Chapter 25
Metric Spaces

There are many ways to measure distance in the spaces in which we live and work. For example, if you want the shortest distance between two geographical places (the distance "as the crow flies"), you follow the line segment joining them. But in real life this isn't always possible. If you are driving your car through a city or across your campus, you need to go around solid objects and not through them. So how do we calculate distance in those cases? Measuring distance in a set X is a very small (but interesting) part of a branch of mathematics known as "point set topology," and we will look at it in detail in this chapter. We will now often refer to the elements of X as points.

So let's go back to the first time you measured distance. It was probably in $\mathbb{R}$, on a number line, and you learned that the distance between two points x and y was the absolute value of the difference of the two numbers. If we write $d(x,y) = |x-y|$, then d is a function and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. That's straightforward enough, but now we want to generalize our concept of distance. So let's turn to the essential properties of a distance function.

First, distance shouldn't be negative, so $d(x,y) \geq 0$ for two points x and y, and if the distance satisfies $d(x,y) = 0$, then you didn't move anywhere, so $x = y$. You also surely believe that distance from x to y should be the distance from y to x. And finally, in Theorem 5.8 (Problem 5.14) you learned the triangle inequality, which said that "if x and y are two real numbers, then $|x+y| \leq |x| + |y|$." In Problem 20.2, you showed how to switch the triangle inequality into a statement about distances. We recall the result of that problem here: For real numbers x, y, and z,

$$|x-y| \leq |x-z| + |z-y|.$$

In English, this means that our path will be shorter if we go directly from x to y as opposed to taking a detour through z, which is as it should be. So we would want our general distance function to satisfy something like this too; that is, in our new "d" notation we want $d(x,y) \leq d(x,z) + d(z,y)$ for arbitrary points x, y, and z. So now we will define something that acts like a distance on an arbitrary set X and does all the important things that a distance should do.

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Let X be a nonempty set. Then a **metric** on X is a function $d : X \times X \to \mathbb{R}$ satisfying (i)–(iv) below.

(i) (Nonnegativity) For all $x, y \in X$, the function d satisfies $d(x,y) \geq 0$.
(ii) (Definiteness) For all $x, y \in X$, the function d satisfies $d(x,y) = 0$ if and only if $x = y$.
(iii) (Symmetry) For all $x, y \in X$, the function d satisfies $d(x,y) = d(y,x)$.
(iv) (Triangle inequality) For all $x, y, z \in X$, the function d satisfies

$$d(x,y) \leq d(x,z) + d(z,y).$$

A metric is also called a distance function. A nonempty set X together with the metric d is called a **metric space** and is denoted by (X,d), or just X when it is clear which distance function we are using.

When you learn the definition, don't forget to say "Let X be a nonempty set. Then a metric on X is a function $d : X \times X \to \mathbb{R} \ldots$." These sentences tell us something about d, and cannot be omitted.

In the introduction, we showed that a metric can be defined on $\mathbb{R}$ by $d_u(x,y) = |x - y|$. Though we outlined how to show that d_u is a metric, you should write out the details to complete the proof. This metric is often called the **usual metric** (hence the subscript u) or the **Euclidean metric** on $\mathbb{R}$, and it is the one upon which your intuition is almost certainly based. A set can have lots of metrics. The next example is a metric on $\mathbb{R}$ that is not the same as the metric given by the absolute value.

Example 25.1. Define a metric $d_d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d_d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}.$$

We will show that d_d is a metric on $\mathbb{R}$. This metric is called the **discrete metric**, and it can really challenge your intuition.

