Chapter 26
Getting to Know Open and Closed Sets

When we work in $\mathbb{R}$ with the usual metric, we think of distance as measured by absolute value. Points are close when the absolute value of the difference is small. We might reasonably argue that points x and y are close when they satisfy $|y-x| < r$, where r is a small positive number; that is to say, y is in the open interval $(x-r, x+r)$. This interpretation allows us to visualize the distance between the points. As it turns out, all metrics have this visual interpretation.

Let x be a point in a metric space (X,d) and let r be a real number with $r > 0$. Then the **open ball of radius** r **about** x is denoted $B_d(x,r)$ and is defined by $B_d(x,r) = \{y \in X : d(y,x) < r\}$. We will call $B_d(x,1)$ the **open unit ball about** x. Note that the radius of the open ball $B_d(x,r)$ is always positive, and $B_d(x,r)$ is centered at x. This is quite an important definition, and we will be able to do a lot with it. But remember, before you work an example, state a theorem, or write a proof, make sure that you and your intended reader are clear on the space you are working on, the metric you are using, and what you want to show.

Example 26.1. Consider the set $\mathbb{R}$ with the usual metric. What does $B_{d_u}(1,1/2)$ mean? Describe $B_{d_u}(x,r)$, for an arbitrary $x \in X$ and $r > 0$ as follows: Describe the set in terms of open intervals and sketch the set on a number line.

By definition, $B_{d_u}(1,1/2) = \{y \in \mathbb{R} : d_u(y,1) < 1/2\}$. The notation is preventing us from seeing something we all know pretty well, so let's get rid of it. Rewriting,

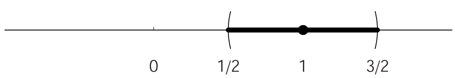

Fig. 26.1 $B_{d_u}(1,1/2)$

U. Daepp and P. Gorkin, *Reading, Writing, and Proving: A Closer Look at Mathematics*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4419-9479-0_26,

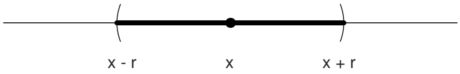

Fig. 26.2 $B_{d_u}(x,r)$

$$\begin{aligned} B_{d_u}(1,1/2) &= \{y \in \mathbb{R} : |y-1| < 1/2\} \\ &= \{y \in \mathbb{R} : -1/2 < y-1 < 1/2\} \\ &= \{y \in \mathbb{R} : 1/2 < y < 3/2\} \\ &= (1/2,3/2). \end{aligned}$$

Figure 26.1 shows $B_{d_u}(1,1/2)$ graphically.

The solution of the general case is the same (see also Figure 26.2): By definition, $B_{d_u}(x,r) = \{y \in \mathbb{R} : d_u(y,x) < r\} = \{y \in \mathbb{R} : |y-x| < r\} = \{y \in \mathbb{R} : -r < y-x < r\}$ $= \{y \in \mathbb{R} : x-r < y < x+r\} = (x-r, x+r)$. Therefore, $B_{d_u}(x,r) = (x-r, x+r)$. ○

The next example shows that the open balls depend on the underlying set X.

Example 26.2. Consider the set $X = [0,1)$ with the usual metric. Find the open ball $B_{d_u}(1/4,2/3)$.

Since the center is $x = 1/4$ and the radius is $r = 2/3$, we find

$$\begin{aligned} B_{d_u}(1/4,2/3) &= \{x \in [0,1) : |x-1/4| < 2/3\} \\ &= \{x \in [0,1) : -2/3 < x-1/4 < 2/3\} \\ &= \{x \in [0,1) : -5/12 < x < 11/12\} \\ &= [0,11/12). \end{aligned}$$

○

Now it's your turn.

Exercise 26.3. Consider the set $\mathbb{R}^2$ with the usual metric (defined in Example 25.2). What is the set $B_{d_u}((0,1),4)$? Describe the set using precise set notation and sketch it. ○

These balls can be used to describe the basic structure of metric spaces. For example, for two distinct points x and y in a metric space, we can always find two disjoint open balls, B_x and B_y, such that $x \in B_x$ and $y \in B_y$. This probably agrees with your intuition. On the other hand, as we shall see, there exist metric spaces in which sets consisting of a single point are open balls! In the remainder of this chapter, we will see how these balls can be used to determine which sequences converge. But before we can develop the connection between balls and convergence, we need to

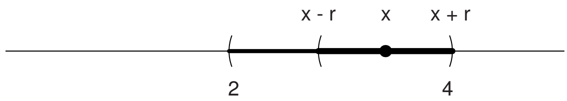

Fig. 26.3 $B_{d_u}(x,r) \subseteq (2,4)$

look at some of the important sets we can make using these balls together with the set operations we studied earlier.

