Chapter 3
Introducing the Contrapositive and Converse

In the last chapter (see Theorem 2.7) we saw that two statement forms, P and Q, that have the same truth table are equivalent. This was also expressed by showing that the equivalence, $P \leftrightarrow Q$, is a tautology. When you are confronted with a mathematical statement that you need to prove, you will often find it helpful to paraphrase it. You will use tautologies to do so, since you don't want to change the truth value of your statement. Some useful tautologies appeared in Theorem 2.9. More appear below and throughout this chapter.

Theorem 3.1. *Let P, Q, and R denote statement forms. Then the following are tautologies:*

(Distributive property) $(P \wedge (Q \vee R)) \leftrightarrow ((P \wedge Q) \vee (P \wedge R))$;
$(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R))$;

(Associative property) $(P \wedge (Q \wedge R)) \leftrightarrow ((P \wedge Q) \wedge R)$;
$(P \vee (Q \vee R)) \leftrightarrow ((P \vee Q) \vee R)$;

(Commutative property) $(P \wedge Q) \leftrightarrow (Q \wedge P)$;
$(P \vee Q) \leftrightarrow (Q \vee P)$.

At this point, you should be able to construct the truth tables for everything above and you should be able to show that all of them are tautologies.

Exercise 3.2. Negate the following:

(a) $(P \wedge Q) \vee (P \wedge R)$;
(b) $P \rightarrow (Q \wedge R)$. ○

Tautologies allow us to replace one statement by another. For example, suppose you want to show that an integer is odd or prime. You can show that the integer is prime or odd; that won't change things because these two statements are equivalent.

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This is a fairly obvious change that usually won't make much of a difference. The same holds if you want to show x is prime and odd; you can show that it is odd and prime if that's easier and you will have accomplished the same thing. Similarly, if you want to show that it is not the case that x is prime and odd, you can show that x is not prime or not odd.

For implications, restating what you want to prove can really make a difference. We need to make sure, however, that what we have is equivalent to our original statement. So recall that we showed, in the last chapter, that $P \to Q$ is equivalent to $\neg P \vee Q$.

Now consider $\neg Q \to \neg P$, which is called the **contrapositive** of the implication $P \to Q$. We need to compare the two truth tables below:

P	**Q**	$\mathbf{P \to Q}$
T	T	T
T	F	F
F	T	T
F	F	T

P	**Q**	$\mathbf{\neg Q \to \neg P}$
T	T	T
T	F	F
F	T	T
F	F	T

So the fact that the truth tables are the same and Theorem 2.7 tell us that the statement forms are logically equivalent. What this means to us is that, if we are trying to prove that an implication is true and we don't see how to do it, we should consider the contrapositive of that statement. Here's how it works in practice.

Theorem 3.3. *Let x be an integer. If x^2 is odd, then x is odd.*

First we need to understand the problem. What does it mean for a number x to be odd? Our definition of "odd number" is the following: An **integer** x **is odd** if there is an integer n such that $x = 2n+1$. So we are assuming that $x^2 = 2n+1$ for some integer n and trying to show $x = 2m+1$ for some integer m. It's hard to see where to go from here, we think.

Remember that Pólya suggests restating the problem, so let's try that. Let P be the sentence "x^2 is odd" and Q be the sentence "x is odd." Then we see that we wish to prove that $P \to Q$ is true. But this is logically equivalent to $\neg Q \to \neg P$, which translates into "If x is not odd, then x^2 is not odd." We can do better than that, since an integer is odd or even. So we can show that "If x is even, then x^2 is even" and that will be equivalent. Let's see if that's easier.

Theorem (Contrapositive of the statement of Theorem 3.3). *Let x be an integer. If x is even, then x^2 is even.*

The first step is to understand the problem. The second step is to prove it. We'll do that here:

"Understanding the problem." When is an integer even? Our definition of "even number" is the following: An **integer** x **is even** if there is an integer n such that $x = 2n$. So we need to show that $x^2 = 2m$, where m is an integer, assuming that $x = 2n$, where n is an integer. We began by understanding the problem, now we are ready to solve it.