Proof. It is clear that d_d is a function from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Now let x and y be points of $\mathbb{R}$. We begin with nonnegativity: $d_d(x,y) = 0$ or $d_d(x,y) = 1$, so clearly $d_d(x,y) \geq 0$. Thus, the nonnegativity condition holds. Furthermore, since $d_d(x,y) = 0$ if and only if $x = y$, the definiteness condition holds. For symmetry, note that if $x \neq y$, then $y \neq x$ and consequently $d_d(x,y) = 1 = d_d(y,x)$. If $x = y$, then $d_d(x,y) = 0 = d_d(y,x)$, establishing symmetry. Finally, we establish the triangle inequality. To this end, note that if z is a point of $\mathbb{R}$, then we have two cases to consider. In the first case, if $x = y$, then $d_d(x,y) = 0$ and the nonnegativity condition implies that $d_d(x,y) = 0 \leq d_d(x,z) + d_d(z,y)$. In the second case, $x \neq y$, which implies that $z \neq x$ or $z \neq y$ (or both). Therefore, either $d_d(x,z) = 1$ or $d_d(z,y) = 1$ (or both). Thus, $d_d(x,y) = 1 \leq d_d(x,z) + d_d(z,y)$, completing the proof of the triangle inequality. □

The discrete metric can be defined on every space: the distance between two distinct points is one, and the distance from a point to itself is necessarily zero. The proof that this is a metric on a set X is indistinguishable from the one above. Thus we have an example of a metric on $\mathbb{R}^2$. Example 25.2 and Exercise 25.3 provide us with some other metrics on $\mathbb{R}^2$.

Example 25.2. On $\mathbb{R}^2$ define a metric by

$$d_u((x_1,x_2),(y_1,y_2)) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$

Using Project 29.10, it can be shown that this is actually a metric on $\mathbb{R}^2$. For now, you may accept this fact. This metric is referred to as the **usual metric** or the **Euclidean metric** on $\mathbb{R}^2$. In fact, one may also define the **usual metric on** $\mathbb{R}^n$ by

$$d_u((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n)) = \sqrt{\sum_{j=1}^{n}(x_j-y_j)^2}.$$ ○

Exercise 25.3. We now have two examples of metrics on $\mathbb{R}$ and two on $\mathbb{R}^2$. Here are two more metrics on $\mathbb{R}^2$. Before you begin the exercise, familiarize yourself with the metrics by computing various distances. For example, try to find the distance from the point $(1,3)$ to the points $(-3,4)$ using the various metrics below.

(a) Show that $d_{tc}((x_1,x_2),(y_1,y_2)) = |x_1-y_1|+|x_2-y_2|$ is a metric on $\mathbb{R}^2$. This metric, d_{tc}, is called the **taxicab** metric. Why would it be called that?
(b) Show that $d_m((x_1,x_2),(y_1,y_2)) = \max\{|x_1-y_1|,|x_2-y_2|\}$ is also a metric on $\mathbb{R}^2$. The metric d_m is sometimes called the **max metric**. ○

The two examples introduced in Exercise 25.3 will appear again in the near future.

A metric tells us when points are close. We studied the notion of "closeness" in Chapter 20 when we studied convergent sequences. You can picture convergence of a sequence to the number L in the following way: a sequence converges to L if for every $\varepsilon > 0$, the sequence eventually lies in the open interval $(L-\varepsilon, L+\varepsilon)$; more precisely, the definition of convergence said, "There exists a real number L such that for every $\varepsilon > 0$, there exists a real number N such that $|x_n - L| < \varepsilon$ for all $n \geq N$." We return to the idea of finding the limit of a sequence, but this time in a metric space. So given a sequence (x_n) of points in a metric space (X,d), then (as we did before) we say that (x_n) **converges** in X if there exists a point $x \in X$ such that for every $\varepsilon > 0$, there exists a real number N such that $d(x_n,x) < \varepsilon$ for all $n \geq N$. As before, in the event that such an x exists, it is also unique. (See Theorem 25.8 below.) The value x is called the **limit** of the sequence, we say that the sequence **converges to** x, and, as before, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$. If the sequence does not converge, we say that it **diverges**. If we consider $X = \mathbb{R}$ with the usual metric, this is exactly the same definition that we had in Chapter 20. Since we allow all sorts of choices for X now, we would like to take this opportunity to point out that the point x must be in the space X—not in some larger space that happens to contain X. If it is clear that x belongs to X, we will often say that the sequence converges, rather than "the sequence converges in X." Also, note that as the metric d changes, the distance between pairs of points changes as well. Therefore, it is conceivable that some sequences will converge in one metric, but not in another.