In a metric space (X,d), a subset U of X is **open** if for every point $x \in U$, there exists an open ball $B_d(x,r)$ satisfying $B_d(x,r) \subseteq U$. Note that r depends on x, so that as x changes, r will too. This definition is very visual, so pictures will help you. A word of caution is in order before we begin our examples: Throw away any preconceived notions you have about what an open ball should look like.

Example 26.4. Show that the interval $(2,4)$ is open in $\mathbb{R}$ with the usual metric.

You may be thinking that there's nothing to show; after all, it's an open interval. But we still need to use the definition to check that this interval is open in the sense we have defined here, so let's briefly review exactly what we have to show.

We will show that *for every* $x \in (2,4)$, there is an open ball $B_{d_u}(x,r) \subseteq (2,4)$. We know our metric is the usual one, so by Example 26.1, we know we need to show that there exists a positive number r with $(x-r, x+r) \subseteq (2,4)$. To find r, you should draw pictures whenever you can. Before you read our solution, find your own following this outline: first, draw and label an appropriate picture, which you will then use as you continue on in this proof. Next, pick an *arbitrary* point $x \in (2,4)$. Now, we need to find a positive number r with $B_{d_u}(x,r) = (x-r, x+r) \subseteq (2,4)$. Look at your picture to find a possible value of r. Then show that it works.

Proof. We will prove that $(2,4)$ is open. Let $x \in (2,4)$. Then $2 < x < 4$, so both $x-2$ and $4-x$ are positive. Let $r = \min\{x-2, 4-x\}$. (See Figure 26.3.) Then $r > 0$. Now we'll check that $B_{d_u}(x,r) \subseteq (2,4)$. So let $y \in B_{d_u}(x,r)$. Then, by definition, $|y-x| < r$. Thus, $-r < y-x < r$. From the upper inequality we obtain $y-x < 4-x$, and hence $y < 4$. From the lower inequality we obtain $-(x-2) < y-x$, and hence $2 < y$. Thus, $y \in (2,4)$, and we conclude that $B_{d_u}(x,r) \subseteq (2,4)$. $\square$

A few questions and comments are in order. First, don't forget to check that $r > 0$. An open ball of 0 radius, or (worse yet) negative radius, is no ball at all. Choosing r as the minimum of finitely many real numbers is a pretty standard thing to do, so it's a good idea to get used to it now. And finally, there are lots of choices for r. We picked one value that worked. Every smaller positive value would work too.

Exercise 26.5. Let (X,d) be a metric space and let $U \subseteq X$. Complete the sentence: The set U is not open in (X,d) if ....

This is one of those exercises where you will want to check your solution against ours before you go on. So here's ours: The set U is not open in (X,d) if there exists a point $x \in U$ such that for every open ball $B_d(x,r)$ about x, there exists a point $y \in B_d(x,r) \cap U^c$. ○

Exercise 26.6. Show that the interval $[2,4]$ is not open in $\mathbb{R}$ with the usual metric. ○

Exercise 26.7. The setting: $\mathbb{R}^2$ with the usual metric. Your mission: to show that the open unit ball, $B_{d_u}((0,0),1)$, about $(0,0)$ is open.

Let the following steps guide you:

(a) Sketch the open ball of radius 1 about the point $(0,0)$.
(b) Without thinking too much about it, choose a point in the open ball (make sure you don't choose $(0,0)$). Make a dot at the point, and label it (a,b).
(c) Draw as large an open ball as you can that is still contained in $B_{d_u}((0,0),1)$ and centered at your dot (a,b). What's the radius of that open ball? (Here's a potentially helpful suggestion: draw the radius of the open ball $B_{d_u}((0,0),1)$ that passes through the point (a,b).)
(d) Now you are ready to start the exercise. Find a positive real number r such that $B_{d_u}((a,b),r)$ appears to be contained in $B_{d_u}((0,0),1)$.
(e) Show that $B_{d_u}((a,b),r) \subseteq B_{d_u}((0,0),1)$. Write out the whole proof carefully. Include your picture; it's very helpful for the writer and the reader. ○

There are lots of interesting sets in a metric space, all building on the notion of open ball. We have already introduced open sets. We come now to closed sets. A set E in a metric space X is **closed** if and only if the complement, E^c, is open. So, since the complement of an arbitrary open set is a closed set, we can immediately write down several closed sets. For example, in $\mathbb{R}$ with the usual metric, the set $(-\infty,2] \cup [4,\infty)$ must be closed, since its complement is the open set $(2,4)$. (See Example 26.4).

