Chapter 16
Inverses

Given functions $f : A \to B$ and $g : C \to D$ with $\operatorname{ran}(f) \subseteq C$, we can define a third function called the **composite function** from A to D. (We will usually call this the **composition**, rather than the composite function.) This composition is the function $g \circ f : A \to D$ defined by $(g \circ f)(x) = g(f(x))$. So, for example, if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = x^2$ and $g(x) = \sin(x)$, then $(g \circ f)(x) = g(f(x)) = g(x^2) = \sin(x^2)$. Note that the order really matters here. Using f and g as above, for example, $(f \circ g)(x) = f(g(x)) = f(\sin(x)) = (\sin(x))^2$. You can check pretty easily that these two functions are different. (Check this pretty easily.) So composition of functions is not commutative.

Consider the two functions in Figure 16.1. Here $f : A \to B$, where $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$, while $g : C \to D$, where $C = \{2, 3, 4, 5, 6\}$ and $D = \{\alpha, \beta, \gamma\}$. Then $\operatorname{ran}(f) \subset C$, so the composition $g \circ f$ is defined. To determine the action of $g \circ f$ algebraically, use the definition of each. For example, $(g \circ f)(a) = g(f(a)) = g(2) = \beta$. To determine the action visually, follow the arrows, remembering that f goes first.

Take this opportunity to check that the composition of three functions satisfies the associative property. In other words, if we have three functions f, g, and h so that the composition makes sense (what would that mean?), then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

We'll use this result in this chapter.

Sometimes it is useful to "undo" the action of a function f. If f maps 3 to 5, we might wish to "undo" that by finding a function that takes 5 back to 3. This is most useful when we can undo the action of f on the whole range, not at just one point, because then every element ends up back where it started. For example, if f cubes all the values in its domain, we can "reverse" that action by taking the cube root. Mathematically what this means is that if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, then $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^{1/3}$ satisfies two things: $(g \circ f)(x) = x$ for all $x \in \operatorname{dom}(f)$, and $(f \circ g)(y) = y$ for all $y \in \operatorname{dom}(g)$.

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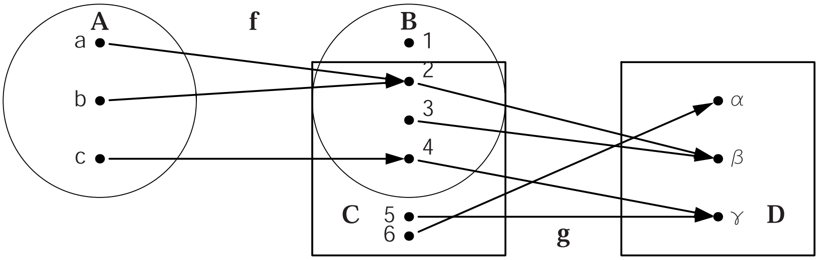

Fig. 16.1 $g \circ f : A \to D$

But what happens if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$? If we want g to "undo" this action, then we want g to satisfy $(g \circ f)(x) = x$ for all $x \in \mathbb{R}$. But if $x = 2$, we need $(g \circ f)(2) = g(4) = 2$ and, if $x = -2$, we need $(g \circ f)(-2) = g(4) = -2$ (see Figure 16.2). What's the problem here? Well, g is not allowed to assign two different values to the number 4. So we can't do this for all functions. When can we do it? (Think first, read on later.)

Suppose that a function is bijective. Then, rather than looking in the domain and asking what x gets mapped to, we can look in the range at y and ask where it came from. Since the function is onto, y came from some x. Since the function is one-to-one, y came from exactly one x. So we can define an inverse function as follows.

