Chapter 21
Equivalent Sets

If you were asked how many people are in your class, you would do the natural thing and count them. Thinking about this carefully, we see that what you are doing is assigning each person in the room one and only one number. If we asked whether or not there are more people in this class than were in your high school geometry class, you could certainly answer that question by comparing the two numbers.

Now suppose we look at the positive integers and the natural numbers. Which set has more elements? What should that even mean? This is a more difficult question to answer correctly than you might think. The following mathematical folktale (often attributed to Hilbert) illustrates the problem and a solution.

Suppose there is a hotel with infinitely many rooms. The hotel is completely booked when the coach of a Davis Cup team arrives. The clever manager accommodates her by moving all of the other guests to the room numbered one higher than the room they previously occupied, which clears the first room for the coach. Then the four members of the team arrive. Each must have his own room, of course. The very clever manager moves everyone up four rooms, making enough room for the four athletes. Finally, the team's infinitely many fans arrive (this happens all the time at really good hotels). The very, very clever manager accommodates all guests by moving the residents of room number n to the room with number $2n$. Now all the new people can go in the odd-numbered rooms.

An interesting commentary of this problem was given by Smilla in the book *Smilla's Sense of Snow*. She says, "What delights me about this story is that everyone involved, the guests and the owner, accept it as perfectly natural to carry out an infinite number of operations so that one guest can have peace and quiet in a room of his own. That is a great tribute to solitude."[1] [53, p. 11].

What does "Hotel Infinity" really show us? It is our aim in the next few sections to discuss and answer these questions.

To make precise what it means for two sets (even two infinite sets) to have the same number of elements, we need a definition. We say that a set A is **equivalent** to

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[1] Excerpt from *Smilla's Sense of Snow* by Peter Høeg, translated by Tiina Nunnally. Translation copyright ⓒ1993 by Farrar, Straus and Giroux, LLC. Reprinted by permission of Farrar, Straus and Giroux, LLC.

a set B if there exists a bijection $f : A \to B$. We write $A \approx B$ for A is equivalent to B. (Other authors use the words equipotent or equinumerous.)

You actually know a lot about this concept from previous chapters (particularly Chapter 15). The next result summarizes information we already have.

Theorem 21.1. *Let $\mathscr{A}$ be a nonempty collection of sets. Equivalence between elements A and B of $\mathscr{A}$, as defined by $A \approx B$ above, is an equivalence relation on $\mathscr{A}$.*

You will be asked to prove this theorem in Problem 21.5.

Example 21.2. Show that the open interval $(0,1)$ is equivalent to the open interval $(0,3)$.

Proof. Define a function $f : (0,1) \to (0,3)$ by $f(x) = 3x$. We leave it to you to show that this function is bijective. Thus $(0,1) \approx (0,3)$. □

Since the relation $\approx$ is symmetric, we will usually say that A and B are equivalent, rather than A is equivalent to B. This concept allows us to give a precise definition of a finite set. We say that a set S is **finite** either if $S = \emptyset$ or if S is equivalent to the set $\{1,2,3,\ldots,n\}$ for some positive integer n. Thus, to prove that a nonempty set is finite, we need to find a bijection between S and a set $\{1,2,3,\ldots,n\}$ for some $n \in \mathbb{Z}^+$. Since the relation is symmetric, either set can serve as the domain. The bijection is the mathematical analog of what we usually describe as counting. A set is said to be **infinite** if it is not finite.

Example 21.3. By negating the definition of finite, say what it means for a set to be infinite.

A set S is infinite if it is nonempty and for every positive integer n there does not exist a bijection from S to the subset $\{1,2,3,\ldots,n\}$. ○

What are some examples of finite sets? By our definition, every set of the form $\{1,2,3,\ldots,n\}$ is a finite set. What about a set like $\{2,4,6\}$? It certainly feels finite, but to prove that it is finite we would have to construct a bijective function. It is easy here; the function $f : \{1,2,3\} \to \{2,4,6\}$ defined by $f(n) = 2n$ certainly works. You could also define a function g on $\{1,2,3\}$ by $g(1) = 4, g(2) = 6$, and $g(3) = 2$. In fact, if our set has more than one element, then there is more than one choice for the bijection.

