

Chapter 8

More on Operations on Sets

Most of what we did in the last two chapters was concerned with operations on two sets. In Exercise 6.14 we defined unions and intersections of three sets. In general, we may have two or three sets, as many sets as there are integers, or even more sets than that. We'll need a new definition and special notation. In this chapter, we will introduce the notation that will allow us to keep track of these sets. Unfortunately, a rigorous definition will have to wait until Chapter 14.

Let n be a positive integer and suppose that we have sets A_1, A_2, \dots, A_n . How can we talk about the union of these n sets? the intersection? For example, when we have three sets, if we write $\bigcup_{j=1}^3 A_j = A_1 \cup A_2 \cup A_3$, we would be referring to the set of x in our universe that lie in at least one of our sets, A_1, A_2 , or A_3 . Of course, there is nothing special about three sets; that is, for every positive integer, n , we can write

$$\bigcup_{j=1}^n A_j = A_1 \cup A_2 \cup \dots \cup A_n \text{ and } \bigcap_{j=1}^n A_j = A_1 \cap A_2 \cap \dots \cap A_n.$$

The first set would be the set of all x in the universe that lie in at least one of the A_j for $j = 1, 2, \dots, n$, while the second would be the set of x that lie in all of the sets A_j . If we have a set A_j for each positive integer j and we want to take the union and intersection over all positive integers, then we write

$$\bigcup_{j=1}^{\infty} A_j = A_1 \cup A_2 \cup \dots \text{ and } \bigcap_{j=1}^{\infty} A_j = A_1 \cap A_2 \cap \dots.$$

This is probably a good time to look at some examples.

Example 8.1. We can write the union in different ways. For example,

$$\bigcup_{j=1}^{10} [0, j] = [0, 1] \cup [0, 2] \cup \dots \cup [0, 10] = [0, 10].$$

Similarly,

$$\bigcap_{j=1}^{10} [0, j] = [0, 1] \cap [0, 2] \cap \cdots \cap [0, 10] = [0, 1].$$

○

In Example 8.1, we had unions and intersections of finitely many sets (ten, to be precise). We now take a look at what can happen when we take unions and intersections of even more sets.

Example 8.2. (a) For each $q \in \mathbb{Z}^+$ define the set $A_q = \{p/q : p \in \mathbb{Z}\}$. These sets can be used to define the union $\bigcup_{q \in \mathbb{Z}^+} A_q$.

(b) This time we define, for each $i \in \mathbb{N}$, the set $B_i = \{p/3^i, p \in \mathbb{Z}\}$. These sets may be used to define the intersection $\bigcap_{i \in \mathbb{N}} B_i$. ○

Exercise 8.3. Write the sets $\bigcup_{j=1}^{\infty} [j, j+1]$ and $\bigcap_{j=1}^{\infty} [j, j+1]$ in their simplest form, by listing the first few sets in the union or intersection until the pattern is established, and then stating your guess. (You don't have to prove that your guess is correct.) ○

Let X denote our universe and let \mathcal{A} be a collection of subsets of X . Then the **union of the collection** and **intersection of the collection** are defined just as you might expect: The union is defined by

$$\bigcup_{A \in \mathcal{A}} A = \{x \in X : x \in A \text{ for some } A \in \mathcal{A}\}$$

and, for a nonempty collection \mathcal{A} , the intersection by

$$\bigcap_{A \in \mathcal{A}} A = \{x \in X : x \in A \text{ for all } A \in \mathcal{A}\}.$$

In some situations, it will be helpful to label our sets. We often label objects in real life; for example, we label runners in order of their finish in a race. Why not do this with sets? We could label the set we think of first as A_1 , the runner-up would be set A_2 , and so on. Sometimes this is possible, as in Example 8.2 above, but other times it is not. For example, sometimes we do not know how many sets we have and other times we will have so many sets that it is impossible to tell which “came first.” While this may seem odd, it happens all the time.

