Chapter 14
Functions, Domain, and Range

What is a function? You've probably gotten a definition of this somewhere along the way. We will state the definition of function in terms of relations, which is probably different than the way you have seen it stated.

Let A and B be sets. A **function f from A to B** is a relation f from A to B satisfying

(i) for all $a \in A$, there exists $b \in B$ such that $(a,b) \in f$, and
(ii) for all $a \in A$, and all $b,c \in B$, if $(a,b) \in f$ and $(a,c) \in f$, then $b = c$.

A function is often called a **map** or **mapping**. We usually write $f : A \to B$ to indicate that f is a function from A to B. The sets A and B may be explicitly identified, but they are often understood from the context. (See Problem 14.4.)

When we know what the two sets are and that the two conditions are satisfied, we say f is a well-defined function. If the object we try to define does not satisfy these properties, it isn't a function, and we often say that f (which we shouldn't call a function) is not well-defined.

Condition (i) makes sure that each element in A is related to some element of B, while condition (ii) makes sure that no element of A is related to more than one element of B. Note that it may be the case that an element of B has no element of A to which it is related; or an element of B could be related to more than one element of A. The set A is called the **domain**, and denoted by $\text{dom}(f)$, and the set B is called the **codomain**, and denoted $\text{cod}(f)$.

As a first example, consider the function that assigns to a citizen of the United States his or her height measured in inches on a particular day at a particular time. We'll assume that people are at most 20 feet tall. This is a function because we have a *domain* (the set of the citizens of the United States on a particular day at a particular time), a *codomain* (the set of real numbers between 0 and 240 inches), *condition (i)* (each person has a height), and *condition (ii)* (each person has exactly one height on that day, at that time). Now let's turn to a nonexample. We still consider the domain to be the set of citizens of the United States, but this time let the codomain consist of all the countries in the world (on a particular day, at a particular time). Consider the relation that assigns to each person in the domain his or her country

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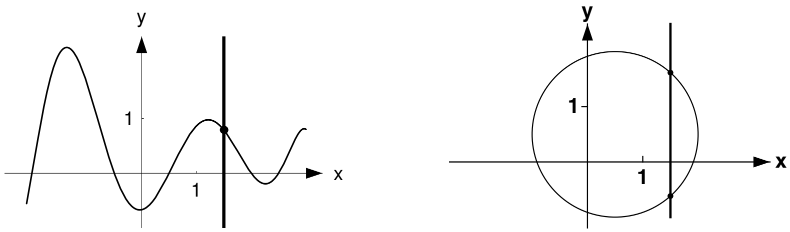

Fig. 14.1

(countries) of citizenship. This is not a function because a United States citizen can be a citizen of more than one country. Though (i) is satisfied because each person in the domain is a U.S. citizen, (ii) is not.

Exercise 14.1. Let $A = \{1,2,3\}$ and $B = \{2,4,6\}$. Which of the following are functions from A to B? If they are not functions, explain which rule is violated.

(a) The relation f is $\{(1,2),(2,4),(3,4)\}$.
(b) The relation f is $\{(1,2),(1,4),(2,2),(3,6)\}$.
(c) The relation f is $\{(1,2),(3,4)\}$.
(d) The relation f is $\{(2,4),(1,2),(3,6)\}$. ○

Exercise 14.2. You probably learned that a function $f : \mathbb{R} \to \mathbb{R}$ can be represented by a graph, and that there is a vertical line test to determine whether or not f is a function (see Figure 14.1). Which condition in the definition corresponds to the vertical line test? Why? ○

In Exercise 14.1, you probably recognized (d) as a function from A to B. It is more usual to write $f(1) = 2$, $f(2) = 4$, and $f(3) = 6$. Since each x in the domain is related to a unique y in the codomain, we will write $f(x) = y$ rather than $(x,y) \in f$.

Exercise 14.3. Rewrite the definition of a function using the notation introduced in the paragraph above. ○

Here are some more examples and nonexamples of functions.