Exercise 25.4. Complete the sentences.

(a) Let (x_n) be a sequence in a metric space X with metric d. Let $x \in X$. Then (x_n) does not converge to x if ....
(b) Let (x_n) be a sequence in a metric space (X,d). Then (x_n) does not converge if ....

We'll break tradition and give you the answer to part (a) of the above exercise here, because we need it: A sequence (x_n) does not converge to x if there exists an $\varepsilon > 0$ such that for every real number N, there exists $m \in \mathbb{N}$ such that $m \geq N$ and $d(x_m, x) \geq \varepsilon$. While an answer to (b) might read "a sequence (x_n) does not converge if for every $x \in X$, the sequence does not converge to x," this will probably not be the most useful formulation of the answer. We leave the more useful version to you. ○

Example 25.5. We know that $1/n \to 0$ in $\mathbb{R}$ with the usual metric. Show that $(1/n, 1/n) \to (0,0)$ in $\mathbb{R}^2$ with the usual metric.

Proof. Let $\varepsilon > 0$, and let N be a real number with $N > \sqrt{2}/\varepsilon$. If n is an integer with $n \geq N$, then

$$\begin{aligned} d_u((1/n,1/n),(0,0)) &= \sqrt{(1/n-0)^2+(1/n-0)^2} \\ &= \sqrt{2}/n \\ &\leq \sqrt{2}/N && \text{(since } n \geq N\text{)} \\ &< \sqrt{2}(\varepsilon/\sqrt{2}) && \text{(as } N > \sqrt{2}/\varepsilon\text{)} \\ &= \varepsilon. \end{aligned}$$

See Figure 25.1 for a graphical illustration of this convergent sequence. □

You may wonder where we came up with $\sqrt{2}/\varepsilon$. We did it by understanding the problem and devising a plan by working backwards. So what you see here is what happened after we went to a separate sheet of paper, and started with the inequality $\sqrt{2}/n < \varepsilon$.

Example 25.6. In Chapter 20, we showed that the sequence $(1/n)$ converges to 0 in $\mathbb{R}$ with the usual metric. Does $(1/n)$ converge to 0 in the discrete metric?

We claim that the sequence (in $\mathbb{R}$ with the discrete metric) does not converge to 0. To see this, let $\varepsilon = 1/2$. For every $N \in \mathbb{R}$, there exists an integer $n \geq N$. Since $1/n \neq 0$, we know that $d_d(1/n, 0) = 1$. Hence for $\varepsilon = 1/2$, and for every $N \in \mathbb{R}$, there exists an integer $n \geq N$ such that $x_n = 1/n$ satisfies $d_d(x_n, 0) = d_d(1/n, 0) = 1 \geq 1/2$. Thus $(1/n)$ does not converge to 0. ○

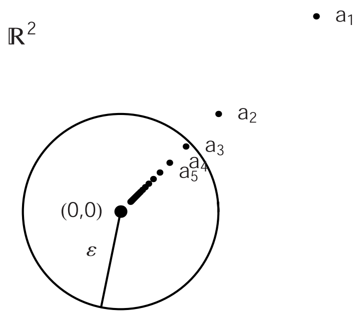

Fig. 25.1 $(1/n, 1/n) \to (0,0)$

In the discrete metric, every point is "far" from every other point. This makes it very hard to converge.

Exercise 25.7. Consider $\mathbb{R}$ with the discrete metric. Describe the convergent sequences in this metric space. ○

Sequences have many important properties, some of which we discuss in the problems. The proofs are often quite similar to the proofs we did in Chapter 20. At this point, we give one example of a theorem with such a proof.