Let $f : A \to B$ be a bijective function. The **inverse** of f is the function $f^{-1} : B \to A$ defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

Whenever we define a function, we have to ask ourself: "Is it well-defined?" Is it? The domain is defined to be B. By definition, the value of an element of B under f^{-1} is some $x \in A$. Hence A qualifies as a codomain of f^{-1}. Now we check condition (i) of the definition of a function. Let $b \in B$. Since f is onto, there exists an element $a \in A$ such that $f(a) = b$. Hence $f^{-1}(b) = a$ is defined and property (i) holds. For property (ii) we assume that there is an element $b \in B$, and elements a and c in A such that $f^{-1}(b) = a$ and $f^{-1}(b) = c$. By the definition of f^{-1} we have $f(a) = b$

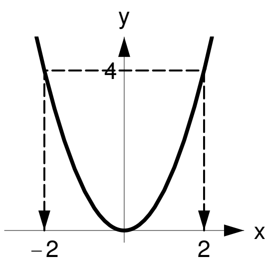

Fig. 16.2 $f(x) = x^2$

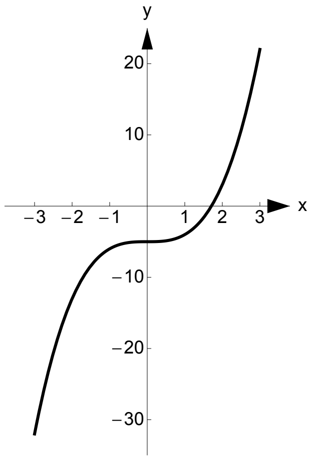

Fig. 16.3 $f(x) = x^3 - 5$

and $f(c) = b$. Hence $f(a) = f(c)$ and since f is one-to-one, $a = c$. This shows that property (ii) holds and we conclude that f^{-1} is well-defined. Note that this function is only defined in the case when f is bijective.

The discussion in the last paragraph shows that f^{-1} is indeed a function. Thus we have shown that if $f : A \to B$ is a bijective function, then there exists an inverse function $g : B \to A$. The remainder of this chapter will be spent understanding inverse functions. In particular, we will show that an inverse function is unique and we will speak of "the" inverse of f.

Example 16.1. We define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 5$. Graph the function f. Then prove that f is one-to-one and onto. Once you have done that, decide what f^{-1} is.

"Devising a plan." Assume for the moment that we know that f is bijective, so that we know that f^{-1} exists. To find f^{-1}, we use what we know: $f^{-1}(y) = x$ if and only if $f(x) = y$. Thus we must solve $x^3 - 5 = y$ for x. Once we solve this equation, we find that $x = (y+5)^{1/3}$. Now we are ready to solve this problem.

Proof. We first prove that f is one-to-one. So let x_1 and x_2 be real numbers. If $f(x_1) = f(x_2)$, then $x_1^3 - 5 = x_2^3 - 5$. Hence $x_1^3 = x_2^3$. We factor the difference to get $x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)$. Thus, $x_1 = x_2$ or $x_1^2 + x_1x_2 + x_2^2 = 0$. In the latter case, we write $x_1^2 + x_1x_2 + x_2^2 = (x_1 + x_2)^2 - x_1x_2 = 0$. So, we would have $x_1x_2 = (x_1 + x_2)^2 \geq 0$, and this would mean that all three terms in $x_1^2 + x_1x_2 + x_2^2$ are nonnegative. The only way this sum can be zero is if each summand is zero. Thus $x_1 = x_2 = 0$. Therefore $x_1 = x_2$ and we may conclude that f is one-to-one. (Alternatively, you can use the remark following the discussion and proof of Theorem 13.2.)

Now we show that f is onto. Let $y \in \mathbb{R}$ and set $x = (y+5)^{1/3}$. (See the remark on the existence of nth roots following the discussion and proof of Theorem 13.2.) Then $x \in \mathbb{R}$, and $f(x) = \left((y+5)^{1/3}\right)^3 - 5 = y$. Since $f : \mathbb{R} \to \mathbb{R}$ is a well-defined function, Lemma 15.1 implies that f is onto.