So again let X be our universe and suppose we have a set I . Suppose further that for each $\alpha \in I$ there is a unique set $A_\alpha \subseteq X$ corresponding to it. The set I is called an **index set**, each $\alpha \in I$ is called an **index**, and the set $\{A_\alpha : \alpha \in I\}$ is called an **indexed set** or an indexed collection. Thus, an index set is a set that labels the members of another set \mathcal{A} . We may also write $\{A_\alpha\}_{\alpha \in I}$ for an indexed collection of sets.

In this setting, our definition of union and intersection will look a bit different from the one presented above. So recall the notation: X denotes our universe and we let $\{A_\alpha : \alpha \in I\}$ be an indexed collection of sets with $A_\alpha \subseteq X$ for all α in an index set I . Then the union of the indexed collection $\{A_\alpha : \alpha \in I\}$ is just

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X : x \in A_\alpha \text{ for some } \alpha \in I\},$$

and for $I \neq \emptyset$, the intersection of the indexed collection $\{A_\alpha : \alpha \in I\}$ is

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

Exercise 8.4. Find the simplest way to describe the following sets (you may find sketches helpful):

- (a) $\bigcup_{x \in \mathbb{R}^+} (0, x)$;
- (b) $\bigcup_{n \in \mathbb{N}} [0, n]$;
- (c) $\bigcap_{n \in \mathbb{N}} [0, n]$.

○

Note that the index notation and the general definition of union and intersection given here include the cases in Chapter 6 and the ones we mentioned in the beginning of this chapter. For instance, if $I = \{1, 2\}$, then $\bigcap_{i \in I} A_i = A_1 \cap A_2$.

Some more practice with this notation will probably be very helpful at this point.

Exercise 8.5. (a) Write $\bigcup_{j=0}^{\infty} [0, j]$ using an appropriate index set.

- (b) Write $\bigcup_{j=1}^{\infty} (0, j)$ using an appropriate index set.

○

Some sets are more easily described with index notation, others without such notation. Let's go back and look at the sets in Example 8.2.

Example 8.6. Consider the indexed collection of sets $\{A_q\}_{q \in \mathbb{Z}^+}$ defined in Example 8.2 (a). Then $\bigcup_{q \in \mathbb{Z}^+} A_q = \mathbb{Q}$.

Proof. If $x \in \bigcup_{q \in \mathbb{Z}^+} A_q$, then $x \in A_q$ for some $q \in \mathbb{Z}^+$. Thus, there exist $q \in \mathbb{Z}^+$ and $p \in \mathbb{Z}$ such that $x = p/q$. Hence $x \in \mathbb{Q}$, and we have shown that $\bigcup_{q \in \mathbb{Z}^+} A_q \subseteq \mathbb{Q}$.

Conversely, if $x \in \mathbb{Q}$, then $x = p/q$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$. Now (for reasons that you will explain) we may choose q so that $q > 0$. For this q we have $q \in \mathbb{Z}^+$ and therefore $x \in A_q$. Hence $x \in \bigcup_{q \in \mathbb{Z}^+} A_q$. So, $\mathbb{Q} \subseteq \bigcup_{q \in \mathbb{Z}^+} A_q$, and therefore $\bigcup_{q \in \mathbb{Z}^+} A_q = \mathbb{Q}$. □

Example 8.7. Consider the indexed collection of sets $\{B_i\}_{i \in \mathbb{N}}$ defined in Example 8.2 (b). We claim that $\bigcap_{i \in \mathbb{N}} B_i = \mathbb{Z}$.

Proof. If $x \in \bigcap_{i \in \mathbb{N}} B_i$, then $x \in B_i$ for all $i \in \mathbb{N}$. In particular, $x \in B_0$. Therefore $x = p/3^0 = p$ for some $p \in \mathbb{Z}$. So $x \in \mathbb{Z}$, and thus $\bigcap_{i \in \mathbb{N}} B_i \subseteq \mathbb{Z}$.