Proof. Let x be even. Then there is an integer n such that $x = 2n$. Therefore, $x^2 = (2n)^2 = 2(2n^2)$. Let $m = 2n^2$. Then $x^2 = 2m$ and m is an integer. Therefore x^2 is even. □

Of course, this takes care of the original theorem, since it is equivalent to the one we proved. Thus, using the contrapositive is one possible way to attempt to prove that an implication is true. We will soon have a number of ways to attack a problem. Try to keep them all in mind.

Some other related remarks: Notation is more important than it may seem. In the theorem above, we assume that x is even and try to show x^2 is even. If we assume that $x = 2n$ and accidentally try to show $x^2 = 2n$ (rather than $x^2 = 2m$), we're stuck because we assumed erroneously that $x = x^2$. In other words, our notation would force us to show that $x = 0$ or $x = 1$, which is not what we should be doing. We introduced an error because of poor notation. So it's important that one symbol be an n and one be an m.

Also, note that we begin the proof by saying what we are assuming, and end the proof by saying what we are concluding. That helps the reader too. Finally, we keep checking that m and n are integers. That's because that is very important; if they weren't integers, x wouldn't have to be even.

So the contrapositive was very helpful here. You do need to be careful though. It must be the contrapositive and not the converse. The converse of an implication $P \to Q$ is the statement form $Q \to P$. Looking at the truth tables for each of these given below,

P	**Q**	**P → Q**
T	*T*	*T*
T	*F*	*F*
F	*T*	*T*
F	*F*	*T*

and

P	**Q**	**Q → P**
T	*T*	*T*
T	*F*	*T*
F	*T*	*F*
F	*F*	*T*

we see that they are different. Unfortunately, though the contrapositive and converse of a statement are really very different, students often confuse them. We'll take just a moment to convince you that it is very important not to do this.

Suppose our statement is, "If I am a Hobbit, then I am under 5 feet tall." This is a true statement, as every Tolkien reader knows. The converse is "If I am under 5 feet tall, then I am a Hobbit." This latter statement is not true, since lots of children are under 5 feet tall, but most of them are not Hobbits. As a mathematical example, consider the sentence about integers "If x is seven, then x is prime," and its converse "If x is prime, then x is seven." Recall that an integer p is **prime** if $p > 1$ and p cannot be written as a product of two positive integers, both different from p. Thus, the original sentence is true for all x, while the converse above is not. On the other hand, you agree that for all x the contrapositive "If x is not prime, then x is not seven," is true, as it must be. But this is trickier when we don't really understand what we are saying as well as we understand this statement. Remember, make sure you understand the problem. We present some examples and exercises for you to try your hand at.

Example 3.4. Consider the slightly odd sentence: "If the sky is green, then $2+2$ $= 4$." What are the converse and the contrapositive of this implication? Which of the following (if any) is true: the statement, the converse, or the contrapositive?

The converse is: "If $2+2=4$, then the sky is green." The contrapositive is: "If $2+2 \neq 4$, then the sky is not green."

To decide on the validity of the statements we have to agree on the truth of each part. We are quite confident that $2+2=4$, so this part is true. We abbreviate the statement with A (for arithmetic). Since we have never seen a (natural) green sky before, we suggest that "the sky is green," abbreviated by S, should be considered as false. With this in place, we look at the relevant parts of the truth tables.

S	**A**	$\mathbf{S \to A}$	$\mathbf{A \to S}$	$\mathbf{\neg A \to \neg S}$
F	T	T	F	T

So the original statement and its contrapositive are true, while the converse is false. Of course we knew beforehand that the original statement and the contrapositive would have the same answer because they are equivalent statements. ○

Now it's your turn.

Exercise 3.5. Consider the sentence "If n is odd, then n^2-n-6 is even."

(a) State the contrapositive.
(b) State the converse. ○

We should also mention one more possibility that comes up frequently: the inverse of an implication $P \to Q$ is the statement form $\neg P \to \neg Q$. As you will show in the exercise below, this form is nothing new. The inverse of an implication can be expressed in terms of our earlier definitions.

