Chapter 7
Operations on Sets

By an operation on sets we mean the construction of a new set from the given ones. As we saw in the last chapter, these new sets may be formed using unions, intersections, set differences, or complements of given sets. In this section, we will look at many important properties of operations on sets. We end the chapter with a summarizing list of identities. In the exercises and problems you will be given the opportunity to prove the most important ones and then commit them to memory, so you don't have to re-prove them every time you need them.

The Venn diagrams introduced in the previous chapter can be helpful in deciding what is true and what is false, and they can be part of understanding the problem. If the arrangement of your sets in the diagram is a so-called "typical" one (see [103]), it is even possible to use the Venn diagram as a proof. However, in this text, we (and you) will use Venn diagrams to guide us, but we will write our proofs without relying on them. When you prove these properties you may not always need to start from the definition. Sometimes you can use what you know, and once you have proven everything in Theorem 7.4, you will know a lot.

The first theorem is a good example of a proof in cases. It keeps things tidy. Now remember, if we use the definition to show two sets A and B are equal, then we must show that if $x \in A$, then $x \in B$ *and* if $x \in B$, then $x \in A$.

Theorem 7.1 (The distributive property). *Let A, B, and C be sets. Then*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Before reading the proof, let's use Pólya's method.

"Understanding the problem." Draw two Venn diagrams representing the left and right sides of the equality above. Each diagram will have three sets, appropriately labeled A, B, and C. Shade in the area described by the left side of the equation in one diagram and then shade the right side in the other diagram. They should look the same. While this should convince you that you are on the right track, it is not enough to convince someone else.

"Devising a plan." We wish to show that two sets are equal. Using the definition of equality of sets, we know that we must show two things. The first thing to show

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is that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. So our first line will begin

$$\text{If } x \in A \cup (B \cap C),$$

and our last line (for this part of the proof) will look like

$$\text{Thus } x \in (A \cup B) \cap (A \cup C).$$

Now we just have to figure out how to get from the first line to the last one. Let's fill in some things, making sure that each line follows logically from the previous one. Working down from the top we get

$$x \in A \cup (B \cap C),$$

$$x \in A \text{ or } x \in B \cap C,$$

and working up from the bottom leads to

$$x \in A \cup B \text{ and } x \in A \cup C,$$

$$x \in (A \cup B) \cap (A \cup C).$$

Looking at what we are missing in our proof suggests that we use a proof in cases; one that depends on whether $x \in A$ or $x \in B \cap C$.

Once we are done with the proof above, we must show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. We use the same method to devise our plan for a proof of this set containment: We write down our first line and look to see where it takes us. Then we'll write down our last line and try to figure out how to get there. That leads to

$$x \in (A \cup B) \cap (A \cup C),$$

$$x \in A \cup B \text{ and } x \in A \cup C,$$

[stuff]

$$x \in A \text{ or } x \in B \cap C,$$

$$x \in A \cup (B \cap C).$$

It looks like if $x \in A$, we have our proof. But what if $x \notin A$? This again suggests a proof in cases; one that depends on whether $x \in A$ or $x \notin A$. If you see what to do now, you can write up the proof. If you still do not see what to do, continue using this method until you see the solution.

Once you see the solution, fill in the missing steps and write the proof up carefully using complete sentences, as we do below.

Proof. If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. Suppose first that $x \in A$. Then $x \in A \cup B$ and $x \in A \cup C$. In this first case, we see that $x \in (A \cup B) \cap (A \cup C)$. Now suppose that $x \in B \cap C$. Then $x \in B$ and $x \in C$. Since $x \in B$, we see that $x \in A \cup B$. Since we also have $x \in C$, we see that $x \in A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$

in this case as well. In both cases $x \in (A \cup B) \cap (A \cup C)$ and we may conclude that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

To complete the proof, we must now show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. So if $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$. It is, once again, helpful to break this into two cases, since we know that $x \in A$ or $x \notin A$. Now if $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then the fact that $x \in A \cup B$ implies that x must be in B. Similarly, the fact that $x \in A \cup C$ implies that x must be in C. Therefore, $x \in B \cap C$. Hence $x \in A \cup (B \cap C)$. In both cases $x \in A \cup (B \cap C)$ and we may conclude that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Since we proved containment in both directions we may conclude that the two sets are equal. □

Look at the proof above. It has complete sentences, variables are identified, we know when we are in one case and then the other, and we know when the proof is complete. You should use the form as a model, but remember that each proof will be unique.

The following theorem involves an "if and only if" statement. Remember that an "if and only if" statement requires that you prove *both* the "if" *and* the "only if."

Theorem 7.2. *Let A and B be sets. Then $A \cup B = A$ if and only if $B \subseteq A$.*

Proof. First we'll show that if $A \cup B = A$, then $B \subseteq A$. So assume $A \cup B = A$. If $x \in B$, then $x \in A \cup B$. Using the assumption that $A \cup B = A$ we have $x \in A$. This shows that $B \subseteq A$.