Now let $x \in \mathbb{Z}$. For each $i \in \mathbb{N}$, we may write $x = (3^i x)/3^i$. Of course, $3^i x \in \mathbb{Z}$, since $x \in \mathbb{Z}$. Hence $x \in B_i$ for all $i \in \mathbb{N}$. This shows that $x \in \bigcap_{i \in \mathbb{N}} B_i$ and therefore $\mathbb{Z} \subseteq \bigcap_{i \in \mathbb{N}} B_i$.

Combining the two arguments we obtain the desired equality, $\bigcap_{i \in \mathbb{N}} B_i = \mathbb{Z}$. □

Exercise 8.8. What's the difference between "an infinite union of sets" and "a union of infinite sets"? Give an example of each, showing how these two phrases differ. (While we haven't given a rigorous definition of infinite here, your intuition should suffice to solve this problem.) ○

You already know that one of DeMorgan's laws for two sets can be stated as

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

This can be rephrased in words as "the complement of a union is the intersection of the complements." DeMorgan's laws do not depend on the number of sets that we have, and that is the point of the next exercise.

Exercise 8.9. Show that the general DeMorgan's laws hold: For every universe X , nonempty index set I , and indexed collection of sets $\{A_\alpha : \alpha \in I\}$, we have

- (i) $X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$ and
 (ii) $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$. ○

Exercise 8.10. Suppose $A_\alpha \subseteq B$ for each $\alpha \in I$. Show that $\bigcup_{\alpha \in I} A_\alpha \subseteq B$. ○

Definitions

Definition 8.1. Let X denote the universe and \mathcal{A} a collection of subsets of X . Then the **union of the collection** is defined as

$$\bigcup_{A \in \mathcal{A}} A = \{x \in X : x \in A \text{ for some } A \in \mathcal{A}\}.$$

Definition 8.2. Let X denote the universe and \mathcal{A} a nonempty collection of subsets of X . Then the **intersection of the collection** is defined as

$$\bigcap_{A \in \mathcal{A}} A = \{x \in X : x \in A \text{ for all } A \in \mathcal{A}\}.$$

Definition 8.3. A set I is called an **index set for a set** \mathcal{A} , if for each $\alpha \in I$ there is an element $A_\alpha \in \mathcal{A}$ corresponding to α , and every element in \mathcal{A} is labeled in this way.

Definition 8.4. Given an index set I and corresponding sets A_α that are defined for each $\alpha \in I$, the set $\{A_\alpha : \alpha \in I\}$ is said to be an **indexed set** or **indexed collection** of sets.

Definition 8.5 (for Problem 8.18). A collection of sets \mathcal{A} is said to be **pairwise disjoint** if the following is satisfied: For all $X, Y \in \mathcal{A}$, if $X \cap Y \neq \emptyset$, then $X = Y$.

Solutions to Exercises

Solution (8.3). You can see that $\bigcup_{j=1}^{\infty} [j, j+1] = [1, 2] \cup [2, 3] \cup [3, 4] \cup \dots = [1, \infty)$ and $\bigcap_{j=1}^{\infty} [j, j+1] = [1, 2] \cap [2, 3] \cap [3, 4] \cap \dots = \emptyset$.

Solution (8.4). You can check the following:

- (a) $\bigcup_{x \in \mathbb{R}^+} (0, x) = (0, \infty)$;
- (b) $\bigcup_{n \in \mathbb{N}} [0, n] = [0, \infty)$;
- (c) $\bigcap_{n \in \mathbb{N}} [0, n] = \{0\}$.

Solution (8.5). You can check the following:

- (a) $\bigcup_{j=0}^{\infty} [0, j] = \bigcup_{j \in \mathbb{N}} [0, j]$;
- (b) $\bigcup_{j=1}^{\infty} (0, j) = \bigcup_{j \in \mathbb{Z}^+} (0, j)$.