It's important to know many ways to show that sets are open, closed, or neither. Here's a useful result that should have a one-line proof.

Theorem 26.8. *Let (X,d) be a metric space. A subset U of X is open if and only if its complement is closed.*

Exercise 26.9. Find the one- (or two-) line proof of Theorem 26.8. ○

We now have examples of open sets and closed sets. But, as you will see as you work the next two exercises, things often get a bit complicated.

Exercise 26.10. Give an example of a set that is neither open nor closed in $\mathbb{R}^2$ with the usual metric. ○

Exercise 26.11. Let (X,d) be a metric space. Is the empty set open? closed? both? neither? ○

You might have found yourself concluding that if a set is not open, then it is closed. This is normal, because in ordinary English if a door is not open, then it is closed. Unfortunately, in mathematics, that's false! In the two exercises above, we have seen examples of sets that are neither open nor closed, and examples of sets that are both open and closed. Don't assume anything when you work the problems: if we didn't prove it, state it, or use it, then it may not be true.

We defined an open ball and an open set. You will show (in Problem 26.13) that every open ball is an open set, but since this is so important, we'll state it as a theorem. It's interesting to note that the proof is very much like the proof of Exercise 26.7.

Theorem 26.12. *Let (X,d) be a metric space. For every point $x \in X$, and every positive real number r, the set $B_d(x,r)$ is open.*

Now we will get to see some ways that we can use open sets and some more odd properties of metric spaces. The first theorem tightens the relationship between open sets and open balls.

Theorem 26.13. *Let (X,d) be a metric space. A set U is open if and only if there is a subset I of X and a set of radii $\{r_y \in \mathbb{R}^+ : y \in I\}$ such that $U = \bigcup_{y\in I} B_d(y,r_y)$.*

There are some things that might be confusing to you in this statement, but it's much easier to see what it means to be an open set if you understand Theorem 26.13. The index set is a way of saying that we don't know how many y we have; there could be finitely many or not, countably many or not, and this way we don't have to deal with that issue. Next, the r_y might confuse you. Each ball has a (positive) radius, and if we wrote $B_d(y,r)$ for all y, we would be saying that all the balls have the same radius, r. That's not what the theorem says, so we shouldn't say that either. By using the notation r_y, we allow each y in I to have its own radius, r_y. Having said all this, we now begin the proof.

Proof. Suppose first that there is a subset I of X such that

$$U = \bigcup_{y\in I} B_d(y,r_y).$$

By the definition of open set, we need to show that for an arbitrary $x \in U$, there exists an open ball $B_d(x,r_x)$ contained in U. Now if $x \in U$, then there exists an element $z \in I$ such that $x \in B_d(z,r_z)$. By Theorem 26.12, the ball $B_d(z,r_z)$ is an open set, and therefore there exists a positive real number s_x such that $B_d(x,s_x) \subseteq B_d(z,r_z)$. Since $B_d(z,r_z) \subseteq U$, we know that $B_d(x,s_x) \subseteq U$. Hence the set U is open.

Now suppose that U is open. We have to find a collection of open balls such that U is the union of those open balls. By the definition of open set, if $x \in U$, there exists an open ball $B_d(x,r_x)$ with $B_d(x,r_x) \subseteq U$. Now we claim that $U = \bigcup_{x\in U} B_d(x,r_x)$. If we establish this claim, our proof will be complete. To see that U is contained in the union, note that if $y \in U$, then $y \in B_d(y, r_y)$, and therefore $y \in \bigcup_{x \in U} B_d(x, r_x)$. Thus, $U \subseteq \bigcup_{x \in U} B_d(x, r_x)$. To show that U contains the union, note that $B_d(x, r_x) \subseteq U$ for each x. From this it is easy to see[1] that $\bigcup_{x \in U} B_d(x, r_x) \subseteq U$, completing the proof.

