

Chapter 15

Functions, One-to-One, and Onto

Functions can map elements from the domain to the codomain in many ways. A function may “hit” every element in the codomain, or it may “miss” some. It may assign more than one x to a y or it may assign exactly one x to each y . We will understand our function better if we know which of these things it does. Precise formulations of these ideas will be given in a moment. It’s a mouthful, though, and really requires practice.

To say that a function $f : A \rightarrow B$ is **one-to-one** means that for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$. A function $f : A \rightarrow B$ is **onto** if $\text{ran}(f) = B$. If a function has this property, then we say that f maps A onto B . Some authors use the word **injective** rather than one-to-one and **surjective** rather than onto. If a function is both one-to-one and onto (or injective and surjective), then we say the function is **bijective**.

Diagrams using small sets may help illustrate the ideas involved in these definitions. We sketch two functions that are not one-to-one in [Figure 15.1](#) and two functions that are not onto in [Figure 15.2](#). Can you make a diagram for a function that is bijective?

Before moving on, let’s think about these definitions. The definition of one-to-one is an implication with quantifiers on elements of the domain A . It moves “forward,” in the sense that we start with elements in A and see where f maps them. The definition of onto, on the other hand, requires that we show something about every element in the set B . It will require us to move “backward,” in the sense that we will start with something in B and see what element of A is mapped to it under f .

If you want to show that a function is onto, we said that you must check that $\text{ran}(f) = B$. Technically that would mean showing $\text{ran}(f) \subseteq B$ and $B \subseteq \text{ran}(f)$. But if we are showing that a function $f : A \rightarrow B$ is onto, we already know that $\text{ran}(f) \subseteq B$; that is, we get that half for free. So if we know we have a function $f : A \rightarrow B$ (maybe it was given to us; maybe we showed it), to show that f maps A onto B , we only have to check that $B \subseteq \text{ran}(f)$. And what does this mean? The answer is given in the lemma below.

Lemma 15.1. *Let $f : A \rightarrow B$ be a function. Then f maps A onto B if and only if for all $b \in B$ there exists (at least one) $a \in A$ such that $f(a) = b$.*

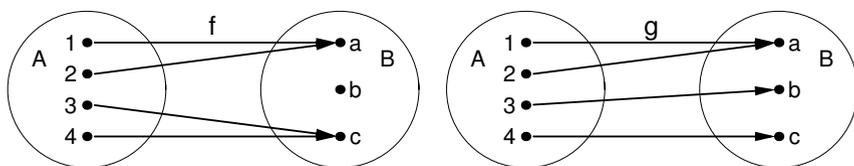


Fig. 15.1 The function f does not map onto B , but the function g does. Neither function is one-to-one

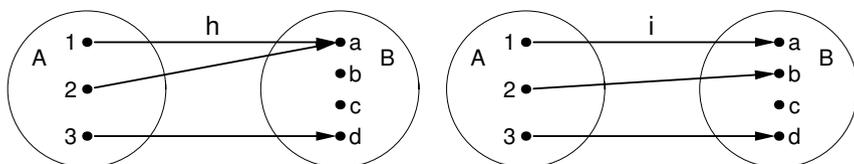


Fig. 15.2 The function h is not one-to-one, but the function i is. Neither function maps onto B

Since this lemma is just a reformulation of the definition of onto, we will often use it without explicitly referencing it.

Let's run through some of these ideas in slow motion. We'll begin with a simple example, and move on to a more challenging one.

Example 15.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 2x + 5$. Then f is one-to-one.

Proving a function is one-to-one is often easier than proving it is onto, because one-to-one doesn't have that backwards quality. To prove it, we assume that (for two arbitrary points x_1 and x_2 in the domain) $f(x_1) = f(x_2)$, and show that $x_1 = x_2$. So now we'll just dive in here. Don't forget that the definition of one-to-one is an implication and therefore we expect to use our assumption.

Proof. Let $x_1, x_2 \in \mathbb{R}$. If $f(x_1) = f(x_2)$, then $2x_1 + 5 = 2x_2 + 5$. Simplification yields $x_1 = x_2$, as desired. Therefore f is one-to-one. \square

We turn to a more interesting example. Note that even though the functions in these two examples are quite different, the proofs that they are one-to-one are quite similar.