Solution (8.8). An infinite union of sets would mean that we take the union over infinitely many sets (but the sets themselves may be finite); in other words, the index set is infinite. For example, $\bigcup_{n \in \mathbb{N}} \{n\}$ would be an infinite union of (finite) sets. On the other hand, a union of infinite sets would mean that the sets themselves must be infinite (while the index set may be finite). For example, the union of the even integers, $2\mathbb{Z}$, with the odd integers, $2\mathbb{Z} + 1$, would be a union of two infinite sets.

Solution (8.9). We will show part (i) and will leave part (ii) for you to do. So we need to show that

$$X \setminus \left(\bigcup_{\alpha \in I} A_{\alpha} \right) = \bigcap_{\alpha \in I} (X \setminus A_{\alpha}).$$

If $x \in X \setminus \left(\bigcup_{\alpha \in I} A_{\alpha} \right)$, then $x \in X$ and $x \notin \bigcup_{\alpha \in I} A_{\alpha}$. By the definition of union this means that $x \in X$ and $x \notin A_{\alpha}$ for every $\alpha \in I$. Hence, $x \in X \setminus A_{\alpha}$ for all $\alpha \in I$, and therefore $x \in \bigcap_{\alpha \in I} (X \setminus A_{\alpha})$. Thus, $X \setminus \left(\bigcup_{\alpha \in I} A_{\alpha} \right) \subseteq \bigcap_{\alpha \in I} (X \setminus A_{\alpha})$.

Now if $x \in \bigcap_{\alpha \in I} (X \setminus A_{\alpha})$, then $x \in X \setminus A_{\alpha}$ for all $\alpha \in I$. This implies that $x \in X$ and $x \notin A_{\alpha}$ for every $\alpha \in I$. Hence, we have $x \in X$ and $x \notin \bigcup_{\alpha \in I} A_{\alpha}$. It follows that $x \in X \setminus \left(\bigcup_{\alpha \in I} A_{\alpha} \right)$ and thus $\bigcap_{\alpha \in I} (X \setminus A_{\alpha}) \subseteq X \setminus \left(\bigcup_{\alpha \in I} A_{\alpha} \right)$.

The two subset relations give the desired equality between the sets.

Solution (8.10). If $x \in \bigcup_{\alpha \in I} A_{\alpha}$, then there exists α_0 such that $x \in A_{\alpha_0}$. Since we suppose that $A_{\alpha_0} \subseteq B$, we know that $x \in B$. Thus $\bigcup_{\alpha \in I} A_{\alpha} \subseteq B$.

Problems

Problem 8.1. For positive integers n , consider the intervals of real numbers given by $A_n = [0, 1/n)$, $B_n = [0, 1/n]$, and $C_n = (0, 1/n)$.

- (a) Find $\bigcup_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} B_n$, and $\bigcup_{n=1}^{\infty} C_n$.
- (b) Find $\bigcap_{n=1}^{\infty} A_n$, $\bigcap_{n=1}^{\infty} B_n$, and $\bigcap_{n=1}^{\infty} C_n$.
- (c) Does $\bigcup_{n \in \mathbb{N}} A_n$ make sense? Why or why not?

Problem 8.2. If $A_x = [-x, x]$, find $\bigcup_{x \in \mathbb{R}^+} A_x$ and $\bigcap_{x \in \mathbb{R}^+} A_x$.

Problem 8.3. Find simpler notation for the two sets

$$A = \bigcup_{j=0}^{\infty} [j, j+1] \quad \text{and} \quad B = \bigcap_{j \in \mathbb{Z}} (\mathbb{R} \setminus (j, j+1)).$$

Problem 8.4. Let I be a nonempty index set and suppose that $B \subseteq A_\alpha$ for all $\alpha \in I$. Show that $B \subseteq \bigcap_{\alpha \in I} A_\alpha$.

