Chapter 17
Images and Inverse Images

In the last chapter, we looked at where points in the domain are mapped to under a function f and where points in the range come from under f. But sometimes we need to look at where f maps a whole set, or where an entire set comes from. So here are two definitions that are waiting to be understood.

Let $f : X \to Y$ be a function and let $A \subseteq X$. Then the **image** of A under f is the set

$$f(A) = \{f(a) : a \in A\}.$$

Note that $f(A)$ is the notation we use for this set, and that this set is a subset of Y. In "street talk" the image of A under f is where the elements of A were taken by f.

Exercise 17.1. It's good to start small. So let's begin with the two sets $A = \{1, 2, 4\}$ and $B = \{-1, 1, -2, 3\}$. Find each of the requested images under the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

(a) What is $f(A)$?
(b) What is $f(B)$?
(c) What is $f(A \cap B)$?
(d) What is $f(A) \cap f(B)$?

We'll solve (a) for you here, so you can see what we are asking you to do. We claim that $f(A) = \{1, 4, 16\}$. To see this, we use the definition:

$$f(A) = \{f(a) : a \in A\} = \{f(1), f(2), f(4)\} = \{1, 4, 16\}. \qquad \bigcirc$$

Small sets are easier because you can often list the values, just as we did above. This won't be possible, in general, as you will see below.

We are also interested in where sets in the codomain come from. This is called the inverse image of a set (because we are going backwards) and there is one unfortunate thing about it: the notation involves the symbol f^{-1}, which we have used only when f is bijective. Well, here f may not be bijective, and therefore, f^{-1} may not be a function. Though this may be confusing at first, this is generally agreed upon

U. Daepp and P. Gorkin, *Reading, Writing, and Proving: A Closer Look at Mathematics*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4419-9479-0_17,

notation and you (the reader) must check carefully on the context. Having said all that, we now define the inverse image.

Let $f : X \to Y$ be a function and let $B \subseteq Y$. Then the **inverse image** of B under f is the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

In other words, the inverse image of B is *the subset of* X consisting of all the elements in the domain that get mapped into B. Note that when f is *not* bijective, the notation $f^{-1}(y)$ makes no sense (why?). If you want to talk about the inverse image of a set with just one element, say so by writing $f^{-1}(\{y\})$. (You may find texts in which the authors use the notation $f^{-1}(y)$, but we find that it often introduces unnecessary confusion.)

Exercise 17.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find:

(a) $f^{-1}(\{4\})$;
(b) $f^{-1}(\{1,2,4\})$;
(c) $f^{-1}(f(A))$, where $A = \{1,2\}$.

Again, we will do (a) here, so you can see what we are asking you to do. By definition,

$$f^{-1}(\{4\}) = \{x \in \mathbb{R} : f(x) \in \{4\}\} = \{x \in \mathbb{R} : f(x) = 4\}.$$

Replacing f by what it equals, we have

$$f^{-1}(\{4\}) = \{x \in \mathbb{R} : x^2 = 4\} = \{-2,2\}.$$

○

Since the sets above are small, we can list all the elements. We ask that you now check your understanding with more challenging sets, but still using the same function as in the previous exercises.

By carefully writing out the definitions of the sets in Exercise 17.3, it is possible to guess what the answers are. We provide rigorous proofs for several parts at the end of this chapter. If you wish to try them yourself first (which you are certainly encouraged to do), make sure that you work from the inside out on parts (e)–(h). So in part (e), for example, first find $f([0,1])$ (which works just like (a)) and call that set A. Then find $f^{-1}(A)$ (which works just like (d)).

Exercise 17.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find:

(a) $f([-1,1])$;
(b) $f(\mathbb{Z})$;
(c) $f^{-1}(\mathbb{N})$;
(d) $f^{-1}([-1,0])$;
(e) $f^{-1}(f([0,1]))$;
(f) $f(f^{-1}([-1,0]))$;
(g) $f^{-1}([0,1] \cup [2,4])$;
(h) $f([0,1] \cap [-1,0])$;

(i) $f([0,1]) \cap f([-1,0])$. ○

Your experience with concrete sets will help you work with abstract sets.