Example 15.3. In Example 14.8, we considered $f : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R}$ defined by $f(x) = (x + 1)/(x - 3)$. We started with an element y and solved for x . We found an x that corresponded to y , but there may be others. Is the function one-to-one (meaning there aren't any others)?

We claim that $f : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R}$ defined by $f(x) = (x + 1)/(x - 3)$ is one-to-one.

Proof. Let $x_1, x_2 \in \mathbb{R} \setminus \{3\}$. If $f(x_1) = f(x_2)$, then

$$\frac{x_1 + 1}{x_1 - 3} = \frac{x_2 + 1}{x_2 - 3}.$$

Multiplying through yields $x_1x_2 + x_2 - 3x_1 - 3 = x_2x_1 + x_1 - 3x_2 - 3$. Canceling, combining, and dividing by 4, we find that $x_1 = x_2$, as desired. Therefore f is one-to-one. \square

Note that $f : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R}$ is not onto. This follows from our work in Example 14.8, where we showed that $\text{ran}(f) = \mathbb{R} \setminus \{1\} \neq \mathbb{R}$. The work of that example together with the proof above show that $f : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R} \setminus \{1\}$ is a bijection. \circ

Exercise 15.4. What do you need to do in order to show that a function is not one-to-one? Use what you just decided to show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is not one-to-one. \circ

Exercise 15.5. In Exercise 14.9, which of the functions are one-to-one? If they are not one-to-one, show that carefully as well. \circ

It is now time to investigate what it really means when we say that a function maps a set A onto a set B .

Example 15.6. Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined in Example 15.2 by $f(x) = 2x + 5$ is onto.

We first devise our plan.

“*Devising a plan.*” We have checked that $f : \mathbb{R} \rightarrow \mathbb{R}$ is well-defined, so (in view of Lemma 15.1) we let $y \in \mathbb{R}$. We must show that $y = f(x)$ for some $x \in \mathbb{R}$. Thus, we must show that $y = 2x + 5$ for some $x \in \mathbb{R}$. It is now easy to see that $x = (y - 5)/2$ will work. Remember, when we write this up, we will act as though the reader has not seen this work.

Proof. Let $y \in \mathbb{R}$ and let $x = (y - 5)/2$. Then $x \in \mathbb{R} = \text{dom}(f)$ and

$$f(x) = 2 \left(\frac{y-5}{2} \right) + 5 = y.$$

Since $f : \mathbb{R} \rightarrow \mathbb{R}$ is a well-defined function, f is onto (by Lemma 15.1). \square

Functions that are defined in cases will play an important role in the rest of this course. They are also illuminating examples, because they show just how much one-to-one depends on “what goes into f ” and just how much onto depends on “what comes out.” Showing that they are one-to-one and onto is a bit tricky. We’ll go through one example carefully first.

Example 15.7. We will show that $f : \mathbb{Z} \rightarrow \mathbb{N}$ as defined below and graphed in [Figure 15.3](#) is a bijective function:

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}.$$

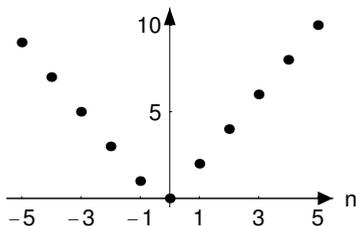


Fig. 15.3

We should mention something before we begin. We, the authors of this text, checked that the f in this example is really a function from \mathbb{Z} to \mathbb{N} . But don't trust us; check it again. Remember to check that f assigns a value in \mathbb{N} to each element of \mathbb{Z} , and that f assigns at most one value in \mathbb{N} to each integer. We now begin our example.

“Devising a plan.” Now to prove that f is one-to-one, we must show that if m and n are integers, and $f(m) = f(n)$, then $m = n$. But it's sort of confusing here, since we have more than one choice for $f(m)$ and $f(n)$. The problem seems to be that the function is defined in cases, and it's not immediately clear which case to use. The choice depends on whether m and n are negative or nonnegative. So we'll break this into all possible cases. Let's see. Both could be nonnegative (case 1), both negative (case 2), or one nonnegative, one negative (case 3). So let's check them all.

