Chapter 22
Finite Sets and an Infinite Set

We have proved that sets are finite, but we have not yet shown rigorously that a set is infinite. It is not as easy as you might think to do so, nor do we have an exact notion of what it means for a finite set "to have n elements." Our proof of the former and the definition of the latter will depend on a principle known as the *pigeonhole principle*. The pigeonhole principle is something that is familiar to all of us. As a very simple example of this, recall a childhood birthday party in which you played musical chairs. In case you weren't invited to any parties, we'll remind you of the rules behind the game. Let's say there were 10 children at the party. Someone, say the child's father, would set up 9 chairs in a row. Someone else, say the child's mother, would play a song on the piano stopping unexpectedly at some point. When the music ended, the 10 children would scramble for the 9 seats. If everyone sat down, two people would sit in the same chair. This game is our first example of the pigeonhole principle. Now we turn to a more elaborate one.

Theorem 22.1. *Suppose that n people (n ≥ 2) are at a party. Then there exist at least two people at the party who know the same number of people present.*

First you need to know the rules. We will assume that no one knows him- or herself. We will also assume that if x claims to know y, then y also knows x.

The idea behind the proof is this, and you can try it out at your next party. You will put n boxes on the board numbered 0 through $n-1$. Each person counts up the number of people he or she knows at the party. You ask them that number and write their name in the box with the same number. Note that each person's answer corresponds to exactly one of the boxes 0 through $n-1$. The theorem claims that at least two people's names will end up in the same box.

Proof. We imagine n boxes that are numbered 0 through $n-1$. For an integer m with $0 \leq m \leq n-1$, box m contains the names of the people who know m people at the party.

We break this proof into two cases. First, suppose that there is someone at the party who doesn't know anyone. We'll call this party crasher Ms. X. Now if we pick some other party attendee, he doesn't know himself and, since Ms. X doesn't know

him, he doesn't know Ms. X either. This implies that he knows at most $n-2$ people at the party. The point is this: No one's name can be in the box labeled $n-1$. This means that the names of the n people are in the $n-1$ boxes labeled 0 through $n-2$. Obviously then, there is a box with at least two names in it, indicating that two of those people know the same number of people and we are done in this case.

Now suppose that everyone knows at least one person at the party. Then no one's name can be in the box marked 0; everyone's name will be in one of the $n-1$ boxes marked $1,\ldots,n-1$. Once again we have n names in $n-1$ boxes, and thus at least two must be in the same box.   □

This theorem and its proof illustrate the idea behind the pigeonhole principle. In its popular form, the principle says that *if there are more pigeons than holes, then at least one hole is the home of more than one pigeon.* There are many wonderful applications of the pigeonhole principle (and many can be found at the website [13] under algebra).

We now turn to the more precise statement of the pigeonhole principle and its proof. The principle is attributed to Peter Gustav Lejeune Dirichlet and is also known as the Dirichlet principle or the Dirichlet drawer principle. (There are other Dirichlet principles; see Spotlight: Minimum or Infimum? in Chapter 17.) Curiously, this intuitively obvious principle has a rather intricate proof.

Theorem 22.2 (Pigeonhole principle). *Let m and n be positive integers with $m > n$, and let f be a map satisfying $f : \{1,\ldots,m\} \to \{1,\ldots,n\}$. Then f is not one-to-one.*

Proof. We will prove this theorem by induction on n. The assertion that we will prove is: For every integer m such that $m > n$, if $f : \{1,\ldots,m\} \to \{1,\ldots,n\}$, then f is not one-to-one.

For the base case, $n = 1$. Our assumption that $m > n$ implies that $m > 1$. For every map $f : \{1,\ldots,m\} \to \{1\}$, it is clear that $f(1) = 1 = f(m)$. Since $1 \neq m$, we may conclude that f is not one-to-one, completing the base step.

For the induction step, let $n \in \mathbb{Z}^+$ and suppose that if m is an integer greater than n and f is a function $f : \{1,\ldots,m\} \to \{1,\ldots,n\}$, then f is not one-to-one. We will show that if $m > n+1$ and $f : \{1,\ldots,m\} \to \{1,\ldots,n+1\}$, then f is not one-to-one.