Exercise 17.4. Looking back at the examples in the exercises above, decide which of the following you think are true for all functions $f : X \to Y$, all subsets A and B of X, and all subsets C and D of Y:

(a) $f(f^{-1}(C)) = C$;
(b) $f^{-1}(f(A)) = A$;
(c) $f(A \cap B) = f(A) \cap f(B)$;
(d) $f(A) = f(B)$ implies that $A = B$;
(e) $f^{-1}(C) = f^{-1}(D)$ implies that $C = D$. ○

All of the statements above may look reasonable, yet they are all false. Nevertheless, there are many similar statements that are true. You can prove them all with the tools you have developed at this point. To emphasize the accepted writing techniques, we provide an example below.

Theorem 17.5. *Let $f : X \to Y$ and let A_1 and A_2 be subsets of X. Then*

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

"Understanding the problem." Remember that you won't get anywhere if you don't know the definitions. So we need to figure out what the sets in the statement are. We begin by making sure we know what $f(A_1 \cap A_2)$, $f(A_1)$, and $f(A_2)$ are. First, $f(A) = \{f(x) : x \in A\}$. So that should make it pretty clear. Things in $f(A_1 \cap A_2)$ look like $f(x)$ where $x \in A_1 \cap A_2$. Now it should occur to you that you must write out what it means to be in $f(A_1) \cap f(A_2)$. Once you have done that, you have done the preliminaries.

"Devising a plan." When we worked with sets with a special form (like the Cartesian product of two sets) we emphasized that if we never used the special form of the elements, we would most likely never prove the desired result. The same is true here—if we never use the fact that the elements have the form $f(x)$ where $x \in A_1 \cap A_2$ we shouldn't expect to be able to prove the result. The next step is to note that what we want to do is to show that one set is contained in another set. We know how to do that, too. So our plan is to start with an element in the set on the left side, use the special form of this set, and show the element is in the set on the right. As we *"carry out our plan,"* note how quickly we move to the special form of the element.

Proof. If $y \in f(A_1 \cap A_2)$, then $y = f(x)$ for some $x \in A_1 \cap A_2$. Since $x \in A_1 \cap A_2$, we have $x \in A_1$ and $x \in A_2$. Since $x \in A_1$ and $y = f(x)$, we see that $y \in f(A_1)$. Similarly, since $x \in A_2$ and $y = f(x)$, we see that $y \in f(A_2)$. Therefore $y \in f(A_1) \cap f(A_2)$. Thus $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. □

Exercise 17.6. We already have an example to show that, with the notation from the theorem above, we need not have $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. But what is wrong with the following "proof" of this "nonfact"?

Not a proof. It follows from Theorem 17.5 that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. To show the reverse set inclusion, we let $y \in f(A_1) \cap f(A_2)$. By definition of intersection, $y \in f(A_1)$ and $y \in f(A_2)$. Therefore, $y = f(x)$ for some x in A_1 and $y = f(x)$ for some $x \in A_2$. Since $x \in A_1$ and $x \in A_2$, we see that $x \in A_1 \cap A_2$. Thus $y = f(x)$ where $x \in A_1 \cap A_2$, so $y \in f(A_1 \cap A_2)$. This proves that $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$, and the nonfact is established! ⍰

We know there's something wrong above since the assertion isn't always true. But it isn't always false either. Find the error and see if you can think of another hypothesis we might place on f that would help us to determine the functions for which the assertion is true. ○

The next theorem is one you will use repeatedly. You really can do all the proofs yourself.