Proof. [Proof that f is one-to-one] Let $m, n \in \mathbb{Z}$ and suppose that $f(m) = f(n)$.

Case 1. Suppose that $m \geq 0$ and $n \geq 0$. Then $f(m) = 2m$ and $f(n) = 2n$. Thus $2m = 2n$, and therefore $m = n$.

Case 2. Suppose that $m < 0$ and $n < 0$. Then $f(m) = -2m - 1$ and $f(n) = -2n - 1$. Thus, $-2m - 1 = -2n - 1$, and therefore $m = n$.

Case 3. Suppose that one of the two, say m , is nonnegative, and the other is negative. Then $f(m) = 2m$ and $f(n) = -2n - 1$. Thus $2m = -2n - 1$. But this means that an even number, $2m$, is equal to an odd number, $-2n - 1$, which is impossible.

Therefore, if $f(m) = f(n)$, only cases 1 and 2 can occur. In either of these cases, we have shown that $m = n$. Thus f is one-to-one. \square

Now we will show that f maps \mathbb{Z} onto \mathbb{N} .

“Devising a plan.” Recall that to show f is onto, we must show that $\text{ran}(f) = \mathbb{N}$. We (you, actually) already checked that f is a well-defined function from \mathbb{Z} to \mathbb{N} , so $\text{ran}(f) \subseteq \mathbb{N}$. So let $k \in \mathbb{N}$. We wish to find m such that $f(m) = k$ and $m \in \mathbb{Z}$. Again, because there are cases, this is a bit confusing. So let's try to think about this before we really begin. We'll try to do this for a few specific points. Though this won't prove anything, it may tell us how to begin our proof. So moving along in the spirit of showing something is onto, if $f(m) = 4$, what's m ? (It's 2.) If $f(m) = 3$, what's m ? (It's -2 .) Now maybe you see it. If you don't, keep trying until you do. After a while, you should see that our choice of m depends on whether k is even or odd.

Proof. [Proof that f maps \mathbb{Z} onto \mathbb{N}] Let $k \in \mathbb{N}$. If k is even, then $k = 2m$ for some $m \in \mathbb{Z}$ with $m \geq 0$. Thus, $m \in \mathbb{Z}$ and $f(m) = 2m = k$. If k is odd, then $k + 1$ is even. Hence $m = (k + 1)/(-2) \in \mathbb{Z}$. Since $k \geq 1$, we have $m < 0$. Thus, $f(m) = -2m - 1 = -2((k + 1)/(-2)) - 1 = k$. We conclude that for all $k \in \mathbb{N}$, there exists $m \in \mathbb{Z}$ such that $f(m) = k$. Since $f : \mathbb{Z} \rightarrow \mathbb{N}$ is a well-defined function, f maps \mathbb{Z} onto \mathbb{N} . \square

Note that though the functions in the last two examples are quite different, the proofs that they are onto are quite similar.

What about an example of a function $f : A \rightarrow B$ that does not map A onto B ? Our Lemma 15.1 is quite handy here: Let $f : A \rightarrow B$ be a function. Then f is not onto if there exists $b \in B$ such that $f(a) \neq b$ for every $a \in A$.

For example, consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$. Then to show that f does not map onto \mathbb{R} , we note that $-1 \in \mathbb{R}$, but there exists no $a \in \mathbb{R}$ such that $f(a) = -1$. You should be able to show that f does map \mathbb{R} onto the nonnegative real numbers.

Exercise 15.8. In Exercise 14.9, which of the functions map onto their codomain? If they do not map onto, show that carefully as well. \circ

We now turn to a concrete example of a function f and a special way to construct a function (called the restriction) that is related to f in a very special way.

Suppose that the set A contains all the buses in the city of Bern, Switzerland, and the function $f : A \rightarrow \mathbb{Z}^+$ enumerates them by assigning a positive integer to each one. We will let C denote the subset of A that contains all the buses that run on biogas. There are now enough biogas buses available to serve the city on a regular day and, therefore, we are interested in enumerating just the biogas buses. If we wish to select only the biogas buses, we would be interested in the enumeration of the set C ; that is, we use the old function but consider only buses in the set C . This is easily done, by restricting the function f from A to C . Formally speaking, the restriction is the following: Let $f : A \rightarrow B$ be a function and $C \subseteq A$. The **restriction** of f to C is the function $f|_C : C \rightarrow B$ defined by $f|_C(x) = f(x)$ for all $x \in C$.