Finally, we claim that $f^{-1}:\mathbb{R}\to\mathbb{R}$ is defined by $f^{-1}(y)=(y+5)^{1/3}$. So let $g(y)=(y+5)^{1/3}$. If we can show that $g(y)=x$ if and only if $f(x)=y$, then g satisfies the definition of inverse function and we can conclude that $g=f^{-1}$. So let $y\in\mathbb{R}$. Then $x=g(y)=(y+5)^{1/3}$ if and only if $x^3=y+5$. The last equality holds if and only if $y=x^3-5=f(x)$. Therefore $x=g(y)$ if and only if $y=f(x)$ and $f^{-1}(y)=g(y)=(y+5)^{1/3}$. □

We'll have another way to show that a function g is the inverse of a function f as soon as we prove Theorem 16.4 below. That theorem will make our life easier. However, until we prove that theorem, we'll have to resort to the definition because it's all we have.

The example above brings up an important point. Students often confuse the notation f^{-1} with $1/f$. In the example above $1/f$ would be the function defined for $x\neq 5^{1/3}$ by $1/(x^3-5)$, while we have seen that f^{-1} is defined on all of $\mathbb{R}$ by $f^{-1}(x)=(x+5)^{1/3}$. These two functions are really quite different! In fact, f^{-1} and $1/f$ are rarely the same. (See Project 29.7 for more information.)

You may also be wondering whether the original function will always be defined in terms of x and the inverse function in terms of y. The answer is: Of course not. For one thing, there is nothing "original" about the first function; we might just as well have started with the "inverse" function in the example above (see Problem 16.9). What we decide to call the variable is irrelevant: For example, the functions $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=(x+5)^{1/3}$ and $h(y)=(y+5)^{1/3}$ are the same function.

Example 16.2. Let $f:\mathbb{R}\setminus\{3\}\to\mathbb{R}\setminus\{1\}$ be defined by $f(x)=(x+1)/(x-3)$. (You should graph the function f, and compare it to our graph in Figure 16.4.) We'll find $f^{-1}:\mathbb{R}\setminus\{1\}\to\mathbb{R}\setminus\{3\}$.

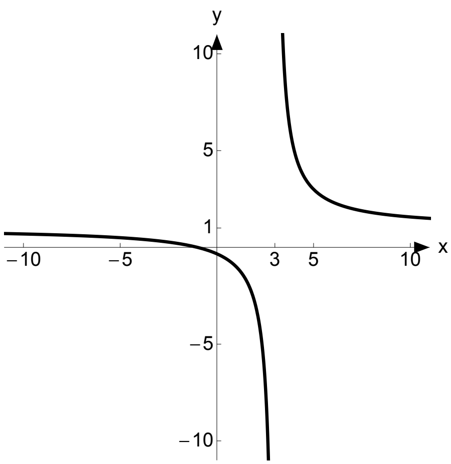

Fig. 16.4 $f(x)=(x+1)/(x-3)$

Before you read our solution, use Examples 14.8 and 15.3 to check that this function is bijective and that the domain and range are also appropriate for f^{-1}.

To find an expression for f^{-1}, let $y \in \mathbb{R} \setminus \{1\}$. Then there exists (exactly one) $x \in \mathbb{R} \setminus \{3\}$ such that $f(x) = y$. Further, $f(x) = y$ if and only if $(x+1)/(x-3) = y$, and this happens if and only if $x = (3y+1)/(y-1)$.

Therefore $f^{-1}(y) = x = (3y+1)/(y-1)$. ○

In our examples and exercises thus far, you probably noticed us repeating the same steps: We first check that $f : A \to B$ is bijective. If it is, then we know f^{-1} exists. To find f^{-1}, we choose $y \in B$ and solve for the unique x such that $f(x) = y$. Then, by definition, $f^{-1}(y) = x$, and we are done.

Now you should be ready to do a more challenging example as an exercise.

