Chapter 2
Logically Speaking

> "I know what you're thinking about," said Tweedledum; "but it isn't so, nohow." "Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."—Lewis Carroll, [17, p. 139]

Suppose your friend tells you that Mr. Hamburger is German or Swiss. You happen to know that Mr. Hamburger is not Swiss. Using your powers of reasoning, you decide that Mr. Hamburger is German. Note that this argument can be generalized, because it doesn't really depend on Mr. Hamburger being Swiss or German. If your friend said that "A or B is true" and you happened to know that "B is not true," you would conclude that "A is true." This is an example of a valid argument. Now suppose your friend tells you that Mr. French eats only pickles on Wednesday, and only chocolate on Monday. You know that Mr. French is eating chocolate that day. Now what can you say? While you may conclude that Mr. French has odd eating habits, you would not have used a logically valid argument to do so. In this example, there is really only one thing you can conclude. We'll return to this at the end of this chapter.

In order to understand an argument, we must be able to read and comprehend the sentences that compose it. We need to be able to tell whether the sentences in our argument are true or false, and whether they follow logically from the previous ones. So now for a definition. A statement is a sentence that is true or false, but not both. "Two is not a prime number" is an example of a (false) statement. "Do you love me?" is not a statement. Below are some examples and some nonexamples of statements. These will be your first examples of nonexamples.

Exercise 2.1. Which of the sentences below are statements and which are not?

(a) It is raining outside.
(b) The professor of this class is a woman.
(c) Two plus two is five.
(d) $X+6=0$.
(e) Seven is a prime number.
(f) All odd numbers are prime.

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(g) This sentence is false. ○

Because English usage and mathematical usage may differ slightly, we must be certain that we understand our statements before we construct arguments. We now carefully study the truth or falsity of statements. Our treatment is brief. (See [69] for a more detailed study of mathematical logic.)

The rules of logic that we present in this chapter should work for all statements, and not just particular ones. For this reason, we introduce letters such as P, Q, R, or S to represent statements. Thus P will have two possible truth values: true, denoted T, or false, denoted F. We can negate P or combine it with Q by saying things like:

Not P.
P and Q .
P or Q.
If P, then Q.
P if and only if Q.

Such symbolic sentences will be called statement forms. A precise definition of statement form will be given once we have precise definitions of the connectives "not," "and," "or," "if ..., then ...," and "if and only if."

In the English language we might say

It is raining.
It is not raining.
If it is raining, the sky is gray.
It is raining or it is snowing.
It is cold and it is snowing.
It is snowing if and only if it is cold.

Let's start with the simplest case. Suppose your teacher says, "This book has a blue cover." Taking a quick glance at the cover, you can decide on the truth value of that statement; namely, that it is false. In order to have a true statement, you could say, "This book does not have a blue cover." If we have a statement form P, the negation of P is the statement form "not P." Under what circumstances should the negation of P be true or false? We will always use the notation $\neg P$ for "not P." If P is true, then $\neg P$ should be false. If P is false, then $\neg P$ should be true. We can summarize all the possibilities in a truth table as follows:

P	**¬P**
T	F
F	T

What about combining two statement forms, P and Q, into one statement form as "P or Q"? In this sentence, it is particularly important to distinguish between mathematical usage of the word "or" and everyday speech. For example, if we say, "You can have cake or ice cream," it could be that you can have both. If we say, "The door is open or closed," it cannot be that the door is both open and closed. English statements involving the word "or" are often ambiguous; in mathematics, ambiguity is generally frowned upon. The statement form "P or Q" is called a disjunction and is denoted $P \vee Q$. In mathematics, a disjunction is true when P alone is true, Q alone is true, or both P and Q are true. So in mathematics, you can always have your cake and ice cream.

Exercise 2.2. Complete the truth table for $P \vee Q$.

P	**Q**	**P ∨ Q**
T	*T*	
T	*F*	
F	*T*	
F	*F*	

○

The statement form "P and Q" is called a conjunction and is denoted $P \wedge Q$. We will have you fill in the truth table for "P and Q" below. It should be clear that this will be true when both P and Q are true, and false otherwise.