Now we will prove that if $B \subseteq A$, then $A \cup B = A$. So let us assume that $B \subseteq A$. We must show that $A \cup B \subseteq A$ and $A \subseteq A \cup B$. To prove the first containment, we have that if $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then x is where it needs to be and we have nothing more to prove. If $x \in B$, then we use the assumption that $B \subseteq A$ to conclude that $x \in A$. In both cases we get $x \in A$ and therefore have $A \cup B \subseteq A$. To prove the second containment, let $x \in A$. Then $x \in A \cup B$ and we conclude that $A \subseteq A \cup B$. Together we have proven that $A \cup B = A$. □

The structure of the proof of Theorem 7.2 is more complicated than the proof of the distributive property. First, as we said above, there are two things to prove: the "if" and the "only if." Next, both of these statements have hypotheses and conclusions. In each case, you must be aware of what you are assuming and what you are proving. What's even more important, though, is that you *use* what you are assuming to get to your desired conclusion. If you don't use your assumption, then your original statement was poorly constructed, you proved more than you thought you did, or your proof was in error. In fact, in the proof above, we did not use our assumption that $B \subseteq A$ to prove $A \subseteq A \cup B$. Did we make an error, or did we prove more than we said we did?

Now that you have seen two examples of how to write such a proof, it is time for you to try it by yourself. Try proving one of the two DeMorgan's laws below.

Exercise 7.3. Let A and B be subsets of the set X. Then

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

(a) Devise your plan. (Include a Venn diagram.)
(b) Write up your proof. ○

We now give the promised list of some of the properties of set operations. We proved three of them above. In the problems you will be asked to work more of the proofs.

Theorem 7.4. *Let X denote a set, and A, B, and C denote subsets of X. Then*

1. $\emptyset \subseteq A$ *and* $A \subseteq A$.
2. $(A^c)^c = A$.
3. $A \cup \emptyset = A$.
4. $A \cap \emptyset = \emptyset$.
5. $A \cap A = A$.
6. $A \cup A = A$.
7. $A \cap B = B \cap A$. *(Commutative property)*
8. $A \cup B = B \cup A$. *(Commutative property)*
9. $(A \cup B) \cup C = A \cup (B \cup C)$. *(Associative property)*
10. $(A \cap B) \cap C = A \cap (B \cap C)$. *(Associative property)*
11. $A \cap B \subseteq A$.
12. $A \subseteq A \cup B$.
13. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *(Distributive property)*
14. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *(Distributive property)*
15. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$. *(DeMorgan's law)*
    (When X is the universe we also write $(A \cup B)^c = A^c \cap B^c$.*)*
16. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. *(DeMorgan's law)*
    (When X is the universe we also write $(A \cap B)^c = A^c \cup B^c$.*)*
17. $A \setminus B = A \cap B^c$.
18. $A \subseteq B$ *if and only if* $(X \setminus B) \subseteq (X \setminus A)$.
    (When X is the universe we also write $A \subseteq B$ *if and only if* $B^c \subseteq A^c$.*)*
19. $A \subseteq C$ *and* $B \subseteq C$ *if and only if* $A \cup B \subseteq C$.
20. $C \subseteq A$ *and* $C \subseteq B$ *if and only if* $C \subseteq A \cap B$.
21. $A \cup B = A$ *if and only if* $B \subseteq A$.
22. $A \cap B = B$ *if and only if* $B \subseteq A$.

Many results can be proved using the methods demonstrated thus far in this chapter. Once you have proven these statements, though, it is a good idea to use them in other proofs. Practice using the results in Theorem 7.4 in the next exercise.

Exercise 7.5. Let A, B, and C be sets. Prove the following using relevant statements from Theorem 7.4: If $C^c \subseteq B$, then $(A \setminus B) \cup C = C$. ○

Definition

Definition 7.1 (for Problems 7.6 through 7.10). The **symmetric difference** of sets A and B is the set $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Solutions to Exercises

Solution (7.3). First we show that

$$X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B).$$

If $x \in X \setminus (A \cup B)$, then $x \notin A \cup B$. Therefore $x \notin A$ and $x \notin B$. Consequently, $x \in X \setminus A$ and $x \in X \setminus B$. Thus $x \in (X \setminus A) \cap (X \setminus B)$. We conclude that $X \setminus (A \cup B) \subseteq (X \setminus A)$ $\cap (X \setminus B)$.

We now show that

$$(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B).$$

If $x \in (X \setminus A) \cap (X \setminus B)$, then $x \in X \setminus A$ and $x \in X \setminus B$. Thus, $x \in X$ and $x \notin A$, and $x \in X$ and $x \notin B$. So, $x \in X$ and $x \notin A$ and $x \notin B$. This implies that $x \in X$ and $x \notin A \cup B$. Therefore, $x \in X \setminus (A \cup B)$, and we see that $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$. Thus, the two sets are equal.

Solution (7.5). Since $C^c \subseteq B$, statements 18 and 2 of Theorem 7.4 imply $B^c \subseteq C$, and thus $B^c \cup C = C$ by statement 19 of the same theorem. The rest of the proof now follows from the following string of equalities (numbers indicate the relevant statements from Theorem 7.4):

$$\begin{aligned} (A \setminus B) \cup C &= (A \cap B^c) \cup C && \text{(by 17)} \\ &= (A \cup C) \cap (B^c \cup C) && \text{(by 8 and 13)} \\ &= (A \cup C) \cap C && \text{(since } B^c \cup C = C \text{ as shown)} \\ &= C && \text{(by 8, 12, and 20).} \end{aligned}$$

Problems

In all the problems below, X denotes a set; A, B, and C denote subsets of X.