Exercise 16.3. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}.$$

We showed in Example 15.7 that this function is bijective. Find a formula for f^{-1}. (You might find it helpful to reexamine the graph of f in Figure 15.3.) ○

If A is a set, one very important function mapping A to itself is the identity function. So, the **identity function** i_A is the function $i_A : A \to A$ defined by $i_A(x) = x$ for all $x \in A$. You should check that i_A is well-defined, is both one-to-one and onto, and is its very own inverse. In addition, this function is easy to use. For example, if A and B are sets and f is a function such that $f : A \to B$, then $f \circ i_A = f$, while $i_B \circ f = f$.

Theorem 16.4. *Let $f : A \to B$ be a bijective function. Then*

(i) $f \circ f^{-1} = i_B$; that is, $(f \circ f^{-1})(y) = y$ for all $y \in B$.
(ii) $f^{-1} \circ f = i_A$; that is, $(f^{-1} \circ f)(x) = x$ for all $x \in A$.
(iii) f^{-1} is a bijective function.
(iv) If $g : B \to A$ is a function satisfying $f \circ g = i_B$ or $g \circ f = i_A$, then $g = f^{-1}$.

The last part of this theorem says that if we know that our function has an inverse, then f^{-1} is the one and only function satisfying the identities in (iv). This can come in quite handy. Consider the following.

Sometimes, as in Exercise 16.3, it is difficult to compute f^{-1}. In these cases it is nice to check your answer. Theorem 16.4 tells you one way to do so: Suppose you know that f is bijective, and you are claiming that g is the inverse. If you find that $g \circ f = i_A$ or $f \circ g = i_B$, you know you have the right answer!

We also remark here that (i) and (ii) above really follow from the definition of inverse function: $f(x) = y$ if and only if $f^{-1}(y) = x$.

Proof. (i) If $y \in B$, let $z = f^{-1}(y)$. By definition $f^{-1}(y) = z$ if and only if $f(z) = y$. Therefore

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(z) = y.$$

(ii) If $x \in A$, let $z = f(x)$. By definition $f(x) = z$ if and only if $f^{-1}(z) = x$. Therefore,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(z) = x.$$

(iii) We leave this for you to do in Problem 16.9.

(iv) We note first that $\mathrm{dom}(g) = \mathrm{dom}(f^{-1}) = B$.

First suppose that $g \circ f = i_A$. Then, using the associative property of composition and (i) above, we have

$$f^{-1} = i_A \circ f^{-1} = (g \circ f) \circ f^{-1} = g \circ (f \circ f^{-1}) = g \circ i_B = g.$$

In exactly the same way (except we use (ii) in place of (i)), we can show that if $f \circ g = i_B$, then $g = f^{-1}$. □

Before applying Theorem 16.4 make sure that you check that f really is bijective. It is one of the hypotheses, after all!

Exercise 16.5. For each of the functions and their inverses in Example 16.1, Example 16.2, and Exercise 16.3 check that $f^{-1} \circ f = i_{\mathrm{dom}(f)}$ and $f \circ f^{-1} = i_{\mathrm{ran}(f)}$. ○

The theorem above includes the basic facts about inverses. But there are more theorems that will be useful as we move along.

Theorem 16.6. *Let $f : A \to B$ and $g : B \to C$ be bijective functions. Then $g \circ f$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Before we begin, let's make sure we understand the function and think about what we need to prove. We note that $g \circ f : A \to C$. To show that the composition is bijective, we must show that it is one-to-one and onto. To find the inverse, if we have a guess for what it should be, we can use Theorem 16.4 (iv), which (in this case) says that if $g \circ f$ is bijective and h is a function such that $(h \circ (g \circ f))(x) = x$ for all x in A or $((g \circ f) \circ h)(y) = y$ for all y in B, then h must be $(g \circ f)^{-1}$. So all we need to do is think of a good candidate for h (the object we want to show is the inverse) and show it works. But the statement of the theorem gives us a candidate for h. Now that we have the plan, we can try to carry it out.

Proof. First we'll show that the composition $g \circ f$ is one-to-one. So let $x_1, x_2 \in A$. If $(g \circ f)(x_1) = (g \circ f)(x_2)$, then $g(f(x_1)) = g(f(x_2))$. Now since g is one-to-one, $f(x_1) = f(x_2)$. But f is also one-to-one, and therefore $x_1 = x_2$, as desired.