Exercise 14.4. Decide which of the following are functions and which are not, giving reasons for your answers.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 2$.
(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/x^2$.
(c) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x,x)$.
(d) Let $f : \mathbb{Q} \to \mathbb{Q}$ be defined by $f(p/q) = 1/q$, where p and q are integers and $q \neq 0$.
(e) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (x,3)$. ○

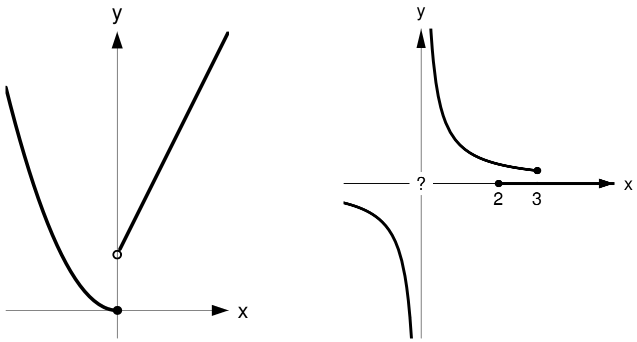

Fig. 14.2 The graph on the left is of f, the one on the right is of g

You have seen many examples of functions. One particular type of example, that of a function defined in cases, allows us to explicitly illustrate many of the ideas discussed in this section. Before you begin working with a function that is defined in cases, make sure that you understand the function. If you can, graph it. Remember that the best thing to do is to work with concrete objects (like trying $x = 2$ or $x = -3$) until you get a feel for what is happening. For functions that are defined in cases we have to be particularly careful to check that the cases don't overlap; or if they do, that the function is defined in a unique way for all the elements in the domain that are in the overlap. Of course, we are not changing the rules here. All you really have to do is check that you know what the domain and codomain are, and that conditions (i) and (ii) of the definition hold. Here are some examples.

Example 14.5. We will check to see whether each of the objects defined below and graphed in Figure 14.2 is a function.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ 2x+1 & \text{if } x > 0 \end{cases}.$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x \ge 2 \\ 1/x & \text{if } x \le 3 \end{cases}.$$

For (a) note that the domain is $\mathbb{R}$ and the codomain is also $\mathbb{R}$. From the definition of f it is easy to see that f is defined for all $x \in \mathbb{R}$. Hence condition (i) of the definition of a function holds.

Now let $a \in \mathbb{R}$ and suppose that there exist real numbers b and c with $f(a) = b$ and $f(a) = c$. The most orderly way to check condition (ii) is the following: If $a \le 0$, then $b = f(a) = a^2$ and $c = f(a) = a^2$, so $b = c$. If $a > 0$, then $b = f(a) = 2a+1$ and $c = f(a) = 2a+1$. Hence $b = c$. In either case, $b = c$. So condition (ii) holds. Since both (i) and (ii) are satisfied, f is well-defined.

The formula given in part (b) does not define a function for two reasons. First note that 0 is in the domain. Since $0 \leq 3$, we see that $g(0)$ is not defined to be an element in the codomain. Hence condition (i) is violated, and we conclude that g is not a function.

We mention here that there is a second problem with the definition of g: consider a real number a such that $2 \leq a \leq 3$. For instance, let $a = 2.5$. Then $g(2.5) = 0$ (since $2.5 \geq 2$) and $g(2.5) = 2/5$ (since $2.5 \leq 3$). This violates condition (ii) of the definition of a function. Thus g does not satisfy condition (i) or (ii). Therefore, the object defined above is not a function, for two reasons.

Although the example above violates both (i) and (ii), keep in mind that it is enough that (i) or (ii) alone be violated to ensure that f is not a function. ○

Exercise 14.6. For each of the two examples below decide whether or not the object so defined is a function. Give reasons for your answers.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -(x^2) & \text{if } x \leq 0 \end{cases}.$$

(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in 2\mathbb{Z} \\ 2 \text{ if } x \text{ is prime} \\ 3 \text{ otherwise} \end{cases}.$$

○

One very important example of a function defined in cases is the familiar absolute value function.

Example 14.7. The absolute value function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x|$. It is easy to check that this does define a function on $\mathbb{R}$. ○

When you define a new mathematical concept, it's always a good idea to think about it and pose questions. Of course, it's also a good idea to answer those questions, if you can. We now turn to some questions that we find interesting. See if you can think of some questions on your own.

What does it mean to say that two functions $f : A \longrightarrow B$ and $g : A \longrightarrow B$ are equal? Since this is a very important concept that we will need again later, we provide the answer here. But try to think about how this answer follows from the definition of a function.

> Two functions $f : A \longrightarrow B$ and $g : A \longrightarrow B$ are equal if and only if $f(x) = g(x)$ for all $x \in A$.