Problem 7.1. In this problem we refer to statements of Theorem 7.4.

(a) Prove statement 2.
(b) Prove statement 14.
(c) Prove statement 16.
(d) Prove statement 18.

(e) Prove statement 20.
(f) Prove statement 22.

Problem 7.2. Prove that $A \cap B = \emptyset$ if and only if $B \subseteq (X \setminus A)$.

Problem 7.3. Prove that $A = B$ if and only if $(X \setminus A) = (X \setminus B)$. Make sure you use statements from Theorem 7.4 rather than going back to the definition.

Problem 7.4. Prove the following using the results stated in Theorem 7.4:

(a) $(A \cup B) \cap B = B$;
(b) $(A \cap B) \cup B = B$.

Problem 7.5. Show that $(A \setminus B) \cup (B \cap A^c) = (A \cup B) \setminus (B \cap A)$ in two differents ways, by completing the two parts below.

(a) Prove this equality using only the definition of set containment.
(b) Using theorems from the text, give a different proof of the equality above. If you use a result that has a name, state the name of the result.

Problem 7.6. Draw the Venn diagram for the symmetric difference $A \triangle B$ of two sets A and B and prove that

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Problem 7.7. For sets A and B prove the following.

(a) $A \triangle A = \emptyset$;
(b) $A \triangle \emptyset = A$;
(c) $A \triangle B = B \triangle A$. (The symmetric set difference is commutative.)

Problem 7.8. Prove that for sets A, B, and C, we have $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. (The symmetric set difference is associative.)

Problem 7.9. Prove that for sets A and B, we have $A \triangle B = A \setminus B$ if and only if $B \subseteq A$.

Problem 7.10. Prove that for sets A and B, we have $(A \cup B) \triangle (A \cap B) = A \triangle B$.

Problem 7.11. Sketch Venn diagrams of the set on the left and the set on the right side of the equation

$$(A \setminus (B \cap C)) \cup (B \setminus C) = (A \cup B) \setminus (B \cap C).$$

Once you have done that, prove that the equality above holds.

Problem 7.12. Consider the following sets:

(i) $A \setminus (A \cup B \cup C)$,
(ii) $A \setminus A \cap B \cap C$,
(iii) $A \cap B^c \cap C^c$,
(iv) $A \setminus (B \cup C)$, and
(v) $(A \setminus B) \cap (A \setminus C)$.

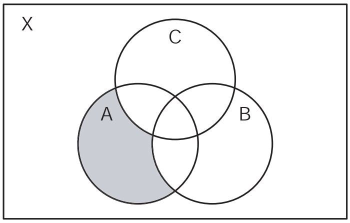

Fig. 7.1

(a) Which of the sets above are written ambiguously, if any?
(b) Of the ones that make sense, which of the sets above agree with the shaded set in Figure 7.1?
(c) Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Problem 7.13. Consider the following sets:

(i) $(A \cap B) \setminus (A \cap B \cap C)$,
(ii) $A \cap B \setminus (A \cap B \cap C)$,
(iii) $A \cap B \cap C^c$,
(iv) $(A \cap B) \setminus C$, and
(v) $(A \setminus C) \cap (B \setminus C)$.

(a) Which of the sets above are written ambiguously, if any?
(b) Of the sets above that make sense, which ones equal the set sketched in Figure 7.2?
(c) Prove that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

Problem# 7.14. In this problem you will prove that the union of two sets can be rewritten as the union of two disjoint sets.

(a) Prove that the two sets $A \setminus B$ and B are disjoint.
(b) Prove that $A \cup B = (A \setminus B) \cup B$.

Problem 7.15. Prove that $A^c \cup B^c = X$ if and only if A and B are disjoint.

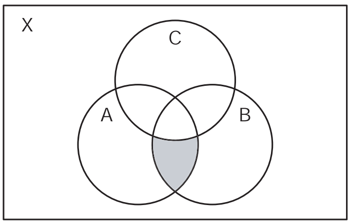

Fig. 7.2

Problem 7.16. Prove that $(A \setminus B) \cup (B \setminus C) \subseteq A \setminus C$ if and only if $A \cap C \subseteq A \cap B$ and $B \cap C^c \subseteq A$.

Problem 7.17. Prove or disprove: If $A \cup B = A \cup C$, then $B = C$.

Problem 7.18. Prove or give a counterexample for the following statement.
Let X be the universe and $A, B \subseteq X$. If $A \cap Y = B \cap Y$ for all $Y \subseteq X$, then $A = B$.

Problem 7.19. Prove that $A \cap (B^c \cap C^c) = \emptyset$ if and only if $A \subseteq B \cup C$.

Problem 7.20. Prove that $(A \cup B) \setminus (C \cup D) = (A \setminus (C \cup D)) \cup (B \setminus (C \cup D))$.