To see that the composition is onto, let $z \in C$. Since g is onto, there exists a $y \in B$ such that $g(y) = z$. Since $y \in B$ and f is onto, there exists $x \in A$ such that $f(x) = y$. Therefore, $x \in A$ and

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$

Since $g \circ f : A \to C$, we conclude that $g \circ f$ is onto.

Now we will show that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$ by applying (iv) of Theorem 16.4 to $g \circ f$. We just showed that $g \circ f$ is bijective. Now we check the hypotheses of part (iv) of the theorem. First, note that the domain is correct; that is, $f^{-1} \circ g^{-1} : C \to A$.

We now show that $((f^{-1} \circ g^{-1}) \circ (g \circ f))(z) = z$ for all $z \in A$. By (ii) of Theorem 16.4 applied twice (as well as the associative property of composition), for every $z \in A$ we have

$$((f^{-1} \circ g^{-1}) \circ (g \circ f))(z) = f^{-1}(g^{-1}(g(f(z)))) = f^{-1}(f(z)) = z.$$

Using (iv) of Theorem 16.4 we may conclude that $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. □

Remember that Pólya suggests that after solving a problem, we should look back and see whether we can use the result or the method to solve a different problem. Here's a good chance to try that out: Use the proof above to establish the following.

Theorem 16.7. *Let $f : A \to B$ and $g : B \to C$ be functions.*

(i) If f and g are one-to-one, then $g \circ f$ is one-to-one.
(ii) If f and g are onto, then $g \circ f$ is onto.

The converses of the two statements in the theorem above are not true. However, two corresponding weaker statements can be made. In addition, part (iii) of Theorem 16.8 provides a useful characterization of the inverse.

Theorem 16.8. *Let $f : A \to B$ and $g : B \to C$ be functions.*

(i) If $g \circ f$ is onto, then g is onto.
(ii) If $g \circ f$ is one-to-one, then f is one-to-one.
(iii) Suppose now that $f : A \to B$ and $g : B \to A$. If $f \circ g = i_B$ and $g \circ f = i_A$, then $g = f^{-1}$.

How does (iii) in Theorem 16.8 differ from part (iv) in Theorem 16.4? Well, both are implications, but the antecedent in one is a disjunction and the antecedent in the other is a conjunction. In addition, in Theorem 16.8, we do not assume that f or g is bijective. You will need to show that the conditions in (iii) imply that f and g are, in fact, bijective. If you already know that one of your functions f and g is bijective, Theorem 16.4 will usually be easier to use than Theorem 16.8.

Exercise 16.9. Prove Theorem 16.8. ○

Definitions

Definition 16.1. Given functions $f : A \to B$ and $g : C \to D$ with $\mathrm{ran}(f) \subseteq C$, we define the **composition** of f and g as the function $g \circ f : A \to D$, where $(g \circ f)(x) = g(f(x))$. The composition is also called the **composite function** from A to D.

Definition 16.2. Let $f : A \to B$ be a bijective function. The **inverse** of f is the function $f^{-1} : B \to A$ defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

Definition 16.3. The **identity function** on a set A is the function $i_A : A \to A$, defined by $i_A(x) = x$.

Solutions to Exercises

Solution (16.3). This problem only asks for a formula for f^{-1}, which we will give here. You need to think about how we obtained this formula. You should check that $f^{-1} : \mathbb{N} \to \mathbb{Z}$ defined by

$$f^{-1}(m) = \begin{cases} m/2 & \text{if } m \text{ is even} \\ -(m+1)/2 & \text{if } m \text{ is odd} \end{cases}$$

really is the inverse of f.