So let us suppose that $m > n+1$ and we have a map

$$f : \{1,\ldots,m\} \to \{1,\ldots,n+1\}.$$

There are three cases to consider: Either f maps nothing to $n+1$, more than one element to $n+1$, or exactly one element to $n+1$.

For the first case, $n+1$ is not in the range of f, so f actually defines a map $f : \{1,\ldots,m\} \to \{1,\ldots,n\}$. Since $m > n+1 > n$, our induction hypothesis tells us that f is not one-to-one, and we are done in this case.

In case two, there exist $j,k \in \{1,\ldots,m\}$ with $j \neq k$, and $f(j) = n+1 = f(k)$. Then f is not one-to-one, and we are done in this case, too.

For the last case, we may assume that $j \in \{1,\ldots,m\}$ is the only integer for which $f(j) = n+1$. We now define the function $g : \{1,\ldots,m\} \to \{1,\ldots,m\}$ that interchanges m with j and leaves all other elements of $\{1,\ldots,m\}$ fixed:

$$g(k) = \begin{cases} k & \text{if } k \neq j, m \\ j & \text{if } k = m \\ m & \text{if } k = j \end{cases}.$$

Then g is one-to-one. Furthermore, since j is the only integer that f maps to $n+1$ we know that $(f \circ g)(k) = f(g(k)) = n+1$ if and only if $g(k) = j$. Since g is one-to-one, this happens if and only if $k = m$. Thus $(f \circ g)|_{\{1,\ldots,m-1\}}$ maps $\{1,\ldots,m-1\}$ to $\{1,\ldots,n\}$. Since we assume that $m > n+1$, we know that $m-1 > n$. Our induction hypothesis now applies, and we conclude that $(f \circ g)|_{\{1,\ldots,m-1\}}$ is not one-to-one. By Problem 15.19 $f \circ g$ is not one-to-one. Since g is one-to-one, by Theorem 16.7 f is not one-to-one in this case either.

By the principle of mathematical induction, if $m, n \in \mathbb{Z}^+$ and $m > n$, then no function $f : \{1,\ldots,m\} \to \{1,\ldots,n\}$ is one-to-one.   □

The proof of the pigeonhole principle summarizes much of what you learned: mathematical induction, proof in cases, and one-to-one functions.

We are now in a position to prove that a set is infinite.

Theorem 22.3. *The set $\mathbb{N}$ is infinite.*

Proof. Suppose to the contrary that $\mathbb{N}$ is finite. Since $\mathbb{N} \neq \emptyset$ there exists an integer m and a one-to-one mapping, g, of $\mathbb{N}$ onto $\{1, 2, \ldots, m\}$. Now $\{1, 2, \ldots, m+1\} \subseteq \mathbb{N}$, so we may consider the restriction $g|_{\{1,2,\ldots,m+1\}} : \{1, 2, \ldots, m+1\} \to \{1, 2, \ldots, m\}$. The pigeonhole principle (Theorem 22.2) implies that $g|_{\{1,2,\ldots,m+1\}}$ is not one-to-one. This, in turn, implies (as you surely showed in Problem 15.19) that g is not one-to-one, contradicting our choice of g. Therefore, it must be the case that $\mathbb{N}$ is infinite.   □

Exercise 22.4. Prove that $\mathbb{Z}$ is infinite.   ○

The next exercise is similar to the one above, but requires more work.

Exercise 22.5. Prove that if X is an infinite set, then the power set of X is infinite.   ○

We carefully defined what it means for a set to be finite, but so far we have not described what it means for a set to "have n elements." After the following theorem, we will be ready to do that.

Theorem 22.6. *Let A be a nonempty finite set. There is a unique positive integer n such that $A \approx \{1,\ldots,n\}$.*

Before we begin, note that there are two things to show: there exists a positive integer n with certain properties and that there is only one such integer. One of these should be easy. Which one?