Exercise 2.3. Complete this truth table.

P	**Q**	**P ∧ Q**
T	*T*	
T	*F*	
F	*T*	
F	*F*	

○

Now consider the statement form "If P, then Q." This statement form is called an implication and is often stated as "P implies Q" and written $P \to Q$. (Note that though English usage of the word "implies" may suggest a relationship between P and Q, our analysis of truth values has assumed no connection at all between P and Q.) There are equivalent ways of stating an implication, and some will require careful thinking on the reader's part. Remember as you read on that "If P, then Q" may also be stated as

Q if P.
P is sufficient for Q (meaning P is enough to make Q happen).
Q is necessary for P (if P happened, then Q must have happened).
P only if Q (same as above; if P happened, then Q must have happened).
Q whenever P.

The statement form P in each of these formulations is called the antecedent, and Q is called the conclusion. Under what conditions is an implication true? false? Let's begin with an example you are all familiar with. Suppose we say to our son,

"If you clean your room, then you can go to Henry's house."

Under what conditions would he feel that we had lied? In the example, the antecedent, P, is "you clean your room." The conclusion, Q, is "you can go to Henry's house." Well, if our son cleans his room and we let him go to Henry's, everybody is happy. That implication should be true. So, if P is true and Q is true, the whole statement should be true. Also, it should be as clear to you as it will be to our son, that if he cleans his room and we do not let him go to Henry's, we lied. So, if P is true, and Q is false, the implication should be false. Now what if he doesn't clean his room? We never discussed this possibility. So, no matter what we decide here, we have not lied. In this situation, the statement is not false; hence we consider it to be true. So, if P is false, no matter what the truth value is of the conclusion, we will consider the implication to be true.

Summarizing this discussion, the only way that the implication "If P, then Q" can be false is if P is true and Q is false. In the exercise below you will sum up this discussion in the form of a truth table.

Exercise 2.4. Complete this truth table.

P	**Q**	$\mathbf{P \rightarrow Q}$
T	T	
T	F	
F	T	
F	F	

○

It is often helpful to rephrase a statement, making sure that you maintain the same true and false values. The statement form "P if and only if Q" is called an equivalence, and we will write this as $P \leftrightarrow Q$. This is the same statement form as *"(P only if Q) and (P if Q)."* In view of the discussion above, we see that this is also $(P \rightarrow Q) \wedge (Q \rightarrow P)$. Thus the truth table for the equivalence is

P	**Q**	$\mathbf{P \rightarrow Q}$	$\mathbf{Q \rightarrow P}$	$\mathbf{P \leftrightarrow Q}$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Look down the final column and you'll see that the equivalence is true precisely when P and Q are both true or both false.

The statement form "If P, then Q" is also written as $P \Rightarrow Q$, and "P if and only if Q" might be written as $P \Leftrightarrow Q$ or "P iff Q."

Having studied the connectives, we are ready for our definition of a statement form. A statement form is a letter representing an unspecified statement or an expression built from such letters using connectives. Statement forms can be quite complicated. Assigning truth values to them can be done in a step-by-step fashion. The exercise below illustrates this.

Exercise 2.5. Find the truth table for the statement form

$$(P \to (\neg Q \vee R)) \wedge (R \vee Q)\,.$$

To solve this you must break the complicated form $(P \to (\neg Q \vee R)) \wedge (R \vee Q)$ into simpler parts. Once you have done this, you should find the truth value of each of the parts using the truth values of P, Q, and R. ○

Now consider the two statement forms $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$. In the next exercise, you will find the truth table for each of these expressions and compare them.