Solution (16.5). For the function in Example 16.1 we calculate

$$(f^{-1} \circ f)(x) = f^{-1}(x^3 - 5) = ((x^3 - 5) + 5)^{1/3} = \left(x^3\right)^{1/3} = x = i_{\mathbb{R}}(x)$$

and

$$(f \circ f^{-1})(x) = f((x+5)^{1/3}) = \left((x+5)^{1/3}\right)^3 - 5 = x + 5 - 5 = x = i_{\mathbb{R}}(x).$$

For the function in Example 16.2 the calculations are

$$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{x+1}{x-3}\right) = \frac{3\frac{x+1}{x-3} + 1}{\frac{x+1}{x-3} - 1} = \frac{4x}{4} = x = i_{\mathbb{R}\setminus\{3\}}(x)$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{3x+1}{x-1}\right) = \frac{\frac{3x+1}{x-1} + 1}{\frac{3x+1}{x-1} - 3} = \frac{4x}{4} = x = i_{\mathbb{R}\setminus\{1\}}(x).$$

Finally, for the function in Exercise 16.3 the calculations are as follows.

First, for $f^{-1} \circ f$, we use the two cases that f forces upon us. So suppose that $n \in \mathbb{Z}$ and $n \geq 0$. Then

$$(f^{-1} \circ f)(n) = f^{-1}(2n) = (2n)/2 = n,$$

since $2n$ is even.

Now suppose that $n < 0$. Then

$$(f^{-1} \circ f)(n) = f^{-1}(-2n-1) = -((-2n-1)+1)/2 = n,$$

since $m = -2n-1$ is odd. Therefore $(f^{-1} \circ f)(n) = n = i_{\mathbb{Z}}(n)$, completing the first part of the problem.

Now, for $f \circ f^{-1}$, we use the two cases f^{-1} forces upon us. So suppose that $m \in \mathbb{N}$ is even. Then

$$(f \circ f^{-1})(m) = f(m/2) = 2(m/2) = m,$$

since $m/2 \geq 0$. If m is odd, then

$$(f \circ f^{-1})(m) = f(-(m+1)/2) = -2(-(m+1)/2) - 1 = m,$$

since $-(m+1)/2 < 0$. Therefore $(f \circ f^{-1})(m) = m = i_{\mathbb{N}}(m)$, as required.

Solution (16.9). (i) If $c \in C$, then the fact that $g \circ f$ is onto implies that there exists $a \in A$ such that $(g \circ f)(a) = c$. Therefore $g(f(a)) = c$. Since $f(a) \in B$, we have shown that there is an element $b = f(a)$ in B such that $g(b) = c$. Since $g : B \to C$, we conclude that g is onto.

(ii) If a_1 and a_2 are in A and $f(a_1) = f(a_2)$, then $g(f(a_1)) = g(f(a_2))$. Therefore $(g \circ f)(a_1) = (g \circ f)(a_2)$. Since $g \circ f$ is one-to-one, $a_1 = a_2$ and f is one-to-one, as desired.

(iii) Since $g \circ f$ is one-to-one, (ii) implies that f is one-to-one. Since $f \circ g$ is onto, (i) implies that f is onto. Thus f is bijective. Consequently, (iii) follows from Theorem 16.4 (iv). □

Problems

Problem 16.1. Find the compositions $f \circ g$ and $g \circ f$ assuming the domain of each is the largest set of real numbers for which the functions f, g, $f \circ g$, and $g \circ f$ make sense. In your solution to each of the following, give the compositions and the corresponding domain and range:

(a) $f(x) = 1/(1+x)$, $g(x) = x^2$;
(b) $f(x) = x^2$, $g(x) = \sqrt{x}$ (simplify this one);
(c) $f(x) = 1/x$, $g(x) = x^2 + 1$;
(d) $f(x) = |x|$, $g(x) = f(x)$.

Problem 16.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 4$. Use Theorem 16.8 to show that if $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = (x-4)^{1/3}$, then $g = f^{-1}$.

Problem 16.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x+y$. Prove that there is no function $g : \mathbb{R} \to \mathbb{R}^2$ such that $g \circ f = i_{\mathbb{R}^2}$.

