let $b_1 = 3$. If, on the other hand, $a_{11} \neq 2$, define $b_1 = 2$. Then the first digits of $f(1)$ and b are different, so b is not $f(1)$. For b_2, if $a_{22} = 2$, let $b_2 = 3$. If, on the other hand, $a_{22} \neq 2$, define $b_2 = 2$. Then the second digits of b and $f(2)$ are different. So b is not $f(2)$. Now compare the element b we have constructed with the list below:

$$\begin{aligned}
f(1) &= 0.\mathbf{a_{11}}a_{12}a_{13}\ldots \\
f(2) &= 0.a_{21}\mathbf{a_{22}}a_{23}\ldots \\
f(3) &= 0.a_{31}a_{32}\mathbf{a_{33}}\ldots \\
&\vdots \\
f(n) &= 0.a_{n1}a_{n2}a_{n3}\ldots\mathbf{a_{nn}}\ldots \\
\\
b &= 0.\mathbf{b_1 b_2 b_3}\ldots.
\end{aligned}$$

We constructed b so that $b_n \neq a_{nn}$, and therefore $b \neq f(n)$ for every n. Then b can't be in our list, which is a bit bizarre since we claim to have numbered all the elements in $(0,1)$, and b is certainly one of the things we numbered. This contradiction must mean that we have assumed falsely that $\mathbb{R}$ is countable. □

A word of caution: students often forget that some of the theorems proved in previous chapters and some of the definitions only apply to finite sets. Since the notion of finite and infinite is often counterintuitive, you really must make sure that you check the hypotheses of the theorems you wish to apply *before* you apply the theorems.

Reactions to Cantor's work in set theory were mixed (see, for example, [59, p. 1003]). Leopold Kronecker opposed Cantor's theory and so did Henri Poincaré (see Spotlight: The Continuum Hypothesis on page 270). In a discussion of Cantor's work, Poincaré [83] said, "For my part, and I am not alone, I think that the important thing is never to introduce objects other than those that can be completely defined in a finite number of words."[1] Hilbert and Russell praised Cantor's work. In his memorial speech for Hermann Minkowski, Hilbert points out that Minkowski was the first mathematician of their time who understood the importance of Cantor's work. He quotes Minkowski as saying, "History will call Cantor one of the most profound mathematicians of our time; it is truly regretful that a very prominent mathematician [here Hilbert tells us that this mathematician is Kronecker] led an opposition not based entirely upon factual grounds, which spoiled Cantor's pleasure in his scientific investigations."[1]

Definitions

Definition 23.1. An infinite set A is said to be **countably infinite** if $A \approx \mathbb{N}$.

[1] The translation is ours.

Definition 23.2. A set is **countable** if it is either finite or countably infinite. A set is said to be **uncountable** if it is not countable.

Solutions to Exercises

Solution (23.1).

Proof. Since A and B are countably infinite, we know that $A \approx \mathbb{N}$ and $B \approx \mathbb{N}$. By the transitivity (and symmetry) of the relation $\approx$, we may conclude that $A \approx B$. By the definition of this equivalence relation, there exists a bijective function f mapping A onto B. □

Solution (23.3).

(a) For the demo, $f : T \to T$ is defined by $f(t) = \min\{x \in T : x > t\}$.
(b) $g(0) = 2, g(1) = 3, g(2) = 5, g(3) = 7, g(4) = 11, g(5) = 13$. The subset of T used is $\{x \in T : x > g(5)\} = \{x \in T : x > 13\} = \{17, 19, 23, \ldots\}$. Thus $g(6) = \min\{17, 19, 23, \ldots\} = 17$.
(c) Suppose that $j \neq \ell$. We may assume that $j < \ell$. We showed that this implies that $g(j) < g(\ell)$. Hence $g(j) \neq g(\ell)$. This is the contrapositive of the statement in the question, and therefore the statement "if $g(j) = g(\ell)$, then $j = \ell$" is true as well.
(d) From the list in part (b) we deduce that $19 = g(7)$.