Exercise 15.9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function.

- Give an example of a function f and a nonempty subset A of \mathbb{R} such that f is not one-to-one, but $f|_A$ is one-to-one.
- Give an example of a function f and a nonempty subset A of \mathbb{R} such that $f|_A : A \rightarrow \mathbb{R}$ is not onto, but f is onto.
- Are there examples of f and $A \subseteq \mathbb{R}$ such that f is a bijection, but $f|_A : A \rightarrow \mathbb{R}$ is not? Give an example or explain why none exists.
- Are there examples of f and $A \subseteq \mathbb{R}$ such that $f|_A : A \rightarrow \mathbb{R}$ is a bijection, but f is not? Give an example or explain why none exists.

Definitions

Definition 15.1. A function $f : A \rightarrow B$ is **one-to-one** or **injective** if for all $a_1, a_2 \in A$, whenever $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Definition 15.2. A function $f : A \rightarrow B$ is **onto** or **surjective** if $\text{ran}(f) = B$.

Definition 15.3. A function $f : A \rightarrow B$ is **bijective** if it is injective and surjective (or, equivalently, one-to-one and onto).

Definition 15.4. Let $f : A \rightarrow B$ be a function and $C \subseteq A$. The **restriction** of f to C is the function $f|_C : C \rightarrow B$ defined by $f|_C(x) = f(x)$ for all $x \in C$.

Solutions to Exercises

Solution (15.4). Carefully negating the definition, we see that to show a function $f : A \rightarrow B$ is not one-to-one, we must show that there exist x_1 and x_2 in A with $f(x_1) = f(x_2)$ and $x_1 \neq x_2$. For the particular function $f(x) = x^2$ given above, we note that the numbers 1 and -1 are both real numbers with $f(1) = f(-1)$, but $1 \neq -1$. Therefore f is not one-to-one.

Solution (15.5). Only the function defined in (a) is one-to-one: If $x_1, x_2 \in \mathbb{R} \setminus \{0\}$ and $f(x_1) = f(x_2)$, then $1/x_1 = 1/x_2$. Therefore, $x_1 = x_2$, showing that f is one-to-one.

For part (b) consider $(1, 1), (2, 2) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \text{dom}(f)$. Then $(1, 1) \neq (2, 2)$, but $f(1, 1) = 1 = f(2, 2)$. This shows that f is not one-to-one.

For part (c), let $-1, -3 \in \mathbb{R} = \text{dom}(f)$. Then $f(-1) = 2 = f(-3)$ and hence f is not one-to-one.

Solution (15.8). None of the functions maps onto their codomain. The range of each function was presented in the solution to Exercise 14.9.

Solution (15.9). We will need only two functions to solve all four parts of this problem. Of course, many other examples are possible.

(a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x - 1 & \text{if } x > 0 \end{cases}$$

and let $A = \mathbb{R} \setminus (0, 1] \subseteq \mathbb{R}$. Clearly f is not one-to-one, but $f|_{\mathbb{R} \setminus (0, 1]}$ is.

(b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x$ and let $A = \mathbb{Z}$. Then f is onto but $f|_{\mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{R}$ is not.

(c) The example presented in part (b) works here also.

(d) The example presented in part (a) works here also.

Problems

Problem 15.1. For each of the following, you are asked to give an example of a function. (You should always state the domain, codomain, and the associated rule of your function.)

- Give an example of a function that is both one-to-one and onto.
- Give an example of a function that is one-to-one, but not onto.
- Give an example of a function that is not one-to-one, but is onto.
- Give an example of a function that is neither one-to-one nor onto.

Problem 15.2. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. Show that f is not onto.

- Show that f , as defined above, maps \mathbb{R} onto $\{x \in \mathbb{R} : x \geq 0\}$.
- Consider the function $g : \mathbb{Z} \rightarrow \mathbb{N}$ defined by $g(x) = x^2$. Is g onto?
- Both g and f take elements of the domain and square them. Why did we use the letter g in the previous part of this problem, rather than the letter f ?

Problem 15.3. Is the absolute value function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = |x|$ one-to-one? Why or why not?