Problem 16.4. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x,0)$. Show that there is no function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ g = i_{\mathbb{R}^2}$.

Problem 16.5. Functions $f : A \to B$ are given below. For each of them find the range of f. Further, if possible, find $f^{-1} : B \to A$. Rigorous proofs are not required, but you should provide explanations for each of your statements.

(a) The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is defined by $f(x) = 1/x$.
(b) The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = x+y$.
(c) The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x,y) = (y,x)$.
(d) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sin x$.
(e) The function $f : \{x \in \mathbb{R} : -\pi/2 < x < \pi/2\} \to \mathbb{R}$ is defined by $f(x) = \tan x$.

Problem 16.6. The functions $f : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{1\}$ and $g : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-2\}$ defined by

$$f(x) = \frac{x-3}{x+2} \text{ and } g(x) = \frac{3+2x}{1-x}$$

are well-defined (you need not check this).

(a) Calculate $f \circ g$ and $g \circ f$.
(b) What can you conclude about f and g from your result in part (a)? If you use a theorem, give a reference.

Problem 16.7. (a) If possible, find examples of functions $f : A \to B$ and $g : B \to A$ such that $f \circ g = i_B$ when:

(i) $A = \{1,2,3\}$, $B = \{4,5\}$;
(ii) $A = \{1,2\}$, $B = \{4,5\}$;
(iii) $A = \{1,2,3\}$, $B = \{4,5,6,7\}$.

Draw diagrams of A and B in each case above.

(b) Give an example of sets A and B, and functions $f : A \to B$ and $g : B \to A$ such that $f \circ g = i_B$, but $g \circ f \neq i_A$. (Thus the existence of a function g such that $f \circ g = i_B$ is *not* enough to conclude that f has an inverse!) Why doesn't this contradict Theorem 16.4, part (iv)?
(c) Give an example of sets A and B, and functions $f : A \to B$ and $g : B \to A$ such that $g \circ f = i_A$, but $f \circ g \neq i_B$. (Thus the existence of a function g such that $g \circ f = i_A$ is not enough to conclude that f has an inverse!) Why doesn't this contradict Theorem 16.4, part (iv)?
(d) Let A and B be two sets, and let $f : A \to B$ be a function. Assume further that there exists a function $g : B \to A$ such that $f \circ g = i_B$. Must f be one-to-one? onto?

(e) Looking over your work above, what should be your strategy in solving a question like (d) above? Whatever you decide, use it to solve the following: Let f and g be as above and suppose $g \circ f = i_A$. Must f be one-to-one? onto?

Problem 16.8. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x, x+y)$. Show that f has an inverse and find the inverse function.

Problem# 16.9. Let $f : A \to B$ be a bijective function. Prove part (*iii*) of Theorem 16.4 and show that $(f^{-1})^{-1} = f$.

The following theorem is quite often useful. It provides an alternate way to prove that a function is bijective.

Theorem 16.10. *If $f : A \to B$, then the following are equivalent.*

1. *The function f is a bijection.*
2. *The function f has an inverse.*
3. *There exists a function $h : B \to A$ such that $h \circ f = i_A$ and $f \circ h = i_B$.*

Problem 16.10. Prove Theorem 16.10. (One way to do this is to prove that 1 implies 2; 2 implies 3; and 3 implies 1.)

Problem 16.11. Prove that $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln x$ is bijective. (You may use properties of the logarithm function and exponential function.)

Problem 16.12. (a) Give an example of a function $f : A \to A$ such that $f \neq i_A$, but $f \circ f = i_A$. Must such a function f be one-to-one? onto?

(b) Give an example of a nonzero function $f : \mathbb{R} \to \mathbb{R}$ such that $(f \circ f)(x) = 0$ for all $x \in \mathbb{R}$. Can such a function be one-to-one? onto?

Problem 16.13. Let $f : A \to A$ be a function. Suppose that $f \circ f : A \to A$ is a bijection. Must such a function f be a bijection? (Prove this or give a counterexample.)