Theorem 25.8. *If a sequence (x_n) in a metric space (X,d) converges, then the limit is unique.*

The proof of this is, with an appropriate change in notation, the same as the proof of Theorem 20.7.

Definitions

Definition 25.1. Let X be a nonempty set. Then a **metric** on X is a function $d : X \times X \to \mathbb{R}$ satisfying (i)–(iv) below.

(i) (Nonnegativity) For all $x, y \in X$, the function d satisfies $d(x,y) \geq 0$.
(ii) (Definiteness) For all $x, y \in X$, the function d satisfies $d(x,y) = 0$ if and only if $x = y$.
(iii) (Symmetry) For all $x, y \in X$, the function d satisfies $d(x,y) = d(y,x)$.
(iv) (Triangle inequality) For all $x, y, z \in X$, the function d satisfies

$$d(x,y) \leq d(x,z) + d(z,y).$$

Definition 25.2. A nonempty set X together with a metric d is called a **metric space** and is denoted by (X,d) or, when it is clear which distance function we are using, we will simply refer to the space X.

Definition 25.3. The **usual metric** or **Euclidean metric** on the reals is defined by $d_u(x,y) = |x-y|$ for $x,y \in \mathbb{R}$.

Definition 25.4. The **discrete metric** on $\mathbb{R}$ is defined by

$$d_d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases} \quad \text{for } x,y \in \mathbb{R}.$$

Definition 25.5. The **usual metric** or **Euclidean metric** on $\mathbb{R}^n$ for $n \in \mathbb{Z}^+$ is defined by

$$d_u((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2},$$

for $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \mathbb{R}^n$.

Definition 25.6. The **taxicab** metric on $\mathbb{R}^2$ is defined by

$$d_{tc}((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2| \text{ for } (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2.$$

Definition 25.7. The **max metric** on $\mathbb{R}^2$ is defined by

$$d_m((x_1,x_2),(y_1,y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \text{ for } (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2.$$

Definition 25.8. A sequence (x_n) of points in a metric space (X,d) **converges** in X if there exists a point $x \in X$ such that for every $\varepsilon > 0$, there exists a real number N such that $d(x_n,x) < \varepsilon$ for all $n \geq N$. The value x is called the **limit** of the sequence (x_n), we say that the sequence **converges to** x, and we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 25.9. A sequence (x_n) **diverges** in a metric space (X,d) if it does not converge in (X,d).

Definition 25.10 (for Problem 25.7). A set F in a metric space (X,d) is **bounded** if there exists a positive real number M such that $d(x,y) \leq M$ for all $x,y \in F$.

Definition 25.11 (for Problem 25.8). Let (X,d) be a metric space. The metric $d_b : X \times X \to \mathbb{R}$ defined by

$$d_b(x,y) = \min\{d(x,y), 1\}$$

is called the **bounded metric associated with** d on X.

Solutions to Exercises

Solution (25.3). Parts (a) and (b) are very similar, so we will work part (a) only.

By definition, d_{tc} is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Now let (x_1, x_2) and (y_1, y_2) be elements of $\mathbb{R}^2$. We will first show the nonnegativity. Because $|a| \geq 0$ for all real numbers a, we know that $d_{tc}((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2| \geq 0$, showing that nonnegativity of d_{tc} holds. For definiteness, note that $d_{tc}((x_1,x_2),(y_1,y_2)) = 0$ if and only if $|x_1 - y_1| + |x_2 - y_2| = 0$. This last equality holds if and only if $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. This, in turn, holds if and only if $x_1 = y_1$ and $x_2 = y_2$; in other words, $(x_1,x_2) = (y_1,y_2)$. This string of equivalences establishes the definiteness of d_{tc}. Symmetry is shown as follows:

$$\begin{aligned} d_{tc}((x_1,x_2),(y_1,y_2)) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |y_1 - x_1| + |y_2 - x_2| \\ &= d_{tc}((y_1,y_2),(x_1,x_2)). \end{aligned}$$