Exercise 2.6. Write out the truth tables for $\neg(P \vee Q)$, $\neg P \wedge \neg Q$, and $(\neg(P \vee Q)) \leftrightarrow (\neg P \wedge \neg Q)$. What can you conclude? ○

A statement form for which the final column in the truth table consists of all T's is called a tautology. A statement form for which the final column is all F's is called a contradiction. Two *statement forms*, P and Q, are said to be (logically) equivalent if $P \leftrightarrow Q$ is a tautology, and two *statements* are equivalent if they can be obtained from two equivalent statement forms by consistently replacing the letters by English statements.

In view of Exercise 2.6, we see that $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are equivalent statement forms. Thus the statement "It is not the case that Rachel or Leah won the race" is equivalent to "Rachel did not win the race and Leah did not win the race." (Why?)

In Exercise 2.6 you noticed that the two statement forms $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ have the same truth table, and $(\neg(P \vee Q)) \leftrightarrow (\neg P \wedge \neg Q)$ is a tautology. This isn't something that happens in this one particular example. Whenever we notice that a statement is always true, we can state that fact as a theorem. Of course, we will need to give a conclusive argument for the statement's truth, and this is called a proof of the theorem. We'll present a theorem and a proof below, but remember: The statement forms, P and Q, might very well be complicated constructions with many connectives. For instance, P could be $R \to (S \vee (\neg T \wedge R))$. In fact, the theorem we present below is most interesting when P is complicated!

Theorem 2.7. *Two statement forms P and Q are equivalent if and only if they have the same truth table.*

Proof. Consider the truth table for the equivalence $P \leftrightarrow Q$:

P	**Q**	**P ↔ Q**
T	T	T
T	F	F
F	T	F
F	F	T

We know that $P \leftrightarrow Q$ is a tautology if and only if $P \leftrightarrow Q$ has the truth value T (in the table above, this is row 1 and row 4). Finally, $P \leftrightarrow Q$ has the truth value T if and only if P and Q have the same truth value. Since P and Q are equivalent if and only if $P \leftrightarrow Q$ is a tautology, this establishes the theorem. □

While it is very important to be able to restate something in an equivalent form, it is equally important that you be able to negate a statement. Some useful negations appear in the exercises and problems. The negation of an implication is particularly important in mathematics. If you think about integers and the sentence "If x is prime, then x is odd or $x = 2$," you can see that even a relatively simple implication might be difficult to negate. Let's begin with something simpler.

Exercise 2.8. Construct the truth table for $P \to Q$, and the truth table for $\neg P \vee Q$. What do you notice? Now construct a truth table for $(P \to Q) \leftrightarrow (\neg P \vee Q)$. What conclusion can you make? Finally, find an equivalent way to write $\neg(P \to Q)$. ○

If all went well, you noticed that $P \to Q$ is equivalent to $\neg P \vee Q$, and therefore the negation of "If P, then Q" is "P and not Q." Let's return to

$$\text{“If } \underbrace{x \text{ is prime}}_{P}\text{, then } \underbrace{x \text{ is odd or } x = 2}_{Q}\text{.”}$$

Negating this leads to

$$\text{“}\underbrace{x \text{ is prime}}_{P} \text{ and it is } \underbrace{\text{not the case that } x \text{ is odd or } x = 2}_{\neg Q}\text{.”}$$

While this is the negation, it isn't really as helpful as it might be. So we now negate the disjunction "x is odd or $x = 2$" and combine it with our previous work to obtain

$$\text{“}x \text{ is prime and } x \text{ is not odd and } x \neq 2.\text{”}$$

Refining this further, we would probably say something like "x is prime, even, and not equal to two."

The negation of an implication is something you should learn well now because it arises frequently. In the theorem below, we summarize the five most important equivalences that we have covered so far. The first two are often referred to as DeMorgan's laws.

Theorem 2.9. *Let P and Q denote statement forms. The following are tautologies:*

(DeMorgan's laws) $\neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q)$;
$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$;

(Implication and its negation) $(P \to Q) \leftrightarrow (\neg P \vee Q)$;
$\neg(P \to Q) \leftrightarrow (P \wedge \neg Q)$;

(Double negation) $\neg(\neg P) \leftrightarrow P$.