Proof. The existence of some $n \in \mathbb{Z}^+$ such that there is a bijection $f : A \to \{1, \ldots, n\}$ is guaranteed by the definition of a nonempty finite set. So all we have to do is show that there is no other $m \in \mathbb{Z}^+$ with an associated bijection $g : A \to \{1, \ldots, m\}$.

Suppose to the contrary that there does exist such a positive integer m with $m \neq n$, and bijective function g. Since $m \neq n$, one of these integers must be larger than the other, so we assume that $m > n$. Since g is a bijection, it has an inverse. Composing the two bijective functions f and g^{-1}, we obtain a bijective function $f \circ g^{-1} : \{1, \ldots, m\} \to \{1, \ldots, n\}$. But this contradicts the pigeonhole principle, and we conclude that n is unique.   □

The integer n in the above theorem is exactly what we mean by the "size of A" or "number of elements in A." We will say that the **cardinality** of a finite set A is 0 if A is empty and n if $A \approx \{1, \ldots, n\}$. In symbols, we write $|A| = n$, where $n \in \mathbb{N}$. (Why couldn't we define the cardinality of a set before we proved the theorem?) While it is possible to define cardinality in more generality, we have only defined it in the case that A is finite.

Definition

Definition 22.1. The **cardinality** of a finite set A is 0 if A is empty and n if $A \approx \{1, \ldots, n\}$. In symbols, we write $|A| = n$, where $n \in \mathbb{N}$.

Solutions to Exercises

Solution (22.4).

Proof. We know that $\mathbb{N} \subset \mathbb{Z}$, and Theorem 22.3 tells us that $\mathbb{N}$ is infinite. Since Corollary 21.10 says that every subset of a finite set is finite, our set $\mathbb{Z}$ must be infinite.   □

Solution (22.5).

Proof. Define a map $f : X \to \mathscr{P}(X)$ by $f(x) = \{x\}$ for all $x \in X$. It is clear that f is well-defined and one-to-one. Therefore, f maps X onto a subset, $\mathscr{A}$, of $\mathscr{P}(X)$. Since X is infinite and f is a bijection of X onto $\mathscr{A}$, we know that $\mathscr{A}$ must be infinite as well. Thus, $\mathscr{A}$ is an infinite subset of $\mathscr{P}(X)$, and we may use Corollary 21.10 to conclude that $\mathscr{P}(X)$ is infinite.   □

Problems

Problem 22.1. Consider the story of n people at a party in Theorem 22.1. Suppose someone else has a rival party the same evening, and no one can attend both. Some-

one takes a picture of the people at the rival party and shows it to everyone at your party. Your party isn't that much fun, so you each look at the picture and say how many people you know at the other party. No one says the same number. What can you conclude about the number of people attending the other party?

Problem 22.2.   (a) Suppose there are 15 people in a class. Show that two people must be born in the same month.
  (b) A conductor has just taken on a new job in a small town where he has five trumpet players in his orchestra. He has a concert every other evening for his first year. Traditionally the players are seated from left to right in order of decreasing musical ability. The conductor does not want to offend the players, so he has decided to seat them differently at each performance. Can he do it? Why or why not?

Problem 22.3. Suppose 21 numbers are chosen from the set $\{1,\ldots,40\}$. Show that among the chosen numbers there are (at least) two of them, n and m, such that $n - m = 1$.

Problem 22.4. Let S be a region in the plane bounded by a square with sides of length two. Prove that if we put five points in S, there exist (at least) two of these points that are at most a distance of $\sqrt{2}$ apart.

Problem 22.5. Let $n \in \mathbb{N}$ and let $f : \{1,2,\ldots,2n+1\} \to \{1,2,\ldots,2n+1\}$ be a bijective function. Prove that for some odd integer $k \in \{1,2,\ldots,2n+1\}$, the integer $f(k)$ is also odd.

Problem 22.6. Let $f : \{1,2,\ldots,99\} \to \{1,2,\ldots,99\}$ be an injective function. Define $g : \{1,2,\ldots,99\} \to \mathbb{N}$ by $g(n) = n + f(n)$. Prove that there exists an integer n such that $g(n)$ is even.