□

Theorem 26.13 can be restated as follows: A set U in a metric space X is open if and only if U is a union of open balls.

The proofs of many of the theorems in this chapter provide an excellent opportunity for you to apply all the techniques that you have learned in this course. For this reason, we have left many as problems. Here's another useful theorem.

Theorem 26.14. *An arbitrary union of open sets is open.*

The proof of this is left as a problem (Problem 26.11) for you, the reader. By "arbitrary union" we mean that we don't know how many sets we have. So make sure that you don't accidentally assume that there are finitely many sets, or even countably many.

Theorem 26.15. *An arbitrary intersection of closed sets is closed.*

The proof of this is left for you to do (Problem 26.12). If you have been paying close attention to the theorems and definitions presented thus far, this should follow from Theorem 26.14. What about an intersection of open sets? a union of closed sets? The results are given below and the proofs are outlined in the problems.

Theorem 26.16. *Let $U_1, \ldots, U_n$ be open sets. Then $\bigcap_{j=1}^{n} U_j$ is an open set.*

Theorem 26.17. *Let $F_1, \ldots, F_n$ be closed sets. Then $\bigcup_{j=1}^{n} F_j$ is a closed set.*

We'll conclude this chapter with the metric we promised would challenge your intuition.

Example 26.18. Consider $\mathbb{R}$ with the discrete metric, d_d. Prove the following.

(a) For each point $x \in \mathbb{R}$, the set $\{x\}$ is an open ball.
(b) Every set in $(\mathbb{R}, d_d)$ is open.
(c) Every set in $(\mathbb{R}, d_d)$ is closed.

For part (a), note that for $x \in \mathbb{R}$, the set $\{x\} = B_{d_d}(x, 1/2)$. By Theorem 26.12, the set $B_{d_d}(x, r)$ is an open set and, consequently, $\{x\}$ is open.

For part (b), let S be a subset of $\mathbb{R}$. Since $S = \bigcup_{s \in S} \{s\}$, from part (a) we see that S is a union of open sets. By Theorem 26.14, S is open.

For part (c), let T be a subset of $\mathbb{R}$. Then T^c is also a subset of $\mathbb{R}$, and it follows from part (b) that T^c is open. But a set is closed if and only if its complement is open, and therefore T is closed. ○

There's lots more that we can do here, and we will do it in the problems.

[1] If this isn't easy to see, show it using an element-chasing argument. In fact, when you worked Exercise 8.10, you already showed it.

Definitions

Definition 26.1. Let x be a point in a metric space (X,d) and let r be a real number with $r > 0$. Then the **open ball of radius** r **about** x is denoted $B_d(x,r)$ and is defined by $B_d(x,r) = \{y \in X : d(y,x) < r\}$.

Definition 26.2. The **open unit ball about** x in a metric space (X,d) is the open ball $B_d(x,1)$.

Definition 26.3. In a metric space (X,d), a subset U of X is **open** if for every point $x \in U$, there exists an open ball $B_d(x,r)$ satisfying $B_d(x,r) \subseteq U$.

Definition 26.4. A set E in a metric space X is **closed** if the complement, E^c, is open.

Definition 26.5 (for Problem 26.18). Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f : (X,d_X) \to (Y,d_Y)$ **preserves distances** if

$$d_Y(f(x), f(x')) = d_X(x,x')$$

for all x,x' in X.

Definition 26.6 (for Problems 26.19 and 26.20). Let E be a subset of a set X with metric d. A point x is said to be an **interior point** of E if there exists an open ball $B_d(x,r)$ with $B_d(x,r) \subseteq E$. The set of all interior points is called **the interior** of E and is denoted by E^o.

Definition 26.7 (for Problems 26.21 through 26.24). Let (X,d) be a metric space. Let $E \subseteq X$. A point $x \in X$ is a **limit point** of E if every open set containing x contains a point $y \in E$ with $y \neq x$. We denote the set of all limit points of the set E by E_l.

Definition 26.8 (for Problem 26.24). Let E be a subset of a metric space X. The **closure** of E is denoted by $\overline{E}$ and is defined by $\overline{E} = E \cup E_l$.