Problem 15.4. (a) Is there a one-to-one function from the set $\{1, 2, 3\}$ to the set $\{1, 3\}$? Why or why not?

- Is there a function mapping $\{1, 2, 3\}$ onto the set $\{1, 3\}$? Why or why not?
- Is there a one-to-one function mapping the open interval $(0, 2)$ to the open interval $(0, 1)$?
- Is there a one-to-one function mapping the set $\{x \in \mathbb{R} : x > 0\}$ to the open interval $(0, 1)$?
- Is there a function mapping the set $\{x \in \mathbb{R} : x > 0\}$ onto the open interval $(0, 1)$?

Problem 15.5. Recall Definition 15.1 of one-to-one.

- State the contrapositive of the definition.
- By negating the contrapositive, complete the following definition. A function $f : X \rightarrow Y$ is not one-to-one if . . .

Problem 15.6. Criticize the following “definition” of onto. “A function $f : X \rightarrow Y$ is onto if there exists an $x \in X$ such that for each $y \in Y$ we have $f(x) = y$.”

Problem 15.7. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 5 + (x - 3)^2$.

- Prove that f is not injective.
- Find $\text{ran}(f)$ and prove that your conjecture is correct.

Problem 15.8. We define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 - 4x + 7 & \text{if } x \leq 1 \\ 5 - x^2 & \text{if } x > 1 \end{cases}.$$

This function is well-defined and you may assume this without proving it. Prove that this function is bijective.

Problem 15.9. We define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 + 2x + 4 & \text{if } x \leq -2 \\ -2x & \text{if } -2 < x < 2 \\ -2 - x & \text{if } x \geq 2 \end{cases} .$$

This function is well-defined and you may assume this without proving it. Prove that this function is bijective.

Problem 15.10. For each of the functions below, determine whether or not the function is one-to-one and whether or not the function is onto. If the function is not one-to-one, give an explicit example to show what goes wrong. If it is not onto, determine the range.

- Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 1/(x^2 + 1)$.
- Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \sin(x)$. (Assume familiar facts about the sine function.)
- Define $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(n, m) = nm$.
- Define $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f((x, y), (u, v)) = xu + yv$. (Do you recognize this function?)
- Define $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f((x, y), (u, v)) = \sqrt{(x - u)^2 + (y - v)^2}$. (Do you recognize this function?)
- Let A and B be nonempty and $b \in B$. Define $f : A \rightarrow A \times B$ by $f(a) = (a, b)$.
- Let X be a nonempty set. Define $f : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by $f(A) = X \setminus A$.
- Let B be a fixed proper subset of a nonempty set X . We define a function $f : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by $f(A) = A \cap B$.
- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 - x & \text{if } x < 1 \\ 1/x & \text{otherwise} \end{cases} .$$

Problem 15.11. Definition 14.4 introduced the characteristic function χ_A of a subset A of a set X .

- Under what conditions on the sets A and X is the function $\chi_A : X \rightarrow \{0, 1\}$ onto? Justify your answer.
- Under what conditions on the sets A and X is the function $\chi_A : X \rightarrow \{0, 1\}$ one-to-one? Justify your answer.
- Under what conditions on the sets A and X is the function $\chi_A : X \rightarrow \{0, 1\}$ bijective? Justify your answer.

Problem 15.12. For each of the following, determine whether or not $f : A \rightarrow B$ is a function from the set A to the set B . If it is, prove that f is one-to-one, or give an example to show that f is not one-to-one. Then prove that f is onto, or give an example of an element in the codomain that is not in the range to show that f is not onto.

- (a) Define $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ by $f(x, y) = (y, x)$.
 (b) Define $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(x, y) = x^2 + y^2$.
 (c) Let $y \in \mathbb{R}$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = y \cdot x$. (Does your answer depend on y ?)
 (d) Define $f : \mathcal{P}(\mathbb{Z}) \rightarrow \mathbb{Z}$ by $f(S) = \max S$.

Problem# 15.13. Let $f : \mathbb{R} \rightarrow (-1, 1)$ be defined by

$$f(x) = \frac{x}{1 + |x|}.$$

Prove that f is a bijective function, mapping \mathbb{R} onto the open interval $(-1, 1)$.