Here's a second question: What is the function's relationship to elements of the domain, and how does this differ from the function's relationship to elements of the

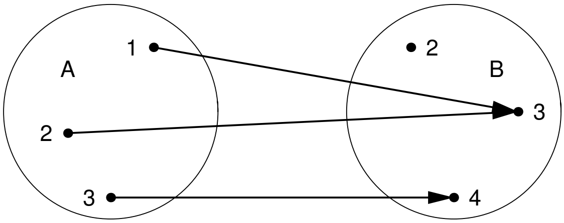

Fig. 14.3

codomain? We must be able to evaluate f for every element in the domain, while elements in $\mathrm{cod}(f)$ may or may not be associated with elements of the domain. The elements of the codomain that are related to elements of $\mathrm{dom}(f)$ are obviously important in understanding the function. For this reason, we look at the set called the range of f, which consists precisely of these points.

Given a function $f : A \to B$, the **range** of f, denoted $\mathrm{ran}(f)$, is defined by

$$\mathrm{ran}(f) = \{b \in B : \text{there exists at least one } a \in A \text{ such that } f(a) = b\}.$$

Sometimes it is fairly easy to determine the range, but it generally requires a method (demonstrated below) that we think of as working backwards. You'll start with $b \in B$ and try to find $a \in A$. Then, to show that $b \in \mathrm{ran}(f)$, you have two things to check: The element a must map to b under f (that is, $f(a) = b$), *and* a must be an element of A. This latter statement is often obvious, but don't forget to check it!

It's always easier to start with small sets and see if you understand what is happening. You can do this visually as well. For example, say $A = \{1,2,3\}$, $B = \{2,3,4\}$, and the function $f : A \to B$ is defined by $f(1) = 3, f(2) = 3$, and $f(3) = 4$. We can "see" the action of f by drawing a little picture as in Figure 14.3

From this picture we can see easily that f sends two things to 3, one thing to 4, and nothing to 2. So we can "see" that though 2 is in the codomain, it is not in the range. If you are asked for examples or counterexamples, remember that small sets will sometimes do the trick!

Our next example is really a method. Once we complete the example, we will review exactly what you must do in similar circumstances. But one thing we will mention in advance: you will always need to devise a plan as we do below.

Example 14.8. Let $f : \mathbb{R} \setminus \{3\} \to \mathbb{R}$ be defined by $f(x) = (x+1)/(x-3)$. Determine the range of f.

"Devising a plan." We need to figure out which $y \in \mathbb{R}$ come from something under f. It's a bit difficult to simply gaze at f, or even the graph of f, and see what comes out of it, so we'll try working backwards to see what y might be. (Though the graph of f provides a good way to see if your answer is reasonable, it does not provide a proof.) So, suppose $y \in \mathbb{R}$ did come from something in the domain. That would mean

$$y = f(x), \text{ for some } x \in \mathbb{R} \setminus \{3\};$$

in other words, $y = (x+1)/(x-3)$. Since we need to figure out what x is, we should solve for it. Multiplying through by $x-3$, we get $(x+1) = yx - 3y$. Collecting all terms involving x yields $x - yx = -3y - 1$. Factoring out x, dividing, simplifying, and ignoring potential problems (like what?), we get $x = (3y+1)/(y-1)$. So if y came from some x at all, y had to come from $x = (3y+1)/(y-1)$. That's fine, as long as $y \neq 1$ (that was a potential problem). So ran(f) "appears to be"

$$\{y \in \mathbb{R} : y \neq 1\} = \mathbb{R} \setminus \{1\}.$$

The reason for saying "appears to be" is that we started by assuming y came from something called x, and then found out what x had to be. But the definition of range really requires us to start with an x and show that $f(x) = y$. So we need to check that everything we did above is reversible, and that the two sets ran(f) and $\mathbb{R} \setminus \{1\}$ are equal. All of this was helpful in deciding what the range is, but the actual proof is still to come. The proof below is the form you should follow. When we write it, we need to pretend the reader has not seen the work we just completed.

Proof. We will show that $\text{ran}(f) = \mathbb{R} \setminus \{1\}$. Let $y \in \text{ran}(f)$. Then, clearly, $y \in \mathbb{R}$. So $\text{ran}(f) \subseteq \mathbb{R}$. To show that $y \neq 1$, suppose that this is not the case; so we will suppose $y = 1 \in \text{ran}(f)$ and see what happens. Since $y \in \text{ran}(f)$, there exists a point x in the domain with $f(x) = y = 1$. Using the definition of f, we find that $1 = f(x) = (x+1)/(x-3)$. Therefore, $x+1 = x-3$. This would mean that $1 = -3$, which is not possible. So $y \in \text{ran}(f)$ implies $y \in \mathbb{R}$ and $y \neq 1$. Thus, $\text{ran}(f) \subseteq \mathbb{R} \setminus \{1\}$.