Solutions to Exercises

Solution (26.3). We first calculate $B_{d_u}((0,1),4)$. The graphical representation is shown in Figure 26.4.

$$\begin{aligned} B_{d_u}((0,1),4) &= \{(x,y) \in \mathbb{R}^2 : d_u((x,y),(0,1)) < 4\} \\ &= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2 + (y-1)^2} < 4\} \\ &= \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 16\}. \end{aligned}$$

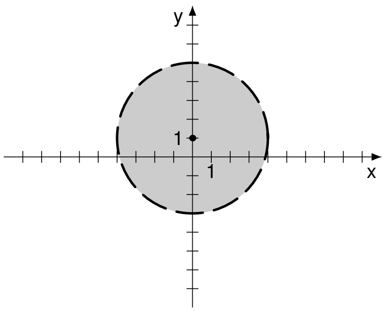

Fig. 26.4 $B_{d_u}((0,1),4)$

Solution (26.6). We will use the solution to Exercise 26.5: Choose $x = 2$ and note that $2 \in [2,4]$. Let $B_{d_u}(2,r)$ be an arbitrary open ball about 2. Then r is a positive real number. We claim that $2 - r/2 \in B_{d_u}(2,r) \cap [2,4]^c$. Since $d_u(2 - r/2, 2) = |(2 - r/2) - 2| = r/2 < r$, we conclude that $2 - r/2 \in B_{d_u}(2,r)$. Also, since $2 - r/2 < 2$ we know that $2 - r/2 \notin [2,4]$. This establishes the claim and we have shown that $[2,4]$ is not open in $\mathbb{R}$ with the usual metric.

Solution (26.7). We follow the outline provided beginning with the illustration in Figure 26.5.

For $(a,b) \in B_{d_u}((0,0),1)$, let $r = 1 - \sqrt{a^2+b^2}$. Now we claim that $B_{d_u}((a,b),r)$ is an open ball about (a,b) satisfying $B_{d_u}((a,b),r) \subseteq B_{d_u}((0,0),1)$.

For the first part of this claim, note that since $(a,b) \in B_{d_u}((0,0),1)$, we have $d_u((a,b),(0,0)) = \sqrt{a^2+b^2} < 1$. Hence $r = 1 - \sqrt{a^2+b^2} > 0$ and $B_{d_u}((a,b),r)$ is an open ball about (a,b).

To prove the set inclusion, let $(x,y) \in B_{d_u}((a,b),r)$. Then

$$\begin{aligned} d_u((x,y),(0,0)) &\leq d_u((x,y),(a,b)) + d_u((a,b),(0,0)) \\ &\qquad \text{(by the triangle inequality for the metric } d_u\text{)} \\ &< r + \sqrt{a^2+b^2} \quad \text{(since } (x,y) \in B_{d_u}((a,b),r)\text{)} \\ &= 1 - \sqrt{a^2+b^2} + \sqrt{a^2+b^2} = 1. \end{aligned}$$

Hence $(x,y) \in B_{d_u}((0,0),1)$. This establishes the second part of the claim.

The definition of an open set implies that $B_{d_u}((0,0),1)$ is open.

Solution (26.9). By the definition of closed set, the set U^c is closed in a metric space (X,d) if and only if $(U^c)^c = U$ is open in X.

Solution (26.10). The set $A = \{(x,y) : 0 < x \leq 1\}$ is neither open nor closed in $\mathbb{R}^2$.

To show that A is not open, consider $(1,0) \in A$. Let $r \in \mathbb{R}$ with $r > 0$. Then $(1 + r/2, 0) \in B_{d_u}((1,0),r)$ and $(1 + r/2, 0) \notin A$. Since r was an arbitrary positive real number we conclude that for each $r \in \mathbb{R}^+$ we have $B_{d_u}((1,0),r) \not\subseteq A$. Hence A is not open.

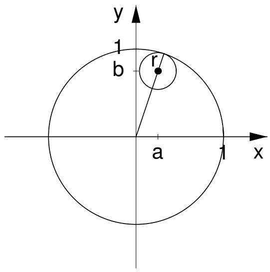

Fig. 26.5 Sketch to find r, the radius of the open ball centered at (a,b)

To show that A is not closed, we will show that A^c is not open. To this end, consider $(0,0) \in A^c = \{(x,y) \in \mathbb{R}^2 : x \leq 0 \text{ or } x > 1\}$. We may assume that $r < 2$. For every $r \in \mathbb{R}^+$ with $r < 2$ we have $(r/2,0) \in B_{d_u}((0,0),r)$ and $(r/2,0) \notin A^c$. Thus A^c is not open and hence A is not closed.