Exercise 21.4. Show that the set $\{6,8,10,14\}$ is finite. ○

The rest of this chapter is a paraphrasing of much of the work we did in Chapters 14 through 17. We isolate the important ideas below and we guide you through the proofs, but you have all the techniques to prove everything yourself.

Theorem 21.5. *The sets $\mathbb{Z}$ and $\mathbb{N}$ are equivalent.*

This theorem is really the essence of the story behind "Hotel Infinity." An infinite set, $\mathbb{Z}$, can have a proper subset, $\mathbb{N}$, that has the "same number of elements" in it!

Proof. Define $f : \mathbb{Z} \to \mathbb{N}$ explicitly as follows:

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -(1+2x) & \text{otherwise} \end{cases}.$$

In Example 15.7, we showed that this function is bijective. From this we conclude that $\mathbb{Z} \approx \mathbb{N}$.   □

The same techniques that were used to prove Theorem 21.5 can be used to prove the next theorem.

Theorem 21.6. *Let A, B, C, and D be nonempty sets. If $A \cap B = \emptyset$, $C \cap D = \emptyset$, $A \approx C$, and $B \approx D$, then $A \cup B \approx C \cup D$.*

Proof. [Outline of proof] Since $A \approx C$, there exists a bijective function $f : A \to C$. Similarly, since $B \approx D$, there exists a bijective function $g : B \to D$. We define a function $H : A \cup B \to C \cup D$ in cases by

$$H(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

We leave it to you to show that H is well-defined and bijective.   □

Exercise 21.7. What happens to our proof if $A \cap B \neq \emptyset$? If we do not assume that $A \cap B = \emptyset$, is the theorem still true?   ○

For finite sets, Theorem 21.6 has an interesting consequence.

Corollary 21.8. *Let A and B be disjoint sets. If A and B are finite, then $A \cup B$ is finite.*

The proof of Corollary 21.8 is left to you in Problem 21.9.

Since the definition of equivalent sets uses bijective functions, many of the theorems you have had will come in quite handy. If you have forgotten the definition of the restriction of a function (see Definition 15.4), you will want to review it before proceeding. We'll use restrictions in many of the proofs about finite and infinite sets.

Theorem 21.9. *Let n be a positive integer. Then every subset of $\{1, 2, 3, \ldots, n\}$ is finite.*

Proof. The proof will be by induction on n.

If $n = 1$, then our set is $\{1\}$, and there are only two subsets; $\{1\}$ and $\emptyset$. Therefore, the result holds if $n = 1$.

We now turn to the induction step. So let $n \in \mathbb{Z}^+$ and suppose that every subset of $\{1, 2, \ldots, n\}$ is finite. We must show that every subset of $\{1, 2, \ldots, n, n+1\}$ is finite.

So consider the set $\{1, 2, \ldots, n, n+1\}$, and a subset S of this set. If $S \subseteq \{1, 2, \ldots, n\}$, then S is finite by our induction hypothesis. Otherwise, $n+1 \in S$. In this case, notice that $\{n+1\}$ is a finite set. Since $S \setminus \{n+1\} \subseteq \{1, \ldots, n\}$ we also know from the induction hypothesis that $S \setminus \{n+1\}$ is finite. Applying Corollary 21.8 we may conclude that $S = (S \setminus \{n+1\}) \cup \{n+1\}$ is finite, completing the proof.   □

Corollary 21.10. *Let S be a finite set. Then every subset of S is finite.*

Proof. If S is empty, the result is clear. So suppose that S is nonempty, and let T denote a subset of S. Again, we may assume that T is nonempty.

By our assumption there exists $n \in \mathbb{Z}^+$ and a bijection $f : S \to \{1, 2, \ldots, n\}$. By Problem 15.19, the restriction function, $f|_T$, is a bijective mapping from T onto a nonempty subset B of $\{1, 2, \ldots, n\}$. From Theorem 21.9, the set B is finite, and therefore there exists a positive integer m and a bijection $g : B \to \{1, 2, \ldots, m\}$. By Theorem 16.6, the composition $h = g \circ (f|_T)$ of the two bijective functions g and $f|_T$ is a bijection of T onto $\{1, 2, \ldots, m\}$. Thus T is finite, completing the proof.   □

We assumed in Corollary 21.8 that the finite sets A and B were disjoint, but our intuition tells us that the union of two finite sets should be finite. How do we prove this? The idea is that the union of two sets can be expressed as the union of two disjoint sets: the intersection appears in the union twice, so to speak, so if we remove it once (from one of the sets), we haven't changed the union. That is the key to the next result.