Problem 15.14. Let a, b, c , and d be real numbers with $a < b$ and $c < d$. Define a bijection from the closed interval $[a, b]$ onto the closed interval $[c, d]$ and prove that your function is a bijection.

Problem 15.15. Let $F([0, 1])$ denote the set of all real-valued functions defined on the closed interval $[0, 1]$. Define a new function $\phi : F([0, 1]) \rightarrow \mathbb{R}$ by $\phi(f) = f(0)$. Is ϕ a function from $F([0, 1])$ to \mathbb{R} ? Is it one-to-one? Is it onto? Remember to prove all claims, and to provide examples where appropriate.

Problem 15.16. Find a function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ that is one-to-one.

Problem 15.17. Let f be a function, $f : \mathbb{R} \rightarrow \mathbb{R}$. Define a new function $f \cdot f : \mathbb{R} \rightarrow \mathbb{R}$ by $(f \cdot f)(x) = f(x) \cdot f(x)$. Prove that $f \cdot f$ is a function. Then do the remaining parts of the problem. (You may wish to work Problem 15.16, if you haven't already done so.)

- (a) Does there exist a function f for which $f \cdot f$ is one-to-one? If not, why not? If there is, what is an example?
 (b) Does there exist a function f for which $f \cdot f$ maps onto \mathbb{R} ? If not, what is $\text{ran}(f \cdot f)$? Your answer will be in terms of $\text{ran}(f)$.

Problem 15.18. Let $f : A \rightarrow B$ be a function and $C \subseteq A$. Prove that $f|_C$ is a well-defined function.

Problem# 15.19. Let $f : A \rightarrow B$ be a function and $C \subseteq A$.

- (a) Prove that if f is one-to-one, then $f|_C$ is one-to-one.
 (b) Prove that if $f|_C$ is onto, then f is onto.

Problem 15.20. Let $f : X \rightarrow Y$ and $g : X \rightarrow Y$ be functions and let \mathcal{A} denote a partition of X . Show that if $f|_A = g|_A$ for all $A \in \mathcal{A}$, then $f = g$.

Problem 15.21. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $f(x) = x^3 - 3x$ and $A = (-\infty, -\sqrt{3}) \cup [\sqrt{3}, \infty)$. Is f a bijection? Is $f|_A : A \rightarrow \mathbb{R}$ a bijection? (Your solution may involve the intermediate value theorem.)

Problem 15.22. Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = |x| + 2x$ is bijective. You might find it more convenient to write f as a function defined in cases.

Problem# 15.23. Define $f : (-1, 1) \rightarrow \mathbb{R}$ by

$$f(x) = \frac{x}{x^2 - 1}.$$

- (a) Show that f is injective.
- (b) Show that f is surjective.

Problem 15.24. Let a, b, c , and d be real numbers, not both of c and d zero. Further, let $X = \{x \in \mathbb{R} : cx + d \neq 0\}$. The function $f : X \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{ax + b}{cx + d},$$

is well-defined (you need not prove that).

- (a) Under what conditions (i.e., restrictions on a, b, c, d) will f be one-to-one? Prove your conjecture.
- (b) Assume that the conditions that you proposed in part (a) hold and that, in addition, $c = 0$. Show that in this case, f is a bijection from \mathbb{R} to \mathbb{R} .

Problem 15.25. Let \sim be an equivalence relation on a nonempty set X . We denote the set of all equivalence classes of \sim by \mathcal{C} . Define $f : X \rightarrow \mathcal{C}$ by $f(x) = E_x$. Thus, an element $x \in X$ is mapped to the equivalence class containing x .

- (a) Prove that f is a well-defined function.
- (b) Prove that f is surjective.
- (c) Give necessary and sufficient conditions on the equivalence relation for f to be bijective.

Problem 15.26. Let $f : A \rightarrow B$ be a surjective function. For $x, y \in A$, we say $x \sim y$ if and only if $f(x) = f(y)$.

- (a) Prove that \sim is an equivalence relation on A .

We denote the collection of all equivalence classes of \sim by \mathcal{C} . Define $g : \mathcal{C} \rightarrow B$ by $g(E_x) = f(x)$.

- (b) Prove that g is a well-defined function.
- (c) Prove that g is bijective.