Solution (23.5). First suppose that A is countable. If A is finite, then since $A \neq \emptyset$ there exists an integer n and a bijection $f : A \to \{1, 2, \ldots, n\}$. In particular, f is a one-to-one mapping of A into $\mathbb{N}$. So we have found our f, if A is finite. If A is infinite, then A is countably infinite. Therefore, there is a bijection $f : A \to \mathbb{N}$. Thus, in both cases, we have a one-to-one mapping $f : A \to \mathbb{N}$.

Now suppose that we have a one-to-one mapping f of A into $\mathbb{N}$. Then f maps A onto its range. Therefore $A \approx \operatorname{ran}(f)$. But $\operatorname{ran}(f)$ is a subset of $\mathbb{N}$, and by Theorem 23.2 we know that it must be countable. Thus A is countable, as desired.

Solution (23.9). Here's one way to prove this:

Proof. If $A = \emptyset$, then $A \times B$ is empty and we are done. Otherwise there exists a positive integer n and a bijective function $f : \{1, 2, \ldots, n\} \to A$. Thus we may write $A \times B$ as a union: $A \times B = \bigcup_{j=1}^{n} (\{f(j)\} \times B)$. It is easy to check that for each j the function $g_j : B \to \{f(j)\} \times B$ defined by $g_j(b) = (f(j), b)$ is a bijection. Therefore, $\{f(j)\} \times B$ is countable for each j. Thus, we have written $A \times B$ as a finite union of countable sets, and by Corollary 23.7 we know that a finite union of countable sets is countable. Thus $A \times B$ is countable. □

Problems

Problem 23.1. Give an example, if possible, of each of the following:

(a) a countably infinite collection of pairwise disjoint finite sets whose union is countably infinite (see Problem 8.18 for the definition of pairwise disjoint);
(b) a countably infinite collection of nonempty sets whose union is finite;
(c) a countably infinite collection of pairwise disjoint nonempty sets whose union is finite.

Problem 23.2. Which of the following sets are finite? countably infinite? uncountable? (Be careful—don't apply theorems for finite sets to infinite sets and don't apply theorems for countable sets to uncountable sets!) Give reasons for your answers for each of the following:

(a) $\{1/n : n \in \mathbb{Z} \setminus \{0\}\}$;
(b) $\mathbb{R} \setminus \mathbb{N}$;
(c) $\{x \in \mathbb{N} : |x-7| > |x|\}$;
(d) $2\mathbb{Z} \times 3\mathbb{Z}$.

Problem 23.3. For each of the following sets decide whether it is countable or uncountable and justify your answer:

(a) The set of all lines with rational slopes;
(b) $\mathbb{Q} \setminus \{0\}$;
(c) $\mathbb{N} \setminus \{1,3\}$;
(d) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x+y = 1\}$;
(e) $[0,\infty)$.

Problem 23.4. Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable? Guess and then prove, please.

Problem 23.5. Suppose that $A \subseteq B \subseteq C$, that the sets A and C are equivalent, and that C is countable. Is $A \approx B$? Prove or give a counterexample.

Problem 23.6. (a) Give an example of two sets A and B, such that $B \subseteq A$ and $B \approx A$, but $B \neq A$.
(b) Prove that if A is a countably infinite set, then there is always a subset B of A such that $B \subset A$ and $B \approx A$.
(c) Prove that if A is an uncountable set, then there is always a subset B of A such that $B \subset A$ and B is also uncountable.
(d) Prove that if A is a finite set, $B \subseteq A$, and $B \approx A$, then $B = A$.

Problem 23.7. Prove that a set A is uncountable if there is an injective function $f : (0,1) \to A$.

Problem 23.8. (a) Let $X, Y \subseteq \mathbb{R}$. Suppose that $f : X \to Y$ is a function with the property that for all $x, y \in X$, if $x < y$, then $f(x) < f(y)$. Prove that f is one-to-one.