Now let $y \in \mathbb{R} \setminus \{1\}$. Let $x = (3y+1)/(y-1)$. Since $y \neq 1$, we see that $x \in \mathbb{R}$. Remember that we need to check that $x \in \text{dom}(f)$. We know that $x \in \mathbb{R}$. Could we possibly have $x = 3$? Suppose we do, then $3 = (3y+1)/(y-1)$ which implies $3y - 3 = 3y + 1$. Thus we would have $-3 = 1$, which is impossible. So $x \in \text{dom}(f)$ and we can evaluate f at x to obtain

$$f(x) = \frac{\frac{3y+1}{y-1} + 1}{\frac{3y+1}{y-1} - 3} = \frac{3y+1+y-1}{3y+1-3y+3} = y.$$

It follows that $\mathbb{R} \setminus \{1\} \subseteq \text{ran}(f)$. Therefore $\text{ran}(f) = \mathbb{R} \setminus \{1\}$, completing the proof.

□

Before going on, we will make two remarks. If you hadn't read *"Devising a plan"* above the proof, the definition of $x = (3y+1)/(y-1)$ would probably look bizarre. Remember that we didn't guess it; we worked backwards to see what x had to be. One other thing to note is that $\text{ran}(f) \neq \mathbb{R}$, but $\text{ran}(f) = \mathbb{R} \setminus \{1\}$. So f maps into $\mathbb{R}$ but it doesn't "hit" the value 1. We'll come back to this in the next chapter.

○

So what must we do when we have to find the range of a function? First, we need to take out a different sheet of paper and figure out what the set should be. Let's say we decide the range is a set called B. Then we need to show the reader that the two sets are equal. There are often many ways to do it, but one way is to start with an element in the range (tell the reader you are doing this) and show it is in B. Then start with an element y in B (tell the reader you are doing this, too) and find an x (which you found somewhere else, but the reader doesn't necessarily need to see that) that satisfies two things: x is in the domain of your function and $f(x) = y$. Write your proof up carefully, identifying variables before you use them, and always checking that your variables are in the appropriate sets.

Exercise 14.9. What is the range of each of the functions below? A picture, when appropriate, is a lovely addition and is heartily encouraged. It does not, however, substitute for the real thing. Write out everything explicitly.

(a) The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$.
(b) The function $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{R}$ defined by $f(x,y) = x/y$.
(c) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4x + 5$. ○

Definitions

Definition 14.1. Let A and B be sets. A **function f from A to B** is a relation f from A to B satisfying

(i) for all $a \in A$, there exists $b \in B$ such that $(a,b) \in f$, and
(ii) for all $a \in A$, and all $b, c \in B$, if $(a,b) \in f$ and $(a,c) \in f$, then $b = c$.

The standard notation is $f : A \longrightarrow B$ and $b = f(a)$ for $(a,b) \in f$. A function is also called a **map** or a **mapping**.

Definition 14.2. Given a function $f : A \longrightarrow B$, the set A is called the **domain** of f, denoted by $\text{dom}(f)$, and the set B is called the **codomain** of f, and denoted by $\text{cod}(f)$.

Definition 14.3. Given a function $f : A \to B$, the **range** of f, denoted $\text{ran}(f)$, is defined by

$$\text{ran}(f) = \{b \in B : \text{there exists at least one } a \in A \text{ such that } f(a) = b\}.$$

Definition 14.4 (for Problems 14.6 through 14.9). Let X be a nonempty set and let A be a subset of X. The **characteristic function** or **indicator function** of the set A in X is

$$\chi_A : X \to \{0,1\} \text{ defined by } \chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \in X \setminus A \end{cases}.$$

Definition 14.5 (for Problems 14.7 through 14.9). Let I be an interval of $\mathbb{R}$ (open, closed, half open, or all of $\mathbb{R}$). A **step function** is a function $f : I \to \mathbb{R}$ of the form

$$f = \sum_{k=1}^{n} a_k \chi_{A_k},$$

where, for $k = 1, \ldots, n$, we have $a_k \in \mathbb{R}$ and intervals A_k (any type, including I itself or containing just one point) such that $\{A_k : k \in \{1, \ldots, n\}\}$ forms a partition of I and has the additional property that $A_j \cap A_\ell = \emptyset$ if $j \neq \ell$.

Definition 14.6 (for Problems 14.11 through 14.13). For $x \in \mathbb{R}$ we define the **greatest integer** of x by $\lfloor x \rfloor = n$, where $n \in \mathbb{Z}$ and $n \leq x < n+1$. The **greatest integer function** or **floor function** is the function $f : \mathbb{R} \to \mathbb{Z}$, defined by $f(x) = \lfloor x \rfloor$.