Exercise 3.6. Consider the implication $P \to Q$.

(a) Write the truth table for the inverse of $P \to Q$.
(b) Express the inverse of an implication in terms of the converse and contrapositive in two different ways.
(c) State the relation between the inverse of $P \to Q$ and the converse of $P \to Q$, and give a reason for your answer. ○

Definitions

Definition 3.1. An **integer** x **is odd** if there is an integer n such that $x = 2n+1$.

Definition 3.2. An **integer** x **is even** if there is an integer n such that $x = 2n$.

Definition 3.3. An integer p is **prime** if $p > 1$ and p cannot be written as a product of two positive integers, both different from p.

Solutions to Exercises

Solution (3.2). The equivalences are given below.

(a) The negation may be stated as $(\neg P \vee \neg Q) \wedge (\neg P \vee \neg R)$, since

$$\begin{aligned} \neg((P \wedge Q) \vee (P \wedge R)) &\leftrightarrow (\neg(P \wedge Q) \wedge \neg(P \wedge R)) \\ &\leftrightarrow ((\neg P \vee \neg Q) \wedge (\neg P \vee \neg R)). \end{aligned}$$

(b) The negation may be stated as $P \wedge (\neg Q \vee \neg R)$, since

$$\begin{aligned} \neg(P \rightarrow (Q \wedge R)) &\leftrightarrow (P \wedge \neg(Q \wedge R)) \\ &\leftrightarrow (P \wedge (\neg Q \vee \neg R)). \end{aligned}$$

Solution (3.5).

(a) The contrapositive is "If $n^2 - n - 6$ is odd, then n is even."
(b) The converse is "If $n^2 - n - 6$ is even, then n is odd."

Solution (3.6).

(a)

P	**Q**	**$\neg$P $\rightarrow$ $\neg$Q**
T	T	T
T	F	T
F	T	F
F	F	T

(b) The inverse of an implication is the converse of the contrapositive of the implication and this, in turn, is equivalent to the contrapositive of the converse of the implication.
(c) The inverse of the implication $P \rightarrow Q$ is equivalent to the converse of the implication $P \rightarrow Q$. To see this, compare the truth table of the inverse given in (a) with the truth table of the converse from page 27. They are the same and thus, by Theorem 2.7, the inverse is equivalent to the converse.

Problems

Problem 3.1. (a) Write a tautology involving only logical symbols, the implication $P \rightarrow Q$, its converse, and $P \leftrightarrow Q$.
(b) Can you write a tautology involving only $P \rightarrow Q$ and its contrapositive? If so, how? If not, why not?

Problem# 3.2. (a) Let x be an integer. Prove that if x is odd, then x^2 is odd. Make sure you state your assumption as the first line and your conclusion as the last line.

(b) State the contrapositive of what you just proved.
(c) Combining the result of part (a) with Theorem 3.3 gives a stronger result. Say precisely what that result is.

Problem 3.3. For each of the following, write out the contrapositive and the converse of the sentence.

(a) If you are the President of the United States, then you live in a white house.
(b) If you are going to bake a soufflé, then you need eggs.
(c) If x is a real number, then x is an integer.
(d) If x is a real number, then $x^2 < 0$.

Problem 3.4. State the contrapositive of each of the following.

(a) If it rains, then it pours.
(b) If I had a bell, I would ring the bell in the morning.
(c) The house is red, if the house is not blue.
(d) Dinner is cooked only if I make it.

Problem 3.5. State the converse of each of the following.

(a) If it rains, then it pours.
(b) If I am young, then I am restless.
(c) I am alone, if it is Saturday.
(d) I eat fish only if it is cooked.

Problem 3.6. State the inverse of each of the following.

(a) If it rains, then it pours.
(b) If I am living abroad, then I need brownies.
(c) To run quickly, it is sufficient to have long legs.
(d) To make good chocolate chip cookies, it is necessary to have baking soda.

Problem 3.7. Consider the statement form $P \to Q$.

(a) Write the negation of the converse of this statement form in as simple a form as possible.
(b) Write the negation of the contrapositive of this statement form in as simple a form as possible.
(c) Write the negation of the inverse of this statement form in as simple a form as possible.