(b) Suppose that $g : \mathscr{P}(\{1,2\}) \to \mathscr{P}(\{1,2,3\})$ is a function with the property that for all $A,B \in \mathscr{P}(\{1,2\})$, if $A \subset B$, then $g(A) \subset g(B)$. Is g necessarily one-to-one? Prove it or give an example of g that is not one-to-one.

Problem 23.9. Prove Corollary 23.4.

Problem 23.10. Prove Corollary 23.7. To do this, note that the corollary can be restated a bit more formally as follows. If for some positive integer n, we have n sets $A_1,\ldots,A_n$, and each one is countable, then $\bigcup_{i=1}^{n} A_i$ is countable.

Problem 23.11. Prove Corollary 23.10.

Problem 23.12. There is another way to show that $\mathbb{Q}$ is countable. Turn the outline below into a proof by describing the counting process. (Don't try to find a formula for the function.)

Proof. [Outline of proof] The proof is simplest if we show that the set of positive rationals, $\mathbb{Q}^+$, is countably infinite. You showed in the exercises (and it is easy to see) that $\mathbb{Q}^- \approx \mathbb{Q}^+$. Then $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is infinite and countable, so $\mathbb{Q} \approx \mathbb{N}$. So we will restrict our attention to $\mathbb{Q}^+$. To see that $\mathbb{Q}^+$ is countable, we will make a chart of all the fractions of the form m/n where m and n are positive integers; that is, we consider the following array of numbers:

$$\begin{array}{ccccc}
1 & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Try counting the elements in the array in an orderly fashion. Make sure you don't count numbers twice! □

Problem 23.13. The set $\mathbb{Z}[\sqrt{-5}]$ is a subset of the complex numbers defined by $\mathbb{Z}[\sqrt{-5}] = \{a + \sqrt{5}b\mathrm{i} : a,b \in \mathbb{Z}\} \subseteq \mathbb{C}$, and it is called a quadratic integer ring. Is $\mathbb{Z}[\sqrt{-5}]$ countable or uncountable? Prove your answer.

Problem 23.14. Let $n \in \mathbb{Z}^+$ and denote by $\mathbb{N}^n = \mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}$ (n times). Prove that $\mathbb{N}^n$ is countable for all $n \in \mathbb{Z}^+$.

Problem 23.15. We denote by $\mathbb{N}^\infty = \{(a_1,a_2,a_3,\ldots) : a_j \in \mathbb{N} \text{ for all } j \in \mathbb{Z}^+\}$. Prove that $\mathbb{N}^\infty$ is uncountable.

Problem 23.16. Let X and Y be two nonempty finite sets and denote by $\mathscr{F}(X,Y)$ the set of all functions $f : X \to Y$. Is $\mathscr{F}(X,Y)$ finite, countably infinite, or uncountable? Prove your answer.

Problem 23.17. Prove that the set $A = \{m^2 : m \in \mathbb{Z}\}$ is countably infinite.

Problem 23.18. Prove that if $A \cap B = \emptyset$, then $\mathscr{P}(A \cup B) \approx \mathscr{P}(A) \times \mathscr{P}(B)$.

Problem 23.19. Prove the following equivalences:

(a) $(0,1) \approx [0,1)$ and
(b) $(0,1) \approx (0,1/2) \cup [1,2)$.

Problem 23.20. Prove the following generalization of Exercise 21.12.

Theorem 23.13. *If A_j is finite for all $j \in \mathbb{Z}^+$, then $\bigcup_{j \in \mathbb{Z}^+} A_j$ is countable.*

Note that induction will not work here. We suggest that you adapt the ideas of the alternate proof of Theorem 23.11 outlined in Problem 23.12. As a second note, we mention that this theorem could be generalized to allow the sets A_j to be countable. However, a proof of this has a subtlety and requires something we have not yet discussed—something known as the axiom of choice. See Project 29.12.

Problem 23.21. Prove that the set of all decreasing functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.