Solutions to Exercises

Solution (14.1). The relations in (a) and (d) are functions, those in (b) and (c) are not.

Solution (14.2). Condition (ii) corresponds to the vertical line test, since it says that if we draw the vertical line $x = a$, it should pass through the graph of f at most once.

Solution (14.3). A **function f from a set A to a set B** is a relation $f : A \longrightarrow B$ satisfying

(i) for all $a \in A$, there exists $b \in B$ such that $f(a) = b$, and
(ii) for all $a \in A$, and all $b, c \in B$, if $f(a) = b$ and $f(a) = c$, then $b = c$.

Solution (14.4). Parts (a), (c), and (e) define functions. The others do not. In (b), we have not defined $f(0)$ as an element of $\mathbb{R}$. In (d) note that if, for example, we consider $a = 2/1 = 4/2$, then $f(2/1) = 1$, while $f(4/2) = 1/2$. Thus $(2,1)$ and $(2,1/2)$ are both elements of the relation and condition (ii) is violated.

Solution (14.6). For part (a), the domain and codomain are both $\mathbb{R}$ and both conditions in the definition of a function are satisfied. Note that though $x = 0$ appears twice in the definition of f, in both cases $f(0) = 0$. For part (b), consider $x = 2$. Since $x = 2 \in 2\mathbb{Z}$, we have $f(2) = 1$. On the other hand, 2 is also prime, so $f(2) = 2$. Thus $(2,1)$ and $(2,2)$ are both elements of the relation, but $1 \neq 2$, and condition (ii) is violated.

Solution (14.9). For (a) we claim that $\text{ran}(f) = \mathbb{R} \setminus \{0\}$. Clearly, $\text{ran}(f) \subseteq \mathbb{R} \setminus \{0\}$. So suppose that $y \in \mathbb{R} \setminus \{0\}$. Let $x = 1/y$. Then $x \in \mathbb{R}$ and $x \neq 0$. Thus, $x \in \text{dom}(f)$. Furthermore, $f(x) = 1/(1/y) = y$. Therefore, $y \in \text{ran}(f)$ and $\mathbb{R} \setminus \{0\} \subseteq \text{ran}(f)$, completing the proof.

For (b) we claim that $\text{ran}(f) = \mathbb{Q}$. If $z \in \text{ran}(f)$, then there exists $(x,y) \in \text{dom}(f) = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ with $z = f(x,y) = x/y \in \mathbb{Q}$. Thus $\text{ran}(f) \subseteq \mathbb{Q}$. Conversely, if $z \in \mathbb{Q}$, then $z = p/q$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. Hence $(p,q) \in \text{dom}(f)$ and $f(p,q) = p/q = z$. Thus $z \in \text{ran}(f)$ and $\mathbb{Q} \subseteq \text{ran}(f)$. The two parts together establish the claim.

For (c) we claim that $\text{ran}(f) = \{z \in \mathbb{R} : z \geq 1\}$. (We went to another sheet of paper to come up with this claim. A sketch (see Figure 14.4) is also helpful here, but it is *not* a proof.)

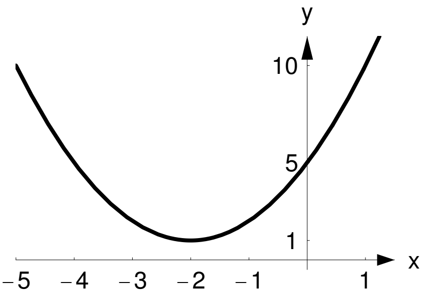

Fig. 14.4 $f(x) = x^2 + 4x + 5$

Proof. First note that if $y \in \text{ran}(f)$, then there exists $x \in \mathbb{R}$ such that $y = x^2 + 4x + 5$. Completing the square, we get $y = (x+2)^2 + 1$. Since $(x+2)^2 \geq 0$, we see that $y \geq 1$. Therefore, $y \in \{z \in \mathbb{R} : z \geq 1\}$, and hence $\text{ran}(f) \subseteq \{z \in \mathbb{R} : z \geq 1\}$.