Proof. In Exercise 2.6 we showed that the first tautology holds. You will establish the second in Problem 2.3. The third and fourth tautologies were the content of Exercise 2.8. Finally, you will establish the last tautology when you work Problem 2.2. □

Here are some examples for you to try.

Exercise 2.10. Negate the following. It's interesting to note that you can negate a statement even if you don't understand what it says. It is easier to get it right, though, if you understand the statement.

(a) If I go to the party, then he is there.
(b) If x is even, then x is divisible by 2.
(c) If a function is differentiable, then it is continuous.
(d) If x is a natural number, then x is even or x is odd. ○

Exercise 2.11. Which of the following are equivalent to each other? All the answers have appeared in this chapter.

$$P \to Q, \neg(P \vee Q), \neg(P \wedge Q), P \wedge \neg Q, \neg(P \to Q)\ ,$$
$$P \vee \neg Q, \neg P \vee \neg Q, \neg P \wedge \neg Q, \neg P \vee Q.$$ ○

So let's apply what we have learned in this chapter to Mr. French, who eats only pickles on Wednesday and only chocolate on Monday. One statement is that "if it is Wednesday, then Mr. French eats only pickles." We let W represent the statement "it is Wednesday," and P the statement "Mr. French eats only pickles." Thus, we know that $W \to P$ is true. (If you thought we should have said $W \wedge P$ is true, note that we do not know that the statement W is true, so we must use the implication here.) The second is "if it is Monday, then Mr. French eats only chocolate." Letting M denote "it is Monday" and C the statement that "Mr. French eats only chocolate" we may write what we are given as $M \to C$. Finally we are told that "Mr. French is eating chocolate." From this we can conclude that $\neg P$ is true. Let's put this together.

1. $W \to P$,
2. $M \to C$, and
3. $\neg P$.

Now, it's fairly clear that the second statement is irrelevant. So let us look at the truth tables for the first and third statements (for convenience, we combine the two tables):

W	**P**	**W → P**	**¬P**
T	*T*	*T*	*F*
T	*F*	*F*	*T*
F	*T*	*T*	*F*
F	**F**	**T**	**T**

We know that both $W \to P$ and $\neg P$ are true, and from our truth table we see that there is only one time that this happens: when both W and P are false. So there you have it. All we can conclude is that it is not Wednesday.

People differ in their approaches to problems. In the example above, you might have found it easier not to rewrite the problem. That's fine. On the other hand, when a problem starts to confuse you, looking at it as we have here will often help you figure out how to attack a problem.

Solutions to Exercises

Solution (2.1). All sentences are statements except (d) and (g).

Part (d) is not a statement because its truth depends on X, and X is a variable. So the sentence is sometimes true and sometimes false.

Part (g) is tricky. Suppose it were a statement. Then it would have to be true or false, but not both. Suppose “This sentence is false” were true. Then it would have to be false, but it cannot be both true and false. From this we conclude that the sentence has to be false. But reading the sentence tells us that if it is false, it must again be true as well. We conclude that it cannot be a statement, because we cannot assign a unique truth value to it.

Solution (2.2). The truth table for $P \vee Q$ is

P	**Q**	**P ∨ Q**
T	T	T
T	F	T
F	T	T
F	F	F

Solution (2.3). The truth table for $P \wedge Q$ is

P	**Q**	**P ∧ Q**
T	T	T
T	F	F
F	T	F
F	F	F

Solution (2.4). The truth table for $P \to Q$ is

P	**Q**	**P → Q**
T	T	T
T	F	F
F	T	T
F	F	T

This is the same as the truth table for $\neg P \vee Q$.

Solution (2.5). We will break the problem into parts in such a way that at each step we apply only one additional connective (except for the negation, which is handled easily in general).