Problem 22.7. Prove the following alternate form of the pigeonhole principle.
   Let A and B be nonempty finite sets, and suppose that $|A| > |B|$. If $f : A \to B$ is a function, then f is not one-to-one.

Problem# 22.8. Show that $\mathbb{Q}$ is infinite.

Problem 22.9. Using only the definition of finite and the pigeonhole principle, prove that $\mathbb{R}$ is infinite.

Problem 22.10. Let A be a set, and suppose that B is an infinite subset of A. Show that A must be infinite.

Problem 22.11. Suppose that A is an infinite set, B is a finite set, and $f : A \to B$ is a function. Show that there exists $b \in B$ such that $f^{-1}(\{b\})$ is infinite.

Problem 22.12. Let X be an infinite set, and A and B be finite subsets of X. Answer each of the following, giving reasons for your answers:

  (a) Is $A \cap B$ finite or infinite?

(b) Is $A \setminus B$ finite or infinite?
(c) Is $X \setminus A$ finite or infinite?
(d) Is $A \cup B$ finite or infinite?
(e) If $f : A \rightarrow X$ is a one-to-one function, is $f(A)$ finite or infinite?

Problem 22.13. Let A, B, and C be finite sets.

(a) Recall that we showed that if A and B are disjoint, then $A \cup B$ is finite. Look over the proof outlined in Problem 21.9 and determine $|A \cup B|$ in terms of $|A|$ and $|B|$, assuming that A and B are disjoint.
(b) Suppose that A and B are not disjoint. Show that $|A \cup B| = |A| + |B| - |A \cap B|$.
(c) Find a formula that works for three sets A, B, and C. (You don't need to prove that your formula works.)

Problem 22.14.   (a) Suppose that A and B are finite sets with $|A| = m$ and $|B| = n$. In Problem 21.11 you showed that if A and B are finite, then $A \times B$ is finite. Look over the proof and determine $|A \times B|$ in terms of m and n.
(b) Suppose that $A_1, A_2, \ldots, A_k$ are finite sets. Guess a formula for the cardinality of $A_1 \times A_2 \times \cdots \times A_k$ (in terms of $|A_1|, |A_2|, \ldots$, and $|A_k|$). Prove that your formula is correct.

Problem 22.15. Prove that if X is a finite set with $|X| = n \in \mathbb{N}$, then $|\mathscr{P}(X)| = 2^n$.

Problem 22.16. Each of the problems below is an application of one of the counting principles given in Problems 22.13 and 22.14. Decide which part of that problem applies, and use it to answer the problem.

(a) Thirty second-graders, twenty-five third-graders, and fifteen fourth-graders entered an art contest. Three prizes were awarded, one for each grade. In how many ways can the prizes be awarded to three of the children? (Don't forget to say which formula from Problem 22.13 or 22.14 applies.)
(b) Suppose that there are 100 people in a room. Of these 55 are men, 33 are Swiss, 10 are Swiss males. How many are Swiss or male (or both)? (Don't forget to say which formula from Problem 22.13 or 22.14 applies.)

   The rest of the problems are interrelated. If you can't see how to do the problem you are working on, look at the results from the previous problems and Problem 22.13.

Problem 22.17. Let A be a nonempty finite set with $|A| = n$ and let $a \in A$. Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

Problem 22.18.   (a) Suppose that A is a finite set and $B \subseteq A$. We showed that B is finite. Show that $|B| \leq |A|$.
(b) Suppose that A is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.
(c) Show that if two finite sets A and B satisfy $B \subseteq A$ and $|A| \leq |B|$, then $A = B$.

Problem 22.19. Suppose that A and B are finite sets and $f : A \rightarrow B$ is one-to-one. Show that $|A| \leq |B|$.

Problem 22.20. Let A and B be sets with A finite. Let $f : A \to B$. Prove that $|\mathrm{ran}(f)| \leq |A|$.

Problem 22.21. Let A be a finite set. Show that a function $f : A \to A$ is one-to-one if and only if it is onto. Is this still true if A is infinite?