Now suppose that $y \in \{z \in \mathbb{R} : z \geq 1\}$. Let $x = \sqrt{y-1} - 2$. (We worked backwards to get this, of course.) Since $y \geq 1$, we have $x \in \mathbb{R}$. So $x \in \text{dom}(f)$. Furthermore, $f(x) = f(\sqrt{y-1} - 2) = (\sqrt{y-1} - 2)^2 + 4(\sqrt{y-1} - 2) + 5$. Thus (as the reader can check) $f(x) = y - 1 - 4\sqrt{y-1} + 4 + 4\sqrt{y-1} - 8 + 5 = y$. Therefore, $y \in \text{ran}(f)$ and $\{z \in \mathbb{R} : z \geq 1\} \subseteq \text{ran}(f)$, as desired. □

Spotlight: The Definition of Function

It's probably difficult to imagine that there could be any debate about the definition of function. In fact, the development of the definition of function is quite interesting. For example, Leonhard Euler first defined a function as follows [94, p. 72]: "A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities." Euler later revised his definition because of work on a problem known as the vibrating string problem. Discussion ensued, and Dirichlet is now often credited with providing us with roughly the definition we use today.

Once this discussion appeared to be settled, people could then concentrate on studying various kinds of functions; including, for example, continuous, discontinuous, differentiable, or even nowhere differentiable functions. Dirichlet also introduced the following example (now called the Dirichlet function):

$$D(x) = \begin{cases} c & \text{if } x \in \mathbb{Q} \\ d & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

where c and d are distinct real numbers. This was the first example of many things, including the first example of a function that is discontinuous everywhere (see [58]). In a very interesting article written around 1940 (or, rather, the English translation of this article), Luzin points out that not everyone agreed that Dirichlet had completely answered the question of what a function is. According to Luzin [65, p. 263], some mathematicians found the definition perfect, others found it too broad, and still others found it meaningless. Even as late as 1928, Hermann Weyl [109, p. 22] stated that no one can explain what a function is; then Weyl finishes the paragraph by telling us what a function is: "A function is given if by some definite rule to each real number a there is assigned a real number b (as e.g. by the formula $b = 2a+1$). One then says that b is the value of the function f for the value a of the argument."[1]

For an overview of the definition of the concept of function, we recommend Rüthing's entertaining paper [94], where definitions (from 1718 to 1939) attributed to various authors are presented in their original language, with translation and without comment. You will notice that the final definition, due to N. Bourbaki and given in 1939, agrees with our definition.

The history of the vibrating string problem is described in [59, pp. 503–518]. In [57, p. 724], Katz presents the definition of function used by Johann Bernoulli, an earlier and later definition used by Euler, and definitions attributed to Lacroix, Fourier, Heine, and Dedekind. For a complete and readable overview on this topic, we recommend the papers of Luzin (both [64] and [65]), Youschkevitch [113], and Kleiner [58]. Kleiner's paper also has an extensive bibliography.

Problems

Problem 14.1. Complete the following: A relation $f : A \to B$ is not a function if ...

Problem 14.2. Suppose that $f : X \to Y$. Recall that the definition of $\operatorname{ran}(f)$ was stated in the text. State carefully what it means when we say $y \in Y$ is not in the range of f.

Problem 14.3. Which of the following are functions from the set A to the set B? Give reasons for your answers.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by the relation $f = \{(x,y) : x^2 + y^2 = 4\}$ on $\mathbb{R}$.
(b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(x+1)$.
(c) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x + y$.
(d) The domain of f is the set of all closed intervals of real numbers of the form $[a,b]$, where $a,b \in \mathbb{R}$, $a \le b$, the codomain of f is $\mathbb{R}$, and f is defined by $f([a,b]) = a$.
(e) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by $f(n,m) = m$.
(f) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 \text{ if } x \ge 0 \\ x \text{ if } x \le 0 \end{cases}.$$

[1] The translation is ours.

(g) Define $f : \mathbb{Q} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x+1 & \text{if } x \in 2\mathbb{Z} \\ x-1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases}.$$

(See Example 6.1 for the definitions of $2\mathbb{Z}$ and $3\mathbb{Z}$.)

(h) The domain of f is the set of all circles in the plane $\mathbb{R}^2$, the codomain is $\mathbb{R}$, and if c is a circle in the domain, define f by $f(c) =$ the circumference of c.

(i) *(For students with a background in calculus.)* The domain and codomain of f are the set of all polynomials with real coefficients, and f is defined by $f(p) = p'$. (Here p' is the derivative of p.)

(j) *(For students with a background in calculus.)* The domain of f is the set of all polynomials, the codomain is $\mathbb{R}$, and f is defined by $f(p) = \int_0^1 p(x)\,dx$. (Here $\int_0^1 p(x)\,dx$ is the definite integral of p.)

Problem 14.4. A function $f : A \to \mathbb{R}$ is often called a real-valued function. Thus, authors will often say, "Let f be a real-valued function." Sometimes, A is explicitly defined. Other times, it is understood that the codomain is $\mathbb{R}$ and that the domain is the largest set A for which all values under f result in real numbers. For all real-valued functions below, specify the implied domain, assuming that $A \subseteq \mathbb{R}$.