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{R}$	$\neg\mathbf{Q}\vee\mathbf{R}$	$\mathbf{P}\rightarrow(\neg\mathbf{Q}\vee\mathbf{R})$	$\mathbf{R}\vee\mathbf{Q}$	$(\mathbf{P}\rightarrow(\neg\mathbf{Q}\vee\mathbf{R}))\wedge(\mathbf{R}\vee\mathbf{Q})$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

Solution (2.6). We note that the last statement form is always true. (Note that the first and second statement forms have the same truth table.)

Solution (2.8). In the solution to Exercise 2.4, we noted that $P \rightarrow Q$ and $\neg P \vee Q$ are equivalent. Thus $\neg(P \rightarrow Q)$ is equivalent to $\neg(\neg P \vee Q)$, which is, as we have seen in Exercise 2.6, equivalent to $P \wedge \neg Q$. In words, the negation of "If P, then Q" is "P and not Q."

Solution (2.10). More than one answer is possible but they must be equivalent, of course.

(a) I go to the party and he is not there.
(b) One answer is: x is even and x is not divisible by 2.
(c) A function is differentiable and it is not continuous.
(d) One answer is: x is a natural number and x is not even and x is not odd. Equivalently, we could say: x is a natural number, and x is neither even nor odd.

Problems

Problem 2.1. In the following implications, identify the antecedent and the conclusion. (Don't worry about whether the implication is true or false.)

(a) If it is raining, I will stay home.
(b) I wake up if the baby cries.
(c) I wake up only if the fire alarm goes off.
(d) If x is odd, then x is prime.
(e) The number x is prime only if x is odd.
(f) You can come to the party only if you have an invitation.
(g) Whenever the bell rings, I leave the house.

Problem# 2.2. Construct a truth table for $\neg(\neg P)$. Is this what you expect? Why?

Problem# 2.3. Write out the truth tables for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$. What can you conclude?

Problem# 2.4. Find a statement form, S, equivalent to $\neg(P \vee Q)$ and show that it is logically equivalent by constructing the truth table for "S if and only if $\neg(P \vee Q)$" and showing that this statement form is a tautology.

Problem 2.5. Write out the truth table for the statement form $P \rightarrow \neg(Q \wedge \neg P)$. Is this statement form a tautology, a contradiction, or neither?

Problem 2.6. Write out the truth table for the statement form $(P \rightarrow (\neg R \vee Q)) \wedge R$. Is this statement form a tautology, a contradiction, or neither?

Problem 2.7. Negate the sentences below and express the answer in a sentence that is as simple as possible.

(a) I will do my homework and I will pass this class.
(b) Seven is an integer and seven is even.
(c) If T is continuous, then T is bounded.
(d) I can eat dinner or go to the show.
(e) If x is odd, then x is prime.
(f) The number x is prime only if x is odd.

Problem 2.8. Negate the following.

(a) If I am not home, then Sam will answer the phone and he will tell you how to reach me.
(b) If the stars are green or the white horse is shining, then the world is eleven feet wide.
(c) If we go swimming or bowling, then dinner will be late or Bob will bring veggie burgers.

Problem 2.9. Consider the statement form $(P \wedge \neg Q) \rightarrow R$.

(a) Find the truth table for this statement form.
(b) Construct a different statement form using P, Q, and R such that if you call your construction S, then $((P \wedge \neg Q) \rightarrow R) \leftrightarrow S$ is a tautology.

Problem 2.10. Consider the statement form $(P \vee \neg Q) \rightarrow (R \wedge Q)$.

(a) Write out the truth table for this form.
(b) Give a statement in English that is in this form.
(c) Write the negation of your English statement, and simplify the sentence as much as possible.

Problem 2.11. For each of the cases below, write a tautology using the given statement form.

For example, if you are given $P \vee \neg Q$, you might write $(P \vee \neg Q) \leftrightarrow (Q \rightarrow P)$.

(a) $\neg(\neg P)$;
(b) $\neg(P \vee Q)$;
(c) $\neg(P \wedge Q)$;

(d) $P \to Q$.