Problem 3.8. Consider the sentence: “The horses eat the grass only if they are led to the pasture.”

(a) Write the negation of the converse of this sentence. Your answer has to be simple and as eloquent as possible.
(b) Is the sentence in (a) the same as the converse of the negation of the original sentence? Explain your answer.

Problem 3.9. Let x and y be real numbers. Show that if $x \neq y$, then $2x+4 \neq 2y+4$. (Hint: Use the contrapositive.)

Problem 3.10. Matilda always eats at least one of the following for breakfast: cereal, bread, or yogurt. On Monday, she is especially picky.

If she eats cereal and bread, she also eats yogurt. If she eats bread or yogurt, she also eats cereal. She never eats both cereal and yogurt. She always eats bread or cereal.

Can you say what Matilda eats on Monday? If so, what does she eat?

Problem 3.11. Consider the following statement.

If the coat is green, then the moon is full or the cow jumps over it.

(a) This odd statement is composed of several substatements. Identify each substatement, assign a letter to it, and write down the original statement as a statement form using these letters and logical connectives.
(b) Find the contrapositive of the original statement form from part (a). Use this to write the contrapositive of the *original* statement as an English sentence.
(c) Find the converse of the *original* statement form from part (a). Use this to write the converse of the *original* statement as an English sentence.
(d) Find the negation of the *original* statement form from part (a). Use this to write the negation of the *original* statement as an English sentence.
(e) Are some of the statements in this problem (the original or the ones you obtained) equivalent? If so, which ones?

Problem 3.12. Consider the two statement forms $P \to Q$ and $P \to (Q \vee \neg P)$.

(a) Make a truth table for each of these statement forms.
(b) What can you conclude from your solution to part (a)?

Problem 3.13. Karl's favorite brownie recipe uses semisweet chocolate, very little flour, and less than $1/4$ cup sugar. He has four recipes: one French, one Swiss, one German, and one American. Each of the four has at least two of the qualities Karl wants in a brownie recipe. Exactly three use very little flour, exactly three use semisweet chocolate, and exactly three use less than $1/4$ cup sugar.

The Swiss and the German recipes use different kinds of chocolate. The American and the German recipes use the same amount of flour, but different kinds of chocolate. The French and the American recipes use the same amount of flour. The German and American recipes do not both use less than $1/4$ cup sugar.

Karl is very excited because one of these is his favorite recipe. Which one is it?

Problem 3.14. Let n be an integer. Prove that if $3n$ is odd, then n is odd.

Problem 3.15. Let x be a natural number. Prove that if x is odd, then $\sqrt{2x}$ is not an integer.

Problem 3.16. Let x and y be real numbers. Show that if $x \neq y$ and $x, y \geq 0$, then $x^2 \neq y^2$.

Problem 3.17. In the statement below, G is a group and H is a normal subgroup of G. (You need not know what "group," "normal subgroup," or "p-group" mean to do this problem!)

"If H and G/H are p-groups, then G is a p-group."

(a) State the converse of this statement.
(b) State the contrapositive of this statement.
(c) Consider the following: "G is a p-group if and only if H and G/H are p-groups." Write this in terms of your answers to the first two parts of this problem.

Problem 3.18. Prove that if the product of two integers x and y is odd, then both integers are odd. Describe your method of proof.

Problem 3.19. Consider the statement "If Simon takes German or French, then he cannot take Russian."

(a) State the contrapositive of this implication.
(b) State the converse of this implication.

For parts (c) and (d), assume the original statement is true.

(c) Suppose someone tells you that Simon did not take German. What, if anything, can you conclude about Simon? Why?
(d) Suppose someone tells you that Simon took French. What, if anything, can you conclude about Simon? Why?

Problem 3.20. Consider the statement form $(P \vee \neg Q) \rightarrow (R \wedge Q)$.

(a) Write out the truth table for this form.
(b) Make up a meaningful English statement that has this form.
(c) Write the contrapositive of your English statement. Simplify the sentence as much as you can.