(a) $f(x) = \frac{3+x}{x-2}$;

(b) $g(x) = \ln(2x^2 + x - 6)$;

(c) $h(x) = \sqrt{8x - 15 - x^2}$;

(d) $k(x) = \sqrt{\frac{(x-3)(2x+5)}{x(x+2)(x-5)}}$.

Problem 14.5. Let $f : \mathscr{P}(\mathbb{R}) \to \mathbb{Z}$ be defined by

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}.$$

Prove that f above is a well-defined function.

Problem 14.6. Let X be a nonempty set and let A be a subset of X. The **characteristic function** or **indicator function** of the set A in X is

$$\chi_A : X \to \{0,1\} \text{ defined by } \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A \end{cases}.$$

(a) Since this is called the characteristic function, it probably is a function, but check this carefully anyway.

(b) Determine the domain and range of this function. Make sure you look at all possibilities for A and X.

Problem 14.7. Using the characteristic function from Problem 14.6 we can define another useful type of function:

Let I be an interval of $\mathbb{R}$ (open, closed, half open, or all of $\mathbb{R}$). A **step function** is a function $f : I \to \mathbb{R}$ of the form

$$f = \sum_{k=1}^{n} a_k \chi_{A_k},$$

where, for $k = 1, \ldots, n$, we have $a_k \in \mathbb{R}$ and intervals A_k (any type, including I itself or containing just one point) such that $\{A_k : k \in \{1, \ldots, n\}\}$ forms a partition of I and has the additional property that $A_j \cap A_\ell = \emptyset$ if $j \neq \ell$.

(a) Prove that a step function as defined above is a function.
(b) Find the range of the step function.
(c) Can the range of a step function have infinitely many elements? Explain.
(d) Suppose that we have a function $g : I \to \mathbb{R}$, where I is an interval of $\mathbb{R}$ and the range of g is finite. Is g necessarily a step function? If it is, prove it; if it isn't, give a counterexample.

Problem 14.8. For each of the step functions below, sketch the graph and identify the range. (Step functions are introduced in Problem 14.7.)

(a) $f : [-5,5] \to \mathbb{R}$ defined by
$f(x) = 2\chi_{[-5,-1]} - 3\chi_{(-1,1)} + 5\chi_{\{1\}} + \chi_{(1,3)} + 2\chi_{[3,5]}$;
(b) $g : [-3,4) \to \mathbb{R}$ defined by $g(x) = \sum_{k=-3}^{3} 2^k \chi_{[k,k+1)}$.

Problem 14.9. Write each of the following functions as a sum of products of characteristic functions and other well-known functions. Say whether or not your function is a step function and explain your answer. (Characteristic functions are introduced in Problem 14.6 and step functions in Problem 14.7.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 2 - x & \text{if } x \leq 0 \\ 2 & \text{if } 0 < x < 2 \\ x & \text{else} \end{cases}$;
(b) $g : [0,12] \to \mathbb{R}$ given by the graph of Figure 14.5 below;

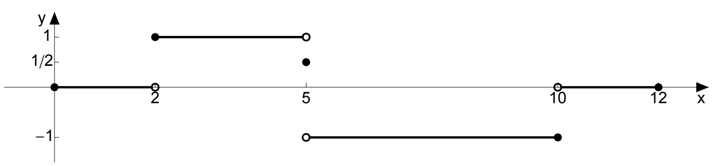

Fig. 14.5

(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$.

Problem 14.10. Let X be a bounded nonempty subset of $\mathbb{R}$. Suppose that we define $g : \mathscr{P}(X) \setminus \{\emptyset\} \to \mathbb{R}$ by $g(S) = \sup S$. Is g a well-defined function? Why or why not?

Problem 14.11. For $x \in \mathbb{R}$ we define the **greatest integer** of x by $\lfloor x \rfloor = n$, where $n \in \mathbb{Z}$ and $n \leq x < n+1$. The **greatest integer function** or **floor function** is the function $f : \mathbb{R} \to \mathbb{Z}$, defined by $f(x) = \lfloor x \rfloor$.

(a) Find $\lfloor 1/2 \rfloor$, $\lfloor \pi \rfloor$, $\lfloor -3.5 \rfloor$, and $\lfloor -10 \rfloor$.
(b) Prove that f is a well-defined function.
(c) What is the range of f? (Show all work!)
(d) Sketch the graph of the greatest integer (floor) function if it is restricted to $-5 \leq x \leq 5$.