Theorem 17.7. *Let $f : X \to Y$. Let A, A_1, and A_2 be subsets of X and B, B_1, and B_2 subsets of Y. Then*

1. *if $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$;*
2. *$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$;*
3. *$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$;*
4. *in general, $f(X \setminus A) \neq Y \setminus f(A)$;*
5. *if $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$;*
6. *$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;*
7. *$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$;*
8. *$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$;*
9. *$A \subseteq f^{-1}(f(A))$;*
10. *$f(f^{-1}(B)) \subseteq B$.*

We have already presented a proof of (3) in Theorem 17.5, and we will provide a proof of (9) in Example 17.8. The other parts of the theorem are left to the reader (that's you) in the problems. Remember that before beginning the proof of each part you must make sure you know what the left-hand side is, and what the right-hand side is. We suggest that you write out the element definition of both sides carefully (as we do in Example 17.8 below), and then show that appropriate relations hold using acceptable mathematical and writing techniques.

If additional conditions are placed on the function f, then some of the conclusions in Theorem 17.7 can be strengthened. We look at such a case in the following example. Some of the problems will ask you to consider similar restrictions.

Example 17.8. We will prove part 9 of Theorem 17.7. Then we will show that the inclusion is, in general, proper. We conclude this example by showing that if f is required to be one-to-one, then the two sets are actually equal.

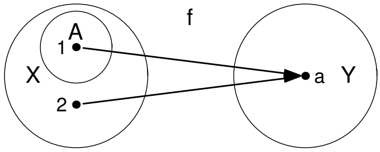

Fig. 17.1 $f:\{1,2\}\to\{a\}$

(a) First we prove that if $f:X\to Y$ and $A\subseteq X$, then $A\subseteq f^{-1}(f(A))$. Before we begin, we note that the right side is a bit complicated. Let's make sure we understand it: Since $f^{-1}(B)=\{x\in X: f(x)\in B\}$ replacing B by $f(A)$, we see that $f^{-1}(f(A))=\{x\in X: f(x)\in f(A)\}$. So we must show that if $z\in A$, then $z\in\{x\in X: f(x)\in f(A)\}$; in other words, we must show that $z\in X$ and $f(z)\in f(A)$.

Proof. If $z\in A$, then since $A\subseteq X$, we know that $z\in X$. By the definition of $f(A)$, we have $f(z)\in f(A)$. Thus, $z\in f^{-1}(f(A))$, and $A\subseteq f^{-1}(f(A))$. □

(b) Figure 17.1 indicates why, for an arbitrary function and an arbitrary set A, we cannot expect that the two sets A and $f^{-1}(f(A))$ are equal. From the diagram we see that if we let $A=\{1\}$ and define $f:\{1,2\}\to\{a\}$ by $f(1)=f(2)=a$, then $A=\{1\}$ while $f^{-1}(f(A))=f^{-1}(f(\{1\}))=f^{-1}(\{a\})=\{1,2\}$. Thus $A\neq f^{-1}(f(A))$.

(c) However, if the function $f:X\to Y$ is one-to-one and $A\subseteq X$, then we can conclude that $A=f^{-1}(f(A))$.

Proof. The inclusion $A\subseteq f^{-1}(f(A))$ is proven in (a) above. For the reverse inclusion, suppose $z\in f^{-1}(f(A))$. Then $z\in X$ and $f(z)\in f(A)$. Thus, there exists $x\in A$ such that $f(z)=f(x)$. Now f is one-to-one and so $z=x$. But $x\in A$, so $z\in A$. Hence $f^{-1}(f(A))\subseteq A$, and we conclude that the two sets are equal. □

Definitions

Definition 17.1. Let $f:X\to Y$ be a function and let $A\subseteq X$. Then the **image** of A under f is the set

$$f(A)=\{f(a): a\in A\}.$$

Definition 17.2. Let $f:X\to Y$ be a function and let $B\subseteq Y$. Then the **inverse image** of B under f is the set

$$f^{-1}(B)=\{x\in X: f(x)\in B\}.$$

Solutions to Exercises

Solution (17.1). You should be able to check that:

(b) $f(B) = \{1,4,9\}$;
(c) $f(A \cap B) = \{1\}$;
(d) $f(A) \cap f(B) = \{1,4\}$.