To prove that the triangle inequality holds for d_{tc}, let $(z_1,z_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} d_{tc}((x_1,x_2),(y_1,y_2)) &= |x_1 - y_1| + |x_2 - y_2| \\ &\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &\quad \text{(by the triangle inequality in } \mathbb{R}) \\ &= (|x_1 - z_1| + |x_2 - z_2|) + (|z_1 - y_1| + |z_2 - y_2|) \\ &= d_{tc}((x_1,x_2),(z_1,z_2)) + d_{tc}((z_1,z_2),(y_1,y_2)). \end{aligned}$$

This shows that d_{tc} is a metric on $\mathbb{R}^2$. The taxicab metric between two points measures the distance you have to travel from one point to the next in a city built with rectangular blocks, assuming you stay on the streets, do not take detours, and do not have to worry about one-way streets.

Solution (25.4). The solution to part (a) was given earlier following the exercise, so here is the solution to part (b).

The sequence (x_n) does not converge in the metric space (X,d) if for every $x \in X$ there exists a real number $\varepsilon > 0$ such that for every real number N, there exists m such that $m \geq N$ and $d(x_m, x) \geq \varepsilon$.

Solution (25.7). We claim that a sequence (x_n) in $(\mathbb{R}, d_d)$ converges if and only if there exist real numbers x and M such that $x_n = x$ for all $n \geq M$. (Such a sequence is called an eventually constant sequence.)

First assume that (x_n) is a sequence for which there exist real numbers x and M satisfying $x_n = x$ for all $n \geq M$. We will show that $x_n \to x$. Let $\varepsilon > 0$ and let $N = M$. Then for $n \geq N$ we know that $d_d(x_n, x) = d_d(x, x) = 0 < \varepsilon$, which shows that (x_n) converges.

For the converse, assume that (x_n) converges. Consider $\varepsilon = 1/2$. Then there exists N such that $d_d(x_n, x) < 1/2$ for $n \geq N$. By the definition of d_d, the only way that this can happen is if $x_n = x$ for $n \geq N$. Taking $M = N$, we have shown that there exist x and M such that $x_n = x$ for all $n \geq M$, as desired.

Problems

Unless otherwise specified, assume that you are working in a general metric space (X,d).

Problem 25.1. (a) Suppose a student writes the following: A metric is a function satisfying (i)–(iv) below.

(i) (Nonnegativity) $d(x,y) \geq 0$,
(ii) (Definiteness) $d(x,y) = 0$, if and only if $x = y$,
(iii) (Symmetry) $d(x,y) = d(y,x)$, and
(iv) (Triangle inequality) if z is a point in X, then

$$d(x,y) \leq d(x,z) + d(z,y).$$

Write this student a letter indicating what was omitted from the definition, what must be inserted, and what else (if anything) needs to be changed to make it a correct definition.

(b) Suppose the student had exactly the same definition as in the text, except for the triangle inequality, where the student has "$d(x,y) \leq d(x,z) + d(z,y)$ for some $z \in X$."
Write a correct, careful, and complete response to this student.

Problem 25.2. (a) In $\mathbb{R}$, find the distance of the number 1 to the number 3 in the usual metric and in the discrete metric.

(b) In $\mathbb{R}^2$, find the distance of the point $(1,3)$ to the point $(2,5)$ in the usual metric, the taxicab metric, the max metric, and the discrete metric.

Problem 25.3. (a) Sketch the set $\{(x,y) \in \mathbb{R}^2 : d_u((x,y),(0,0)) < 1\}$, where d_u is the usual metric.

(b) Sketch the set $\{(x,y) \in \mathbb{R}^2 : d_{tc}((x,y),(0,0)) < 1\}$, where d_{tc} is the taxicab metric.