Problem 8.5. Let $I \neq \emptyset$ and $\{A_\alpha : \alpha \in I\}$ be an indexed collection. Prove that for all $\beta \in I$

$$\bigcap_{\alpha \in I} A_\alpha \subseteq A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha.$$

Problem 8.6. Prove or give a counterexample: Let $\{A_n : n \in \mathbb{Z}^+\}$ and $\{B_n : n \in \mathbb{Z}^+\}$ be two indexed collections of sets. If $A_n \subset B_n$ for all $n \in \mathbb{Z}^+$, then

$$\bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} B_n.$$

(Recall that $A \subset B$ means strict inclusion; that is, $A \subseteq B$ and $A \neq B$.)

Problem 8.7. Let I be a set and let $\{A_\alpha : \alpha \in I\}$, and $\{B_\alpha : \alpha \in I\}$ be two indexed collections of sets such that $A_\alpha \subseteq B_\alpha$ for all $\alpha \in I$. Prove that

$$\bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\alpha \in I} B_\alpha.$$

Problem 8.8. Prove the following set inclusion.

$$\bigcup_{b \in \mathbb{R}^+} \{(x, y) \in \mathbb{R}^2 : x + y = b\} \subseteq \bigcap_{s \in \mathbb{R}^-} \{(x, y) \in \mathbb{R}^2 : x + y > s\}.$$

Problem 8.9. For $n \in \mathbb{Z}^+$ define

$$A_n = \{x \in \mathbb{R} : \frac{1}{n} < x \leq 2\} \quad \text{and} \quad B_n = \{x \in \mathbb{R} : 0 < x < \frac{3}{n} + 2\}.$$

Prove that $\bigcup_{n \in \mathbb{Z}^+} A_n \subseteq \bigcap_{n \in \mathbb{Z}^+} B_n$.

Problem 8.10. Let I and J be nonempty sets such that $J \subseteq I$, and let $\{A_\alpha : \alpha \in I\}$ be an indexed collection. Prove that

- (a) $\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$ and
- (b) $\bigcap_{\alpha \in I} A_\alpha \subseteq \bigcap_{\alpha \in J} A_\alpha$.

Problem 8.11. Let $\{A_r : r \in \mathbb{R}\}$ and $\{B_r : r \in \mathbb{R}\}$ be two indexed collections of sets. Prove that

$$\left(\bigcap_{r \in \mathbb{R}} A_r\right) \cup \left(\bigcap_{r \in \mathbb{R}} B_r\right) \subseteq \bigcap_{r \in \mathbb{R}} (A_r \cup B_r).$$

Provide an example showing that this inclusion can be proper.

Problem# 8.12. Let I be a nonempty set, $\{A_\alpha : \alpha \in I\}$ an indexed collection of sets, and let B be a set.

(a) Prove the distributive property:

$$\left(\bigcup_{\alpha \in I} A_\alpha\right) \cap B = \bigcup_{\alpha \in I} (A_\alpha \cap B).$$

(b) State and prove a distributive property for $(\bigcap_{\alpha \in I} A_\alpha) \cup B$.

Problem# 8.13. Suppose that $\{A_\alpha : \alpha \in I\}$ is a nonempty indexed collection of subsets of a set X .

- (a) If $A_{\alpha_0} = \emptyset$ for some $\alpha_0 \in I$, prove that $\bigcap_{\alpha \in I} A_\alpha = \emptyset$.
- (b) If $A_{\alpha_0} = X$ for some $\alpha_0 \in I$, prove that $\bigcup_{\alpha \in I} A_\alpha = X$.

Problem 8.14. Define

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus \{-n, -n+1, \dots, 0, \dots, n-1, n\}).$$

The set A should be familiar to you. Guess what it is and then prove that your guess is correct.

Problem 8.15. Guess a simpler way to express the set A defined as

$$A = \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}),$$

and then prove that your guess is correct.

Problem 8.16. Suppose that X is a set with more than one element. What is $\bigcup_{x \in X} \{x\}$? What is $\bigcap_{x \in X} \{x\}$?