Theorem 21.11. *The union of two finite sets is finite.*

Proof. Let A and B denote the two finite sets. Now you have already shown (in Problem 7.14) that

$$A \cup B = (A \setminus B) \cup B,$$

and it should be clear that these two sets are disjoint. By Corollary 21.10, the set $A \setminus B$ is finite. The set B is finite by assumption. Since these are two disjoint sets, we have written $A \cup B$ as the disjoint union of two finite sets and Corollary 21.8 now implies that $A \cup B$ is finite.   □

Note that in the theorem above, we showed that a set is finite without exhibiting a specific bijection. We'll be building up many results that are useful, and we will not always go back to the definition to see how to prove things. This is a great plus—proofs become shorter, and sometimes prettier and more interesting. Of course, to use the theorems, you also have to know what they are!

We conclude this chapter with a useful exercise.

Exercise 21.12. Use induction to prove the following. Let $m \in \mathbb{Z}^+$. If $A_1, A_2, \ldots, A_m$ are finite sets, then the union $\bigcup_{j=1}^m A_j$ is finite.

Definitions

Definition 21.1. A set A is **equivalent** to a set B if there exists a bijection $f : A \rightarrow B$. We write $A \approx B$ for A is equivalent to B.

Definition 21.2. A set S is **finite** if either $S = \emptyset$ or if S is equivalent to the set $\{1, 2, 3, \ldots, n\}$ for some positive integer n.

Definition 21.3. A set is **infinite** if it is not finite.

Solutions to Exercises

Solution (21.4). We define $f : \{6, 8, 10, 14\} \rightarrow \{1, 2, 3, 4\}$ by $f(6) = 1$, $f(8) = 2$, $f(10) = 3$, and $f(14) = 4$. It is clear that f is a bijective function, hence $\{6, 8, 10, 14\}$ is finite.

Solution (21.7). If A and B are not disjoint, the function H may not be well-defined. In addition, if A and B are not disjoint, the conclusion of the theorem may not hold. To see this, take finite sets with $A = B = \{1\}$, $C = \{2\}$, and $D = \{3\}$.

Solution (21.12). We will prove this statement by induction on m. For the base step ($m = 1$), we get $\bigcup_{j=1}^{1} A_j = A_1$, which is finite by assumption.

Now we proceed to the induction step. Let $m \in \mathbb{Z}^{+}$ and suppose that $\bigcup_{j=1}^{m} A_j$ is finite. We need to show that $\bigcup_{j=1}^{m+1} A_j$ is finite. But $\bigcup_{j=1}^{m+1} A_j = \left(\bigcup_{j=1}^{m} A_j \right) \cup A_{m+1}$. By the induction hypothesis, $\bigcup_{j=1}^{m} A_j$ is finite, and A_{m+1} is assumed to be finite. By Theorem 21.11, the union of two finite sets is finite. Thus $\bigcup_{j=1}^{m+1} A_j$ is finite. The result now follows from the principle of mathematical induction.

Problems

Problem 21.1. Show that the following intervals of real numbers are equivalent:

  (a) $[0, 1]$ and $[0, 2]$;
  (b) $[0, 1]$ and $[2, 5]$.

Problem 21.2. Prove that $\{1, \ldots, 10\} \times \{1, \ldots, 15\}$ is finite using only the definition of a finite set; that is, write down the relevant bijection explicitly.

Problem 21.3. Explain, in words, the difference between a "finite union of sets" and a "union of finite sets." Give examples of each that show these really are different.

Problem# **21.4.**   (a) Show that the set $\mathbb{Q}^{+}$ of positive rationals and the set $\mathbb{Q}^{-}$ of negative rationals are equivalent.

(b) Show that the set of even integers and set of odd integers are equivalent.

Problem 21.5. Prove Theorem 21.1.