Solution (17.2). You should be able to check that:

(b) $f^{-1}(\{1,2,4\}) = \{-1,1,-\sqrt{2},\sqrt{2},-2,2\}$;
(c) $f^{-1}(f(A)) = f^{-1}(\{1,4\}) = \{-1,1,-2,2\}$.

Solution (17.3). We give complete solutions to (a), (d), (f), and (h) followed by answers to (b), (c), (e), (g), and (i).

(a) We claim that $f([-1,1]) = [0,1]$. To see this, let $y \in f([-1,1])$. By definition of the image,

$$f([-1,1]) = \{f(x) : x \in [-1,1]\} = \{x^2 : x \in [-1,1]\}.$$

So there exists $x \in [-1,1]$ such that $y = x^2$. Since $x \in [-1,1]$, we know that $0 \leq x^2 \leq 1$. Therefore, $y \in [0,1]$, and $f([-1,1]) \subseteq [0,1]$.
Conversely, if $y \in [0,1]$, then we let $x = \sqrt{y}$. Thus $x \in [0,1] \subset [-1,1]$, and $f(x) = x^2 = (\sqrt{y})^2 = y$. Therefore, there exists $x \in [-1,1]$ such that $y = f(x)$ and $y \in f([-1,1])$. So $[0,1] \subseteq f([-1,1])$, and we conclude that the two sets are equal.

(d) We claim that $f^{-1}([-1,0]) = \{0\}$. To see this, let $x \in f^{-1}([-1,0])$. By definition,

$$f^{-1}([-1,0]) = \{x \in \mathbb{R} : f(x) \in [-1,0]\} = \{x \in \mathbb{R} : x^2 \in [-1,0]\}.$$

Thus $x \in f^{-1}([-1,0])$ implies that $x^2 \in [-1,0]$. This is only possible if $x = 0$. Therefore $f^{-1}([-1,0]) \subseteq \{0\}$. Now suppose that $x \in \{0\}$. Then $f(x) = f(0) = 0$. Therefore, $f(x) \in [-1,0]$, and $x \in f^{-1}([-1,0])$. Consequently, $\{0\} \subseteq f^{-1}([-1,0])$, and we conclude that $f^{-1}([-1,0]) = \{0\}$.

(f) We claim that $f(f^{-1}([-1,0])) = \{0\}$. By (d), we know $f^{-1}([-1,0]) = \{0\}$. Therefore, we need to find

$$f(f^{-1}([-1,0])) = f(\{0\}).$$

Thus

$$f(f^{-1}([-1,0])) = f(\{0\}) = \{f(0)\} = \{0\},$$

as desired.

(h) We work from the inside out. Thus,

$$f([0,1] \cap [-1,0]) = f(\{0\}) = \{f(0)\} = \{0\}.$$

The answer to (b) is $\{z^2 : z \in \mathbb{N}\}$; to (c) is $\{\sqrt{n} : n \in \mathbb{N}\} \cup \{-\sqrt{n} : n \in \mathbb{N}\}$; to (e) is $[-1,1]$; to (g) is $[-1,1] \cup [-2,-\sqrt{2}] \cup [\sqrt{2},2]$; and to (i) is $[0,1]$.

There are many possible ways to solve these problems. Though each problem seems to be different, what remains the same in every problem is that you need to understand the notation and the definitions.

Solution (17.4). None of the five statements holds for all functions and sets. Thus we will give a counterexample for each case. In all cases we will use the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

(a) Exercise 17.3 (f) provides a counterexample.
(b) Exercise 17.3 (e) provides a counterexample.
(c) Exercise 17.3 (h) and (i) provide a counterexample.
(d) Using f as defined above, $f(\{-1,1\}) = \{1\} = f(\{1\})$ but $\{-1,1\} \neq \{1\}$.
(e) Again using the function defined above, $f^{-1}(\{-1,0\}) = \{0\} = f^{-1}(\{0\})$ but $\{-1,0\} \neq \{0\}$.