Problem 8.17. For every $\alpha \in \mathbb{R}$ and for every $m, n \in \mathbb{Z}^+$ define

$$A_\alpha = \{x \in \mathbb{Z}^+ : x < \alpha\} \quad \text{and} \quad B_{mn} = \{x \in \mathbb{R} : |x - m| < \frac{1}{n}\}.$$

Prove that

$$\bigcup_{\alpha \in \mathbb{R}} A_\alpha \subseteq \bigcup_{m \in \mathbb{Z}^+} \left(\bigcap_{n \in \mathbb{Z}^+} B_{mn}\right).$$

Problem# 8.18. A collection of sets \mathcal{A} is said to be **pairwise disjoint** if the following is satisfied: For all $X, Y \in \mathcal{A}$, if $X \cap Y \neq \emptyset$, then $X = Y$.

A comment about this definition may be in order: Speaking informally, a collection of sets is pairwise disjoint if whenever we choose two sets from the collection, they are disjoint or they are equal.

- Give an example of a pairwise disjoint collection of infinitely many sets.
- What is the contrapositive of “if $X \cap Y \neq \emptyset$, then $X = Y$ ”?
- What is the converse of “if $X \cap Y \neq \emptyset$, then $X = Y$ ”?
- If \mathcal{A} is a pairwise disjoint collection of sets, does the assertion you found in (b) hold for all $X, Y \in \mathcal{A}$?
- If the assertion that you found in (b) holds for all X and Y in some set \mathcal{A} , is \mathcal{A} a pairwise disjoint collection of sets?
- Suppose that \mathcal{B} is a pairwise disjoint collection of sets. Can we conclude that $\bigcap_{X \in \mathcal{B}} X = \emptyset$?
- Suppose that $\bigcap_{X \in \mathcal{B}} X = \emptyset$. Is \mathcal{B} necessarily a pairwise disjoint collection of sets?

Problem 8.19. For $n \in \mathbb{N}$ we define $A_n = \{x \in \mathbb{N} : \frac{n}{2} - 1 < x \leq \frac{n}{2}\}$. We use these sets for two collections of sets: $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ and $\mathcal{B} = \{A_n : n \in 2\mathbb{N}\}$ (where $2\mathbb{N}$ denotes the set of even nonnegative integers).

- Is the collection \mathcal{A} pairwise disjoint? Prove or give a counterexample.
- Is the collection \mathcal{B} pairwise disjoint? Prove or give a counterexample.
- Is \mathcal{A} contained in \mathcal{B} ? Are the sets equal?

Problem 8.20. An indexed collection of sets $\{A_n : n \in \mathbb{Z}^+\}$ is said to be *increasing* if $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{Z}^+$. The indexed collection is said to be *strictly increasing* if $A_n \subset A_{n+1}$ for each $n \in \mathbb{Z}^+$.

- Give an example of an increasing indexed collection of sets (consisting of subsets of \mathbb{R}) that is not strictly increasing.
- Find an increasing indexed collection of sets $\{A_n : n \in \mathbb{Z}^+\}$ (consisting of subsets of \mathbb{R}) such that $\bigcup_{n \in \mathbb{Z}^+} A_n = [0, 1]$ and $\bigcap_{n \in \mathbb{Z}^+} A_n = \{0\}$.
- Find a strictly increasing indexed collection of sets $\{B_n : n \in \mathbb{Z}^+\}$ (consisting of subsets of \mathbb{R}) such that $\bigcup_{n \in \mathbb{Z}^+} B_n = [0, 1]$.

Problem 8.21. Find an example of an indexed collection of sets $\{A_j : j \in \mathbb{Z}^+\}$ such that $A_{j+1} \subset A_j$ for each $j \in \mathbb{Z}^+$, and $\bigcap_{j=1}^{\infty} A_j \neq \emptyset$.