Problem 21.6. (a) Prove that $(0,1) \approx \mathbb{R}$. (If you choose to use Problem 15.13, make sure you solve that problem too!)
(b) Prove that $\mathbb{R} \approx \mathbb{R}^+$.

Problem 21.7. (a) Show that $\mathbb{Z} \approx 2\mathbb{Z}$.
(b) Using theorems from this chapter (don't define functions!) show that $2\mathbb{Z} \approx \mathbb{N}$.

Problem 21.8. Prove Theorem 21.6 working with the outline given in the text.

Problem 21.9. (a) Suppose that A and B are nonempty finite sets and $A \cap B = \emptyset$. Show that there exist positive integers n and m such that $A \approx \{1,2,\ldots,n\}$ and $B \approx \{n+1,\ldots,n+m\}$.
(b) Prove Corollary 21.8.

Problem 21.10. Prove Theorem 21.13 below. We suggest that you start by working Problem 16.17 if you have not already done so.

Theorem 21.13. *Let A,B,C, and D be nonempty sets with $A \approx C$ and $B \approx D$. Then $A \times B \approx C \times D$.*

Problem 21.11. Prove the following corollary of Theorem 21.13 above.

Corollary 21.14. *Let A and B be finite sets. Then $A \times B$ is a finite set.*

Problem 21.12. Let $\mathscr{A} = \{[a,b) : a,b \in \mathbb{R} \text{ and } a < b\}$ be the collection of bounded half-open intervals of real numbers.

(a) Prove that $\mathscr{A} \approx \mathbb{R} \times \mathbb{R}^+$.
(b) Prove that $\mathscr{A} \approx \mathbb{R} \times \mathbb{R}$.

Problem 21.13. For $j \in \mathbb{Z}^+$, let $A_j \subseteq \{1,\ldots,j\}$. Suppose that for some $n \in \mathbb{Z}^+$, we have $B \subseteq \bigcup_{j=1}^n A_j$. Is B necessarily finite? Prove it or give a counterexample.

Problem 21.14. Let A be a subset of the set of all functions $f : \mathbb{R} \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a bijective function. We use the notation $gAg^{-1} = \{g \circ f \circ g^{-1} : f \in A\}$. Prove that $A \approx gAg^{-1}$.

Problem 21.15. Prove that $((\mathbb{Z}^+ \times \mathbb{N}) \cup (\mathbb{Z}^- \times \mathbb{N})) \approx \mathbb{Z} \times \mathbb{Z}$.

Problem 21.16. By Theorem 21.5, $\mathbb{Z} \approx \mathbb{N}$ and hence there is a bijection $g : \mathbb{Z} \to \mathbb{N}$.

(a) We define $G : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N} \times \mathbb{N}$ by $G(n,m) = (g(n), g(m))$. Prove that G is a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N} \times \mathbb{N}$.
(b) Explain why part (a) shows that $\mathbb{Z} \times \mathbb{Z} \approx \mathbb{N} \times \mathbb{N}$.
(c) Show that $\mathbb{Z} \times \mathbb{Z} \approx \mathbb{N} \times \mathbb{N}$ using theorems from this chapter and without constructing a bijective function.

Problem 21.17. Let A be a nonempty finite set of integers and let $\mathscr{L}$ be the set of all linear polynomials with coefficients in A; that is, $\mathscr{L} = \{ax + b : a \in A \text{ and } b \in A\}$. Prove that $\mathscr{L}$ is finite.

Problem 21.18. Let $A = \{(a_1, a_2, a_3, a_4, a_5) : a_j \in \{0, 1\} \text{ for } j \in \mathbb{Z} \text{ and } 1 \le j \le 5\}$. Prove that $\mathscr{P}(\{1, 2, 3, 4, 5\}) \approx A$.

Problem 21.19. Let $n \in \mathbb{Z}^+$ and define $A_n = \{(a_1, \ldots, a_n) : a_j \in \{0, 1\} \text{ for } j \in \mathbb{Z}$ and $1 \le j \le n\}$. Prove that for all $n \in \mathbb{Z}^+$ we have $\mathscr{P}(\{1, \ldots, n\}) \approx A_n$. (See Problem 21.18.)

Problem 21.20. Prove that $(0, \infty) \approx [0, \infty)$.

Problem 21.21. Prove that $(0, 1) \approx [0, 1]$.