Solution (17.6). In "not a proof" we correctly establish that "$y = f(x)$ for some $x \in A_1$ and $y = f(x)$ for some $x \in A_2$." From this statement we incorrectly conclude that "$x \in A_1$ and $x \in A_2$." We may only conclude that there exists an $x_1 \in A_1$ such that $y = f(x_1)$, and there exists an $x_2 \in A_2$ such that $y = f(x_2)$. We may not conclude that $x_1 = x_2$. Indeed, Exercise 17.1 shows that there are cases where neither of the two elements is in the intersection.

However, the following is true. Let $f : X \to Y$ be an injective function, and let A_1 and A_2 be subsets of X. Then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. Why? Well, we know that $f(x_1) = y = f(x_2)$ and we now know that f is injective. Thus, $x_1 = x_2$, so $x_1 \in A_1 \cap A_2$. The rest of the "not a proof" is valid.

Spotlight: Minimum or Infimum?

If you've ever forgotten to show that the infimum of a set was the minimum, this historical example should show you that it can happen to the best of us.

Suppose you know the temperature on the surface of the earth (because people on the surface measured it) and you want the temperature on the inside (but no one can get to the place to measure it). How do you get the temperature? This kind of question interested several famous mathematicians. In fact, there's plenty of mathematical research done today that is related to this problem.

This problem is known as the Dirichlet problem. It can be studied in more generality, but we'll stick to looking at functions defined on a sphere. To understand the statement, you need to have studied several variable calculus. At the end of this spotlight, we will state the problem on the sphere, along with the references, for those of you who have the background.

The solution to this problem used something called the "Dirichlet principle." The idea of the principle was to look at a collection of certain integrals with the region of integration fixed, but with different integrands. It was first argued that the values of the integrals were bounded below (and therefore, as we have learned, there was an infimum). From there, the mathematicians assumed that for one of the functions, the integral was the minimum. This principle was used by many excellent mathematicians: by George Green, Georg Friedrich Bernhard Riemann, Sir William Thomson (also known as Lord Kelvin), and others. In 1870, Karl Theodor Wilhelm Weierstrass presented an important paper about the validity of this argument. Even the title of his article "Über das sogenannte Dirichlet'sche Princip" (On the so-called Dirichlet principle) is enough to show what Weierstrass thought of the principle [107]. He began his paper by reconstructing Dirichlet's argument. He then explained that though an expression may have a lower bound that we can get arbitrarily close to, we may never actually reach it. Weierstrass concluded his paper with an example showing how this might happen. Almost thirty years later, David Hilbert supplied a proof of the principle for certain cases when he presented what he called the "resuscitation" of the Dirichlet principle [86, p. 67].

It may seem odd that mathematicians of this calibre would use an unproven principle. There are two things to remember. First, the principle was supported on physical grounds. Second, rigor was still being introduced to mathematics. In spite of its unusual history, the Dirichlet principle served an important purpose. In Kline's words [59, p. 704] "Had the progress made with the use of the principle awaited Hilbert's work, a large segment of nineteenth-century work on potential theory and function theory would have been lost."

More information on this is available in [35], [59], [32], and [71]. Another criticism of the Dirichlet principle, from a different point of view, was published by Friedrich Prym. More information about this can be found in [87]. The statement of the problem in $\mathbb{R}^3$ is the following: Let f be a continuous real-valued function on the sphere of radius one (the unit sphere). A real-valued function g is called harmonic on the open unit ball ($\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1\}$) if g has continuous second partial derivatives satisfying

$$g_{xx} + g_{yy} + g_{zz} = 0$$

throughout the ball. The question is: Does there exist a function F that is continuous on the closed ball of radius one, equal to f on the unit sphere, and harmonic on the open unit ball?

Problems

Problem 17.1. Recall that $[a,b]$ denotes the closed interval from a to b, while (a,b) denotes the open interval. For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x-1$, find:

(a) $f((0,1))$;
(b) $f((a,b))$, where $a, b \in \mathbb{R}$ and $a < b$;
(c) $f^{-1}((-2,-1))$;

(d) $f^{-1}((a,b))$, where $a,b \in \mathbb{R}$ and $a < b$.