Problem 2.12. When we write, we should make certain that we say what we mean. If we write $P \wedge Q \vee R$, you may be confused, since we haven't said what to do when you are given a conjunction followed by a disjunction. Put parentheses in to create a statement form with the given truth table.

P	**Q**	**R**	**$P \wedge Q \vee R$**
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

Problem 2.13. For each of the cases below, write a contradiction using the given statement form.

For example, if you are given $\neg(\neg P)$, you might write $\neg(\neg P) \leftrightarrow \neg P$.

(a) $P \to Q$;
(b) $\neg(P \vee Q)$;
(c) $\neg P \vee \neg Q$;
(d) $P \leftrightarrow Q$.

Problem 2.14. Consider the following statement: If f is not continuous at 1 and -1, then the *group of invariants* is *an infinite cyclic group*, *a cyclic group of order* 2, or *the trivial group*.

You probably do not know what the words in italics mean, but you don't need to know in order to work this problem. Just think of them as describing different objects. This is an exercise in restating things you don't understand—something that might be useful in the future!

(a) Write the form of this statement using P, Q, R, S, and T. (It's possible to use fewer variables and still have a correct solution.) Say precisely what each of your letters represent.
(b) Write the negation of this statement in words. Use a phrase that is as simple and direct as possible.

Problem 2.15. Consider the statement "It snows or it is not sunny."

(a) Find a different statement that is equivalent to the given one.
(b) Find a different statement that is equivalent to the negation of the given one.

Problem 2.16. The following problem is well known. Many different versions of this problem appear in [101].

On a certain island, each inhabitant is a truth-teller or a liar (and not both, of course). A truth-teller always tells the truth and a liar always lies. Arnie and Barnie live on the island.

(a) Suppose Arnie says, "If I am a truth-teller, then each person living on this island is a truth-teller or a liar." Can you say whether Arnie is a truth-teller or liar? If so, which one is he?
(b) Suppose that Arnie had said, "If I am a truth-teller, then so is Barnie." Can you tell what Arnie and Barnie are? If so, what are they?

Problem 2.17. Write a truth table for $(P \wedge (P \to Q)) \to Q$. What can you conclude?

Problem 2.18. Police at *Small Unnamed University* have received a report that a student was skateboarding in the hall. They rush to the scene of the crime to determine who the guilty party is, and they are met by three students: Alan, Bernard, and Charlotte. When questioned, Alan says, "If Bernard did not do it, then it was Charlotte." Bernard says, "Alan and Charlotte did it together or Charlotte did it alone," and Charlotte says, "We all did it together."

(a) If the police know that exactly one person committed the crime, and exactly one person is lying, who is the guilty party?
(b) As it turns out, exactly one person committed the crime and all the students are lying. Who is the guilty party?

Problem 2.19. Show that if two statements, P and Q, are equivalent, then their negations, $\neg P$ and $\neg Q$, are also equivalent.

Problem 2.20. We know that each of the three statements below is correct. What can we conclude? Why?

1. If he was killed before noon, then his body temperature is at most 20°C.
2. His body temperature is at most 20°C and the police know who murdered him.
3. If the police know who murdered him, then he was killed before noon.

Problem 2.21. We have been avoiding the use of "either…or" in this text because the English language uses this construction ambiguously and the interpretation often depends on the context. Consider the following two statements:

1. "Either Peter or Paul ran in the race."
2. "You're either with us or against us."

(a) Write the (intended) statement form for each of the two statements.
(b) Give two more examples of the use of "either A or B" in the English language such that in the first of your examples both A and B occur, and in the second of your examples exactly one of A and B can occur.
(c) The symbol $\dot{\vee}$ is sometimes used for the "exclusive disjunction"; that is, a connective that yields a true statement if and only if exactly one of the two statements is true. Give an equivalent statement form of $P \dot{\vee} Q$ using only the connectives introduced earlier in this text.
(d) The negation "neither…nor" is interpreted exclusively as the negation of the regular disjunction in the English language. Give the truth table for the statement form "neither P nor Q" and give a meaningful English statement that is in this form.