Problem 16.14. Suppose that $f : A \to B$ and g_1 and g_2 are functions from B to A such that $f \circ g_1 = f \circ g_2$. Show that if f is bijective, then $g_1 = g_2$. If $g_1 \circ f = g_2 \circ f$ and f is bijective, must $g_1 = g_2$?

Problem 16.15. Let $f : A \to A$ be a function. Define a relation on A by $a \sim b$ if and only if $f(a) = f(b)$. Is this an equivalence relation? If f is one-to-one, what is the equivalence class of a point $a \in A$?

Problem 16.16. Let $f : A \to A$ be a function. Define a relation on A by $a \sim b$ if and only if $f(a) = b$. Is this an equivalence relation for an arbitrary function f? If not, is there a function for which it is an equivalence relation?

Problem# 16.17. Let A, B, C, and D be nonempty sets. Let $f : A \to B$ and $g : C \to D$ be functions.

(a) Prove that if f and g are one-to-one, then $H : A \times C \to B \times D$ defined by

$$H(a,c) = (f(a), g(c))$$

is a one-to-one function. (Check that it is one-to-one and a function.)

(b) Prove that if f and g are onto, then H is also onto.

Problem# 16.18. Let A, B, C, and D be nonempty sets. Let $f : A \to B$ and $g : C \to D$ be functions. Consider H defined on $A \cup C$ by

$$H(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C \end{cases}.$$

Show that there exist sets A, B, C, and D for which H is *not* a function, but there also exist such sets for which H is a function. What conditions can we place on A and C to ensure that H is a function?

Problem 16.19. Let $a \in \mathbb{R}$ with $|a| < 1$. Define f on the set $\{x \in \mathbb{R} : |x| < 1\}$ by

$$f(x) = \frac{a - x}{1 - ax}.$$

(a) Show that the range of f is contained in the set $\{x \in \mathbb{R} : |x| < 1\}$.
(b) Does f map onto the set $\{x \in \mathbb{R} : |x| < 1\}$?
(c) Prove that f is one-to-one.
(d) Compute $f \circ f$.
(e) Find f^{-1}.

Problem 16.20. Let $\mathbb{R}[x]$ denote the set of all polynomials with real coefficients. (See Problem 10.13.)

(a) Define a function f on $\mathbb{R}[x]$ by $f(p) = p(0)$. What is the range of f? Is f one-to-one?
(b) Define a function g on the nonzero polynomials in $\mathbb{R}[x]$ by $g(p) =$ degree of p. Is g a function? Is it one-to-one? What is the range of g?
(c) Recall that a value z is a root of a polynomial p if $p(z) = 0$. Define F on $\mathbb{R}[x]$ by $F(p) =$ a root of p. Is F a function? Why or why not?
(d) Define h on $\mathbb{R}[x]$ by $(h(p))(x) = xp(x)$. Is h a function? If so, is it one-to-one? What is the range of h?
(e) Define $k : \mathbb{R}[x] \to \mathbb{R}[x]$ by $k(p) = p \circ p$. Show that k is neither one-to-one nor onto.

Problem 16.21. For each part give examples of functions $f : A \to B$ and $g : B \to C$ satisfying the stated conditions.

(a) The composition $g \circ f$ is onto, but f is not onto.
(b) The composition $g \circ f$ is one-to-one, but g is not one-to-one.

Problem 16.22. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is onto and $f \circ f \circ f = f$. Prove that f is bijective.

Problem 16.23. Let A, B, C, and D be nonempty sets with $B \subseteq C$ and $D \subseteq A$. Suppose that both functions $f : A \to B$ and $g : C \to D$ are onto and $f \circ g \circ f = f$. (Note that the compositions $g \circ f$ and $f \circ g$ are both defined.)

(a) Show that $(f \circ g)|_B$ is one-to-one.
(b) Give an example to show that $(g \circ f)|_D$ is not necessarily one-to-one.