Problem 17.2. For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^2$, find:

(a) $f((0,1))$;
(b) $f((-1,3))$;
(c) and (in general) $f((a,b))$, where $a,b \in \mathbb{R}$ and $a < b$;
(d) $f^{-1}((-2,-1))$;
(e) $f^{-1}((0,2))$;
(f) and (in general) $f^{-1}((a,b))$, where $a,b \in \mathbb{R}$ and $a < b$.

Actually, we are really only interested in your answers to (c) and (f). So why did you have to work all the other parts?

Problem 17.3. For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, find:

(a) $f((-1,1))$;
(b) $f(\{-1,1\})$;
(c) $f^{-1}(\{1\})$;
(d) $f^{-1}([-1,0))$;
(e) $f^{-1}(f([0,1]))$.

Problem 17.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 9 - x^2$. Find the two sets $f((-3,1])$ and $f^{-1}((-1,4))$.

Problem 17.5. Consider the function $\chi_{(0,1)} : \mathbb{R} \to \mathbb{R}$ (this is the characteristic of Definition 14.4). Find:

(a) $\chi_{(0,1)}((0,1))$;
(b) $\chi_{(0,1)}((-1,3))$;
(c) and (in general) $\chi_{(0,1)}((a,b))$, where $a,b \in \mathbb{R}$ and $a < b$; prove that the set you found is correct;
(d) $\chi_{(0,1)}^{-1}((-2,-1))$;
(e) $\chi_{(0,1)}^{-1}((0,2))$;
(f) and (in general) $\chi_{(0,1)}^{-1}((a,b))$, where $a,b \in \mathbb{R}$ and $a < b$; prove that the set you found is correct,

Actually, we are really only interested in your answers to (c) and (f). So why did you have to work all the other parts?

Problem 17.6. We denote the characteristic function of $\mathbb{Z}$ in $\mathbb{R}$ by $\chi_{\mathbb{Z}} : \mathbb{R} \to \mathbb{R}$ (see Definition 14.4). In each case below, start by writing out the definition for the particular set and function. Then write the solution to each of the following in as simple a form as possible:

(a) $\chi_{\mathbb{Z}}(\mathbb{Z}^+)$;
(b) $\chi_{\mathbb{Z}}^{-1}(\mathbb{Z}^+)$;
(c) $\chi_{\mathbb{Z}}(\chi_{\mathbb{Z}}^{-1}(\mathbb{Z}^+))$;
(d) $\chi_{\mathbb{Z}}^{-1}(\chi_{\mathbb{Z}}(\mathbb{Z}^+))$.

Problem 17.7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4 + 1$.

(a) Make a careful graph of f.
(b) Using your graph, show how you can guess $f([0,2])$.
(c) Prove that your guess for $f([0,2])$ is correct.
(d) Use your graph to find $f^{-1}([2,17])$.
(e) Prove that your guess for $f^{-1}([2,17])$ is correct.

Problem 17.8. Let p and q be two polynomials of degree two with real coefficients. (See Problem 10.13 for definitions.) Suppose $p^{-1}(\{0\}) = q^{-1}(\{0\})$.

(a) Give an example of such p and q, with $p \neq q$.
(b) Suppose that $p^{-1}(\{0\}) = \{0,1\} = q^{-1}(\{0\})$. Must $p = q$? Either prove this or give a counterexample.

Problem 17.9. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(n) = \begin{cases} -2n & \text{if } n \leq 0 \\ 2n-1 & \text{if } n > 0 \end{cases}.$$

Find $f(2\mathbb{Z})$ and prove that your answer is correct.

Problem 17.10. Prove Theorem 17.7 part 1.

Problem 17.11. Using Theorem 17.7 part 1, rather than element-chasing, prove Theorem 17.7 part 3.

Problem 17.12. Prove Theorem 17.7 part 2.

Problem 17.13. (a) Establish part 4 of Theorem 17.7. We suggest the following strategy: Try to prove that the two sets are equal. If you do this carefully, you may start to wish for restrictions on f that you don't have. This should help you think of examples to show that the two sets need not be equal.
(b) If f is onto, does the statement in Theorem 17.7 part 4 become an equality? What if f is one-to-one?
(c) Show that if f is bijective, then equality holds.