Solution (26.11). The empty set is open in every metric space (X,d). The reason is that the antecedent "$x \in \emptyset$" is always false. This means that the defining implication is always true for $\emptyset$.

The whole space X is also always open: For all $x \in X$, we know that $B_d(x,r) \subseteq X$ for every positive real r. Since $\emptyset^c = X$ and X is open, the definition of closed implies that $\emptyset$ is closed.

Thus we have two examples of sets that are both open and closed: X and $\emptyset$.

Problems

We continue to assume that X is a metric space with metric d unless otherwise stated.

Problem 26.1. Show that the set $(1,3) \cup (4,5)$ is open in $\mathbb{R}$ with the usual metric.

Problem 26.2. Consider $\mathbb{R}$ with the usual metric. In each case, give an example of a nonempty closed set C and a nonempty open set U such that

(a) $U \cap C$ is open.
(b) $U \cup C$ is closed.
(c) $U \cap C$ is neither open nor closed.
(d) $U \cup C$ is neither open nor closed.

Problem 26.3. Consider the set $A = \{(x,y) \in \mathbb{R}^2 : -1 < x < 0, -1 < y < 1\}$. Prove that A is open in $\mathbb{R}^2$ with respect to the max metric, d_m. Include a sketch with your proof.

Problem 26.4. Decide whether the statements below are true or false. If the statement is true, give a brief reason why. If the statement is false, give a counterexample.

(a) In $\mathbb{R}$ with the usual metric, the interval $[0,\infty)$ is a closed set.
(b) In $\mathbb{R}$ with the discrete metric, the interval $[0,\infty)$ is an open set.
(c) A finite union of open sets is an open set.
(d) An arbitrary union of closed sets is a closed set.

Problem 26.5. Consider the set $\mathbb{R}^+$ with the usual metric.

(a) Show that the set $(1,\infty)$ is an open set in $(\mathbb{R}^+, d_u)$.
(b) Show that the set $(0,1]$ is a closed set in $(\mathbb{R}^+, d_u)$.

Problem 26.6. Let (X,d) be a metric space.

(a) Prove that for all x and y in X, if $y \neq x$, then there exists an open ball centered at y, say $B_d(y,r)$, such that $x \notin B_d(y,r)$.
(b) Prove that if $x \in X$, then $\{x\}$ is a closed set.

Problem 26.7. Let (X,d) be a metric space. Let $(U_j)_{j=1}^{\infty}$ be a sequence of open sets.

(a) Give an example to show that $\bigcap_{j=1}^{\infty} U_j$ may not be open.
(b) Is it ever true that $\bigcap_{j=1}^{\infty} U_j$ is open?

Problem 26.8. Let $E = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$. Prove that E is open in $\mathbb{R}^2$ with respect to the usual metric.

Problem 26.9. Complete the following definition. Let X be a metric space with metric d and let F be a subset of X. Then F is not closed if ....

Problem 26.10. (a) Give three examples of sets that are both open and closed in $\mathbb{R}$ with the discrete metric.
(b) Give two examples of sets that are both open and closed in $\mathbb{R}$ with the usual metric.
(c) Give an example of a set that is neither open nor closed in $\mathbb{R}^2$ with the max metric. Prove that it is neither open nor closed!
(d) Give an example of a set that is closed and not open in $\mathbb{R}^2$ with the usual metric. Prove that it is closed and not open!

Problem 26.11. Prove Theorem 26.14. (In other words, show that if $\{O_\alpha : \alpha \in I\}$ is a collection of open sets, then $\bigcup_{\alpha \in I} O_\alpha$ is open.)

Problem 26.12. Prove Theorem 26.15.

Problem 26.13. Prove Theorem 26.12.

Problem 26.14. Let (X,d) be a metric space, $x, y \in X$, and let r_1 and r_2 be positive real numbers with $r_1 < r_2$. In what follows, do not use Theorem 26.16.

(a) Show that $B_d(x,r_1) \subseteq B_d(x,r_2)$.
(b) From the previous problem we know that every open ball is open. Show that if $B_d(x,r_1)$ and $B_d(y,r_2)$ are open balls, then $B_d(x,r_1) \cap B_d(y,r_2)$ is an open set. Is it an open ball? (Justify your answer to this last question, please.)