(c) Sketch the set $\{(x,y) \in \mathbb{R}^2 : d_m((x,y),(0,0)) < 1\}$, where d_m is the max metric.

(d) Sketch the set $\{(x,y) \in \mathbb{R}^2 : d_d((x,y),(0,0)) < 1\}$, where d_d is the discrete metric.

(e) Sketch the set $\{(x,y,z) \in \mathbb{R}^3 : d_u((x,y,z),(0,0,0)) < 1\}$, where d_u is the usual metric.

Problem 25.4. (a) We defined the max metric on $\mathbb{R}^2$. Define the max metric on $\mathbb{R}^n$ and prove that it is a metric.

(b) We defined the taxicab metric on $\mathbb{R}^2$. Define the taxicab metric on $\mathbb{R}^n$ and prove that it is a metric.

Problem 25.5. (a) Show that $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1|$$

is not a metric on $\mathbb{R}^2$.

(b) Is $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d((x_1,x_2),(y_1,y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2$$

a metric on $\mathbb{R}^2$? If you think it is a metric, prove it. If you think it is not a metric, find points in $\mathbb{R}^2$ for which one of the conditions is violated.

Problem 25.6. Let (X,d) be a metric space. Let α be a real number and define a new function d_α on $X \times X$ by $d_\alpha(x,y) = \alpha d(x,y)$. Is d_α a metric on X? If not, what assumptions must be placed on α to assure that d_α is a metric? Prove your answer.

Problem 25.7. A set F in a metric space (X,d) is **bounded** if there exists a positive real number M such that $d(x,y) \leq M$ for all $x,y \in F$.

(a) Consider the following "not a definition" of a bounded set.
"A set is bounded if for each $x,y \in F$ there exists a positive real number M such that $d(x,y) \leq M$."
Give a complete, clear, concise explanation of the problems with this definition.
(b) Give an example of a metric space and an infinite set that is bounded in that metric. Prove that it is bounded.
(c) Complete the following definition: Let X be a metric space with metric d and let F be a subset of X. Then F is not bounded if ....
(d) Give an example of a metric space and a set that is not bounded in that metric. Prove that it is not bounded.

Problem 25.8. Let X be a set with a metric d. Define a function $d_b : X \times X \to \mathbb{R}$ by

$$d_b(x,y) = \min\{d(x,y), 1\}.$$

(a) Show that d_b is a metric on X. This metric is called the **bounded metric associated with** d on X.
(b) (This part uses Problem 25.7.) Consider the metric space (X,d_b). Show that in this space, every subset of X is bounded.

Problem 25.9. Let (X,d) be a metric space and let A be a finite subset of X. Must A be bounded? A complete answer to this question will either be a proof that the set A must be bounded, or an explicit example of a metric space X and a finite unbounded subset A. Justify all assertions!

Problem 25.10. Show that in a metric space (X,d) the metric satisfies

$$|d(x,z) - d(y,z)| \leq d(x,y),$$

for all $x,y,z \in X$.

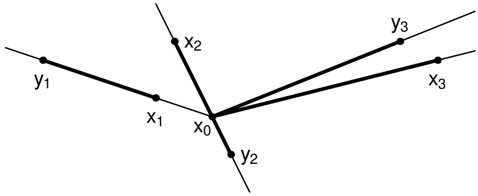

Fig. 25.2 The metric: $d(x_i, y_i)$ for $i = 1, 2, 3$

Problem 25.11. Let X be the space of polynomials with real coefficients. Define a function d from $X \times X \to \mathbb{R}$ by $d(p,q) = |p(0) - q(0)|$. Is d a metric? If so, prove it. If not, why not?

Problem 25.12. *The following problem is appropriate only if you have had integration in calculus.*

Let X be the space of real-valued continuous functions defined on the interval $[0,1]$. Define a function $d : X \times X \to \mathbb{R}$ by

$$d(f,g) = \int_0^1 |f(t) - g(t)|dt,$$

for all $f, g \in X$.