Problem 17.14. (a) Prove Theorem 17.7 part 5.
(b) In the same context, what can you conclude if $B_1 = B_2$? State your result and prove it.

Problem 17.15. Prove Theorem 17.7 part 6.

Problem 17.16. Prove Theorem 17.7 part 7.

Problem 17.17. Prove Theorem 17.7 part 8.

Problem 17.18. (a) Prove Theorem 17.7 part 10.
(b) Give an example to show that the two sets may not be equal.
(c) If f is onto, must the two sets be equal?

(d) If f is one-to-one, must the two sets be equal?

Problem 17.19. Let X and Y be nonempty sets and $f : X \to Y$ a function.

(a) Prove or give a counterexample to the statement: If A and B are subsets of X, then $f(A \setminus B) = f(A) \setminus f(B)$.
(b) Find necessary and sufficient conditions on the function f such that for all subsets A and B of X, we have $f(A \setminus B) = f(A) \setminus f(B)$.

Problem 17.20. Let $f : X \to Y$ be a function satisfying $f(A) \cap f(B) = \emptyset$ whenever A and B are sets with $A \cap B = \emptyset$.

(a) Give an example of such a function. Prove that your example satisfies the condition above.
(b) Prove that such a function must be one-to-one.

Problem 17.21. Let $f : A \to B$ be a function. Prove that if f is onto, then the collection $\{f^{-1}(\{b\}) : b \in B\}$ partitions the set A.

Problem 17.22. Suppose that $f : X \to Y$ is a function, and let A_1 and A_2 be subsets of X.

(a) If $f(A_1) = f(A_2)$, must $A_1 = A_2$?
(b) Let f be a bijective function. Show that if $f(A_1) = f(A_2)$, then $A_1 = A_2$. Indicate clearly where you use one-to-one or onto. Did you use both?

Problem 17.23. Suppose that $f : X \to Y$ is a function, and let B_1 and B_2 be subsets of Y.

(a) If $f^{-1}(B_1) = f^{-1}(B_2)$, must $B_1 = B_2$?
(b) Let f be a bijective function. Show that if $f^{-1}(B_1) = f^{-1}(B_2)$, then $B_1 = B_2$. Indicate clearly where you use one-to-one or onto. Did you use both?

Problem 17.24. Let X be a nonempty set and let A_1 and A_2 be subsets of X. Recall the characteristic function of A in X of Definition 14.4.

(a) If $\chi_{A_1} = \chi_{A_2}$, must $A_1 = A_2$?
(b) We define the product $\chi_{A_1} \cdot \chi_{A_2}$ pointwise by $(\chi_{A_1} \cdot \chi_{A_2})(x) = \chi_{A_1}(x) \cdot \chi_{A_2}(x)$ for all $x \in X$. Prove that $\chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2}$.
(c) Show that $\chi_{A_1} + \chi_{A_2} - \chi_{A_1 \cap A_2} = \chi_{A_1 \cup A_2}$. (In other words, for each $x \in X$, we have $\chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = \chi_{A_1 \cup A_2}(x)$.)
(d) Can you find a similar result for $\chi_{X \setminus A_1}$?

Problem 17.25. *(For students with a background in calculus.)* For real numbers c, d with $c < d$ we denote the open interval in $\mathbb{R}$ by $(c,d) = \{x \in \mathbb{R} : c < x < d\}$. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing if for all $x, y \in \operatorname{dom}(f)$ whenever $x < y$, then $f(x) < f(y)$.

(a) Show that there are continuous functions $f : \mathbb{R} \to \mathbb{R}$ and real numbers a, b with $a < b$ such that $f((a,b)) \neq (f(a), f(b))$.
(b) Prove that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing if and only if $f((a,b)) = (f(a), f(b))$ for all $a, b \in \mathbb{R}$ with $a < b$.