Problem 14.12. See Problems 14.6, 14.7, and 14.11 for the definitions.

(a) Write the greatest integer function as a sum of characteristic functions (there may be more than one way to do this). Depending on your solution, the sum will "appear to have infinitely many terms," but to calculate a particular value you will be adding only finitely many nonzero terms.
(b) Is the greatest integer function a step function? Explain your answer.

Problem 14.13. We use the notation of Problem 14.11 and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \lfloor x \rfloor + \lfloor -x \rfloor$. Find $\operatorname{ran}(g)$, and prove that your answer is correct.

Problem 14.14. (a) If possible, give an example of a function $f : \mathbb{N} \longrightarrow \mathbb{R}$, with $\operatorname{ran}(f) = \operatorname{dom}(f)$.
(b) If possible, give an example of a function $f : \mathbb{N} \longrightarrow \mathbb{R}$, with $\operatorname{ran}(f) = \mathbb{Z}^+$.
(c) If possible, give an example of a function $f : \mathbb{N} \longrightarrow \mathbb{R}$, with $\operatorname{ran}(f) = \mathbb{Z}$.

Problem 14.15. Let $f : \mathbb{R} \setminus \{2\} \longrightarrow \mathbb{R}$ be defined by $f(x) = 1/(x-2)$. Find $\operatorname{ran}(f)$ and prove that your answer is correct.

Problem 14.16. Consider the (well-defined) function $f : \mathbb{R} \setminus \{3/2\} \to \mathbb{R}$ defined by $f(x) = (x-5)/(2x-3)$. Carefully prove that $\operatorname{ran}(f) = \mathbb{R} \setminus \{1/2\}$.

Problem 14.17. Consider the (well-defined) function $f : \mathbb{R} \setminus \{3/7\} \to \mathbb{R}$ defined by $f(x) = (x+2)/(7x-3)$. Find $\operatorname{ran}(f)$ and prove that your solution is correct.

Problem 14.18. (a) Give an example of a function f from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}^+$.
(b) Give an example of a function f from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N}$ such that $\operatorname{ran}(f) = \mathbb{N}$. (Prove that f is a function and $\operatorname{ran}(f) = \mathbb{N}$.)
(c) Give an example of a function f from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N}$ such that $\operatorname{ran}(f) \neq \mathbb{N}$. (Prove that f is a function and $\operatorname{ran}(f) \neq \mathbb{N}$.)

Problem 14.19. Let a, b, c, and d be real numbers with $a < b$ and $c < d$. Let $[a,b]$ and $[c,d]$ be two closed intervals. Find a function f such that $f : [a,b] \to [c,d]$ and $\operatorname{ran}(f) = [c,d]$. Prove everything.

Problem 14.20. (a) Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(x) = |x|$. Is f a function? If so, determine $\operatorname{ran}(f)$.

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$. Is f a function? If so, determine $\operatorname{ran}(f)$.

Problem 14.21. Suppose that f is a function from a set A to a set B. Thus, we know that f is a subset of $A \times B$. Is the relation $\{(y,x) : (x,y) \in f\}$ necessarily a function from B to A? Why or why not? (Say as much as is possible to say with the given information.)

Problem 14.22. Which of the following functions equal $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = |x|$? Prove your answers (make sure you show that the functions below either equal f or do not equal f).

(a) The function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.
(b) The function $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(x) = \sqrt{x^2}$.
(c) The function $k : \mathbb{Z} \to \mathbb{Z}$ defined by $k(x) = \begin{cases} x^2/|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Problem 14.23. Let X be a nonempty set. Find all relations on X that are both equivalence relations on X and functions from X to X.

Problem 14.24. We can now define the indexing process more rigorously than we were able to in Chapter 8. Let I and X be sets and $f : I \to X$ be a function. Then we call the domain I an index set and an element of I an index. The range of the function, denoted here $\{f(j) : j \in I\}$, is called an indexed set. If the set $X \subseteq \mathscr{P}(Y)$ for some set Y (and thus $f : I \to \mathscr{P}(Y)$), then $\{f(j) : j \in I\}$ is usually called an indexed collection of sets.

As a specific example, consider

$$f : \mathbb{Z}^+ \to \mathscr{P}(\mathbb{R}) \text{ defined by } f(n) = \{x \in \mathbb{R} : \pi - 2n \leq x \leq \pi + 2/n\}.$$

(a) Find $\bigcup_{n \in \mathbb{Z}^+} f(n)$.
(b) Find $\bigcap_{n \in \mathbb{Z}^+} f(n)$.