(a) Show that d is a metric.
(b) Find the distance between e^x and $\sin(\pi x/2)$.

Problem 25.13. *This problem is appropriate only if you have had integration in calculus.*

Here (X,d) denotes the space of real-valued continuous functions defined on the interval $[0,1]$ with the metric introduced in Problem 25.12. For each $n \in \mathbb{Z}^+$, we define a function $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Prove that the sequence (f_n) converges in X and find $\lim_{n\to\infty} f_n$.

Problem 25.14. Choose a fixed point x_0 in $\mathbb{R}^2$. If d_u denotes the usual (or Euclidean) metric on $\mathbb{R}^2$, then we define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} d_u(x,y) & \text{if } x \text{ and } y \text{ are on a straight line through } x_0 \\ d_u(x,x_0) + d_u(x_0,y) & \text{otherwise} \end{cases}.$$

Figure 25.2 illustrates this function for three pairs of points in the plane. Prove that d is a metric on $\mathbb{R}^2$.

This metric is sometimes called the "French railway system metric" or the "SNCF metric" (**S**ociété **N**ationale des **C**hemins de fer **F**rançais, see [49, p. 56]). The reason for this is the following: Think of x_0 as Paris, and you'll note that all trains pass through Paris, whether they need to or not.

Problem 25.15. Let d denote the "SNCF metric" on $\mathbb{R}^2$ as introduced in Problem 25.14. For $x_1 \in \mathbb{R}^2$ and a positive real number r, sketch the circle $C = \{x \in \mathbb{R}^2 : d(x,x_1) = r\}$. You may find it helpful to consider the three cases, $d(x_0,x_1) = 0$, $0 < d(x_0,x_1) < r$, and $d(x_0,x_1) \geq r$, separately.

Problem 25.16. Prove each of the following.

(a) Consider $\mathbb{R}^2$ with the max metric. Prove that $(1/n, 2/n) \to (0,0)$.
(b) Consider $\mathbb{R}$ with the usual metric. Prove that $(-1)^n n/(3n+1) \not\to 0$.
(c) Consider $\mathbb{R}^2$ with the max metric. Does $((-1)^n, 2/n))$ converge in this space?

Problem 25.17. Consider $\mathbb{Z}$ with the usual metric.

(a) Show that a sequence that is eventually constant converges; that is, if there exist integers m and k such that $x_n = k$ for all $n \geq m$, then the sequence converges.
(b) Can you give other examples of convergent sequences in $(\mathbb{Z}, d_u)$? Explain your answer.

Problem 25.18. Let (X,d) be a metric space, and let (x_n) be a convergent sequence in X.

(a) Prove that there exists $x \in X$ and a natural number K such that

$$d(x_n, x) \leq K \text{ for all} n \in \mathbb{N}.$$

(You should know a similar problem.)
(b) Prove that the set $\{x_n : n \in \mathbb{N}\}$ is bounded; that is, prove that there exists a positive number M such that $d(x_n, x_m) \leq M$ for all $n, m \in \mathbb{N}$.

Problem 25.19. In Problem 20.21 part (c), we defined the term Cauchy sequence and proved some facts about such sequences. This problem asks you to do the same in a general metric space.

(a) Define a Cauchy sequence in a metric space (X,d).
(b) Prove that if (x_n) converges in (X,d), then (x_n) is Cauchy.

Problem 25.20. (This problem uses Problem 25.19.) Let $X = \mathbb{R} \setminus \mathbb{Q}$ with the usual metric d_u. Prove that the sequence (x_n), where $x_n = \sqrt{2}/n$, is a Cauchy sequence in X, but (x_n) does not converge in X.

Problem 25.21. Let (X,d) be a metric space. Define a new function $d_e : X \times X \to \mathbb{R}$ by

$$d_e(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Show that d_e is also a metric on X.