Problem 26.15. Prove Theorem 26.16 by completing both steps below. You may find it very helpful to work Problem 26.14 first.

(a) Show that the intersection of two open sets is an open set.
(b) Show that the intersection of finitely many open sets is an open set.

Problem 26.16. Prove Theorem 26.17. If you did Problem 26.15, you might consider using that result here.

Problem 26.17. (a) Let x and y be two distinct points in X and let $r = d(x,y)/2$. Show that $B_d(x,r)$ and $B_d(y,r)$ are disjoint sets.
(b) Show that for two distinct points x and y in a metric space, there exist disjoint open sets $\mathscr{O}_x$ and $\mathscr{O}_y$ with $x \in \mathscr{O}_x$ and $y \in \mathscr{O}_y$.

Problem 26.18. Let (X, d_X) and (Y, d_Y) be metric spaces. We say that a function $f : (X, d_X) \to (Y, d_Y)$ **preserves distances** if $d_Y(f(x), f(x')) = d_X(x, x')$ for all x, x' in X.

(a) In $\mathbb{R}$ with the usual metric, the function $f : (\mathbb{R}, d_u) \to (\mathbb{R}, d_u)$ defined by $f(x) = x$ obviously preserves distances. Give an example of another function that preserves distances.
(b) Is every function that preserves distances one-to-one? Either prove this statement or give a counterexample.

Problem 26.19. Let E be a subset of a set X with metric d. A point x is said to be an **interior point** of E if there exists an open ball $B_d(x,r)$ with $B_d(x,r) \subseteq E$. The set of all interior points is called **the interior** of E and is denoted by E^o.

(a) In $\mathbb{R}$ with the usual metric, and $E = (2,4]$, show that $4 \notin E^o$. Then show that $(2,4) = E^o$.
(b) In $\mathbb{R}^2$ with the max metric, find E^o if $E = \{(x,y) : |x| \leq 1\}$.

Problem 26.20. This problem is only appropriate if you completed Problem 26.19. Let (X,d) be a metric space, and E be a subset of X.

(a) By the definition of interior point, if $x \in E^o$, then there exists an open ball $B_d(x,r)$ centered at x such that $B_d(x,r) \subseteq E$. Show that, in fact, for each point $x \in E^o$, there exists an open ball $B_d(x,r_x) \subseteq E^o$. Use this to prove that E^o is an open set.
(b) Prove that a set E is open if and only if every point of E is an interior point. Conclude that a set E is open if and only if $E = E^o$.

Problem 26.21. Let (X,d) be a metric space. Let $E \subseteq X$. A point $x \in X$ is a **limit point** of E if every open set containing x contains a point $y \in E$ with $y \neq x$. Let E_l denote the set of all limit points of the set E.

(a) Complete the following definition. A point $x \in X$ is not a limit point of E if ....

(b) What are the limit points of the interval $(2,4]$ in $\mathbb{R}$ with the usual metric? the discrete metric?
(c) What are the limit points of $B_{d_u}((0,0),1)$ in $\mathbb{R}^2$ with the usual metric? the discrete metric?
(d) What are the limit points of $\{1/n : n \in \mathbb{Z}^+\}$ in $\mathbb{R}$ with the usual metric? the discrete metric?

Problem 26.22. (This problem assumes that you have completed Problem 26.21.) Let (X,d) be a metric space. Let E be a subset of X. Show that x is a limit point of a set E if and only if every open ball $B_d(x,r)$ about x contains a point $y \in E$ with $y \neq x$.

Problem 26.23. (This problem assumes that you have completed Problem 26.21.) Let (X,d) be a metric space. Let E be a subset of X.

(a) Prove that x is a limit point of a set E if and only if every open set about x contains infinitely many points different from x.
(b) Prove that a finite set has no limit points.

Problem 26.24. (This problem assumes that you have completed Problem 26.21.) Let (X,d) be a metric space. Let E be a subset of X.

(a) Show that E is closed if and only if E contains all its limit points. In other words, prove that E is closed if and only if $E_l \subseteq E$.
(b) Let E be a set. The **closure** of E is denoted by $\overline{E}$ and is defined by $\overline{E} = E \cup E_l$. Show that if x is a limit point of $\overline{E}$, then x is a limit point of E.
(c) Show that if x is a limit point of $\overline{E}$, then $x \in \overline{E}$ (note that you did the hard part in (b) above). Conclude that $\overline{E}$ is closed.